Novel representation formulae for discrete 2D autonomous systems

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What are discrete 2D systems?

- 2D systems ⇒ the trajectories evolve over two independent variables.
- Discrete ⇒ the independent variables take only integral values (*not* continuous real values). The indexing set is the 2D integer grid, $\mathbb{Z}^2$.
- Example: image/video processing; spatio-temporal signal processing that arises in seismology, radio telescopy, etc; spectrum sensing in cognitive radio systems, and many more.

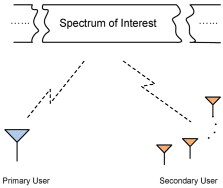

Figure : Figure courtesy: www.research.stevens.edu

Figure : Figure courtesy: www.skatelescope.org

- 2D systems also appear, in a slightly changed form, in repetitive systems, multi-agent systems (platoon of cars).

- We shall consider: Discrete 2D systems that are described by linear 2D partial difference equations with real constant coefficients.

Notation

- Trajectories are doubly indexed, scalar ($\mathbb{R}$) or vector ($\mathbb{R}^n$) valued sequences.
- We shall consider only scalar-valued sequences; the general vector has been dealt with in the paper.

$$\mathbb{R}^{\mathbb{Z}^2} := \{w : \mathbb{Z}^2 \to \mathbb{R}\}$$

- Difference equations are succinctly written using shift operators, σ_1, σ_2. For $w \in \mathbb{R}^{\mathbb{Z}^2}$, then σ_1, σ_2 act on w as

$$\sigma_1 w(\nu_1, \nu_2) = w(\nu_1 + 1, \nu_2) \qquad \sigma_2 w(\nu_1, \nu_2) = w(\nu_1, \nu_2 + 1)$$

- We denote by $\mathcal{A} := \mathbb{R}[\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}]$, the 2-variable Laurent polynomial ring.
- A Laurent polynomial $f(\sigma) = \sum_{\nu \in \mathbb{Z}^2} \alpha_\nu \sigma^\nu$ acts as

$$f(\sigma)w = \sum_{\nu \in \mathbb{Z}^2} \alpha_\nu \sigma^\nu w,$$

this is a finite sum.

Notation (contd.)

- A typical 2D difference equation is of the form $f(\sigma)w = 0$ for some $f(\sigma) \in \mathcal{A}$.

Example

$$w(h+2,k+1) + 10w(h+1,k+2) + 23w(h+1,k+1) - 5w(h,k) = 0$$

$$(\sigma_1^2\sigma_2 + 10\sigma_1\sigma_2^2 + 23\sigma_1\sigma_2 - 5)w = 0.$$

- Following Willems, we call the solution set of a given system of 2D difference equations the behavior, $\mathfrak{B}$.
- Discrete 2D scalar behaviors have the following description

$$\mathfrak{B}(f_1, f_2, \ldots, f_r) := \{w \in \mathbb{R}^{\mathbb{Z}^2} \mid f_1(\sigma)w = f_2(\sigma)w = \cdots = f_r(\sigma)w = 0\}.$$

What is a representation formula, and why is it needed?

- A representation formula spells out solutions to differential/difference equations in terms of initial/boundary conditions and free variables (inputs).

Example: 1D discrete systems

$$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k) + Du(k).$$

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^i Bu(k-1-i) + Du(k).$$

- A wealth of benefits entail such a formula:
  - Systems theoretic questions like stability, characteristic sets can be resolved.
  - Energy-like storage functions/Lyapunov functions may be constructed.
  - In 1D systems, ideas like controllability, observability crucially hinges on the above representation formula.

Unfortunately, such a representation formula for 2D systems has been largely missing!

- The representation formula for 1D systems stems from the first order state-space equation.
- Every 1D system, possibly higher order, can be brought into a first order form[1].
- Such a state-space for 2D systems that is analogous to 1D systems is not yet present!

