# OPTIMAL SINGULAR LQR PROBLEM: A PD FEEDBACK SOLUTION*

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**Abstract.** Unlike regular linear quadratic regulator (LQR) problems, singular LQR problems, in general, cannot be solved using a static state-feedback controller. This work is primarily focused on the design of feedback controllers which solve the singular LQR problem. We show that such problems can be solved using proportional-derivative (PD) state-feedback controllers. It is well known in the literature that the *maximal rank-minimizing* solution of the singular LQR linear matrix inequality (LMI) is pivotal in solving the singular LQR problem. In this paper, we first make use of this maximal rank-minimizing solution to compute the optimal trajectories. Then, we provide a PD feedback controller that restricts the trajectories of the closed-loop system to these optimal ones, and thus solves the singular LQR problem. While numerous solutions to this problem have been proposed over the course of the extensive research efforts in this field, a controller in the form of a PD state-feedback has been long sought after. Our approach is based on the notion of *weakly unobservable (slow)* and *strongly reachable (fast)* subspaces developed in [3]. But unlike [3], we employ these notions to the corresponding Hamiltonian system and not to the plant. This crucial extension of these well-known subspaces to the corresponding Hamiltonian system is key to the optimal PD feedback design that we propose in this paper. It is well-known that an optimal state feedback for the singular LQR problem does not exist; the limiting state feedback controller of the sub-optimal ones (high gain controllers) has unbounded coefficients as optimality is approached. We show in this paper that the limiting high gain controller is in fact a PD controller.

## 1. Introduction.

In this paper, we provide a closed-loop solution for the singular case of the well-known infinite-horizon linear quadratic regulator (LQR) problem.

**Problem 1.1. (Infinite-horizon LQR problem)** *Consider a stabilizable system with the state-space dynamics $\frac{d}{dt}x = Ax+Bu$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition $x_0$, find an input $u$ that minimizes the functional*

$$J(x_0,u) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \;\; \text{with} \;\; \lim_{t\to\infty} x(t) = 0, \tag{1.1}$$

*where* $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$, $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, *and* $R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$.

For *regular* LQR problems, i.e., LQR problems with $R > 0$, the input $u$ that minimizes $J(x_0,u)$ in equation (1.1) can be obtained using a static state-feedback constructed using the *maximal* solution of the algebraic Riccati equation (ARE):

$$A^TK + KA + Q - (KB+S)R^{-1}(B^TK+S^T) = 0. \tag{1.2}$$

Here, by a maximal solution $K_{\mathtt{max}}$, we mean that $K_{\mathtt{max}} - K \geqslant 0$ for any other arbitrary solution $K$ of the ARE. If $K_{\mathtt{max}}$ is the maximal solution of the ARE, then the LQR problem can be solved using the feedback law $u = Fx$, where $F := -R^{-1}(S^T + B^TK_{\mathtt{max}})$. Naturally, a singular LQR problem ($R \geqslant 0$ with $\mathtt{det}\, R = 0$) does not admit an ARE and cannot be solved using this feedback law due to singularity of $R$.

Singular LQR problem has been extensively studied over the past few decades (see, for example, the seminal paper [3]); but, a feedback solution that restricts the system to the optimal trajectories has remained largely elusive. Interestingly, [3] shows existence of a state-feedback controller for every regular relaxation of the problem, but, the limiting controller that is naively expected to work for the singular case fails to exist. Such controllers are known as high gain controllers, for their coefficients grow unbounded in the limit. A polynomial matrix based method for designing a

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PD feedback controller has been put forth in [4], but applicability of this result does not allow the initial condition to be free. It is also built on certain assumptions like controllability of $(A, B)$ and observability of $(Q, A)$. In [5], the notion of deflating subspaces has been used to provide a linear implicit control law of the form $Px + Tu = 0$. But, most often this form does not lead to a feedback law, essentially due to non-invertibility of $T$ (See [6] for the importance of feedback control). Another major drawback of this result is that it assumes the function space to be locally square-integrable. It is well known in the literature that the optimal trajectories for a singular LQR problem, in general, are impulsive in nature. Therefore, the local square-integrability of the signals is an extremely restrictive assumption, for the locally square-integrable functions cannot account for these impulses. Consideration of only square-integrable functions imposes a restriction on the initial condition of the system.

Yet another method of solving the singular LQR problem is via the solution of the constrained generalized continuous algebraic Riccati equations (CGCAREs) (see the recent papers [7], [8], [9]):

$$A^TK + KA + Q - (KB + S)R^{\dagger}(B^TK + S^T) = 0 \text{ and } \mathtt{ker}(R) \subseteq \mathtt{ker}(S + KB), \tag{1.3}$$

where $R^{\dagger}$ is the Moore-Penrose pseudo-inverse of $R$. However, it has been shown in [10] that solvability of CGCARE is equivalent to the corresponding Hamiltonian pencil satisfying a certain rank condition. Hence, CGCARE is generically unsolvable. Thus, in almost all cases of singular LQR problem, this method fails to provide a solution.

In this paper, we provide a method to design a proportional-derivative (PD) state-feedback controller that solves the singular LQR problem. While doing so, we do not put any restriction on the initial condition. Since the initial condition is arbitrary, the optimal trajectories, in general, are impulsive in nature. Hence, the function space assumed in this paper allows impulses.

The first step in computing the optimal solution is to compute the maximal rank-minimizing solution of the following LMI:

$$\mathcal{L}(K) := \begin{bmatrix} A^TK+KA+Q & KB+S \\ B^TK+S^T & R \end{bmatrix} \geqslant 0. \tag{1.4}$$

We call inequality (1.4) the *LQR LMI*. Interestingly, for *every* LQR problem, the optimal cost is given by $x_0^T K_{\mathtt{max}} x_0$, where $K_{\mathtt{max}}$ is the maximal rank-minimizing solution of the LQR LMI (1.4), that is, $K_{\mathtt{max}} - K \geqslant 0$ and $\mathtt{rank}\,\mathcal{L}(K_{\mathtt{max}}) \leqslant \mathtt{rank}\,\mathcal{L}(K)$ for all $K$ that satisfies $\mathcal{L}(K) \geqslant 0$ (see [11]). Hence, in order to compute the optimal cost of a general LQR problem, it is imperative that the maximal rank-minimizing solution of the LQR LMI (1.4) be computed. For regular LQR problems the maximal solution of the ARE given by equation (1.2) is, indeed, the maximal rank-minimizing solution ($K_{\mathtt{max}}$) of the LMI (1.4). For singular LQR problems, if the CGCARE is solvable then $K_{\mathtt{max}}$ can be found by obtaining the maximal solution of the CGCARE (1.3); but, as has been mentioned before, CGCARE is generically unsolvable. There are numerous methods to compute the maximal solution of an ARE: see [12] for different methods. However, these methods cannot be used in the singular case due to nonexistence of an ARE. In [2] we showed that one of the methods to compute $K_{\mathtt{max}}$ for an LQR LMI of the regular case can be extended to the singular case (see [13, Chapter 5] for the regular case). This method, for the regular case, is based on computing a suitable eigenspace of the corresponding *Hamiltonian system.* A direct extension of this method to the singular case fails, since the dimension of the suitable eigenspace of the Hamiltonian system in such a case is less than what is required to compute $K_{\mathtt{max}}$. It has been shown [2] that the Hamiltonian system based method for the regular case can indeed be extended to the singular case by substituting the role of the eigenspace of the Hamiltonian system in the regular case by the subspaces namely the *weakly* *unobservable subspace (slow space)* and the *strongly reachable subspace (fast space)* of the Hamiltonian system. This observation is crucially used for the development of our results. It is worthwhile to mention here that the idea of employing the notion of slow space of the Hamiltonian in the context of the singular LQR problem has also been used in [14], where the authors consider a special case of the problem, namely the *cheap* LQR problem (where $R = 0$).

The paper is structured as follows: Section 2 consists of the notation and a few preliminary results. The idea of weakly unobservable and strongly reachable subspaces have been known to be crucial in singular LQR problems (see [3], [15], [16], [17]). Matrix theoretic characterizations of the weakly unobservable and the strongly reachable subspaces have been provided in [1] and [18], respectively. These works also provide a method to compute the dimensions of these subspaces from the transfer function matrix of the primal. For the sake of completeness we present the results of [2], [1], and [18] in Section 3. In Section 4 we compute the optimal trajectories, while Section 5 provides a PD state-feedback controller that restricts the system to exhibit the optimal trajectories only. We provide an illustrative example in Section 7 to demonstrate the theory presented in this paper. A comparative analysis of this result with the existing results in the literature has been carried out in Section 8. Finally, Section 9 provides a few concluding remarks.

## 2. Notation and Preliminaries.

**2.1. Notation.** The symbols $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ are used for the sets of real numbers, complex numbers, and natural numbers, respectively. We use the symbols $\mathbb{R}_+$ and $\mathbb{C}_-$ for the sets of non-negative real numbers and complex numbers with negative real parts, respectively. The symbol $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}$. We use the symbol $I_{\mathtt{n}}$ for an $\mathtt{n}\times\mathtt{n}$ identity matrix and the symbol $0_{\mathtt{n},\mathtt{m}}$ for an $\mathtt{n}\times\mathtt{m}$ matrix with all entries zero. Symbol $\mathtt{col}(B_1, B_2, \ldots, B_{\mathtt{n}})$ represents a matrix of the form $\begin{bmatrix} B_1^T & B_2^T & \cdots & B_{\mathtt{n}}^T \end{bmatrix}^T$. By $\mathtt{im}\, A$ and $\mathtt{ker}\, A$ we denote the image and nullspace of a matrix $A$, respectively. The symbols $\mathtt{rank}\, A$ and $\mathtt{nullity}\, A$ denote the rank and the dimension of the nullspace of a matrix $A$, respectively. $\mathtt{det}(A)$ represents the determinant of a square matrix $A$. We use the symbols $\mathtt{deg}(p(s))$ and $\mathtt{roots}(p(s))$ to denote the degree and the set of roots (over complex numbers) of a polynomial $p(s)$ with real or complex coefficients (with a root $\lambda$ included in the set as many times as its multiplicity), respectively. The symbol $\mathtt{num}(p(s))$ is used to denote the numerator of a rational function $p(s)$. By $\mathtt{degdet}(A(s))$ we denote the degree of the determinant of a polynomial matrix $A(s)$ and by $\mathtt{numdet}(A(s))$ we denote the numerator of the determinant of a rational function matrix $A(s)$. The symbol $\sigma(A)$ denotes the set of eigenvalues of a square matrix $A$ (with an eigenvalue $\lambda$ included in the set as many times as its algebraic multiplicity). We use the symbol $\sigma(E, H)$ to denote the set of eigenvalues of the matrix pencil $(E, H)$ (with $\lambda \in \sigma(E, H)$ included in the set as many times as its algebraic multiplicity). The symbol $|\Gamma|$ denotes the cardinality of a set $\Gamma$ (counted with multiplicity). We use the symbol $\sigma(A|_{\mathcal{S}})$ to represent the set of eigenvalues of $A$ restricted to an $A$-invariant subspace $\mathcal{S}$. We use the symbol $\mathtt{dim}\,(\mathcal{S})$ to denote the dimension of a space $\mathcal{S}$. The space of all infinitely often differentiable functions and locally square-integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ are represented by the symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ and $\mathcal{L}^2_{\mathtt{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$, respectively. We use the symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})|_{\mathbb{R}_+}$ to represent the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}^{\mathtt{n}}$ that are restrictions of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$. The symbol $\delta$ represents the Dirac delta impulse distribution and $\delta^{(i)}$ represents the $i$-th distributional derivative of $\delta$ with respect to $t$.

**2.2. Weakly unobservable and strongly reachable subspaces.** Consider a system described by $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + Du$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$. Associated with such a system are two important subspaces called the weakly unobservable subspace and the strongly reachable subspace (see [3] for more on these spaces). Before we delve into the definitions of these subspaces, we need to define the space of impulsive-smooth distributions (see [3], [17]).

Definition 2.1. *The set of impulsive-smooth distributions* $\mathfrak{C}^{\mathtt{w}}_{\text{imp}}$ *is defined as:*

$$\mathfrak{C}^{\mathtt{w}}_{\text{imp}} := \left\{ f = f_{\mathtt{reg}} + f_{\mathtt{imp}} \mid f_{\mathtt{reg}} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})|_{\mathbb{R}_+} \text{ and } f_{\mathtt{imp}} = \sum_{i=0}^{k} a_i \delta^{(i)}, \text{ with } a_i \in \mathbb{R}^{\mathtt{w}}, k \in \mathbb{N} \right\}.$$

In what follows, we denote the state-trajectory $x$ and output-trajectory $y$ of the system, that result from initial condition $x_0$ and input $u$, using the symbols $x(t; x_0, u)$ and $y(t; x_0, u)$, respectively. $x(0^+; x_0, u)$ denotes the value of the state-trajectory that can be reached from $x_0$ instantaneously on application of the input $u$ at $t = 0$.

Definition 2.2. *A state* $x_0 \in \mathbb{R}^{\mathtt{n}}$ *is called* weakly unobservable *if there exists an input* $u \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}$ *such that* $y(t; x_0, u) \equiv 0$ *for all* $t \geqslant 0$*. The collection of all such weakly unobservable states is called the* weakly unobservable subspace *of the system and is denoted by* $\mathcal{O}_w$.

The other space of interest is the space of strongly reachable states (see [3]).

Definition 2.3. *A state* $x_1 \in \mathbb{R}^{\mathtt{n}}$ *is called* strongly reachable *(from the origin) if there exists an input* $u \in \mathfrak{C}^{\mathtt{m}}_{\text{imp}}$ *such that* $x(0^+; 0, u) = x_1$ *and* $y(t; 0, u) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{p}})|_{\mathbb{R}_+}$ *(that is, the output is regular). The collection of all such strongly reachable states is called the* strongly reachable subspace *of the state-space and is denoted by* $\mathcal{R}_s$.

Since $\mathcal{O}_w$ deals with inputs from the space of infinitely differentiable functions, we call $\mathcal{O}_w$ the *slow space* of the system. On the other hand, since the space $\mathcal{R}_s$ admits impulsive inputs, we call $\mathcal{R}_s$ the *fast space* of the system. Further, by [3, Theorem 3.10] we know that $\mathcal{O}_w$ is the largest among the subspaces $\mathcal{V}$ for which there exists an $F_{\mathcal{V}} \in \mathbb{R}^{\mathtt{m} \times \mathtt{n}}$ such that

$$(A + BF_{\mathcal{V}})\mathcal{V} \subseteq \mathcal{V} \text{ and } (C + DF_{\mathcal{V}})\mathcal{V} = \{0\}. \tag{2.1}$$

In other words, there exists $F_{\mathcal{O}_w} \in \mathbb{R}^{\mathtt{m} \times \mathtt{n}}$ such that $\mathcal{O}_w$ satisfies the above equation; and for any arbitrary subspace $\mathcal{V}$ that satisfies the above equation, we must have that $\mathcal{V} \subseteq \mathcal{O}_w$. Note that, the class of subspaces that satisfy equation (2.1) also admits a subspace $\mathcal{O}_{wg}$ such that $\sigma((A + BF_{\mathcal{O}_{wg}})|_{\mathcal{O}_{wg}}) \subseteq \mathbb{C}_-$; and $\mathcal{V} \subseteq \mathcal{O}_{wg}$ whenever $\sigma((A + BF_{\mathcal{V}})|_{\mathcal{V}}) \subseteq \mathbb{C}_-$. (see [19, Chapter 4, Chapter 5] for more on this). We call such a space the *good slow space* of the system as defined below (see [20, Chapter 3]).

