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REPRESENTATION FORMULAE FOR DISCRETE 2D AUTONOMOUS SYSTEMS*

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Abstract. In this paper, we provide solution formulae for higher order discrete two-dimensional (2D) autonomous systems. We first consider a special type of 2D autonomous systems. These systems are described by equations that satisfy a certain special property: the module of equations contains elements of the form $(\sigma_2^n + a_{n-1}(\sigma_1)\sigma_2^{n-1} + \cdots + a_0(\sigma_1))w_j = 0$, for each dependent variable w_j, where $a_i(\sigma_1) \in \mathbb{R}[\sigma_1^{\pm 1}]$ and $a_0(\sigma_1)$ is a *unit* in $\mathbb{R}[\sigma_1^{\pm 1}]$. We show that this property is equivalent to the corresponding quotient module being *finitely generated* as a module over the 1-variable Laurent polynomial ring $\mathbb{R}[\sigma_1^{\pm 1}]$. We then show that solutions to these special systems can be viewed as evolutions along the second coordinate direction of certain suitably chosen one-dimensional (1D) trajectories over the first coordinate direction. Consequently, we show that these solutions can be written in terms of various integer powers of a square 1-variable Laurent polynomial matrix $A(\sigma_1)$ acting on suitable 1D trajectories. Following the 1D terminology we call these 1D trajectories *initial conditions*. We call this form of expressing the solutions a *representation formula*. Then, in order to extend this result to general 2D autonomous systems, we obtain an analogue of a classical algebraic result, called *Noether's normalization lemma*, for the Laurent polynomial ring in two variables. Using this result we show that every 2D autonomous system admits a representation formula through a suitable coordinate transformation in the domain $\mathbb{Z}^2$. Further, we analyze the set of initial conditions that appear in our representation formulae and resolve the issue of how freely these initial conditions can be chosen.

Key words. 2D systems, first order representation, time/space relevant systems, Laurent polynomial ring, Noether's normalization

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1. Introduction.

First order representations of systems of partial differential or difference equations, usually termed nD continuous or discrete systems, respectively, have been a topic of active research for the past few decades; see, for example, [4, 5, 14, 15, 13, 23]. For ordinary differential/difference equations (i.e., one-dimensional (1D) systems), a first order representation in input/state/output (or simply i/s/o) form is almost always assumed to be the starting point. This is not the case for nD systems with $n \geqslant 2$ (see [4, 22, 14]). For example, Maxwell's equations are first order, but heat equations or wave equations are not. In [20], Willems demonstrated how a first order representation can be obtained from a general higher order representation for 1D systems. For discrete two-dimensional (2D) systems, a similar construction was provided in [14] using the behavioral description. In [4, 5] i/s/o representations were constructed for 2D systems described in input/output form.

For 1D systems, an i/s/o representation provides a representation formula for the solutions in terms of the "flow" operator acting on the initial conditions plus the "input" convolved with the flow. Unfortunately, an analogous representation formula is absent for nD systems. The main difficulty in obtaining such a formula

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stems from the fact that, unlike the 1D case, nD systems do not have an a priori fixed direction of evolution. In this regard there have been two major approaches. In one of them, one independent variable, namely 'time,' is given preference over the others, and systems which are first order in that particular variable are considered (see [2, 16, 7]). In the other approach, for the case of continuous autonomous systems, a representation formula is given in terms of integrations on the "characteristic variety" of the system. This representation formula is known as the *Ehrenpreis–Palamodov integral representation formula*, see [18, 1].

Unfortunately, a shortcoming of the first approach is that there are many systems which cannot be brought to a first order form in the special variable [7]. For these cases, it is more advantageous to treat both variables equally. On the other hand, a drawback of the integral representation formula is that it first requires a complete knowledge of the points in the characteristic variety, and then an integration to be evaluated on this variety with suitable measures; both of these processes may be computationally very challenging.

In this paper, we shall present a representation theory for discrete 2D autonomous systems, which is similar to the integral representation formula of Ehrenpreis and Palamodov, although, as we shall see, our representation formulae require neither a precise knowledge of the points of the characteristic variety nor integrations to be performed on it. We show that every 2D autonomous system admits a representation formula in terms of a flow matrix acting on the initial conditions. Interestingly, it turns out that the initial conditions are either finite-dimensional vectors or infinite-dimensional trajectories depending upon whether the characteristic variety is zero-dimensional or one dimensional. We also show that for the case when the characteristic variety is one dimensional, the initial condition trajectories can be freely chosen if and only if the system is described as the kernel of a "square" partial difference operator matrix.

In [8], the ramifications of the integral representation formula in the context of discrete systems and nonautonomous systems was posed as an open problem. This paper provides a partial solution to this problem for autonomous systems with $n = 2$.