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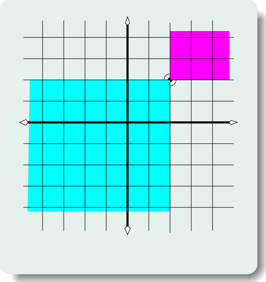
Fornasini and Marchesini[2] showed in 1976

- for $n = 2$,
- single input, single output,
- and systems having a transfer function, which is an input/output map,
- south-west causal,



$$x(h+1, k+1) = A_0 x(h, k) + A_1 x(h+1, k) + A_2 x(h, k+1) + Bu(h, k),$$

$$y(h, k) = Cx(h, k).$$

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- Consider the equation

$$w(h+1,k+1)-w(h+1,k)-w(h,k+1)+w(h,k)=0.$$

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  What if one more equation

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Multivariable polynomial rings are not Principal Ideal Domains (PIDs).

- When w is a function of space and time,

$$\sigma_t w(x,t) = Aw(x,t) + Bu(x,t)$$

  $w(\bullet, t) \in \mathscr{X}$ some suitable Hilbert space, and $A : \mathscr{X} \to \mathscr{X}$ is a linear map.
- Often $\mathscr{X}$ turns out to be infinite dimensional.
- A general higher order system cannot be brought into this form always!

Example: 2D system that cannot be brought into above form

$$(\sigma_t \sigma_x - \sigma_t - \sigma_x + 1)w = 0.$$

- Those which can be brought into the above form has been called time-relevant 2D systems[3].
- A more subtle issue: every time-relevant system may not provide a representation formula because that requires the operator A to be invertible.

[3] D. Napp, P. Rapisarda, and P. Rocha, "Time-relevant stability of 2D systems", Automatica, 47(11), pp 2373-2382, 2011.

- Given a behavior $\mathfrak{B}(f_1, f_2, \dots, f_r) := \{w \in \mathbb{R}^{\mathbb{Z}^2} \mid f_1(\sigma)w = f_2(\sigma)w = \cdots = f_r(\sigma)w = 0\}$. Define the following geometric object

Characteristic variety

$$\mathbb{V} := \{\xi \in \mathbb{C}^2 \setminus \{(0,0)\} \mid f_1(\xi) = f_2(\xi) = \cdots = f_r(\xi) = 0\}.$$

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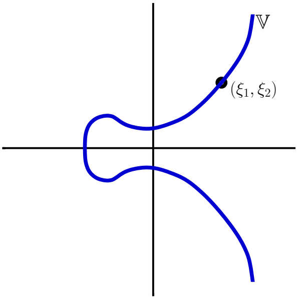

$$w(h,k) = \alpha \xi_1^h \xi_2^k.$$

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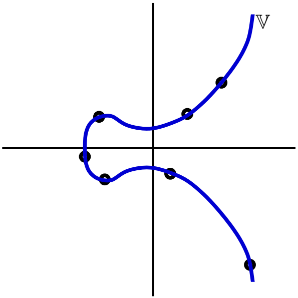

$$w(h,k) = \sum_{i=1}^{n} \alpha_i \xi_{i,1}^h \xi_{i,2}^k.$$

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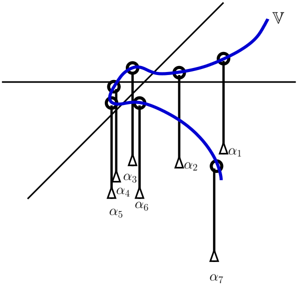

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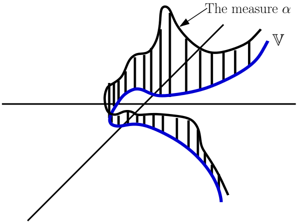

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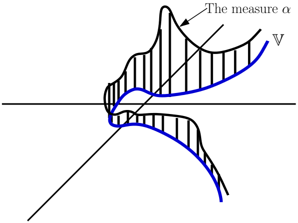

$$w(h,k) = \int_{\mathbb{V}} \xi_1^h \xi_2^k \mathrm{d}\alpha.$$

- Known as Ehrenpreis-Palamodov integral representation formula.
- Not so straightforward: requires full knowledge of the points of $\mathbb{V}$.
- Multiplicity is no longer just a number! It's a structure. Understood using Grothendieck's idea of affine schemes. This leads to Noetherian operators.