Definition 2.4. *The good slow space* $\mathcal{O}_{wg}$ *is the largest subspace* $\mathcal{V}$ *of the state-space for which there exists a feedback* $F_{\mathcal{V}} \in \mathbb{R}^{\mathtt{m} \times \mathtt{n}}$ *such that*

$$(A + BF_{\mathcal{V}})\mathcal{V} \subseteq \mathcal{V}, \; (C + DF_{\mathcal{V}})\mathcal{V} = \{0\}, \; \text{and } \sigma((A + BF_{\mathcal{V}})|_{\mathcal{V}}) \subseteq \mathbb{C}_-.$$

**2.3. Alternative formulation of the singular LQR problem.** Recall from Problem 1.1 that $R \geqslant 0$. Therefore, there exists an orthogonal matrix $U \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}$ such that $U^T R U = \mathtt{diag}(0, \widehat{R})$, where $\widehat{R} \in \mathbb{R}^{\mathtt{r} \times \mathtt{r}}$ and $\mathtt{r} := \mathtt{rank}\, R$. Notice that $\widehat{R} > 0$. This transformation enables us to provide an alternative formulation of the singular LQR Problem 1.1, which separates the regular part from the singular part of the problem. The following lemma is crucial for this purpose.

Lemma 2.5. *Consider the singular LQR Problem 1.1, where* $\mathtt{rank}\, R = \mathtt{r}$*. Let* $U \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}$ *be an orthogonal matrix such that* $U^T R U = \mathtt{diag}(0, \widehat{R})$*, where* $\widehat{R} \in \mathbb{R}^{\mathtt{r} \times \mathtt{r}}$ *and* $\widehat{R} > 0$*. Define* $BU =: \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ *and* $SU =: \begin{bmatrix} S_1 & S_2 \end{bmatrix}$*, where* $B_2, S_2 \in \mathbb{R}^{\mathtt{n} \times \mathtt{r}}$*. Then, the following statements hold:*

1. $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right] \geqslant 0$ *if and only if* $S_1 = 0$, $Q - S_2 \widehat{R}^{-1} S_2^T \geqslant 0$.

2. $u^*$ *is a solution to the singular LQR Problem 1.1 if and only if* $U^T u^* := \texttt{col}(u_1^*, u_2^*)$ *minimizes*

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix} dt. \tag{2.2}$$

3. $K = K^T$ *satisfies* $\mathcal{L}(K) \geqslant 0$ *(equation 1.4) if and only if* $K$ *satisfies the LMI:*

$$\mathcal{L}_{\mathtt{t}}(K) := \begin{bmatrix} A^T K + KA + Q & KB_1 & KB_2 + S_2 \\ B_1^T K & 0 & 0 \\ B_2^T K + S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0. \tag{2.3}$$

4. $K_{\mathtt{max}}$ *is the maximal rank-minimizing solution of the LQR LMI (1.4) if and only if* $K_{\mathtt{max}}$ *is the maximal rank-minimizing solution of the LMI (2.3).*

*Proof* Statement 1 and Statement 2 follow directly from [10, Lemma 2.1].
3. Define the orthogonal matrix $\widehat{U} := \texttt{diag}(I_{\mathtt{n}}, U)$. From the assumptions and Statement 1 of this lemma, it can be verified that $\widehat{U}^T \mathcal{L}(K) \widehat{U} = \mathcal{L}_{\mathtt{t}}(K)$. Thus $\mathcal{L}(K) \geqslant 0$ if and only if $\mathcal{L}_{\mathtt{t}}(K) \geqslant 0$. This proves Statement 3.
4. $\widehat{U}^T \mathcal{L}(K) \widehat{U} = \mathcal{L}_{\mathtt{t}}(K) \Rightarrow \texttt{rank}\, \mathcal{L}(K) = \texttt{rank}\, \mathcal{L}_{\mathtt{t}}(K)$. Also, from Statement 3 of this lemma we know that the solution sets of the LMIs $\mathcal{L}(K) \geqslant 0$ and $\mathcal{L}_{\mathtt{t}}(K) \geqslant 0$ are equal. Thus, $K_{\mathtt{max}}$ is the maximal rank-minimizing solution of the LQR LMI (1.4) if and only if $K_{\mathtt{max}}$ is the maximal rank-minimizing solution of the LMI (2.3). □

Notice that the LMI (2.3) is the LQR LMI corresponding to the singular LQR problem that minimizes the objective function given by equation (2.2). Therefore, Lemma 2.5 allows us to write any singular LQR problem as follows:

**Problem 2.6.** *Let* $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$, *and* $\widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$ *be such that* $\widehat{R} > 0$ *and* $\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0_{\mathtt{d},\mathtt{d}} & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0$, *where* $\mathtt{d} := \mathtt{m} - \mathtt{r}$. *Consider a stabilizable system with state-space dynamics* $\frac{d}{dt}x = Ax + B_1 u_1 + B_2 u_2$, *where* $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{d}}$, *and* $B_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. *Then, for every initial condition* $x_0$, *find an input* $u := \texttt{col}(u_1, u_2)$ *such that* $\lim_{t\to\infty} x(t) = 0$ *and* $u$ *minimizes the functional* (2.2).

This reduction of the original singular LQR problem (Problem 1.1) to its equivalent Problem 2.6 plays a crucial role in the sequel, where we exploit the special structure of the matrices involved in Problem 2.6 to obtain the main results.

**2.4. The primal and the Hamiltonian.** Suppose $\mathtt{p} := \texttt{rank} \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix}$. This matrix being positive semi-definite, admits a factorization given by $\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} = \begin{bmatrix} C & 0 & D_2 \end{bmatrix}^T \begin{bmatrix} C & 0 & D_2 \end{bmatrix}$, where $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, and $D_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{r}}$. Using this factorization in equation (2.2), it can be easily seen that the singular LQR Problem 2.6 can be viewed as an output energy minimization problem of the system $\Sigma$ defined as follows:

$$\Sigma: \quad \frac{d}{dt}x = Ax + B_1 u_1 + B_2 u_2 \text{ and } y = Cx + D_2 u_2. \tag{2.4}$$

We call the system $\Sigma$ the *primal* for the given singular LQR Problem 2.6.

Remark 2.7. The optimal trajectories for the singular LQR problem are impulsive. Therefore, in this paper we consider the trajectory space $\mathfrak{C}^{\mathtt{m}}_{\text{imp}}$ (see Definition 2.1) which allows impulses in trajectories. By equation (2.2) it can be inferred that in order for the objective function to be well-defined, the output $y(t)$ of the primal must be regular. Hence, while searching for an optimal input from the space $\mathfrak{C}^{\mathtt{m}}_{\text{imp}}$, it suffices to restrict our search to the inputs which cause the output $y(t)$ to be regular. We call such inputs the *admissible inputs*. □

By Pontryagin's maximum principle, all the smooth optimal trajectories of Problem 2.6 must necessarily be a trajectory of the following singular descriptor system:

$$(2.5)\qquad \underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt}\begin{bmatrix} x \\ z \\ u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B_1 & B_2 \\ -Q & -A^T & 0 & -S_2 \\ 0 & B_1^T & 0 & 0 \\ S_2^T & B_2^T & 0 & \widehat{R} \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u_1 \\ u_2 \end{bmatrix},$$

where $\mathtt{col}(x,z)$ is the state-costate pair. The system described by equation (2.5) is known in the literature as the *Hamiltonian* system corresponding to the LQR Problem 2.6 and the matrix pair $(E,H)$ is known as the Hamiltonian matrix pair. The Hamiltonian system admits an output-nulling representation given by

$$(2.6)\qquad \tfrac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \widehat{A}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{B}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } 0 = \widehat{C}\begin{bmatrix} x \\ z \end{bmatrix} + \widehat{D}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\text{where}\quad \widehat{A} := \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix}, \widehat{B} := \begin{bmatrix} B_1 & B_2 \\ 0 & -S_2 \end{bmatrix}, \widehat{C} := \begin{bmatrix} 0 & B_1^T \\ S_2^T & B_2^T \end{bmatrix}, \text{ and } \widehat{D} := \begin{bmatrix} 0 & 0 \\ 0 & \widehat{R} \end{bmatrix}.$$

In this paper we show that not only the smooth optimal trajectories, but also the distributional ones must necessarily satisfy the Hamiltonian system's equation.

Due to non-singularity of $\widehat{R}$, we can further reduce the Hamiltonian system to obtain an equivalent system described by the following differential algebraic equations:

$$(2.7)\qquad \underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_r} \frac{d}{dt}\begin{bmatrix} x \\ z \\ u_1 \end{bmatrix} = \underbrace{\begin{bmatrix} A - B_2\widehat{R}^{-1}S_2^T & -B_2\widehat{R}^{-1}B_2^T & B_1 \\ -Q + S_2\widehat{R}^{-1}S_2^T & -(A - B_2\widehat{R}^{-1}S_2^T)^T & 0 \\ 0 & B_1^T & 0 \end{bmatrix}}_{H_r} \begin{bmatrix} x \\ z \\ u_1 \end{bmatrix}.$$

We call the system described by equation (2.7), the *reduced Hamiltonian system*, and the pair $(E_r, H_r)$ the *reduced Hamiltonian matrix pair*. The reduced Hamiltonian system admits an output-nulling representation $\Sigma_{\text{Ham}}$ as follows:

$$(2.8)\qquad \frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A_r & -L \\ -Q_r & -A_r^T \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} u_1 \text{ and } 0 = \begin{bmatrix} 0 & B_r^T \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix},$$

where $A_r := A - B_2\widehat{R}^{-1}S_2^T$, $Q_r := Q - S_2\widehat{R}^{-1}S_2^T$, $L := B_2\widehat{R}^{-1}B_2^T$, and $B_r := B_1$. The reduced Hamiltonian system and the Hamiltonian system are equivalent in the sense that $\mathtt{col}(x,z,u_1)$ is a trajectory of the reduced Hamiltonian system if and only if $\mathtt{col}(x,z,u_1,-\widehat{R}^{-1}(S_2^Tx + B_2^Tz))$ is a trajectory of the Hamiltonian system. But, it is easier to carry out the analysis using the reduced Hamiltonian system. We characterize the slow space and the fast space in terms of the reduced Hamiltonian system, which finally leads to the maximal rank-minimizing solution of the LQR LMI.

The following lemma establishes a few important relations between the primal and the Hamiltonian (see [21, Lemma 4.4]).

LEMMA 2.8. *Consider the primal $\Sigma$, the Hamiltonian matrix pair $(E,H)$, the reduced Hamiltonian matrix pair $(E_r,H_r)$, and the matrices $\widehat{A},\widehat{B},\widehat{C},\widehat{D}$ defined in equation (2.4), equation (2.5), equation (2.7), and equation (2.6), respectively. Define $G(s) := C(sI_{\mathtt{n}} - A)^{-1}\begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} 0 & D_2 \end{bmatrix}$. Then the following statements hold:*

1. *$G(-s)^TG(s) = \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} + \widehat{D}$.*
2. *$\mathtt{numdet}\left(G(-s)^TG(s)\right)$[1]$= \mathtt{det}(sE - H) = (-1)^{\mathtt{r}}\mathtt{det}\,\widehat{R} \times \mathtt{det}(sE_r - H_r)$.*

[1]Here by $\mathtt{numdet}\left(G(-s)^TG(s)\right)$, we mean the numerator of $\mathtt{det}\left(G(-s)^TG(s)\right)$ before any possible pole-zero cancellations.

REMARK 2.9. Throughout this paper, we assume that (i) $(sE_r - H_r)$ is a regular matrix pencil, that is, $\mathtt{det}(sE_r - H_r) \not\equiv 0$; and (ii) $\sigma(E_r, H_r) \cap j\mathbb{R} = \phi$. The assumption that $\mathtt{det}\,(sE_r - H_r) \not\equiv 0$ is a standard assumption in the literature. It means that the Hamiltonian system is autonomous and ensures that, for a given initial condition, the optimal trajectory is unique. It has been shown in [10] that for singular LQR problems, the condition $\mathtt{det}\,(sE_r - H_r) \not\equiv 0$ is generically satisfied. Therefore, this assumption is not restrictive. From Statement 2 of Lemma 2.8, it follows that the condition $\mathtt{det}\,(sE_r - H_r) \not\equiv 0$ is equivalent to the transfer function matrix $G(s)$ of the primal $\Sigma$ being left-invertible. So, in terms of the primal $\Sigma$, this assumption translates to the primal $\Sigma$ being a left-invertible system (see [3, Theorem 3.26]). See [22] for the case when the primal is not a left-invertible system.

Since the primal $\Sigma$ is assumed to be stabilizable, from Statement 2 of Lemma 2.8, it follows that the assumption $\sigma(E_r, H_r) \cap j\mathbb{R} = \phi$ is equivalent to saying that: (a) the primal $\Sigma$ does not have any unobservable eigenvalue on the imaginary axis, and (b) the primal has no transmission zeros on the imaginary axis. Note that, this assumption, too, is not restrictive, because the property that a polynomial has no root on the imaginary axis is generically satisfied. This assumption also is a standard assumption in the literature (see [14], [17]).