1.1. Organization of the paper. In section 2, we provide some basic preliminaries required for the rest of the paper. Then in section 3, we provide our first main result of the paper, Theorem 3.7, which gives a representation formula for a special type of autonomous systems. We call this special type of systems *strongly σ_2-relevant*, for it is a stronger notion of 'time/space-relevant' systems with respect to the one introduced in [7]. Section 4 deals with the discrete version of Noether's normalization lemma. We extend this result to cater to submodules in section 5. With the help of this extension of Noether's normalization, we then show that every autonomous system can be converted to a strongly σ_2-relevant one by a coordinate transformation on the domain $\mathbb{Z}^2$. Consequently, we arrive at a representation formula for a general autonomous system. In section 6, we look at the set of initial conditions and address the issue of initial conditions that can be chosen freely. Finally, we summarize the results in section 7.

1.2. Notation. We use $\mathbb{R}$ and $\mathbb{C}$ to denote the fields of real and complex numbers, respectively. Consequently, $\mathbb{R}^n$, $\mathbb{C}^n$ denote the n-dimensional vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of integers is denoted by $\mathbb{Z}$, and $\mathbb{Z}^2$ denotes the set of two tuples of elements in $\mathbb{Z}$. In this paper, our main object of study is a particular class of doubly indexed sequences of elements in $\mathbb{R}^{\mathtt{w}}$ for some positive integer $\mathtt{w}$. We denote the set of doubly indexed sequences in $\mathbb{R}^{\mathtt{w}}$ by $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, i.e., $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} := \{\mathbb{Z}^2 \to \mathbb{R}^{\mathtt{w}}\}$. The

Laurent polynomial ring in two indeterminates σ_1, σ_2, usually written as $\mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]$, will be denoted by $\mathcal{A}$, and the same in one indeterminate σ_1, written as $\mathbb{R}[\sigma_1^{\pm 1}]$, will be denoted by $\mathcal{A}_1$. We use $\mathcal{A}^{\mathtt{w}}$ to denote the free module of rank $\mathtt{w}$ over $\mathcal{A}$, where the elements of $\mathcal{A}^{\mathtt{w}}$ are written as $\mathtt{w}$-tuples of rows. For a set S, we use $S^{m\times n}$ to denote the set of $(m \times n)$ matrices with entries from the set S. We often use $\mathrm{col}(\bullet)$ to stack up entries one above the other to make a column-vector. The single letter σ is often used to denote the tuple (σ_1, σ_2). Further, for an integer tuple $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$, the symbol σ^ν denotes the monomial $\sigma_1^{\nu_1}\sigma_2^{\nu_2}$. In this paper, we follow the bar notation to denote equivalence classes: for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and a submodule $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$, we use $\overline{r(\sigma)}$ to denote the equivalence class of $r(\sigma)$ in the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$.

2. Background. In this paper, we consider systems described by a set of 2D linear partial difference equations with constant real coefficients. Such partial difference equations are described by using the 2D shift operators σ_1 and σ_2. These shift operators act on a doubly indexed real-valued sequence $w \in \mathbb{R}^{\mathbb{Z}^2}$ as follows: for $\nu' := (\nu_1', \nu_2'), \nu := (\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$(\sigma^{\nu'} w)(\nu_1, \nu_2) = w(\nu_1 + \nu_1', \nu_2 + \nu_2'). \tag{2.1}$$

This definition can be extended naturally to define the action of $\mathcal{A}$, the Laurent polynomial ring in the shifts, on $\mathbb{R}^{\mathbb{Z}^2}$. Let $f(\sigma) \in \mathcal{A}$ be given by $f(\sigma) = \sum_{\nu\in\mathbb{Z}^2} \alpha_\nu \sigma^\nu$ with $\alpha_\nu \in \mathbb{R}$ and only finitely many $\alpha_\nu \neq 0$. Then the action of this $f(\sigma)$ on $w \in \mathbb{R}^{\mathbb{Z}^2}$ is given by

$$(f(\sigma)w)(\nu_1, \nu_2) = \sum_{\nu\in\mathbb{Z}^2} \alpha_\nu (\sigma^\nu w)(\nu_1, \nu_2).$$

Consequently, for a row-vector $r(\sigma) = \begin{bmatrix} r_1(\sigma) & r_2(\sigma) & \cdots & r_{\mathtt{w}}(\sigma) \end{bmatrix}$ and a column-vector $w = \mathrm{col}(w_1, w_2, \dots, w_{\mathtt{w}}) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ we define

$$r(\sigma)w := r_1(\sigma)w_1 + r_2(\sigma)w_2 + \cdots + r_{\mathtt{w}}(\sigma)w_{\mathtt{w}} \in \mathbb{R}^{\mathbb{Z}^2}. \tag{2.2}$$

Equation (2.2) defines the action of the row module $\mathcal{A}^{\mathtt{w}}$ on $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$.