The algebra behind obtaining first order representations of 1D autonomous systems

Equation ideal

$$\mathfrak{a} := \{f(\sigma) \in \mathcal{A} \mid f(\sigma) = q_1(\sigma)f_1(\sigma) + \cdots + q_r(\sigma)f_r(\sigma)\} = \langle f_1(\sigma), \ldots, f_r(\sigma)\rangle.$$

$$\mathfrak{B}(\mathfrak{a}) := \{w \in \mathbb{R}^{\mathbb{Z}^n} \mid f(\sigma)w = 0 \text{ for all } f \in \mathfrak{a}\}.$$

- In a 1D system the equation ideal $\mathfrak{a}$ is always principal, that is, $\mathfrak{a} = \langle f(\sigma)\rangle$.
- Suppose $f(\sigma) = (\sigma)^n + a_{n-1}(\sigma)^{n-1} + \cdots + a_1\sigma + a_0$.
- Let $w \in \mathfrak{B}(\mathfrak{a})$. Define:

$$\mathbf{x} := \begin{bmatrix} w \\ \sigma w \\ \vdots \\ \sigma^{n-1} w \end{bmatrix}.$$

$$\mathbf{x}(k+1) = A\mathbf{x}(k), \quad y(k) = C\mathbf{x}(k),$$

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

- Euclidean division algorithm tells that the quotient ring $\mathcal{A}/\mathfrak{a}$ is a finite dimensional vector space over $\mathbb{R}$, generated by $\{1, \sigma, \dots, \sigma^{n-1}\}$, where $\deg(f(\sigma)) = n$.
- Multiplication by σ is an $\mathbb{R}$-linear map from $\mathcal{A}/\mathfrak{a}$ to itself. If the elements of $\mathcal{A}/\mathfrak{a}$ are written as row-vectors then this map is represented by right-multiplication by the companion matrix A.
- In the basis $\{\overline{1}, \overline{\sigma}, \dots, \overline{\sigma}^{n-1}\}$ we have

$$\sigma \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{n-1} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}}^{A} \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{n-1} \end{bmatrix}.$$

- As we have seen earlier, we get the representation formula

$$w \in \mathfrak{B}(\mathfrak{a}) \quad \Leftrightarrow \quad w(k) = CA^k \mathbf{x}(0) \text{ for all } k \in \mathbb{Z}.$$

A special case

- For scalar 2D systems having $\mathcal{A}/\mathfrak{a}$ to be a finite dimensional vector space over $\mathbb{R}$ it is known[4] [5] that there exist $A_1, A_2 \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{1\times n}$, such that

$$w \in \mathfrak{B}(\mathfrak{a}) \quad \Leftrightarrow \quad w(h,k) = CA_1^h A_2^k \mathbf{x}(0,0) \text{ for all } (h,k) \in \mathbb{Z}^2.$$

- These are called strongly autonomous. Corresponds to the case when $\mathbb{V}$ is a finite set.
- For the general case this is not true. But we can expect

[4] P. Rocha and J. C. Willems, "State for 2-D systems", *Linear Algebra Appl.*, 122/123/124, pp 10031038, 1989.

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- For the general case this is not true. But we can expect

> the quotient ring to be a finitely generated module over a smaller ring.

- Modules are like vector spaces, but scalars come from a ring instead of a field.