Due to Statement 2 of Lemma 2.8 we further infer that if $\lambda$ is a root of $\mathtt{det}(sE_r - H_r)$ (that is, $\lambda \in \sigma(E_r, H_r)$), then $-\lambda$, too, is a root of the same. Of course, the roots also appear in complex conjugate pairs. Therefore, the roots are symmetric about the origin. Consequently, $\mathtt{det}(sE_r - H_r)$ is an even-degree polynomial. Hence, for a singular LQR problem $\mathtt{degdet}(sE_r - H_r) =: 2\mathtt{n_s}$, where $\mathtt{n_s} < \mathtt{n}$ (because $\widehat{D}$ is singular). Hence, the assumption that $\sigma(E_r, H_r) \cap j\mathbb{R} = \phi$ further implies that $|\sigma(E_r, H_r) \cap \mathbb{C}_-| = \mathtt{n_s}$. □

For a quick reference, in Table 1 we have listed some matrices and numbers that have been frequently used throughout this paper.

| **Matrix/Number** | **Definition** | **Remark** |
|---|---|---|
| $A_r$ | $A_r := A - B_2\widehat{R}^{-1}S_2^T$ | Defined in equation (2.8). |
| $B_r$ | $B_r := B_1$ | |
| $L$ | $L := B_2\widehat{R}^{-1}B_2^T$ | |
| $Q_r$ | $Q_r := Q - S_2\widehat{R}^{-1}S_2^T$ | |
| $C_r$ | $C_r := C - D_2\widehat{R}^{-1}S_2^T$ | Defined in Lemma 3.2. Notice that $C_r^TC_r = Q_r$. |
| $\mathtt{r}$ and $\mathtt{d}$ | $\mathtt{r} := \mathtt{rank}\,R$ and $\mathtt{d} := \mathtt{nullity}\,R$ | Notice that $\mathtt{d} = \mathtt{m} - \mathtt{r}$. |
| $E_r$ | $E_r := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{d,d}} \end{bmatrix}$ | $(E_r, H_r)$ is the reduced Hamiltonian matrix pair defined in equation (2.7). |
| $H_r$ | $H_r := \begin{bmatrix} A_r & -L & B_r \\ -Q_r & -A_r^T & 0 \\ 0 & B_r^T & 0 \end{bmatrix}$ | |
| $\mathtt{n_s}$ and $\mathtt{n_f}$ | $2\mathtt{n_s} := \mathtt{degdet}(sE_r - H_r)$ and $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$ | See Remark 2.9 and Lemma 3.2. |

TABLE 1
*Definitions of some matrices and numbers for a quick reference*

## 3. Constructive solution of the singular LQR LMI.

In this section we first provide a characterization of the good slow space of the Hamiltonian system. Then, we present a characterization of the fast space of the primal. We also depict how to get the dimensions of these spaces from the transfer function matrix of the primal.

Finally, we construct the maximal rank-minimizing solution of the LQR LMI 2.3 using these subspaces. These results have already appeared in [2], [1], and [18]. They are being presented here for completeness and ease of referencing in the main results.

**3.1. Characterization of the good slow space of the Hamiltonian system.** The good slow space ($\mathcal{O}_{wg}$) of the Hamiltonian system provides us with the subspace of the state-space, which contains all the initial conditions that result in smooth optimal trajectories for the given singular LQR problem (see Lemma 4.1). In the following lemma we present a characterization of $\mathcal{O}_{wg}$ (see [1, Section 3]).

LEMMA 3.1. *Consider the reduced Hamiltonian matrix pair $(E_r, H_r)$ corresponding to the singular LQR Problem 2.6 as defined in equation (2.7). Assume that $\sigma(E_r, H_r) \cap j\mathbb{R} = \emptyset$. Define $\texttt{degdet}(sE_r - H_r) =: 2\texttt{n}_\texttt{s}$ and $\Lambda := \sigma(E_r, H_r) \cap \mathbb{C}_-$ (recall from Remark 2.9 that $|\Lambda| = \texttt{n}_\texttt{s}$). Let $V_{1\Lambda}$, $V_{2\Lambda} \in \mathbb{R}^{\texttt{n}\times\texttt{n}_\texttt{s}}$ and $V_{3\Lambda} \in \mathbb{R}^{\texttt{d}\times\texttt{n}_\texttt{s}}$ be such that the matrix $\texttt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ is full column-rank and the following holds*[2]*:*

$$\begin{bmatrix} A_r & -L & B_r \\ -Q_r & -A_r^T & 0 \\ 0 & B_r^T & 0 \end{bmatrix}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix} = \begin{bmatrix} I_\texttt{n} & 0 & 0 \\ 0 & I_\texttt{n} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix}\Gamma, \tag{3.1}$$

*where $\sigma(\Gamma) = \Lambda$. Then, the following are true:*

1. *The good slow space of $\Sigma_{\text{Ham}} =: \mathcal{O}_{wg} = \texttt{im}\left[\begin{smallmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{smallmatrix}\right]$.*
2. *$\left[\begin{smallmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{smallmatrix}\right]$ is full column-rank; that is, $\texttt{dim}(\mathcal{O}_{wg}) = \texttt{n}_\texttt{s}$.*
3. *$V_{1\Lambda}$ is full column-rank.*

Statement 3 of Lemma 3.1 gives us an important structural property of the good slow space of the Hamiltonian system. This property is known as *disconjugacy* of the eigenspace of the matrix pair $(E_r, H_r)$ (see [13, Definition 6.1.5]). Columns of the matrix $V_{1\Lambda}$ constitute a basis of a special subspace of the state space. Any initial condition from this subspace results in a smooth optimal trajectory. Moreover, left-invertibility of $V_{1\Lambda}$ plays a crucial role in providing a closed-form expression of the maximal rank-minimizing solution of the singular LQR LMI; it is also pivotal to the design of a PD state-feedback controller.

**3.2. Characterization of the fast space of the primal.** The following lemma presents a closed-form expression for the fast space of the primal ([2, Proposition 3.2], also see [18] for more details). It also enables us to read off the dimension of the fast space from the transfer function matrix of the system.

LEMMA 3.2. *Consider the primal $\Sigma$ and the matrices $A_r, B_r$ as defined in equation (2.4) and equation (2.8), respectively. Define $C_r := C - D_2\widehat{R}^{-1}S_2^T$. Recall that $2\texttt{n}_\texttt{s} = \texttt{deg}\{\texttt{numdet}\, G(-s)^T G(s)\}$, where $G(s)$ is the transfer function matrix of $\Sigma$ and $\texttt{d} = \texttt{nullity}\, R$. Let $\mathcal{R}_s$ denote the fast space of $\Sigma$. Define*

$$\mathcal{M} := \begin{cases} 0_{\texttt{p},\texttt{d}} & \textit{if } \texttt{n}_\texttt{f} = \texttt{d} \\ \begin{bmatrix} 0_{\texttt{p},\texttt{d}} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & C_rB_r \\ 0 & 0 & \dots & C_rB_r & C_rA_rB_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & C_rB_r & \dots & C_rA_r^{\texttt{n}_\texttt{f}-\texttt{d}-2}B_r & C_rA_r^{\texttt{n}_\texttt{f}-\texttt{d}-1}B_r \end{bmatrix} & \textit{if } \texttt{n}_\texttt{f} \geqslant \texttt{d}+1. \end{cases}$$

*Then, the following are true:*

1. *$\texttt{dim}(\texttt{ker}\,\mathcal{M}) = \texttt{n}_\texttt{f}$, where $\texttt{n}_\texttt{f} := \texttt{n} - \texttt{n}_\texttt{s}$.*
2. *$\texttt{dim}\,\mathcal{R}_s = \texttt{n}_\texttt{f}$.*

[2]Such matrices $V_{1\Lambda}, V_{2\Lambda}$, and $V_{3\Lambda}$ always exist. See [1, Section 3.2] for more on this.

3. *Let* $N \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d}+1)\times \mathtt{n_f}}$ *be a matrix such that its columns form a basis for* $\ker \mathcal{M}$. *Define*

$$W := \begin{bmatrix} B_r & A_rB_r & \dots & A_r^{\mathtt{n_f}-\mathtt{d}}B_r \end{bmatrix} N. \tag{3.2}$$

*Then,* $\mathcal{R}_s = \mathrm{im}\, W$.

4. $W$ *is full column-rank, that is, the columns of* $W$ *form a basis for* $\mathcal{R}_s$.

We call $\mathcal{M}$ the *Markov parameter matrix.* It is evident from Lemma 3.2 that $\mathcal{M}$ plays a vital role in providing a closed-form expression of the fast space of the primal. It also plays a crucial role in computation of the optimal trajectories and also in the design of the PD feedback controller.

### 3.3. The maximal rank-minimizing solution of the singular LQR LMI.

The slow space of the Hamiltonian system and the fast space of the primal are intimately related to the maximal rank-minimizing solution $K_{\mathtt{max}}$ of the LQR LMI. The following theorem provides a closed-form expression for $K_{\mathtt{max}}$ by making use of these spaces. See [2, Section IV] for more details.

THEOREM 3.3. *Consider the LQR Problem 2.6 with the corresponding LMI given by equation (2.3). Recall from Lemma 3.1 that the good slow space of the Hamiltonian system* $\Sigma_{\mathrm{Ham}}$ *is given by* $\mathcal{O}_{wg} = \mathrm{im} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}$. *Further recall from Lemma 3.2 that the fast space of the primal* $\Sigma$ *is given by* $\mathcal{R}_s = \mathrm{im}\, W$. *Define* $\begin{bmatrix} V_{1\Lambda} & W \\ V_{2\Lambda} & 0 \end{bmatrix} =: \begin{bmatrix} X \\ Y \end{bmatrix}$, *where* $X, Y \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. *Then, the following statements hold:*

1. $X$ *is invertible.*
2. $K_{\mathtt{max}} := YX^{-1}$ *is symmetric.*
3. $K_{\mathtt{max}}$ *is a rank-minimizing solution of LMI (2.3).*
4. *For any other solution* $K$ *of LMI (2.3),* $K \leqslant K_{\mathtt{max}}$.
5. $K_{\mathtt{max}} \geqslant 0$.

REMARK 3.4. For the regular LQR problem, the relation between a rank minimizing solution of the LQR LMI and its corresponding ARE is a well-known fact [23, Theorem 4.3.1]. For a regular problem, the maximal rank-minimizing solution of the corresponding LMI can be found using the algorithm provided in the seminal paper [24]. Note that, for a regular LQR problem, $\mathtt{n_s} = \mathtt{n}$; and hence by Lemma 3.1, it follows that $V_{1\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is invertible. Further, for such a problem the fast space of the primal, $\mathcal{R}_s = \{0\}$. Thus, by Theorem 3.3, it follows that $K_{\mathtt{max}} = V_{2\Lambda}V_{1\Lambda}^{-1}$; which is in agreement with [24]. So, the algorithm for computation of the maximal rank-minimizing solution of the regular LQR LMI as given in [24] is a special case of Theorem 3.3. However, in this paper Theorem 3.3 provides a recipe to compute the maximal rank minimizing solution of the LQR LMI, both for the regular and the singular case. This eventually leads to a solution of the singular LQR problem. Interestingly, [23] uses special co-ordinate basis (SCB) to show that for the singular LQR case, the rank minimizing solution of the LQR LMI admits a special structure [23, Equation 4.3.20]. Hence, a natural question would be to investigate if the bases of the fast and the slow spaces admit some structure when the primal system is in SCB to start with. Thus, a study on the relation between fast/slow spaces and the SCB might provide valuable insights into the singular LQR problem and its solutions. We do not delve into such a study in this paper, as our primary focus in this paper is the design of a PD state-feedback controller, using the maximal rank minimizing solution of the singular LQR LMI, that solves the singular LQR problem. □

In the following remark we discuss about a certain observation regarding the kernel of $K_{\mathtt{max}}$ and its implication.

REMARK 3.5. In [23, Lemma 4.3.4] it has been shown that an arbitrary solution $K$ of the LQR LMI contains a certain subspace of the state space of the primal inside its kernel (which the authors in [23] call the *detectable strongly controllable subspace*). From Theorem 3.3, we know that $K_{\mathtt{max}} = \begin{bmatrix} V_{2\Lambda} & 0\end{bmatrix}\begin{bmatrix} V_{1\Lambda} & W\end{bmatrix}^{-1}$. From [2, Remark 2.11 and Lemma 2.12], it follows that, without loss of generality, $\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$ can be written as $\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix} = \begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0 & V_{2\mathtt{e}}\end{bmatrix}$, where the columns of the matrix $V_{\mathtt{g}}$ form a basis for the good slow space of the primal and $V_{2\mathtt{e}}$ is full column-rank. Hence, $K_{\mathtt{max}} = \begin{bmatrix} 0 & V_{2\mathtt{e}} & 0\end{bmatrix}\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W\end{bmatrix}^{-1}$. So, $K_{\mathtt{max}}V_{\mathtt{g}} = 0$ and $K_{\mathtt{max}}W = 0$. Also, since $V_{2\mathtt{e}}$ is full column-rank, the kernel of $K_{\mathtt{max}}$ is exactly equal to the direct-sum of the good slow space and the fast space of the primal. This observation gives rise to an interesting conclusion: since, for a given initial condition, the optimal cost of the singular LQR problem is given by $x_0^T K_{\mathtt{max}} x_0$ (see [11, Theorem 2]), any initial condition belonging to the direct-sum of the good slow space and the fast space of the primal incurs zero optimal cost. □

An auxiliary result pertaining to any arbitrary solution $K$ of the LQR LMI (2.3) is required in the sequel. We present this result as a lemma next (see [2, Lemma 4.1]).

LEMMA 3.6. *Let $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be an arbitrary solution of the LQR LMI* (2.3). *Then, $KW = 0$, where $W$ is as defined in equation* (3.2).

REMARK 3.7. Lemma 3.6 shows that the fast space ($\mathcal{R}_s$) of the primal is a subspace of the kernel of any solution $K$ of the LQR LMI (2.3). So, in particular, $\mathcal{R}_s$ is a subspace of $\mathtt{ker}\, K_{\mathtt{max}}$. Hence, for an initial condition from $\mathtt{im}\, W$, the optimal cost must be zero. This conclusion has also been drawn in Remark 3.5. □

**3.4. A few auxiliary results.** The structure of the matrix $\mathcal{M}$ leads to subspaces that follow a chain of inclusions elaborated in the following lemma.