The collection of trajectories $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ that satisfy a given set of partial difference equations is called the *behavior* of the system and is denoted by $\mathfrak{B}$. The above description of the action of $\mathcal{A}^{\mathtt{w}}$ on $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ gives the following representation of behaviors of 2D partial difference equations:

$$\mathfrak{B} := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid R(\sigma)w = 0\}, \tag{2.3}$$

where $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$. Equation (2.3) is called a *kernel representation* of $\mathfrak{B}$ and written as $\mathfrak{B} = \ker(R(\sigma))$. Note that many different matrices can have the same kernel. Importantly, all matrices having the same rowspan over $\mathcal{A}$ result in the same behavior. This leads to the following equivalent definition of behaviors: let $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$ and $\mathcal{R} := \mathrm{rowspan}(R(\sigma))$, then the behavior $\mathfrak{B} = \ker(R(\sigma))$ can be written as

$$\mathfrak{B}(\mathcal{R}) := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid r(\sigma)w = 0 \text{ for all } r(\sigma) \in \mathcal{R}\}. \tag{2.4}$$

In [8] it was shown that for two submodules $\mathcal{R}_1, \mathcal{R}_2$ of $\mathcal{A}^{\mathtt{w}}$, $\mathfrak{B}(\mathcal{R}_1) = \mathfrak{B}(\mathcal{R}_2)$ *if and only if* $\mathcal{R}_1 = \mathcal{R}_2$. This was shown to be a consequence of a general result that $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, as an $\mathcal{A}$-module, is a *large injective cogenerator*. We state this important result in the following proposition.

PROPOSITION 2.1. *Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathcal{A}^{\mathtt{w}}$ be two submodules, and let $\mathfrak{B}(\mathcal{R}_i)$, for $i = 1, 2$, be as defined in* (2.4). *Then $\mathfrak{B}(\mathcal{R}_1) = \mathfrak{B}(\mathcal{R}_2)$ if and only if $\mathcal{R}_1 = \mathcal{R}_2$.*

Proposition 2.1 above shows a one-to-one correspondence between behaviors and submodules of $\mathcal{A}^{\mathtt{w}}$. Following this one-to-one correspondence, we call $\mathcal{R}$ *the equation module* of $\mathfrak{B}$.

In this paper, we provide representation formulae for a special type of 2D systems, namely *autonomous systems*. There are several equivalent definitions of 2D autonomous systems (see [6, 10, 19]). In this paper, we stick to the more algebraic definition of [10], which we state below as Definition 2.2. In Definition 2.2 and later we need the notion of *characteristic ideal* of a behavior. Let $\mathfrak{B}$ be a behavior given by a kernel representation $\mathfrak{B} = \ker(R(\sigma))$ with $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$. The *characteristic ideal* of $\mathfrak{B}$, denoted by $\mathcal{I}(\mathfrak{B})$, is defined as the ideal of $\mathcal{A}$ generated by the $(\mathtt{w} \times \mathtt{w})$ minors of $R(\sigma)$. For $\mathtt{g} < \mathtt{w}$, $\mathcal{I}(\mathfrak{B})$ is defined to be the zero ideal. Although the definition is given in terms of a kernel representation matrix, it was shown in [10] that the ideal generated by the $(\mathtt{w}\times\mathtt{w})$ minors remain unchanged if $R(\sigma)$ is replaced by another equivalent kernel representation matrix. That way, by Proposition 2.1, the characteristic ideal is an invariant of the behavior; this justifies the argument $\mathfrak{B}$ in the notation $\mathcal{I}(\mathfrak{B})$.

DEFINITION 2.2. *A behavior $\mathfrak{B}$ is said to be* autonomous *if its characteristic ideal $\mathcal{I}(\mathfrak{B})$ is nonzero. Further, an autonomous behavior is said to be* strongly autonomous *if the quotient ring $\mathcal{A}/\mathcal{I}(\mathfrak{B})$ is a finite-dimensional vector space over* $\mathbb{R}$.[1]

From the definition of $\mathcal{I}(\mathfrak{B})$ it follows that $\mathfrak{B}$ is autonomous if and only if it admits a kernel representation matrix that is *full column-rank* over the field of fractions of $\mathcal{A}$. It is well known that strongly autonomous systems admit first order representations with a tuple of system matrices in the following manner (see [14, 6]):

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid \exists x \in (\mathbb{R}^n)^{\mathbb{Z}^2} \text{ such that } \sigma_1 x = A_1 x,\ \sigma_2 x = A_2 x,\ w = Cx \right\},$$

where n is a positive integer, $A_1, A_2 \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{\mathtt{w}\times n}$, with A_1, A_2 nonsingular and satisfying $A_1A_2 = A_2A_1$. Consequently, trajectories in a strongly autonomous behavior admit the following representation formula: for all $(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$w(\nu_1, \nu_2) = CA_1^{\nu_1}A_2^{\nu_2}x(\mathbf{0}), \tag{2.5}$$

where $x(\mathbf{0}) \in \mathbb{R}^n$ is an arbitrary initial condition. For this reason, we consider in this paper only those systems which are *not* strongly autonomous. We aim for a representation formula, analogous to (2.5) above, for general autonomous behaviors which are not strongly autonomous.