Example

$$\mathfrak{a} = \langle \sigma_2^2 - 2\sigma_2 + 1, \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1\rangle$$

$\mathcal{A}/\mathfrak{a}$ is a finitely generated module over $\mathbb{R}[\sigma_1, \sigma_1^{-1}]$

Non-example

$$\mathfrak{a} = \langle \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1\rangle$$

$\mathcal{A}/\mathfrak{a}$ is not a finitely generated module over $\mathbb{R}[\sigma_1, \sigma_1^{-1}]$ or $\mathbb{R}[\sigma_2, \sigma_2^{-1}]$

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What is the big deal about finitely generated modules

- Let $\mathcal{A}_1$ denote the Laurent polynomial ring in just σ_1:

$$\mathcal{A}_1 := \mathbb{R}[\sigma_1, \sigma_1^{-1}].$$

- Suppose $\mathcal{A}/\mathfrak{a}$ is a finitely generated module over $\mathcal{A}_1$.
- Fix a generating set, say $\{g_1, g_2, \dots, g_n\} \subseteq \mathcal{A}/\mathfrak{a}$ as an $\mathcal{A}_1$-module. We get a map:

$$\begin{array}{cccc} \psi : & \mathcal{A}_1^n & \twoheadrightarrow & \mathcal{A}/\mathfrak{a}. \\ & e_i & \mapsto & g_i \end{array}$$

- $\ker(\psi)$ is a submodule of the free module $\mathcal{A}_1^n$.
- Since $\mathcal{A}$ is Noetherian, $\ker(\psi)$ is also finitely generated. Let the rows of $X(\sigma_1) \in \mathcal{A}_1^{g\times n}$ generate $\ker(\psi)$.

$$\mathcal{A}_1^n/\text{rowspan}(X(\sigma_1)) \simeq \mathcal{A}/\mathfrak{a}.$$

- Consider the following map:

$$\begin{aligned} \mu : \mathcal{A}/\mathfrak{a} &\rightarrow \mathcal{A}/\mathfrak{a} \\ m &\mapsto \mu(m) := \sigma_2 m. \end{aligned}$$

This is a map of finitely generated $\mathcal{A}_1$-modules.

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$$\mu(g_i) = a_{i1}(\sigma_1)g_1 + a_{i2}(\sigma_1)g_2 + \cdots + a_{in}(\sigma_1)g_n,$$

where $a_{i1}(\sigma_1), a_{i2}(\sigma_1), \ldots, a_{in}(\sigma_1) \in \mathcal{A}_1$.

$$A(\sigma_1) = \begin{bmatrix} a_{11}(\sigma_1) & a_{12}(\sigma_1) & \cdots & a_{1n}(\sigma_1) \\ a_{21}(\sigma_1) & a_{22}(\sigma_1) & \cdots & a_{2n}(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\sigma_1) & a_{n2}(\sigma_1) & \cdots & a_{nn}(\sigma_1) \end{bmatrix}$$

$$\begin{array}{rcl} \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{A}/\mathfrak{a} \\ A(\sigma_1) \downarrow & & \downarrow \mu. \\ \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{A}/\mathfrak{a} \end{array}$$

Lemma

There always exists a generating set $\{g_1, g_2, \ldots, g_n\} \subseteq \mathcal{A}/\mathfrak{a}$ as an $\mathcal{A}_1$-module such that $A(\sigma_1)$ is invertible.

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for all $h \in \mathbb{Z}$.

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First order representation for the special case

- Let $\{g_1(\sigma), g_2(\sigma), \ldots, g_n(\sigma)\} \subseteq \mathcal{A}$ is such that $\left\{\overline{g_1(\sigma)}, \overline{g_2(\sigma)}, \ldots, \overline{g_n(\sigma)}\right\} \subseteq \mathcal{A}/\mathfrak{a}$ generate $\mathcal{A}/\mathfrak{a}$ as an $\mathcal{A}_1$-module.
- For $w \in \mathfrak{B}(\mathfrak{a})$, define

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} g_1(\sigma)w \\ g_2(\sigma)w \\ \vdots \\ g_n(\sigma)w \end{bmatrix} \in \left(\mathbb{R}^{\mathbb{Z}^2}\right)^n.$$

$$\sigma_2 \begin{bmatrix} \overline{g_1(\sigma)} \\ \overline{g_2(\sigma)} \\ \vdots \\ \overline{g_n(\sigma)} \end{bmatrix} = A(\sigma_1) \begin{bmatrix} \overline{g_1(\sigma)} \\ \overline{g_2(\sigma)} \\ \vdots \\ \overline{g_n(\sigma)} \end{bmatrix}.$$