LEMMA 3.8. *Consider the matrix $\mathcal{M}$ as defined in Lemma 3.2 and let $N \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d}+1)\times \mathtt{n_f}}$ be a matrix such that its columns form a basis for $\mathtt{ker}\,\mathcal{M}$. Partition $N$ as $N = \mathtt{col}(N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}})$ with $N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}} \in \mathbb{R}^{\mathtt{d}\times\mathtt{n_f}}$. For all $i \in \{1, 2, \ldots, (\mathtt{n_f}-\mathtt{d})\}$ define $\overline{N}_i := \mathtt{col}(N_i, N_{i+1}, \ldots, N_{\mathtt{n_f}-\mathtt{d}})$. Then,*

$$\mathtt{im}\begin{bmatrix}\overline{N}_{\mathtt{n_f}-\mathtt{d}} \\ 0\end{bmatrix} \subseteq \mathtt{im}\begin{bmatrix}\overline{N}_{\mathtt{n_f}-\mathtt{d}-1} \\ 0\end{bmatrix} \subseteq \cdots \subseteq \mathtt{im}\begin{bmatrix}\overline{N}_{2} \\ 0\end{bmatrix} \subseteq \mathtt{im}\begin{bmatrix}\overline{N}_{1} \\ 0\end{bmatrix} \subseteq \mathtt{im}\, N.$$

*Here the sizes of the zero matrices are such that $\begin{bmatrix}\overline{N}_{i} \\ 0\end{bmatrix} \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d}+1)\times\mathtt{n_f}}$ for all $i \in \{1, 2, \ldots, (\mathtt{n_f}-\mathtt{d})\}$.*

*Proof* Let $\overline{\mathcal{M}}$ be the matrix obtained by removing the first $\mathtt{d}$ columns and the last $\mathtt{p}$ rows of $\mathcal{M}$, that is, $\mathcal{M} = \begin{bmatrix} 0 & \overline{\mathcal{M}} \\ 0_{\mathtt{p},\mathtt{d}} & \overline{m}\end{bmatrix}$ with $\overline{m} := \begin{bmatrix} C_rB_r & C_rA_rB_r & \ldots & C_rA_r^{\mathtt{n_f}-\mathtt{d}-1}B_r\end{bmatrix}$. Then, due to the structure of $\mathcal{M}$ it also follows that $\mathcal{M} = \begin{bmatrix} 0 & 0_{p,\mathtt{d}} \\ \overline{\mathcal{M}} & \overline{n}\end{bmatrix}$, where $\overline{n} := \mathtt{col}(C_rB_r, C_rA_rB_r, \ldots, C_rA_r^{\mathtt{n_f}-\mathtt{d}-1}B_r)$. We use this observation to first show that $\mathtt{im}\begin{bmatrix}\overline{N}_1 \\ 0\end{bmatrix} \subseteq \mathtt{im}\, N$. Since $\mathtt{im}\, N = \mathtt{ker}\,\mathcal{M}$, it follows that

$$\mathcal{M}N = 0 \Leftrightarrow \begin{bmatrix} 0 & \overline{\mathcal{M}} \\ 0_{\mathtt{p},\mathtt{d}} & \overline{m}\end{bmatrix}\begin{bmatrix} N_0 \\ \overline{N}_1\end{bmatrix} = 0 \Rightarrow \overline{\mathcal{M}}\,\overline{N}_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0_{p,\mathtt{d}} \\ \overline{\mathcal{M}} & \overline{n}\end{bmatrix}\begin{bmatrix}\overline{N}_1 \\ 0\end{bmatrix} = 0 \Rightarrow \mathcal{M}\begin{bmatrix}\overline{N}_1 \\ 0\end{bmatrix} = 0$$

$$(3.3)\qquad \Rightarrow \mathtt{im}\begin{bmatrix}\overline{N}_1 \\ 0\end{bmatrix} \subseteq \mathtt{im}\, N \Leftrightarrow \mathtt{im}\begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{\mathtt{n_f}-\mathtt{d}} \\ 0_{\mathtt{d},\mathtt{n_f}}\end{bmatrix} \subseteq \mathtt{im}\begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_{\mathtt{n_f}-\mathtt{d}-1} \\ N_{\mathtt{n_f}-\mathtt{d}}\end{bmatrix}.$$

Let $i \in \{2, 3, \ldots, (\mathtt{n_f}-\mathtt{d})\}$ be arbitrary. Then, we have to show that $\mathtt{im}\begin{bmatrix}\overline{N}_i \\ 0\end{bmatrix} \subseteq \mathtt{im}\begin{bmatrix}\overline{N}_{i-1} \\ 0\end{bmatrix}$, which is equivalent to showing that $\mathtt{im}\begin{bmatrix}\overline{N}_i \\ 0_{\mathtt{d},\mathtt{n_f}}\end{bmatrix} \subseteq \mathtt{im}\,\overline{N}_{i-1}$. This directly follows from equation (3.3), because $\mathtt{im}\,\mathtt{col}(N_i, N_{i+1}, \ldots, N_{\mathtt{n_f}-\mathtt{d}}, 0_{\mathtt{d},\mathtt{n_f}}) \subseteq$

$\mathtt{im}\,\mathtt{col}(N_{i-1}, N_i, \ldots, N_{\mathtt{n_f}-\mathtt{d}})$. This completes the proof. ☐

REMARK 3.9. Define the system given by $\frac{d}{dt}x(t) = A_r x(t) + B_r u_1(t),\ y(t) = C_r x(t)$. Let the initial condition of the system be $x_0 = 0$. Then, it turns out that, the input $u_1(t) := \sum_{i=0}^{\mathtt{n_f}-\mathtt{d}} a_i \delta^{(i)}$ with $a_i \in \mathbb{R}^{\mathtt{d}}$ results in a regular output, that is, $y(t; 0, u_1) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ if and only if $\mathtt{col}(a_0, a_1, \ldots, a_{\mathtt{n_f}-\mathtt{d}}) = \mathtt{col}(N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}})\beta$ for some $\beta \in \mathbb{R}^{\mathtt{n_f}}$ (see [18, Lemma 4.1]). In [18], such an input has been termed as an *admissible impulsive input*. From Lemma 3.8, it can be concluded that if $\sum_{i=0}^{\mathtt{n_f}-\mathtt{d}} a_i \delta^{(i)}$ is an admissible impulsive input, then $\sum_{i=k}^{\mathtt{n_f}-\mathtt{d}} a_i \delta^{(i-k)}$, too, is an admissible impulsive input for all $k \in \{1, 2, \ldots, \mathtt{n_f} - \mathtt{d}\}$. ☐

Using the subspaces in Lemma 3.8, we can form another class of subspaces that follow an inclusion chain as in Lemma 3.8. We present this next.

LEMMA 3.10. *For all* $i \in \{1, 2, \ldots, (\mathtt{n_f} - \mathtt{d})\}$ *define* $W_i := \begin{bmatrix} B_r & A_r B_r & \ldots & A_r^{\mathtt{n_f}-\mathtt{d}} B_r \end{bmatrix} \begin{bmatrix} \overline{N}_i \\ 0_{i,\mathtt{n_f}} \end{bmatrix}$, *where* $\overline{N}_i$ *is as defined in Lemma 3.8. Then, the following filtration follows:*

$$\mathtt{im}\, W_{\mathtt{n_f}-\mathtt{d}} \subseteq \mathtt{im}\, W_{\mathtt{n_f}-\mathtt{d}-1} \subseteq \cdots \subseteq \mathtt{im}\, W_2 \subseteq \mathtt{im}\, W_1 \subseteq \mathtt{im}\, W.$$

The next lemma shows that the subspaces $\mathtt{im}\, W_1, \mathtt{im}\, W_2, \ldots, \mathtt{im}\, W_{\mathtt{n_f}-\mathtt{d}}$ are contained in the kernel of $C_r$.

LEMMA 3.11. *Recall the matrices* $C_r$ *and* $W_1, W_2, \ldots, W_{\mathtt{n_f}-\mathtt{d}}$ *as defined in Lemma 3.2 and Lemma 3.10, respectively. Then,* $C_r W_i = 0$ *for all* $i \in \{1, 2, \ldots, \mathtt{n_f} - \mathtt{d}\}$.

*Proof* By definition, $\mathcal{M}N = 0$. Notice from the definitions of $W_1, W_2, \ldots, W_{\mathtt{n_f}-\mathtt{d}}$ that $\mathcal{M}N = \mathtt{col}(0, C_r W_{\mathtt{n_f}-\mathtt{d}}, \ldots, C_r W_2, C_r W_1)$. Hence the result follows. ☐

REMARK 3.12. Lemma 3.10 implies that if $\delta^{(i)}$ does not appear in the optimal state trajectory, then $\delta^{(i+1)}$ cannot appear in the optimal state trajectory. Lemma 3.11 implies that the optimal output trajectory of the primal due to an initial condition from $\mathtt{im}\, W$ is identically zero. This, further implies that the optimal cost due to an initial condition from the fast space of the primal is zero. Justification of these statements needs a few result, which we present in the sequel. Hence, we justify these statements in Section 5. ☐

## 4. Optimal trajectories.

In this section we evaluate the trajectories of the primal $\Sigma$ (see equation (2.4)) for an arbitrary initial condition, which minimize the cost function given by equation (2.2). Due to Statement 1 of Theorem 3.3, it is evident that the state space $\mathbb{R}^{\mathtt{n}}$ admits a direct-sum decomposition given by $\mathbb{R}^{\mathtt{n}} = \mathtt{im}\, V_{1\Lambda} \oplus \mathtt{im}\, W$. This enables us to compute the optimal trajectories in two steps. First, we compute the optimal trajectories when the initial condition is restricted to the *slow* part, i.e., $\mathtt{im}\, V_{1\Lambda}$. Then, we compute the optimal trajectories for an initial condition in the *fast* part, i.e., $\mathtt{im}\, W$. We achieve these tasks in the following two lemmas.

LEMMA 4.1. *Consider the LQR Problem 2.6 and the matrices* $V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda}$, *and* $\Gamma$ *as defined in equation (3.1). Define* $x_{0s} := V_{1\Lambda}\alpha, z_{0s} := V_{2\Lambda}\alpha, x_s := V_{1\Lambda} e^{\Gamma t}\alpha, z_s := V_{2\Lambda} e^{\Gamma t}\alpha, u_{s_1} := V_{3\Lambda} e^{\Gamma t}\alpha$, *and* $u_{s_2} := -\widehat{R}^{-1}(S_2^T + B_2^T K_{\mathtt{max}}) x_s$, *where* $\alpha \in \mathbb{R}^{\mathtt{n_s}}$ *is arbitrary. Then,*

1. $\mathtt{col}(x_s, z_s, u_{s_1}, u_{s_2})$ *is a trajectory of the Hamiltonian system defined in equation (2.5) corresponding to the initial condition* $\mathtt{col}(x_{0s}, z_{0s})$.
2. $\mathtt{col}(x_s, u_{s_1}, u_{s_2})$ *is a trajectory of the primal* $\Sigma$ *defined in equation (2.4) corresponding to the initial condition* $x_{0s}$.
3. $\int_0^\infty \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix} dt = x_{0s}^T K_{\mathtt{max}} x_{0s}.$

*Proof* 1. Notice from the definition of $K_{\mathtt{max}}$ that $K_{\mathtt{max}} V_{1\Lambda} = V_{2\Lambda}$. Using this identity along with equation (3.1) , it can be easily seen that the trajectory $\mathtt{col}(x_s, z_s, u_{s_1}, u_{s_2})$

satisfies the Hamiltonian system's equation (2.5). Hence, $\mathtt{col}(x_s, z_s, u_{s_1}, u_{s_2})$ is a trajectory of the Hamiltonian system corresponding to the initial condition $\mathtt{col}(x_{0s}, z_{0s})$.
2. It is a matter of simple verification that if $\mathtt{col}(x_s, z_s, u_{s_1}, u_{s_2})$ is a trajectory of the Hamiltonian, then the projection $\mathtt{col}(x_s, u_{s_1}, u_{s_2})$ is a trajectory of the primal.
3. Using the definitions of $x_s, u_{s_1}, u_{s_2}$, and $K_{\mathtt{max}}$ and doing some simple algebraic manipulations with the help of equation (3.1) (see [2, proof of Theorem 4.5]) we get that $\frac{d}{dt}(x_s^T K_{\mathtt{max}} x_s) = -\begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}$. Integrating both sides of this equation, we further get

$$\int_0^\infty \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix} dt = x_s(0)^T K_{\mathtt{max}} x_s(0) - x_s(\infty)^T K_{\mathtt{max}} x_s(\infty).$$

Now, since $\Gamma$ is Hurwitz, from the definition of $x_s$ it is clear that $x_s(\infty) = 0$ and $x_s(0) = x_{0s}$. Therefore, we conclude that

$$\int_0^\infty \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix} dt = x_{0s}^T K_{\mathtt{max}} x_{0s}.$$

□

The following lemma deals with the case when the initial condition is in the fast space.

LEMMA 4.2. *Consider the LQR Problem 2.6 and the matrices $N$ and $W$ as defined in equation (3.2). Also recall the matrices $W_1, W_2, \ldots, W_{\mathtt{n_f}-\mathtt{d}}$ as defined in Lemma 3.10. Define $x_{0f} := W\beta$, $z_{0f} := 0 \in \mathbb{R}^{\mathtt{n}}$, $x_f := -[W_1\delta + W_2\delta^{(1)} + \cdots + W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d}-1)}]\beta$, $z_f := 0 \in \mathbb{R}^{\mathtt{n}}$, $u_{f_1} := -\begin{bmatrix} \delta I_{\mathtt{d}} & \delta^{(1)} I_{\mathtt{d}} & \ldots & \delta^{(\mathtt{n_f}-\mathtt{d})} I_{\mathtt{d}} \end{bmatrix} N\beta$, and $u_{f_2} := -\widehat{R}^{-1}(S_2^T + B_2^T K_{\mathtt{max}}) x_f$, where $\beta \in \mathbb{R}^{\mathtt{n_f}}$ is arbitrary. Then,*

1. *$\mathtt{col}(x_f, z_f, u_{f_1}, u_{f_2})$ is a distributional trajectory of the Hamiltonian system defined in equation (2.5) corresponding to the initial condition $\mathtt{col}(x_{0f}, z_{0f})$.*
2. *$\mathtt{col}(x_f, u_{f_1}, u_{f_2})$ is a distributional trajectory of the primal $\Sigma$ defined in equation (2.4) corresponding to the initial condition $x_{of}$.*
3. $\int_0^\infty \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix} dt = 0.$

*Proof* 1. Partition $N$ as $N = \mathtt{col}(N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}})$ with $N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}} \in \mathbb{R}^{\mathtt{d}\times\mathtt{n_f}}$. Recall from Lemma 3.10 that for all $i \in \{1, 2, \ldots, (\mathtt{n_f} - \mathtt{d})\}$, $W_i$ has been defined as $W_i = \begin{bmatrix} B_r & A_rB_r & \ldots & A_r^{\mathtt{n_f}-\mathtt{d}}B_r \end{bmatrix} \begin{bmatrix} \overline{N}_i \\ 0_{i,\mathtt{n_f}} \end{bmatrix}$, where $\overline{N}_i = \mathtt{col}(N_i, N_{i+1}, \ldots, N_{\mathtt{n_f}-\mathtt{d}})$. Also recall that $W = \begin{bmatrix} B_r & A_rB_r & \ldots & A_r^{\mathtt{n_f}-\mathtt{d}}B_r \end{bmatrix} N$. Clearly,

$$\begin{aligned} W_i &= B_rN_i + A_rW_{i+1} \text{ for all } i \in \{1, 2, \ldots, (\mathtt{n_f} - \mathtt{d} - 1)\}, \\ W_{\mathtt{n_f}-\mathtt{d}} &= B_rN_{\mathtt{n_f}-\mathtt{d}}, \text{ and } W = B_rN_0 + A_rW_1. \end{aligned} \tag{4.1}$$