Given a behavior $\mathfrak{B} = \ker(R(\sigma))$, let $\mathcal{R}$ be the submodule of $\mathcal{A}^{\mathtt{w}}$ spanned by the rows of $R(\sigma)$. We define

$$\mathcal{M} := \mathcal{A}^{\mathtt{w}}/\mathcal{R}$$

and call it the *quotient module* of $\mathfrak{B}$. This quotient module $\mathcal{M}$ plays a central role in this paper. We often let elements from $\mathcal{M}$ act on $\mathfrak{B}$. This action is defined as follows: for $m \in \mathcal{M}$, the action of m on $w \in \mathfrak{B}$ is defined to be the action of a lift of m in $\mathcal{A}^{\mathtt{w}}$ on w. For example, let $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ be such that $\overline{r(\sigma)} = m \in \mathcal{M}$; then

$$mw := r(\sigma)w. \tag{2.6}$$

[1]In this case, when the quotient ring $\mathcal{A}/\mathcal{I}(\mathfrak{B})$ is a finite-dimensional vector space over $\mathbb{R}$, the characteristic ideal $\mathcal{I}(\mathfrak{B})$ is called a *zero-dimensional* ideal and $\mathcal{A}/\mathcal{I}(\mathfrak{B})$ is called an *Artinian* ring (see [3]).

Note that m may have several distinct lifts in $\mathcal{A}^{\mathtt{w}}$, but all of them have the same action on $w \in \mathfrak{B}$. This is because if $r_1(\sigma), r_2(\sigma) \in \mathcal{A}^{\mathtt{w}}$ are two distinct lifts of the same element $m \in \mathcal{M}$, then we have $r_1(\sigma) - r_2(\sigma) \in \mathcal{R}$. Therefore,

$$(r_1(\sigma) - r_2(\sigma))\, w = 0, \quad \text{i.e., } r_1(\sigma)w = r_2(\sigma)w.$$

Thus, the above action of elements in $\mathcal{M}$ on $\mathfrak{B}$ is well defined.

Now note that it follows from Definition 2.2 above that $\mathfrak{B}$ is autonomous *if and only if* the quotient module $\mathcal{M}$ is a *torsion module*; i.e., for every $\overline{m(\sigma)} \in \mathcal{M}$ there exists an $f(\sigma) \in \mathcal{A}$ such that $\overline{f(\sigma)m(\sigma)} = 0 \in \mathcal{M}$. In that case we get the following ideal, called the *annihilator ideal* of $\mathcal{M}$:

$$\operatorname{ann}(\mathcal{M}) := \{f(\sigma) \in \mathcal{A} \mid \overline{f(\sigma)m(\sigma)} = 0 \in \mathcal{M} \text{ for all } m(\sigma) \in \mathcal{A}^{\mathtt{w}}\}.$$

The following is an important relation between the characteristic ideal of an autonomous behavior and the annihilator ideal of the corresponding quotient module. The proof of this result follows from Fitting's lemma (see [3]).

Proposition 2.3. *Let $\mathfrak{B}$ be an autonomous behavior with its equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$. Let $\mathcal{M}$ denote the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$. Then we have*

$$\operatorname{ann}(\mathcal{M})^{\mathtt{w}} \subseteq \mathcal{I}(\mathfrak{B}) \subseteq \operatorname{ann}(\mathcal{M}).$$

In particular,

$$\sqrt{\operatorname{ann}(\mathcal{M})} = \sqrt{\mathcal{I}(\mathfrak{B})}.$$

Remark 2.4. Note that $\mathcal{M}$ is naturally a finitely generated *faithful* module over $\mathcal{A}/\operatorname{ann}(\mathcal{M})$, meaning $\mathcal{M}$, considered as a module over the ring $\mathcal{A}/\operatorname{ann}(\mathcal{M})$, has $\{0\} \subseteq \mathcal{A}/\operatorname{ann}(\mathcal{M})$ for its annihilator. That is,

$$\{f \in \mathcal{A}/\operatorname{ann}(\mathcal{M}) \mid fm = 0 \in \mathcal{M} \text{ for all } m \in \mathcal{M}\} = \{0\}.$$

In other words, $\mathcal{M}$ being faithful over $\mathcal{A}/\operatorname{ann}(\mathcal{M})$ implies that $\mathcal{M}$ contains an isomorphic copy of $\mathcal{A}/\operatorname{ann}(\mathcal{M})$ as a submodule (see [3]).