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$$\sigma_2 \mathbf{x} = A(\sigma_1)\mathbf{x}.$$

$$X(\sigma_1)\mathbf{x} = 0.$$

$$\mathscr{X} := \left\{x \in \left(\mathbb{R}^{\mathbb{Z}}\right)^n \mid X(\sigma_1)x = 0\right\}.$$

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$$\mathbf{x} : \mathbb{Z} \to \mathscr{X}$$

- An 1D trajectory defined on $\mathscr{X}$.
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- Then define

$$\mathbf{x}(k+1) = A(\sigma_1)\mathbf{x}(k),$$

- Suppose $\overline{1} = c_1(\sigma_1)\overline{g_1(\sigma)} + c_2(\sigma_1)\overline{g_2(\sigma)} + \cdots + c_n(\sigma_1)\overline{g_n(\sigma)}$. Define

$$C(\sigma_1) := \begin{bmatrix} c_1(\sigma_1) & c_2(\sigma_1) & \cdots & c_n(\sigma_1) \end{bmatrix},$$

and

$$w = C(\sigma_1)\mathbf{x}$$

$$\Downarrow$$

$$w \in \mathfrak{B}(\mathfrak{a})$$

Representation formula for the special case

Theorem (special representation formula)

- Equation ideal is such that $\mathcal{A}/\mathfrak{a}$ is finitely generated as a module over $\mathcal{A}_1$.
- Define $\mathscr{X}$, $A(\sigma_1)$ and $C(\sigma_1)$ as before.

$w \in \mathfrak{B}(\mathfrak{a}) \quad \Leftrightarrow$ there exists $x \in \mathscr{X}$ such that for all $(h,k) \in \mathbb{Z}^2$,

$$w(h,k) = \Big(C(\sigma_1)A(\sigma_1)^k x\Big)(h).$$

Can we do it for a general ideal?

- The seed of the solution lies in a ninety years old work of Emmy Noether (1882-1935). Noether first presented this result in her 1926 paper, entitled "Der Endlichkeitsatz der Invarianten endlicher linearer Gruppen der Charakteristik p", and used it for studying invariant theory of finite groups over fields of arbitrary characteristics.
- This result, called Noether's Normalization Lemma is considered a touchstone in commutative algebra and algebraic geometry.
- The result says that any quotient ring of a polynomial ring over a field can be viewed as a finitely generated module over a polynomial subring (after a suitable "transformation").
- Unfortunately, the lemma is not directly applicable to the current situation. A rework was needed.

Figure : Emmy Noether

Coordinate change

- Coordinate change in the domain $\mathbb{Z}^2$ is represented by a unimodular matrix $T \in \mathbb{Z}^{2\times 2}$.
- T induces the following two maps:

$$\begin{array}{rcl} \Phi_T : \mathbb{R}^{\mathbb{Z}^2} & \to & \mathbb{R}^{\mathbb{Z}^2} \\ w & \mapsto & w \circ T. \end{array}$$

$$\begin{array}{rcl} & \mathbb{Z}^2 & \xrightarrow{T} & \mathbb{Z}^2 \\ \Phi_T(w) & & \searrow & \downarrow w \\ & & & \mathbb{R} \end{array}$$

commutes.

$$\begin{array}{rcl} \varphi_T : \mathcal{A} & \to & \mathcal{A} \\ \sigma_i & \mapsto & \sigma^{Te_i} \quad \text{for } 1 \leqslant i \leqslant 2. \end{array}$$

Automorphism of $\mathcal{A}$, ideals are mapped to ideals.

Theorem

- $\mathfrak{a} \subseteq \mathcal{A}$ is an ideal and $\mathfrak{B}(\mathfrak{a})$ its behavior.
- $T \in \mathbb{Z}^{2\times 2}$ is a coordinate change.