We need to show that the trajectory $\mathtt{col}(x_f, z_f, u_{f_1}, u_{f_2})$ satisfies equation (2.5) in distributional sense. Using equation (4.1) we get that

$$\begin{aligned} \tfrac{d}{dt}(x_f) &= -x_{0f}\delta - [W_1\delta^{(1)} + W_2\delta^{(2)} + \cdots + W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d})}]\beta \\ &= -W\beta\delta - [W_1\delta^{(1)} + W_2\delta^{(2)} + \cdots + W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d})}]\beta \\ &= -\left[(B_rN_0 + A_rW_1)\delta + \textstyle\sum_{i=1}^{\mathtt{n_f}-\mathtt{d}-1}(B_rN_i + A_rW_{i+1})\delta^{(i)} + B_rN_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d})}\right]\beta \\ &= -A_r[W_1\delta + W_2\delta^{(1)} + \ldots + W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d}-1)}]\beta - B_r[N_0\delta + N_1\delta^{(1)} + \ldots + N_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d})}]\beta \end{aligned}$$

$$\Leftrightarrow \tfrac{d}{dt}(x_f) = A_rx_f + B_ru_{f_1}. \tag{4.2}$$

Now, by Lemma 3.10 we know that $\mathtt{im}\, W_{\mathtt{n_f}-\mathtt{d}} \subseteq \mathtt{im}\, W_{\mathtt{n_f}-\mathtt{d}-1} \subseteq \cdots \subseteq \mathtt{im}\, W_1 \subseteq \mathtt{im}\, W$. Again, by Lemma 3.6, it follows that $K_{\mathtt{max}}W = 0$. Consequently,

$$K_{\mathtt{max}}x_f = 0. \tag{4.3}$$

Using equation (4.2) and equation (4.3), we deduce that

$$Ax_f + B_1 u_{f_1} + B_2 u_{f_2} = A_r x_f - LK_{\mathtt{max}} x_f + B_r u_{f_1} = A_r x_f + B_r u_{f_1} = \frac{d}{dt}(x_f). \tag{4.4}$$

From Lemma 3.11 it directly follows that

$$C_r x_f = 0. \tag{4.5}$$

Next, using the fact that $z_f = 0$ (by definition) along with equation (4.3) and equation (4.5) we get the following equations

$$-Qx_f + A^T z_f - S_2 u_{f_2} = -Q_r x_f - S_2 \widehat{R}^{-1} B_2^T K_{\mathtt{max}} x_f = -C_r^T C_r x_f = 0 = \tfrac{d}{dt}(z_f), \tag{4.6}$$

$$B_1^T z_f = 0, \text{ and} \tag{4.7}$$

$$S_2^T x_f + B_2^T z_f + \widehat{R} u_{f_2} = S_2^T x_f - \widehat{R}\widehat{R}^{-1}(S_2^T + B_2^T K_{\mathtt{max}}) x_f = 0. \tag{4.8}$$

Combining equation (4.4), equation (4.6), equation (4.7), and equation (4.8) together yields equation (2.5). Hence, $\mathtt{col}(x_f, z_f, u_{f_1}, u_{f_2})$ is a trajectory of the Hamiltonian system corresponding to the initial condition $\mathtt{col}(x_{0f}, z_{0f})$.
2. This statement directly follows from equation (4.4).
3. Recall from Section 2.4 that

$$\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} = \begin{bmatrix} C^T \\ 0 \\ D_2^T \end{bmatrix} \begin{bmatrix} C & 0 & D_2 \end{bmatrix}. \tag{4.9}$$

Now, $\begin{bmatrix} C & 0 & D_2 \end{bmatrix} \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix} = Cx_f + D_2 u_{f_2} = C_r x_f - D_2 \widehat{R}^{-1} B_2^T K_{\mathtt{max}} x_f$. Therefore, from equation (4.3) and equation (4.5), it is evident that

$$\begin{bmatrix} C & 0 & D_2 \end{bmatrix} \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix} = 0. \tag{4.10}$$

Combining equation (4.9) and equation (4.10), we have $\begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix} = 0.$
This further implies that $\int_0^\infty \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_f \\ u_{f_1} \\ u_{f_2} \end{bmatrix} dt = 0.$ ☐

Recall from Statement 1 of Theorem 3.3 that $X = \begin{bmatrix} V_{1\Lambda} & W \end{bmatrix}$ is invertible. So, for an arbitrary initial condition $x_0$ there exist $\alpha \in \mathbb{R}^{\mathtt{n_s}}$ and $\beta \in \mathbb{R}^{\mathtt{n_f}}$ such that $x_0 = V_{1\Lambda}\alpha + W\beta$. Therefore, Lemma 4.1 and Lemma 4.2 can be combined to obtain an allowable trajectory of the given system for an arbitrary initial condition. Here, a trajectory being allowable means that the trajectory satisfies the system's equations. In the following theorem, we show that this trajectory, indeed, is the optimal trajectory.

Theorem 4.3. *Consider the LQR Problem 2.6. Recall the definitions of* $x_{0s}, x_{0f}, x_s, x_f, u_{s_1}, u_{s_2}, u_{f_1}$, *and* $u_{f_2}$ *from Lemma 4.1 and Lemma 4.2. Define* $x_0 := x_{0s} + x_{0f}$, $x^* := x_s + x_f$, $u_1^* := u_{s_1} + u_{f_1}$, *and* $u_2^* := u_{s_2} + u_{f_2}$. *Then, the following are true:*

1. $\mathtt{col}(x^*, u_1^*, u_2^*)$ *is an allowable trajectory of the primal* $\Sigma$ *defined in equation (2.4) corresponding to an arbitrary initial condition* $x_0$.
2. $\int_0^\infty \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix} dt = x_0^T K_{\mathtt{max}} x_0.$
3. $\mathtt{col}(x^*, u_1^*, u_2^*)$ *is the optimal trajectory for the initial condition* $x_0$.

*Proof* 1. This statement follows from application of Lemma 4.1 and Lemma 4.2 together with linearity of the system $\Sigma$.

2. Using equation 4.10, it is clear that

$$(4.11)\qquad \begin{bmatrix} C & 0 & D_2 \end{bmatrix} \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} C & 0 & D_2 \end{bmatrix} \begin{bmatrix} x_s + x_f \\ u_{s_1} + u_{f_1} \\ u_{s_2} + u_{f_2} \end{bmatrix} = \begin{bmatrix} C & 0 & D_2 \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}.$$

Combining equation (4.9) and equation (4.11) together, we have

$$(4.12)\qquad \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}.$$

Recall from Theorem 3.3 that $K_{\mathtt{max}}$ is symmetric. Due to Lemma 3.6 we also have $K_{\mathtt{max}} W = 0$. Therefore, it follows that

$$x_0^T K_{\mathtt{max}} x_0 = (V_{1\Lambda}\alpha + W\beta)^T K_{\mathtt{max}} (V_{1\Lambda}\alpha + W\beta) = (V_{1\Lambda}\alpha)^T K_{\mathtt{max}} V_{1\Lambda}\alpha = x_{0s}^T K_{\mathtt{max}} x_{0s}.$$

Hence, using equation (4.12) in Statement 3 of Lemma 4.1, we conclude that

$$\int_0^\infty \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x^* \\ u_1^* \\ u_2^* \end{bmatrix} dt = \int_0^\infty \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x_s \\ u_{s_1} \\ u_{s_2} \end{bmatrix} dt = x_0^T K_{\mathtt{max}} x_0.$$

3. By Theorem 3.3, we know that $K_{\mathtt{max}}$ is the maximal rank-minimizing solution of the LQR LMI 2.3. Therefore, using [11, Theorem 2] we infer that given an initial condition $x_0$, the minimal cost attainable for the singular LQR Problem 2.6 is $x_0^T K_{\mathtt{max}} x_0$. Also, since we have assumed that the primal $\Sigma$ is a left-invertible system, the optimal trajectories must be unique [3]. Hence, from Statement 2 of this theorem, it is evident that $\mathtt{col}(x^*, u_1^*, u_2^*)$ is the optimal trajectory for the singular LQR problem 2.6 corresponding to the initial condition $x_0$. □

**5. PD feedback design.** In this section we design a PD-feedback controller that restricts the system to only the optimal trajectories (Theorem 5.5). Two different direct-sum decompositions of $\mathcal{R}_s$ are crucially used in order to design this feedback. The following lemma provides these direct-sum decompositions.

LEMMA 5.1. *Recall that $\mathcal{R}_s$ denotes the fast space of the primal and $\mathtt{n_f} = \mathtt{dim}\,\mathcal{R}_s$. Also recall that $\mathtt{d} = \mathtt{nullity}\, R$. There exists a subspace $\widetilde{\mathcal{R}}_s \subseteq \mathcal{R}_s$ of dimension equal to $\mathtt{n_f} - \mathtt{d}$ satisfying the following properties:*

1. *$A_r\widetilde{\mathcal{R}}_s \subseteq \mathcal{R}_s$, $\mathtt{dim}\,(A_r\widetilde{\mathcal{R}}_s) = \mathtt{n_f} - \mathtt{d}$, and $\mathcal{R}_s = \mathtt{im}\, B_r \oplus A_r\widetilde{\mathcal{R}}_s$.*
2. *There exists $W_{\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{d}}$ full column-rank such that $\mathcal{R}_s = \widetilde{\mathcal{R}}_s \oplus \mathtt{im}\, W_{\mathtt{e}}$.*

*Proof* By Lemma 3.2, we know that $\mathcal{R}_s = \mathtt{im}\begin{bmatrix} B_r & A_rB_r & \dots & A_r^{\mathtt{n_f}-\mathtt{d}}B_r \end{bmatrix} N$, where $N \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d}+1)\times\mathtt{n_f}}$ is a matrix such that its columns form a basis for the kernel of $\mathcal{M}$. Due to the structure of $\mathcal{M}$, it follows that there exists $\widetilde{N} \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d})\times(\mathtt{n_f}-\mathtt{d})}$ such that the columns of $\begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & \widetilde{N} \end{bmatrix}$ form a basis for $\mathtt{ker}\,\mathcal{M}$. Therefore, $\mathcal{R}_s$ is given by

$$(5.1)\qquad \mathcal{R}_s = \mathtt{im}\, \underbrace{\begin{bmatrix} B_r & A_rB_r & \dots & A_r^{\mathtt{n_f}-\mathtt{d}}B_r \end{bmatrix} \begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & \widetilde{N} \end{bmatrix}}_{\widehat{W}} = \mathtt{im}\, B_r \oplus \mathtt{im}\begin{bmatrix} A_rB_r & A_r^2B_r & \dots & A_r^{\mathtt{n_f}-\mathtt{d}}B_r \end{bmatrix}\widetilde{N}.$$

Recall from Lemma 3.2 that $\widehat{W}$ is full column-rank, which leads to the direct-sum decomposition in the above equation.
Now, by Lemma 3.8, it is evident that $\mathtt{im}\begin{bmatrix} 0 & \widetilde{N} \\ 0 & 0 \end{bmatrix} = \mathtt{im}\begin{bmatrix} \widetilde{N} \\ 0 \end{bmatrix} \subseteq \mathtt{ker}\,\mathcal{M}$. Since $\begin{bmatrix} \widetilde{N} \\ 0 \end{bmatrix}$ is full column-rank, there exists $\widetilde{N}_{12} \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d})\times\mathtt{d}}$ and $\widetilde{N}_{22} \in \mathbb{R}^{\mathtt{d}\times\mathtt{d}}$ such that the columns of the matrix $\begin{bmatrix} \widetilde{N} & \widetilde{N}_{12} \\ 0 & \widetilde{N}_{22} \end{bmatrix}$ form a basis for $\mathtt{ker}\,\mathcal{M}$. So, $\mathcal{R}_s$ is also given by

(5.2) $$\mathcal{R}_s = \mathtt{im}\left[\, B_r \;\, A_rB_r \;\, \dots \;\, A_r^{\mathtt{n_f}-\mathtt{d}}B_r \,\right]\begin{bmatrix} \widetilde{N} & \widetilde{N}_{12} \\ 0 & \widetilde{N}_{22} \end{bmatrix} = \mathtt{im}\, \widetilde{W} \oplus \mathtt{im}\, W_{\mathtt{e}},$$

where $\quad \widetilde{W} := \left[\, B_r \;\, A_rB_r \;\, \dots \;\, A_r^{\mathtt{n_f}-\mathtt{d}-1}B_r \,\right]\widetilde{N}$ and $W_{\mathtt{e}} := \left[\, B_r \;\, A_rB_r \;\, \dots \;\, A_r^{\mathtt{n_f}-\mathtt{d}}B_r \,\right]\begin{bmatrix} \widetilde{N}_{12} \\ \widetilde{N}_{22} \end{bmatrix}.$

Define $\widetilde{\mathcal{R}}_s := \mathtt{im}\, \widetilde{W}$. Then, clearly $\widetilde{\mathcal{R}}_s \subseteq \mathcal{R}_s$ and $\mathtt{dim}\,(\widetilde{\mathcal{R}}_s) = \mathtt{n_f} - \mathtt{d}$. Next, we show that $\widetilde{\mathcal{R}}_s$ satisfies all the required properties.
1. Applying equation (5.2) in equation (5.1) we get that

(5.3) $$\mathcal{R}_s = \mathtt{im}\, B_r \oplus \mathtt{im}\, A_r\widetilde{W} = \mathtt{im}\, B_r \oplus A_r\widetilde{\mathcal{R}}_s.$$

Hence, $A_r\widetilde{\mathcal{R}}_s \subseteq \mathcal{R}_s$, $\mathtt{dim}\,(A_r\widetilde{\mathcal{R}}_s) = \mathtt{n_f} - \mathtt{d}$, and $\mathcal{R}_s = \mathtt{im}\, B_r \oplus A_r\widetilde{\mathcal{R}}_s$.
2. This property trivially follows. □

*Justification of Remark 3.12:* Recall from Lemma 4.2 and Theorem 4.3 that, corresponding to the initial condition $x_0 = W\beta$, where $\beta \in \mathbb{R}^{\mathtt{n_f}}$, the optimal state trajectory is given by $x_f = -[W_1\delta + W_2\delta^{(1)} + \cdots + W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d}-1)}]\beta$. Next, using Lemma 3.10 along with equation (4.1) and equation (5.3), it follows that $W_i\beta = 0 \Rightarrow W_{i+1}\beta = 0$ (note that, columns of $\widetilde{W}$ form a basis for $\mathtt{im}\, W_1$ and $A_r\widetilde{W}$ is full column-rank). Hence, if $\delta^{(i)}$ does nor appear in the optimal state trajectory, then $\delta^{(i+1)}$, too, cannot appear in the same.

From Theorem 4.3 and equation (2.4), it follows that, corresponding to an initial condition $x_0 = W\beta$, where $\beta \in \mathbb{R}^{\mathtt{n_f}}$, the optimal output trajectory of the primal is given by $y^*(t) = C_rx_f - D_2\widehat{R}^{-1}B_2^TK_{\mathtt{max}}x_f$. Then, Lemma 3.11 together with Lemma 3.6 implies that $y^*(t) \equiv 0$. □

REMARK 5.2. Recall from Theorem 4.3 that $\mathtt{col}(x^*, u_1^*, u_2^*)$ is the optimal trajectory for an arbitrary initial condition $x_0$. Further recall that $u_2^* = u_{s_2} + u_{f_2} = -\widehat{R}^{-1}(S_2^T + B_2^TK_{\mathtt{max}})(x_s + x_f) = -\widehat{R}^{-1}(S_2^T + B_2^TK_{\mathtt{max}})x^*$. Thus, the second component of the optimal input, i.e., $u_2^*$, is already given in state-feedback form. Therefore, it remains to show that the first component, i.e., $u_1^*$, admits a formulation in terms of a PD state-feedback. To design this feedback, we need the following assumption.