Change of coordinates in $\mathbb{Z}^2$ plays a crucial role throughout this paper. The idea of change of coordinates and its effects on a behavior are not new (see [22, 19]); however, we use this idea to achieve goals which are different from those achieved in the aforementioned references. By a coordinate change we mean a $\mathbb{Z}$-linear map from $\mathbb{Z}^2$ to itself of the form

$$\begin{array}{rcl} T : \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ \operatorname{col}(\nu_1, \nu_2) =: \nu & \mapsto & T\nu, \end{array}$$

where $T \in \mathbb{Z}^{2\times 2}$ is a *unimodular* matrix (i.e., $\det(T) = \pm 1$). Note that because of unimodularity, the columns of T span the whole of $\mathbb{Z}^2$ as a $\mathbb{Z}$-module. Such a coordinate transformation T induces the two maps

$$(2.7) \qquad \begin{array}{rcl} \varphi_T : \mathcal{A} & \to & \mathcal{A} \\ \sigma^{\nu} & \mapsto & \sigma^{T\nu} \end{array} \qquad \Bigg| \qquad \begin{array}{rcl} \Phi_T : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} & \to & (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \\ w(\nu) & \mapsto & w(T\nu), \end{array}$$

for all $\nu \in \mathbb{Z}^2$. Unimodularity of T makes both of these maps *bijective*. In fact, Φ_T is an automorphism of the $\mathbb{R}$-vector space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, while φ_T is an automorphism of the $\mathbb{R}$-algebra $\mathcal{A}$. As a consequence, an *ideal* $\mathfrak{a} \subseteq \mathcal{A}$ is mapped to another *ideal* $\varphi_T(\mathfrak{a})$. The map φ_T can be extended to a map from $\mathcal{A}^{\mathtt{w}}$ to itself:

$$\begin{aligned}\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} &\to \mathcal{A}^{\mathtt{w}},\\ \begin{bmatrix} f_1(\sigma) & f_2(\sigma) & \cdots & f_{\mathtt{w}}(\sigma)\end{bmatrix} &\mapsto \begin{bmatrix}\varphi_T(f_1(\sigma)) & \varphi_T(f_2(\sigma)) & \cdots & \varphi_T(f_{\mathtt{w}}(\sigma))\end{bmatrix}.\end{aligned} \tag{2.8}$$

The map $\widehat{\varphi}_T$ is an $\mathcal{A}$-module morphism via the automorphism φ_T; i.e., for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and $f(\sigma) \in \mathcal{A}$,

$$\widehat{\varphi}_T(f(\sigma)r(\sigma)) = \varphi_T(f(\sigma))\widehat{\varphi}_T(r(\sigma)).$$

The bijective property of φ_T extends to the module case: as a result, $\widehat{\varphi}_T(\mathcal{R})$, the image of a *submodule* $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ under $\widehat{\varphi}_T$, is also a *submodule*.

In the next two results we bring out the relation between the two maps $\widehat{\varphi}_T$ and Φ_T.

LEMMA 2.5. *Let $v, w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ be related by $v = \Phi_T(w)$. Then for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ we have*

$$r(\sigma)v = \Phi_T(\widehat{\varphi}_T(r(\sigma))w).$$

Proof. It is enough to prove the result for the scalar case; that is, for $v, w \in \mathbb{R}^{\mathbb{Z}^2}$ and $f(\sigma) \in \mathcal{A}$ we must have

$$f(\sigma)v = \Phi_T(\varphi_T(f(\sigma))w). \tag{2.9}$$

This is because $\widehat{\varphi}_T$ is defined as φ_T applied elementwise, and Φ_T is linear. Further, note that every Laurent polynomial is a finite $\mathbb{R}$-linear combination of monomials. Therefore, once again by the linearity of Φ_T, it is enough to prove (2.9) for $\mathcal{A} \ni f(\sigma) = \sigma^\nu$, where $\nu \in \mathbb{Z}^2$ is arbitrary.