Then we have

$$\mathfrak{B}(\mathfrak{a}) = \Phi_T(\mathfrak{B}(\varphi_T(\mathfrak{a}))).$$

Theorem (Discrete Noether's Normalization Lemma)

- Given an ideal $\mathfrak{a} \subseteq \mathcal{A}$.

There exists $T \in \mathbb{Z}^{n \times n}$ a coordinate change such that

$\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1$.

Consequences of Noether's Normalization Lemma

Theorem (general representation formula)

- Given $\mathfrak{a} \subseteq \mathcal{A}$ an ideal and $\mathfrak{B}(\mathfrak{a})$ its behavior.

There exists a coordinate change $T \in \mathbb{Z}^{2\times 2}$ such that

$$\mathfrak{B}(\mathfrak{a}) = \Phi_T(\mathfrak{B}(\varphi_T(\mathfrak{a})))$$

where $\mathfrak{B}(\varphi_T(\mathfrak{a}))$ admits special representation formula.

Consequences of Noether's Normalization Lemma

Theorem (general representation formula)

- Given $\mathfrak{a} \subseteq \mathcal{A}$ an ideal and $\mathfrak{B}(\mathfrak{a})$ its behavior.

There exists a coordinate change $T \in \mathbb{Z}^{2\times 2}$ such that

$$\mathfrak{B}(\mathfrak{a}) = \Phi_T(\mathfrak{B}(\varphi_T(\mathfrak{a})))$$

where $\mathfrak{B}(\varphi_T(\mathfrak{a}))$ admits special representation formula.

In other words, there exist

- a positive integer n,
- a square $(n \times n)$ matrix $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$,
- a $(1 \times n)$ matrix $C(\sigma_1) \in \mathcal{A}_1^{1\times n}$,
- another matrix $X(\sigma_1) \in \mathcal{A}_1^{\bullet\times n}$,

such that

$$w \in \mathfrak{B}(\varphi_T(\mathfrak{a})) \quad \Leftrightarrow \quad \text{there exists } x \in \left(\mathbb{R}^{\mathbb{Z}}\right)^n \text{ satisfying } X(\sigma_1)x = 0, \text{ and}$$

$$w(h,k) = \left(C(\sigma_1)A(\sigma_1)^k x\right)(h).$$

An example

Consider the system

$$\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1).$$

Take the coordinate transformation $T = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$. Then the transformed equation turns out to be

$$\left(\sigma_2^3 - \sigma_2^2 - \sigma_1^{-1}\sigma_2 + \sigma_1^{-1}\right) v = 0.$$

With this we get

- $n = 3$,
- $X(\sigma_1) = 0$,
- $A(\sigma_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma_1^{-1} & \sigma_1^{-1} & 1 \end{bmatrix}$
- $C(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Note that $T \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} h \\ 2h+k \end{smallmatrix}\right]$. Hence, solutions in $\mathfrak{B}$ are given by

$$w(h,k) = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma_1^{-1} & \sigma_1^{-1} & 1 \end{bmatrix}^{2h+k} x \right)(h),$$

where $x \in (\mathbb{R}^3)^{\mathbb{Z}}$ is arbitrary.

Conclusion and future works

- In this paper, we looked into novel representation formulae for discrete 2D autonomous systems.
- These representation formulae generalize the solution formula for 1D autonomous systems given by a flow acting on initial conditions.
- The crucial difference in the 2D case is that here the initial conditions are given by 1D trajectories as opposed to real vectors in the 1D case.
- Moreover, instead of a constant matrix, here in the 2D case the flow operator is a 1-variable Laurent polynomial matrix.
- The techniques involved: manipulations on finitely generated modules, and discrete Noether's normalization lemma.
- The question of how to get minimal size of the 1-variable Laurent polynomial matrix $A(\partial_1)$, or algorithms for computing the matrix.
- The extension of the formulae to non-autonomous systems is also another important unresolved question.

Thank you