ASSUMPTION 5.3. *Zero eigenvalues of $(A_r - LK_{\mathtt{max}})$, if any, are controllable for the pair $(A_r - LK_{\mathtt{max}}, B_r)$, where $A_r, L$, and $B_r$ are as defined in equation* (2.8).[3]

REMARK 5.4. Recall the matrix $\widetilde{W} = \left[\, B_r \;\, A_rB_r \;\, \dots \;\, A_r^{\mathtt{n_f}-\mathtt{d}-1}B_r \,\right]\widetilde{N}$ from equation (5.2). It can be understood from the proof of Lemma 5.1 that the columns of the matrix $\widetilde{N} \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d})\times(\mathtt{n_f}-\mathtt{d})}$ form a basis for $\mathtt{ker}\,\mathcal{M}_t$, where $\mathcal{M}_t$ is obtained by removing the first $\mathtt{d}$ columns and first $\mathtt{p}$ rows from $\mathcal{M}$, that is, $\mathcal{M} = \begin{bmatrix} 0_{\mathtt{p},\mathtt{d}} & 0_{\mathtt{p},\mathtt{d}(\mathtt{n_f}-\mathtt{d})} \\ 0_{\mathtt{p}(\mathtt{n_f}-\mathtt{d}),\mathtt{d}} & \mathcal{M}_t \end{bmatrix}$. It also follows that there exists $W_{\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{d}}$ such that columns of the matrix $[\, \widetilde{W} \;\, W_{\mathtt{e}} \,]$ form a basis for the fast space $\mathcal{R}_s$ of the primal. Furthermore, the columns of the matrix $[\, B_r \;\, A_r\widetilde{W} \,]$, too, form a basis for $\mathcal{R}_s$. Therefore, from Statement 1 of Theorem 3.3, it is evident that $X_1 := [\, V_{1\Lambda} \;\, \widetilde{W} \;\, W_{\mathtt{e}} \,]$ and $X_2 := [\, V_{1\Lambda} \;\, B_r \;\, A_r\widetilde{W} \,]$ are non-singular. □

We now prove the titular main result of this paper, which provides a PD feedback controller that solves the singular LQR problem.

[3]It should be noted here that Assumption 5.3 is not restrictive because of the following reasons: in the statement of Problem 2.6 we have assumed that the system $\frac{d}{dt}x = Ax + B_1u_1 + B_2u_2$ is stabilizable. The feedback $u_2 = -\widehat{R}^{-1}(S_2^T + B_2^TK_{\mathtt{max}})x$ makes sure that $\mathtt{n_s}$ number of eigenvalues of $A$ are stabilized (see Lemma 3.1 and Lemma 4.1). With this feedback the closed-loop system becomes $\frac{d}{dt}x = (A_r - LK_{\mathtt{max}})x + B_1u_1$. Assumption 5.3 *does not* require existence of a feedback $u_1 = Fx$ such that the other $\mathtt{n_f} = \mathtt{n} - \mathtt{n_s}$ eigenvalues are stabilized. It just requires existence of an $F$ such that if there are any zero eigenvalues in the remaining $\mathtt{n_f}$ number of eigenvalues, then those eigenvalues can be made non-zero via a suitable feedback. Thus, the assumption holds generically.

THEOREM 5.5. *Let Assumption 5.3 hold. Recall the matrices* $X_1 := [\, V_{1\Lambda} \; \widetilde{W} \; W_{\mathtt{e}} \,]$ *and* $X_2 := [\, V_{1\Lambda} \; B_r \; A_r\widetilde{W} \,]$ *from Remark 5.4. Then the following are true:*

1. *There exist* $g_0 \in \mathbb{R}^{\mathtt{d}\times(\mathtt{n_f}-\mathtt{d})}$ *and* $g_1 \in \mathbb{R}^{\mathtt{d}\times\mathtt{d}}$ *such that* $(A_r - LK_{\mathtt{max}} + B_rF_p)$ *is non-singular, where* $L$ *is as defined in equation (2.8) and* $F_p := [\, V_{3\Lambda} \; g_0 \; g_1 \,] X_1^{-1}$.
2. *Define* $F_d := [\, 0 \; I_{\mathtt{d}} \; -g_0 \,] X_2^{-1}$ *and* $F_{reg} := -\widehat{R}^{-1}(S_2^T + B_2^T K_{\mathtt{max}})$. *Then, the feedback laws* $u_1 = F_p x + F_d \frac{d}{dt}x$ *and* $u_2 = F_{reg}x$ *solve the singular LQR Problem 2.6.*

*Proof.* 1. We first do a similarity transformation on the matrices $(A_r - LK_{\mathtt{max}})$ and $B_r$ by the matrix $X_2$. From the definition of $X_2$, it is easy to verify that

$$B_{\mathrm{t}} := X_2^{-1}B_r = \begin{bmatrix} 0_{\mathtt{n_s},\mathtt{d}} \\ \widetilde{B} \end{bmatrix}, \text{ where } \widetilde{B} := \begin{bmatrix} I_{\mathtt{d}} \\ 0_{(\mathtt{n_f}-\mathtt{d}),\mathtt{d}} \end{bmatrix}. \tag{5.4}$$

Again, $A_{\mathrm{t}} := X_2^{-1}(A_r - LK_{\mathtt{max}})X_2 = X_2^{-1}(A_r - LK_{\mathtt{max}})[\, V_{1\Lambda} \; \widehat{W} \,]$, where $\widehat{W} := [\, B_r \; A_r\widetilde{W} \,]$. Now, using equation (3.1) and equation (5.4), we deduce that

$$(A_r - LK_{\mathtt{max}})V_{1\Lambda} = A_rV_{1\Lambda} - LV_{2\Lambda} = V_{1\Lambda}\Gamma - B_rV_{3\Lambda} = X_2 \begin{bmatrix} \Gamma \\ A_{21} \end{bmatrix}, \tag{5.5}$$
$$\text{where } A_{21} := -\widetilde{B}V_{3\Lambda} = \begin{bmatrix} -V_{3\Lambda} \\ 0_{(\mathtt{n_f}-\mathtt{d}),\mathtt{n_s}} \end{bmatrix}.$$

Also, using Lemma 3.6 and non-singularity of $X_2$, we have

$$(A_r - LK_{\mathtt{max}})\widehat{W} = A_r\widehat{W} =: X_2 \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}, \text{ for some } A_{12} \in \mathbb{R}^{\mathtt{n_s}\times\mathtt{n_f}}, A_{22} \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}. \tag{5.6}$$

Combining equation (5.5) and equation (5.6), we infer that

$$A_{\mathrm{t}} = X_2^{-1}(A_r - LK_{\mathtt{max}})X_2 = \begin{bmatrix} \Gamma & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{5.7}$$

We claim that the pair $(A_{22}, \widetilde{B})$ is such that the zero eigenvalues of $A_{22}$, if any, are controllable. We prove this claim by contradiction. So, to the contrary, we assume that the claim is false. Thus, by the Popov-Belevitch-Hautus criterion for controllability, there exists $v \in \mathbb{R}^{\mathtt{n_f}} \setminus \{0\}$ such that

$$v^TA_{22} = 0 \text{ and } v^T\widetilde{B} = 0. \tag{5.8}$$

Due to the structure of $\widetilde{B}$ (see equation (5.4)), we must have $v = \begin{bmatrix} 0_{\mathtt{d},1} \\ v_2 \end{bmatrix}$ for some $v_2 \in \mathbb{R}^{(\mathtt{n_f}-\mathtt{d})} \setminus \{0\}$. Further, non-singularity of $X_2$ ensures that there exists $w \in \mathbb{R}^{\mathtt{n}} \setminus \{0\}$ such that $w^TX_2 = [\, 0_{1,\mathtt{n_s}} \; v^T \,] = [\, 0_{1,(\mathtt{n_s}+\mathtt{d})} \; v_2^T \,]$. Therefore, from equation (5.7), we have

$$w^T(A_r - LK_{\mathtt{max}}) = w^TX_2A_{\mathrm{t}}X_2^{-1} = [\, 0_{1,\mathtt{n_s}} \; v^T \,] \begin{bmatrix} \Gamma & A_{12} \\ A_{21} & A_{22} \end{bmatrix} X_2^{-1} = [\, v^TA_{21} \; v^TA_{22} \,] X_2^{-1}.$$

But, $v^TA_{21} = [\, 0_{1,d} \; v_2^T \,] \begin{bmatrix} -V_{3\Lambda} \\ 0_{(\mathtt{n_f}-\mathtt{d}),\mathtt{n_s}} \end{bmatrix} = 0$. Hence, using equation (5.8), we further have $w^T(A_r - LK_{\mathtt{max}}) = 0$. Also, $w^TB_r = w^TX_2B_{\mathrm{t}} = [\, 0_{1,\mathtt{n_s}} \; v^T \,] \begin{bmatrix} 0_{\mathtt{n_s},\mathtt{d}} \\ \widetilde{B} \end{bmatrix} = v^T\widetilde{B} = 0$. This contradicts Assumption 5.3. Hence, the claim that the zero eigenvalues of $A_{22}$, if any, are controllable by $\widetilde{B}$ must be true. This proves the claim.

In view of this claim, it is evident that there exists $\bar{g} \in \mathbb{R}^{\mathtt{d}\times\mathtt{n_f}}$ such that $(A_{22} + \widetilde{B}\bar{g})$ is non-singular. Next, define $F_p := [\, V_{3\Lambda} \; \bar{g} \,] X_2^{-1}$. Then, $A_r - LK_{\mathtt{max}} + B_rF_p = X_2(A_{\mathrm{t}} + B_{\mathrm{t}}[\, V_{3\Lambda} \; \bar{g} \,])X_2^{-1}$, where $B_{\mathrm{t}}$ and $A_{\mathrm{t}}$ are as defined in equation (5.4) and equation (5.7), respectively. Now,

$$A_{\mathrm{t}} + B_{\mathrm{t}}[\, V_{3\Lambda} \; \bar{g} \,] = \begin{bmatrix} \Gamma & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} 0_{\mathtt{n_s},\mathtt{n_s}} & 0_{\mathtt{n_s},\mathtt{n_f}} \\ \widetilde{B}V_{3\Lambda} & \widetilde{B}\bar{g} \end{bmatrix} = \begin{bmatrix} \Gamma & A_{12} \\ A_{21}+\widetilde{B}V_{3\Lambda} & A_{22}+\widetilde{B}\bar{g} \end{bmatrix}.$$

But, from equation (5.4) and equation (5.5), it is clear that $A_{21} + \widetilde{B}V_{3\Lambda} = \begin{bmatrix} -V_{3\Lambda} \\ 0_{(\mathtt{n_f}-\mathtt{d}),\mathtt{n_s}} \end{bmatrix} + \begin{bmatrix} V_{3\Lambda} \\ 0_{(\mathtt{n_f}-\mathtt{d}),\mathtt{n_s}} \end{bmatrix} = 0$. Therefore, $A_{\mathrm{t}} + B_{\mathrm{t}}[\, V_{3\Lambda} \; \bar{g} \,] = \begin{bmatrix} \Gamma & A_{12} \\ 0 & A_{22}+\widetilde{B}\bar{g} \end{bmatrix}$. Since $\Gamma$ is Hurwitz and

$(A_{22}+\widetilde{B}\bar{g})$ is non-singular, we must have that $(A_{\mathtt{t}}+B_{\mathtt{t}}\left[\begin{smallmatrix} V_{3\Lambda} & \bar{g}\end{smallmatrix}\right])$ is non-singular; which further implies that $(A_r - LK_{\mathtt{max}} + B_rF_p)$ is non-singular.

Since $\mathtt{im}\left[\begin{smallmatrix} B_r & A_r\widetilde{W}\end{smallmatrix}\right] = \mathtt{im}\left[\begin{smallmatrix} \widetilde{W} & W_{\mathtt{e}}\end{smallmatrix}\right]$, it follows from the structures of $X_2$ and $X_1$ that there exists a non-singular matrix $T \in \mathbb{R}^{\mathtt{n_f}\times\mathtt{n_f}}$ such that $X_2 = X_1\left[\begin{smallmatrix} I_{\mathtt{n_s}} & 0\\ 0 & T^{-1}\end{smallmatrix}\right]$. Say, $g_0 \in \mathbb{R}^{\mathtt{d}\times(\mathtt{n_f}-\mathtt{d})}$ and $g_1 \in \mathbb{R}^{\mathtt{d}\times\mathtt{d}}$ is defined as $\begin{bmatrix} g_0 & g_1\end{bmatrix} := \bar{g}T$. Thus, $F_p = \left[\begin{smallmatrix} V_{3\Lambda} & \bar{g}\end{smallmatrix}\right]X_2^{-1} = \left[\begin{smallmatrix} V_{3\Lambda} & \bar{g}\end{smallmatrix}\right]\left[\begin{smallmatrix} I_{\mathtt{n_s}} & 0\\ 0 & T\end{smallmatrix}\right]X_1^{-1} = \left[\begin{smallmatrix} V_{3\Lambda} & \bar{g}T\end{smallmatrix}\right]X_1^{-1} = \left[\begin{smallmatrix} V_{3\Lambda} & g_0 & g_1\end{smallmatrix}\right]X_1^{-1}$. But, we have already proved that $(A_r - LK_{\mathtt{max}} + B_rF_p)$ is non-singular. Hence, there exist $g_0 \in \mathbb{R}^{\mathtt{d}\times(\mathtt{n_f}-\mathtt{d})}$ and $g_1 \in \mathbb{R}^{\mathtt{d}\times\mathtt{d}}$ such that $(A_r-LK_{\mathtt{max}}+B_rF_p)$ is non-singular, where $F_p = \left[\begin{smallmatrix} V_{3\Lambda} & g_0 & g_1\end{smallmatrix}\right]X_1^{-1}$.