Now, $v, w \in \mathbb{R}^{\mathbb{Z}^2}$ are assumed to be related by $v = \Phi_T(w)$. First, note that for $\nu, \nu' \in \mathbb{Z}^2$, the action of the monomial σ^ν on v is given by

$$\begin{aligned}(\sigma^\nu v)(\nu') = v(\nu' + \nu) = \Phi_T(w)(\nu' + \nu) = w(T(\nu' + \nu)) &= (\sigma^{T\nu} w)(T\nu')\\ &= \Phi_T(\varphi_T(\sigma^\nu)w)(\nu').\end{aligned}$$

Since ν' was arbitrary, we get for all $\nu \in \mathbb{Z}^2$

$$\sigma^\nu v = \Phi_T(\varphi_T(\sigma^\nu)w). \tag{2.10}$$

This is what we had claimed. ∎

Given a behavior $\mathfrak{B}$, we now define

$$\Phi_T(\mathfrak{B}) := \{v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid v = \Phi_T(w) \text{ for some } w \in \mathfrak{B}\}. \tag{2.11}$$

THEOREM 2.6. *Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule with behavior $\mathfrak{B}(\mathcal{R})$, and let $T \in \mathbb{Z}^{2\times 2}$ be unimodular. Then we have*

$$\mathfrak{B}(\mathcal{R}) = \Phi_T(\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))). \tag{2.12}$$

Proof. We first prove $\mathfrak{B}(\mathcal{R}) \supseteq \Phi_T(\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R})))$. Let $w \in \mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$, we want to show that $\Phi_T(w) = w \circ T =: v \in \mathfrak{B}(\mathcal{R})$. Note that by Lemma 2.5, for any $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$, we have

$$r(\sigma)v = \Phi_T(\widehat{\varphi}_T(r(\sigma))w).$$

In particular, for all $r(\sigma) \in \mathcal{R}$, we have $\widehat{\varphi}_T(r(\sigma)) \in \widehat{\varphi}_T(\mathcal{R})$. Therefore, for all $r(\sigma) \in \mathcal{R}$,

$$r(\sigma)v = \Phi_T(\widehat{\varphi}_T(r(\sigma))w) = 0 \text{ (since } w \in \mathfrak{B}(\widehat{\varphi}_T(\mathcal{R})))$$
$$\Rightarrow v \in \mathfrak{B}(\mathcal{R}).$$

For the converse, note that since T is unimodular it has an inverse in $\mathbb{Z}^{2\times 2}$. Then replacing T by T^{-1} and $\mathcal{R}$ by $\widehat{\varphi}_T(\mathcal{R})$ in the above chain of arguments, we get $\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R})) \supseteq \Phi_{T^{-1}}(\mathfrak{B}(\mathcal{R}))$ because $\widehat{\varphi}_{T^{-1}}(\widehat{\varphi}_T(\mathcal{R})) = \mathcal{R}$. It then follows that $\Phi_T(\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))) \supseteq \mathfrak{B}(\mathcal{R})$ because $\Phi_T \circ \Phi_{T^{-1}} = \mathrm{id}$. ◻

3. Representation formula for a special type of autonomous systems. In [7] the notion of "time/space-relevant" discrete 2D autonomous systems was introduced. For "square"[2] 2D time-relevant systems, it was shown in [7] how a state-variable type of first order representation can be obtained. In this section, we define a stronger notion of relevance with respect to one coordinate axis for discrete 2D autonomous systems. Further, we show how strong relevance of an autonomous system leads to a representation formula for the trajectories in it. This result will prove to be essential for a representation formula of general autonomous systems.

In [7], the notion of time-relevance has been defined in terms of "characteristic sets." A set $\mathcal{S} \subseteq \mathbb{Z}^2$ is said to be a characteristic set of a 2D autonomous behavior $\mathfrak{B}$ if every trajectory in $\mathfrak{B}$ is uniquely determined by its restriction to $\mathcal{S}$ (see [13, 19]). In mathematical terms, for $w_1, w_2 \in \mathfrak{B}$ we have $w_1 = w_2$ if and only if $w_1|_{\mathcal{S}} = w_2|_{\mathcal{S}}$, where $w|_{\mathcal{S}}$ denotes the restriction of w to $\mathcal{S}$. With this notion of characteristic sets, σ_2*-relevant* systems can be defined following the definition of time-relevant systems of [7].

DEFINITION 3.1. *A* 2*D behavior* $\mathfrak{B}$ *is said to be* σ_2-relevant *if for every* $k \in \mathbb{Z}$ *the subset of* $\mathbb{Z}^2$ *of the form*

$$\mathcal{S}_k := \{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid \nu_2 \leqslant k\}$$

is a characteristic set of $\mathfrak{B}$.