2. Recall from Theorem 4.3 that for an arbitrary initial condition $x_0$, the optimal trajectory of the primal $\Sigma$ is given by $\mathtt{col}(x^*, u_1^*, u_2^*)$. Our aim is to show that the feedback laws defined in this theorem restrict the system to exhibit the optimal trajectories only. So, we first show that the optimal trajectories satisfy the system's equation under the proposed feedback laws and then we show that, for a given initial condition, the optimal trajectory is the only trajectory that the system admits.
We show that the given feedback laws admit the optimal trajectory in three steps: first, we show that the trajectory $\mathtt{col}(x_s, u_{s_1}, u_{s_2})$ (defined in Lemma 4.1) corresponding to the initial condition $V_{1\Lambda}\alpha$ is an allowable trajectory by the feedback law. Then, we show that the trajectory $\mathtt{col}(x_f, u_{f_1}, u_{f_2})$ (defined in Lemma 4.2) corresponding to the initial condition $W\beta$ is an allowable trajectory, too. Finally, we show that the optimal trajectory $\mathtt{col}(x^*, u_1^*, u_2^*)$ is an allowable trajectory.
Recall that $x_s = V_{1\Lambda}e^{\Gamma t}\alpha$, $u_{s_1} = V_{3\Lambda}e^{\Gamma t}\alpha$, and $u_{s_2} = -\widehat{R}^{-1}(S_2^T + B_2^TK_{\mathtt{max}})x_s$. So,

$$F_px_s + F_d\frac{d}{dt}x_s = (F_pV_{1\Lambda} + F_dV_{1\Lambda}\Gamma)e^{\Gamma t}\alpha.$$

But, from the definition of $F_p$ and $F_d$, $F_pV_{1\Lambda} = V_{3\Lambda}$ and $F_dV_{1\Lambda} = 0$. Thus,

$$F_px_s + F_d\frac{d}{dt}x_s = V_{3\Lambda}e^{\Gamma t}\alpha = u_{s_1}.$$

Therefore, from Statement 2 of Lemma 4.1, we infer that

$$Ax_s + B_1(F_px_s + F_d\frac{d}{dt}x_s) + B_2F_{reg}x_s = Ax_s + B_1u_{s_1} + B_2u_{s_2} = \frac{d}{dt}x_s.$$

Hence, the feedback law allows the trajectory $\mathtt{col}(x_s, u_{s_1}, u_{s_2})$.
Recall that $x_f := -[W_1\delta+W_2\delta^{(1)}+\cdots+W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d}-1)}]\beta$, $u_{f_1} = -\left[\begin{smallmatrix} \delta I_{\mathtt{d}} & \delta^{(1)}I_{\mathtt{d}} & \dots & \delta^{(\mathtt{n_f}-\mathtt{d})}I_{\mathtt{d}}\end{smallmatrix}\right]N\beta$, and $u_{f_2} := -\widehat{R}^{-1}(S_2^T + B_2^TK_{\mathtt{max}})x_f$, where $N$ is as defined in equation (3.2). Also recall from equation (4.2) that $\frac{d}{dt}x_f = A_rx_f + B_ru_{f_1}$. Hence,

$$\begin{aligned} F_px_f + F_d\tfrac{d}{dt}x_f &= -F_p\textstyle\sum_{i=1}^{\mathtt{n_f}-\mathtt{d}}W_i\beta\delta^{(i-1)} - F_dA_r\sum_{i=1}^{\mathtt{n_f}-\mathtt{d}}W_i\beta\delta^{(i-1)} + F_dB_ru_{f_1}.\\ &= -\textstyle\sum_{i=1}^{\mathtt{n_f}-\mathtt{d}}(F_pW_i + F_dA_rW_i)\beta\delta^{(i-1)} + u_{f_1} \quad (\text{since } F_dB_r = I_{\mathtt{d}}). \end{aligned} \tag{5.9}$$

Partition $N$ as $N = \mathtt{col}(N_0, \overline{N}_1)$ with $N_0 \in \mathbb{R}^{\mathtt{d}\times\mathtt{n_f}}$ and $\overline{N}_1 \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d})\times\mathtt{n_f}}$. Recall from equation (3.2) and equation (5.1) that $\mathtt{im}\left[\begin{smallmatrix} I_{\mathtt{d}} & 0\\ 0 & \widetilde{N}\end{smallmatrix}\right] = \mathtt{ker}\,\mathcal{M} = \mathtt{im}\,N = \mathtt{im}\left[\begin{smallmatrix} N_0\\ \overline{N}_1\end{smallmatrix}\right] \Rightarrow \mathtt{im}\,\overline{N}_1 = \mathtt{im}\,\widetilde{N}$. Hence, from Lemma 3.10 and Remark 5.4, we infer that $\mathtt{im}\,W_1 = \mathtt{im}\,\widetilde{W}$. From Lemma 3.10 we further get that $\mathtt{im}\,W_{\mathtt{n_f}-\mathtt{d}} \subseteq \mathtt{im}\,W_{\mathtt{n_f}-\mathtt{d}-1} \subseteq \cdots \subseteq \mathtt{im}\,W_1 = \mathtt{im}\,\widetilde{W}$. Therefore, for all $i \in \{1, 2, \dots, (\mathtt{n_f}-\mathtt{d})\}$ there exists $T_i \in \mathbb{R}^{(\mathtt{n_f}-\mathtt{d})\times\mathtt{n_f}}$ such that $W_i = \widetilde{W}T_i$. Thus, from equation (5.9) we further get that

$$F_px_f+F_d\frac{d}{dt}x_f = -\sum_{i=1}^{\mathtt{n_f}-\mathtt{d}}(F_p\widetilde{W}+F_dA_r\widetilde{W})T_i\beta\delta^{(i-1)}+u_{f_1} = -\sum_{i=1}^{\mathtt{n_f}-\mathtt{d}}(g_0-g_0)T_i\beta\delta^{(i-1)}+u_{f_1} = u_{f_1}.$$

Therefore, from Statement 3 of Lemma 4.2, it is clear that

$$Ax_f + B_1(F_p x_f + F_d \frac{d}{dt} x_f) + B_2 F_{reg} x_f = Ax_f + B_1 u_{f_1} + B_2 u_{f_2} = \frac{d}{dt}(x_f).$$

Hence, the feedback laws admit the trajectory $(x_f, u_{f_1}, u_{f_2})$. Finally, since $x^* = x_s + x_f, u_1^* = u_{s_1} + u_{f_1}$, and $u_2^* = u_{s_2} + u_{f_2}$, using linearity, we conclude that corresponding to an arbitrary initial condition $x_0 = [\, V_{1\Lambda} \; W \,] \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$, the optimal trajectory $(x^*, u_1^*, u_2^*)$ is an allowable trajectory by the feedback law.
The only thing that remains to be shown is that given an arbitrary initial condition $x_0$, the trajectory of the closed-loop system can be uniquely determined. It can be easily seen that the feedback laws mentioned in Statement 2 of this theorem results in the closed-loop system

$$\underbrace{(I_{\mathtt{n}} - B_r F_d)}_{E_{CL}} \frac{d}{dt} x(t) = \underbrace{(A_r - LK_{\mathtt{max}} + B_r F_p)}_{A_{CL}} x(t). \tag{5.10}$$

So, for a given initial condition, the trajectory of the closed-loop system is uniquely determined if and only if the matrix pencil $(sE_{CL} - A_{CL})$ is regular [25]. In Statement 1 of this theorem, we have already shown that $A_{CL} = (A_r - LK_{\mathtt{max}} + B_r F_p)$ is non-singular. Note that, non-singularity of $A_{CL}$ ensures that $\mathtt{det}(sE_{CL} - A_{CL}) \not\equiv 0$ (see [25, Theorem 1.2.1]). Hence, the matrix pencil $(sE_{CL} - A_{CL})$ is regular. Since, we have already showed that given an arbitrary initial condition $x_0$, $x^*$ satisfies the equation (5.10), we conclude that the closed-loop system admits the optimal trajectories only. Therefore, the feedback laws given in the Statement 2 of this theorem solve the singular LQR Problem 2.6. □

## 6. Regularity and internal stability of the closed-loop system.

The optimal PD feedback law provided in Theorem 5.5 results in the closed-loop system as given by equation (5.10). Note that, Assumption 5.3, which we have made in order to design the optimal PD feedback controller, does not necessitate that the partial closed-loop system $(A_r - LK_{\max}, B_r)$ be stabilizable. Therefore, a natural question that arises is: does the optimal feedback law guarantee that the closed-loop system is internally stable? The answer is affirmative. To explain this, we first note that Assumption 5.3 is made in order to guarantee that there exists a feedback matrix $F_p$ as defined in Theorem 5.5 such that $A_{CL}$ is non-singular. This enables us to write the following theorem.

THEOREM 6.1. *The matrix pencil $(sE_{CL} - A_{CL})$ as defined in equation (5.10) is a regular matrix pencil, that is,* $\mathtt{det}(sE_{CL} - A_{CL}) \not\equiv 0$.

*Proof* Recall from Statement 1 of Theorem 5.5 that $A_{CL}$ is non-singular. Hence, $\mathtt{det}(sE_{CL} - A_{CL}) \not\equiv 0$. □

Since the matrix $E_{CL}$ is singular (because $E_{CL}B_r = 0$), the closed-loop system is a singular descriptor system. So, in order to show that the closed-loop system is internally stable, we need to consider the notion of stability for a singular descriptor system. The following proposition from [25, Theorem 3.1.1] characterizes such systems, which are asymptotically stable.

PROPOSITION 6.2. Consider the singular descriptor system as given in equation (5.10). Then, the system is asymptotically stable if and only if $\sigma(E_{CL}, A_{CL}) \subseteq \mathbb{C}_-$.

Note that, from Proposition 6.2, it follows that the stability of the closed-loop system is not governed by the eigenvalues of $A_{CL}$, but rather, by the eigenvalues of the matrix pair $(E_{CL}, A_{CL})$. We now show that the closed-loop system is asymptotically stable.

THEOREM 6.3. *The closed-loop system as given in equation (5.10) is asymptotically stable.*

*Proof* Recall from the definition of $F_d$ that $F_d V_{1\Lambda} = 0$. Hence, $E_{CL}V_{1\Lambda} = V_{1\Lambda}$. So, by equation (3.1) and the definition of $F_p$, it follows that $A_{CL}V_{1\Lambda} = E_{CL}V_{1\Lambda}\Gamma$. Therefore,

$\sigma(\Gamma) \subseteq \sigma(E_{CL}, A_{CL})$. We now show that $\sigma(\Gamma)$ is, in fact, equal to $\sigma(E_{CL}, A_{CL})$; that is, all the slow modes of the closed-loop singular descriptor system are given by the eigenvalues of the matrix $\Gamma$. We show this indirectly by utilizing the general expression of an arbitrary trajectory of the closed-loop system.

Recall from Theorem 6.1 that the matrix pencil $(sE_{CL} - A_{CL})$ is regular. This further ensures that for an arbitrary initial condition $x_0 = \begin{bmatrix} V_{1\Lambda} & W \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, the trajectory of the closed-loop system is uniquely determined. This trajectory has been shown in Theorem 5.5 to be the optimal trajectory $x^*(t) = V_{1\Lambda} e^{\Gamma t}\alpha - [W_1\delta + W_2\delta^{(1)} + \cdots + W_{\mathtt{n_f}-\mathtt{d}}\delta^{(\mathtt{n_f}-\mathtt{d}-1)}]\beta$. Hence, $\sigma(E_{CL}, A_{CL}) = \sigma(\Gamma) \subseteq \mathbb{C}_-$. Consequently, the closed-loop system is asymptotically stable. Alternatively, since $\sigma(\Gamma) \subseteq \mathbb{C}_-$, we must have that $\lim_{t\to\infty} x^*(t) = 0$. Thus, the closed-loop system is asymptotically stable. $\square$

## 7. An illustrative example.

Consider the system $\frac{d}{dt}x(t) = Ax(t) + B_1u_1(t) + B_2u_2(t)$, where

$$A = \begin{bmatrix} 3 & 0 & -2 & 2 & 0 \\ 1 & -3 & 2 & -1 & 5 \\ -2 & 8 & 3 & -1 & -8 \\ -5 & 3 & 2 & -2 & -4 \\ 1 & -5 & 0 & 0 & 6 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -2 & -1 \\ 0 & 1 \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

For an arbitrary initial condition $x_0 = \mathtt{col}(x_{01}, x_{02}, x_{03}, x_{04}, x_{05})$, our objective is to find an optimal input $u^* = \mathtt{col}(u_1^*, u_2^*)$ that minimizes the functional (2.2), where

$$Q = \begin{bmatrix} 18 & -4 & 0 & 9 & 13 \\ -4 & 15 & 8 & -6 & -5 \\ 0 & 8 & 6 & -3 & 1 \\ 9 & -6 & -3 & 6 & 6 \\ 13 & -5 & 1 & 6 & 13 \end{bmatrix}, S_2 = \begin{bmatrix} -3 & -6 \\ 9 & 2 \\ 3 & 2 \\ -3 & -4 \\ -6 & -2 \end{bmatrix}, \text{ and } \widehat{R} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

We also design a PD state-feedback for the optimal input.

Note that $\mathtt{d} = \mathtt{m} - \mathtt{r} = 4 - 2 = 2$. We first compute the reduced Hamiltonian matrix pair $(E_r, H_r)$ as $E_r = \begin{bmatrix} I_5 & 0 & 0 \\ 0 & I_5 & 0 \\ 0 & 0 & 0_{2,2} \end{bmatrix}$ and $H_r = \begin{bmatrix} A_r & -L & B_r \\ -Q_r & -A_r^T & 0 \\ 0 & B_r^T & 0_{2,2} \end{bmatrix}$, where $A_r := A - B_2\widehat{R}^{-1}S_2^T$, $Q_r := Q - S_2\widehat{R}^{-1}S_2^T$, $L := B_2\widehat{R}^{-1}B_2^T$, and $B_r := B_1$. It can be found out that $\mathtt{det}\,(sE_r - H_r) = 64(s^2 - \frac{4}{9})$. Therefore, $2\mathtt{n_s} = \mathtt{degdet}\,(sE_r - H_r) = 2 \Rightarrow \mathtt{n_s} = 1$. Also. $\sigma(E_r, H_r) \cap \mathbb{C}_- = -\frac{2}{3}$.

**The good slow space of the Hamiltonian system:** Solve $H_rV_\Lambda = E_rV_\Lambda\Gamma$ for a $V_\Lambda \in \mathbb{R}^{12\times 1}$, where $\Gamma = -\frac{2}{3}$. It can be verified that $V_\Lambda = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda} \end{bmatrix}$ with $V_{1\Lambda} = \begin{bmatrix} 2 \\ 1 \\ -2.8 \\ -9 \\ 3 \end{bmatrix}$, $V_{2\Lambda} = -38.4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $V_{3\Lambda} = \begin{bmatrix} 0.4 \\ -\frac{217}{15} \end{bmatrix}$ satisfies the equation. Hence, the good slow space of the Hamiltonian is given by $\mathcal{O}_{wg} = \mathtt{im} \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \end{bmatrix}$ (see Lemma 3.1).

**The fast space of the primal:** Since $\mathtt{rank} \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0_{2,2} & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} = 4$, we obtain the matrices $C \in \mathbb{R}^{4\times 5}$ and $D_2 \in \mathbb{R}^{4\times 2}$ such that $\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0_{2,2} & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} = \begin{bmatrix} C^T \\ 0_{2,4} \\ D_2^T \end{bmatrix} \begin{bmatrix} C & 0_{4,2} & D_2 \end{bmatrix}$. $C = \begin{bmatrix} -2 & 1 & 0 & -1 & -2 \\ -2 & -2 & -2 & 0 & -2 \\ -1 & 3 & 1 & -1 & -2 \\ -3 & 1 & 1 & -2 & -1 \end{bmatrix}$ and $D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \\ 0 & 2 \end{bmatrix}$ provides the desired factorization.

Now, by Lemma 3.2, the dimension of the fast space $\mathcal{R}_s$ of the primal is $\mathtt{dim}\,\mathcal{R}_s = \mathtt{n_f} = \mathtt{n} - \mathtt{n_s} = 5 - 1 = 4$. By following Lemma 3.2, we compute a matrix $N \in \mathbb{R}^{6\times 4}$ which is full column-rank such that $\mathcal{M}N = 0$, where $\mathcal{M} = \begin{bmatrix} 0_{4,2} & 0 & 0 \\ 0 & 0 & C_rB_r \\ 0 & C_rB_r & C_rA_rB_r \end{bmatrix}$ and $C_r = C - D_2\widehat{R}^{-1}S_2^T$. Notice that $N = \begin{bmatrix} N_0 \\ N_1 \\ N_2 \end{bmatrix}$ with $N_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $N_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and

$N_2 = \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ gives the desire result. Compute the matrix $W$ as
$W = \left[\, B_r \;\; A_r B_r \;\; A_r^2 B_r \,\right] \begin{bmatrix} N_0 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & -3 & -8 \\ -2 & -1 & -3 & -6 \\ 0 & 1 & 1 & 2 \end{bmatrix}$. Then, $\mathcal{R}_s = \mathtt{im}\, W$.

**The maximal rank-minimizing solution $K_{\mathtt{max}}$ of the singular LQR LMI:** Following Theorem 3.3, we first compute the matrices $X = \left[\, V_{1\Lambda} \;\; W \,\right]$ and $Y = \left[\, V_{2\Lambda} \;\; 0_{5,4} \,\right]$.
Then, $K_{\mathtt{max}} = YX^{-1} = 9.6 \begin{bmatrix} 4 & 2 & 0 & 2 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

**Optimal trajectories:** We first compute $\alpha \in \mathbb{R}^1$ and $\beta \in \mathbb{R}^4$ such that $x_0 = \left[\, V_{1\Lambda} \;\; W \,\right] \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] = X \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$. It can be verified that

$$\alpha = -\frac{1}{4}(2x_{01} + x_{02} + x_{04}) \text{ and } \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 16.4x_{01}+4.2x_{02}+4x_{03}+4.2x_{04}+4x_{05} \\ 16.4x_{01}+4.2x_{02}+4x_{03}+6.2x_{04}+10x_{05} \\ -14.4x_{01}+0.8x_{02}-4x_{03}-5.2x_{04}-10x_{05} \\ 2x_{01}-x_{02}+x_{04}+2x_{05} \end{bmatrix}.$$

Next, we compute $W_1$ and $W_2$ as defined in Lemma 3.10. They are found out to be

$$W_1 = \left[\, B_r \;\; A_r B_r \,\right] \left[\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } W_2 = B_r N_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, by Theorem 4.3, the optimal state trajectory is given by

$$x^*(t) = V_{1\Lambda} e^{\Gamma t} \alpha - W_1 \beta \delta - W_2 \beta \delta^{(1)} = \begin{bmatrix} 2 \\ 1 \\ -2.8 \\ -9 \\ 3 \end{bmatrix} e^{-\frac{2}{3}t} \alpha - \begin{bmatrix} \beta_3+\beta_4 \\ \beta_4 \\ -\beta_3-3\beta_4 \\ -2\beta_3-3\beta_4 \\ \beta_4 \end{bmatrix} \delta - \begin{bmatrix} \beta_4 \\ 0 \\ -\beta_4 \\ -2\beta_4 \\ 0 \end{bmatrix} \delta^{(1)}.$$

The optimal input is given by $u^*(t) = \mathtt{col}(u_1^*(t), u_2^*(t))$, where

$$u_1^*(t) = V_{3\Lambda} e^{\Gamma t} \alpha - \left[\delta I_2 \quad \delta^{(1)} I_2 \quad \delta^{(2)} I_2\right] N\beta = \begin{bmatrix} 0.4 \\ -\frac{217}{15} \end{bmatrix} e^{-\frac{2}{3}t} \alpha - \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \delta - \begin{bmatrix} \beta_3 \\ 0 \end{bmatrix} \delta^{(1)} - \begin{bmatrix} \beta_4 \\ 0 \end{bmatrix} \delta^{(2)},$$
$$u_2^*(t) = -\widehat{R}^{-1}(S_2^T + B_2^T K_{\mathtt{max}}) x^*(t).$$

**PD feedback design:** Notice that $N = \begin{bmatrix} I_2 & 0 \\ 0 & \widetilde{N} \end{bmatrix}$, where $\widetilde{N} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, we find out matrices $\widetilde{N}_{12} \in \mathbb{R}^{4\times 2}$ and $\widetilde{N}_{22} \in \mathbb{R}^{2\times 2}$ such that $\mathtt{im} \begin{bmatrix} \widetilde{N} & \widetilde{N}_{12} \\ 0 & \widetilde{N}_{22} \end{bmatrix} = \mathtt{im}\, N$. We find out these matrices to be $\widetilde{N}_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\widetilde{N}_{22} = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then, by following Theorem 5.5 we first get the matrices $\widetilde{W} = \left[\, B_r \;\; A_r B_r \,\right] \widetilde{N}, W_{\mathtt{e}} = \left[\, B_r \;\; A_r B_r \;\; A_r^2 B_r \,\right] \begin{bmatrix} \widetilde{N}_{12} \\ \widetilde{N}_{22} \end{bmatrix}$, $X_1 = \left[\, V_{1\Lambda} \;\; \widetilde{W} \;\; W_{\mathtt{e}} \,\right]$, and $X_2 = \left[\, V_{1\Lambda} \;\; B_r \;\; A_r \widetilde{W} \,\right]$. Now, we compute the matrices $F_p = \left[\, V_{3\Lambda} \;\; g_0 \;\; g_1 \,\right] X_1^{-1}$ and $F_d = \left[\, 0 \;\; I_2 \;\; -g_0 \,\right] X_2^{-1}$ with $g_0 = g_1 = 0_{2,2}$ to get

$$F_p = \begin{bmatrix} -0.2 & -0.1 & 0 & -0.1 & 0 \\ \frac{217}{30} & \frac{217}{60} & 0 & \frac{217}{60} & 0 \end{bmatrix} \text{ and } F_d = \left[\begin{smallmatrix} 4.1 & 1.05 & 1 & 1.05 & 1 \\ 4.1 & 1.05 & 1 & 1.55 & 2.5 \end{smallmatrix}\right].$$

$F_{reg} = -\widehat{R}^{-1}(S_2^T + B_2^T K_{\mathtt{max}}) = \begin{bmatrix} \frac{37}{15} & \frac{1}{15} & -\frac{1}{3} & 1.4 & \frac{2}{3} \\ 1.5 & -0.5 & -0.5 & 1 & 0.5 \end{bmatrix}$. Then, the feedback law $u_1 = F_p x(t) + F_d \frac{d}{dt} x(t)$, $u_2 = F_{reg} x(t)$ solves the given singular LQR problem.
The closed-loop system is given by $E_{CL} \frac{d}{dt} x(t) = A_{CL} x(t)$, where $E_{CL} = (I_5 - B_r F_d)$ and $A_{CL} = (A_r - LK_{\mathtt{max}} + B_r F_p)$. It can be verified that $\mathtt{det}(sE_{CL} - A_{CL}) = -\frac{71}{12}(s + \frac{2}{3})$, that is, the matrix pencil $(sE_{CL} - A_{CL})$ is regular.
**Simulation result:** For the given singular LQR problem, we use the feedback law $u_1 = F_p x(t) + F_d \frac{d}{dt} x(t)$, $u_2 = F_{reg} x(t)$ to the primal. Then, for the initial condition $x_0 = \left[\, 0 \;\; -1 \;\; 1.2 \;\; -3 \;\; 1 \,\right]^T$ the system exhibits the trajectory as shown in Figure 1. For

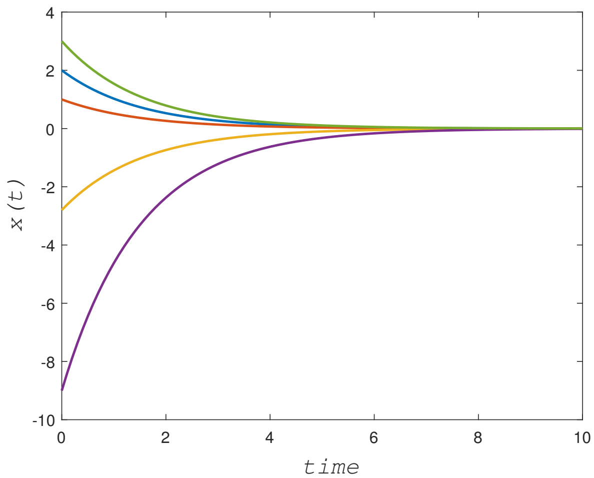


Fig. 1. *The state trajectory under the optimal PD feedback law for the illustrative example*

the given initial condition, the optimal trajectory is analytically found to be $x^*(t) = \begin{bmatrix} 2 \\ 1 \\ -2.8 \\ -9 \\ 3 \end{bmatrix} e^{-\frac{2}{3}t}$. The trajectory shown in the figure matches with this trajectory.

**8. Comparison with the existing results in the literature.** In this section we compare our results with the ones presented in [4] and [5]. We show that the result presented in this paper overcomes the restrictions of the aforementioned works.

**8.1. Comparison with the result presented in [4].** In [4], the authors provide a polynomial matrix based method to design a PD feedback controller that solves a given singular LQR problem. But, unfortunately, the result presented there has several shortcomings which we discuss next.

- The most important shortcoming of [4] is that it cannot account for arbitrary initial conditions, which is not desirable; because the initial condition of a state space system should ideally be free. [4] considers only those initial conditions for which the optimal state does not contain any impulses, while the optimal input may contain $\delta$, but never $\delta^{(1)}$ or any higher derivatives. The authors call such initial conditions which does not satisfy this condition the *inadmissible* initial conditions. Using the results presented in our paper, it can be shown that such a condition is satisfied if and only if the initial condition belongs to the subspace $\texttt{im}\begin{bmatrix} V_{1\Lambda} & B_r \end{bmatrix}$. On the other hand, the result presented in this paper does not impose any restriction on the initial condition of the system.
- The applicability of the result in [4] needs the system to be controllable. However, the result presented here needs only stabilizability of the system, which is a standard assumption in the literature.
- Another assumption of [4] that we do not need in this paper is the observability of the pair $(Q, A)$.

**8.2. Comparison with the result presented in [5].** The deflating subspace based method presented in [5] assumes that the states and the inputs of the system are from the space of locally square-integrable functions, that is, $\mathfrak{L}_2^{\text{loc}}$. This assumption, in turn, imposes a restriction on the initial condition $x_0$ of the system. This is due to the fact that for an arbitrary $x_0$, the optimal trajectory of a singular LQR problem is distributional in nature, that is, it contains impulses and its derivatives [3]. Therefore, the optimal trajectory does not belong to the space $\mathfrak{L}_2^{\text{loc}}$. Even though the cost functional can be made arbitrarily close to the optimal cost, it will never achieve the optimal cost using an input from $\mathfrak{L}_2^{\text{loc}}$. As has been shown in the illustrative example in Section 7 that corresponding to an arbitrary initial condition $x_0 = V_{1\Lambda}\alpha + W\beta$, both the optimal state $x^*$ and the optimal input $u^* = \texttt{col}(u_1^*, u_2^*)$ are distributional in nature and hence do not belong to $\mathfrak{L}_2^{\text{loc}}$. It can be easily verified that the optimal state and the optimal input belongs to $\mathfrak{L}_2^{\text{loc}}$ only if $\beta = 0$, that is, the initial condition is restricted to the subspace $\texttt{im}\, V_{1\Lambda}$.

The most important advantage of the result presented here is the implementability of the optimal input as a PD state-feedback over the implicit control law of the form $Px + Tu = 0$ as presented in [5]. To demonstrate this, we use the same example that has been presented in Section 7. Following the method presented in [5], we evaluate $\mathcal{L}_{\texttt{t}}(K)$ defined in equation 2.3 at $K_{\texttt{max}}$ and then obtain a factorization of $\mathcal{L}_{\texttt{t}}(K_{\texttt{max}})$ as

$$\mathcal{L}_{\texttt{t}}(K_{\texttt{max}}) = \begin{bmatrix} A^T K_{\texttt{max}} + K_{\texttt{max}} A + Q & K_{\texttt{max}} B_1 & K_{\texttt{max}} B_2 + S_2 \\ B_1^T K_{\texttt{max}} & 0 & 0 \\ B_2^T K_{\texttt{max}} + S_2^T & 0 & \widehat{R} \end{bmatrix} = \begin{bmatrix} P^T \\ T_1^T \\ T_2^T \end{bmatrix} \begin{bmatrix} P & T_1 & T_2 \end{bmatrix}$$

with $P \in \mathbb{R}^{4\times 5}$ and $T_1, T_2 \in \mathbb{R}^{4\times 2}$. It can be verified that

$$P = \begin{bmatrix} -9.732429 & -1.4724515 & 0.059265 & -4.895847 & -3.3048654 \\ -0.2425838 & -3.5039419 & -2.4053162 & 1.0813662 & -0.2253242 \\ 0.1039512 & -0.4134482 & 0.2850169 & -0.0905329 & 0.8929492 \\ -0.1008634 & -0.6191429 & 0.3601488 & -0.2305061 & 1.1089344 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } T_2 = \begin{bmatrix} 2.323045 & 0.6121941 \\ -0.9972075 & -0.8357937 \\ 1.6152251 & -1.4021297 \\ 0.0093261 & 0.9801527 \end{bmatrix}$$

achieve the desired factorization. Therefore, the control law proposed in [5] is given by $Px + T_1u_1 + T_2u_2 = 0$, that is, $Px + T_2u_2 = 0$. Note that, the optimal trajectory that has been evaluated in the illustrated example also satisfies this control law. However, this control law, unfortunately, cannot be implemented as a feedback law, because the law does not provide any information about the input $u_1$. On the other hand, using the method presented in this paper, we have provided a PD feedback controller that solves the singular LQR problem given in Section 7. A feedback controller is always advantageous from an engineering point of view, which is bolstered by [6].

**9. Conclusion.** In this paper, we first presented a method to compute the maximal rank-minimizing solution of the LMI arising from a singular LQR problem (Theorem 3.3). We have developed this method using the notions of slow space (weakly unobservable subspace) of the Hamiltonian system and the fast space (strongly reachable subspace) of the primal. We have shown that augmenting the basis of the good slow space of the Hamiltonian system $\Sigma_{\text{Ham}}$ with the basis of the fast space of the primal $\Sigma$ is the crucial idea that leads to the method. Using the maximal rank-minimizing solution, we computed the optimal trajectories for the singular LQR problem. Finally, we provided a feedback law of the form $u = F_p x + F_d \frac{d}{dt}x$, i.e., a PD feedback that solves the singular LQR problem. This work makes use of the ideas introduced in [3], [16], [17] that used impulsive-smooth distributions as the function-space for the states and inputs. Such a setting seems particularly advantageous for differential-algebraic systems, since such systems inherently admit impulsive states. Hence, the approach adapted in this paper to solve singular LQR problems for state-space systems have the potential of being generalized to differential-algebraic systems as well. This will be a matter of our future research.

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