An algebraic criterion equivalent to time-relevance (and therefore σ_2-relevance) of square systems was subsequently presented in [7]. According to this algebraic formulation, a square 2D behavior $\mathfrak{B}$ is σ_2-relevant if and only if it admits a kernel representation matrix $R(\sigma) \in \mathcal{A}^{\mathtt{w}\times\mathtt{w}}$ having the form

$$R(\sigma) = I_{\mathtt{w}} + R_1(\sigma_1)\sigma_2^{-1} + \cdots + R_L(\sigma_1)\sigma_2^{-L}, \tag{3.1}$$

where L is a finite positive integer and $R_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$ for $1 \leqslant i \leqslant L$. Definition 3.2 below defining *strongly* σ_2*-relevant* systems is inspired from this algebraic formulation. Note that it follows from (3.1) that $\mathfrak{B}$ is σ_2-relevant if and only if it admits a kernel representation matrix of the form

$$R(\sigma) = I_{\mathtt{w}}\sigma_2^L + R_1(\sigma_1)\sigma_2^{L-1} + \cdots + R_L(\sigma_1), \tag{3.2}$$

where L is a finite positive integer and $R_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$ for $1 \leqslant i \leqslant L$. This, in turn, is equivalent to saying that every row-vector in $\mathcal{A}^{\mathtt{w}}$ with entries having only *nonnegative* powers in σ_2 is equivalent modulo the equation module $\mathcal{R}$ to a row-vector in $\mathcal{A}^{\mathtt{w}}$ with entries having only *finitely many* powers of σ_2, namely $1, \sigma_2, \sigma_2^2, \ldots, \sigma_2^{L-1}$. Now, let us denote by $\mathcal{A}_1[\sigma_2]$ the ring of polynomials with nonnegative powers in σ_2 with

[2] By "square" it is meant that the system admits a square kernel representation matrix with nonzero determinant (see [19]).

coefficients from $\mathcal{A}_1$. Then the above condition translates, in algebraic terms, to the following:

The $\mathcal{A}_1[\sigma_2]$-module $\mathcal{A}_1[\sigma_2]^{\mathtt{w}}$ quotiented by the rowspan of $R(\sigma)$ over $\mathcal{A}_1[\sigma_2]$ is *finitely generated* as a module over $\mathcal{A}_1$.

In Definition 3.2 we extend this property of the quotient module being a finitely generated module over $\mathcal{A}_1$ to encompass not only nonnegative powers of σ_2, but also negative powers as well. In other words, we call a behavior strongly σ_2-relevant if the quotient module $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Note that we no longer restrict ourselves to systems described by square kernel representation matrices.

DEFINITION 3.2. *Let $\mathfrak{B}$ be an autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$. Then $\mathfrak{B}$ is said to be* strongly σ_2-relevant *if the quotient module $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$.*

Note that strongly autonomous systems are trivially strongly σ_2-relevant. Indeed, for strongly autonomous systems, $\mathcal{M}$ is a finite-dimensional vector space over $\mathbb{R}$, which is trivially a finitely generated module over $\mathcal{A}_1$. However, there are other strongly σ_2-relevant systems which are not strongly autonomous. The following is a scalar (i.e., $\mathtt{w} = 1$) example of one such system.

Example 3.3. Consider the behavior

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2^2 - 2\sigma_2 + 1 \\ \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1 \end{bmatrix}.$$

Since $\mathfrak{B}$ has only one manifest variable, the equation module $\mathcal{R}$ is the ideal $\mathfrak{a} := \langle \sigma_2^2 - 2\sigma_2 + 1, \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1 \rangle$. Consequently, the quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$. The presence of the polynomial $\sigma_2^2 - 2\sigma_2 + 1$ in the equation ideal $\mathfrak{a}$ implies that this $\mathcal{M}$ is a finitely generated module over $\mathcal{A}_1$. Indeed, every element in $\mathcal{M}$ can be written as a linear combination of $\{\overline{1}, \overline{\sigma_2}\}$ with coefficients coming from $\mathcal{A}_1$. First, note that $\sigma_2^2 - 2\sigma_2 + 1 \in \mathfrak{a}$ implies that $\sigma_2 - 2 + \sigma_2^{-1} \in \mathfrak{a}$. Therefore, $(\overline{\sigma_2})^{-1} = -\overline{\sigma_2} + 2 \in \mathcal{M}$. As a result, all negative powers of $\overline{\sigma_2}$ can be written as polynomials in nonnegative powers of $\overline{\sigma_2}$ with coefficients in $\mathbb{R}$. As a consequence, every Laurent polynomial in $\overline{\sigma_1}$ and $\overline{\sigma_2}$ is equal to a polynomial having only *nonnegative* powers of $\overline{\sigma_2}$ with coefficients from $\mathbb{R}[\overline{\sigma_1}^{\pm 1}] = \mathcal{A}_1/(\mathfrak{a} \cap \mathcal{A}_1)$. However, note that $\sigma_2^2 - 2\sigma_2 + 1$ is monic. Therefore, using the Euclidean division algorithm, all positive powers of $\overline{\sigma_2}$ greater than one can be expressed in terms of $\overline{\sigma_2}$ and $\overline{1}$. Therefore, given any polynomial, say $f(\sigma) \in \mathcal{A}$, it can be reduced to a polynomial $f_1(\sigma) \in \mathcal{A}$ having only nonnegative powers of σ_2. Further, one can find $a_1(\sigma_1), a_0(\sigma_1) \in \mathcal{A}_1$ and $q(\sigma) \in \mathcal{A}$ such that

$$f_1(\sigma) = q(\sigma)(\sigma_2^2 - 2\sigma_2 + 1) + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1).$$

In other words, $\overline{f(\sigma)} = \overline{a_1(\sigma_1)}\overline{\sigma_2} + \overline{a_0(\sigma_1)}$. This proves that every element in $\mathcal{M}$ can be written as a linear combination of $\overline{1}$ and $\overline{\sigma_2}$ with coefficients from $\mathcal{A}_1$. That is, $\mathcal{M}$ is finitely generated as a module over $\mathcal{A}_1$. Thus, $\mathfrak{B}$ above is strongly σ_2-relevant.

The notion of strong σ_2-relevance is indeed a stronger notion than the corresponding notion of σ_2-relevance of [7]. Observe that in the situation of (3.2), strong σ_2-relevance requires $R_L(\sigma_1)$ to be invertible in $\mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$. Thus for square autonomous systems, being strongly σ_2-relevant is a sufficient condition for σ_2-relevance but is clearly not necessary. For square autonomous systems, the difference between the σ_2-relevant system and the strongly σ_2-relevant system can also be expressed in terms of the characteristic sets. Whereas σ_2-relevant systems require only half-planes as characteristic sets, strongly σ_2-relevant systems (that are square) require "strips" to be characteristic sets. This fact is be clear from Proposition 3.4 below. In Proposition 3.4, we bring out several structural properties involving the equation module $\mathcal{R}$, the kernel representation matrices, and the annihilator ideal of the quotient module $\mathcal{M}$, which are all equivalent to strong σ_2-relevance. Note that the definition for strongly σ_2-relevant square autonomous systems could have been given in terms of characteristic sets. However, we use the property of $\mathcal{M}$ being finitely generated over $\mathcal{A}_1$ as the definition for strong σ_2-relevance, since this definition applies to general autonomous systems as opposed to square autonomous systems. Further, as an immediate consequence of this definition, we are able to construct certain useful matrices, which finally lead us to our desired representation formulae. We elaborate on this now.

Note that $\mathcal{M}$ being a finitely generated module over $\mathcal{A}_1$ implies that $\mathcal{M}$ admits a finite generating set as an $\mathcal{A}_1$-module. Suppose $\{g_1, g_2, \dots, g_n\} \subseteq \mathcal{M}$ is one such finite generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module, where n is some finite positive integer. This generating set can be used to define the following map of $\mathcal{A}_1$-modules from the free module of row-vectors $\mathcal{A}_1^n$ to $\mathcal{M}$ as

$$\begin{array}{rrcl} \psi: & \mathcal{A}_1^n & \to & \mathcal{M}, \\ & e_i & \mapsto & g_i \text{ for all } 1 \leqslant i \leqslant n, \end{array} \tag{3.3}$$

where e_i is the standard basis row-vector in $\mathcal{A}_1^n$, i.e.,

$$e_i := \overbrace{\begin{bmatrix} 0 & 0 & \cdots & \underset{\substack{\uparrow \\ i\text{th position}}}{1} & \cdots & 0 \end{bmatrix}}^{n \text{ entries}} \in \mathcal{A}_1^n.$$

So a row-vector $r(\sigma_1) = \begin{bmatrix} r_1(\sigma_1) & r_2(\sigma_1) & \cdots & r_n(\sigma_1) \end{bmatrix} \in \mathcal{A}_1^n$ is mapped to an element in $\mathcal{M}$ by ψ as

$$\psi(r(\sigma_1)) = r_1(\sigma_1)g_1 + r_2(\sigma_1)g_2 + \cdots + r_n(\sigma_1)g_n.$$

Now, since $\mathcal{M}$ is originally a module over $\mathcal{A}$, multiplication of elements in $\mathcal{M}$ by $f(\sigma) \in \mathcal{A}$ is well defined. When $\mathcal{M}$ is viewed as a module over $\mathcal{A}_1$, this multiplication defines a map from $\mathcal{M}$ to itself, which is clearly $\mathcal{A}_1$-linear. In mathematical terms, the map

$$\begin{array}{rcrcl} \mu_f & : & \mathcal{M} & \to & \mathcal{M}, \\ & & m & \mapsto & f(\sigma)m \end{array} \tag{3.4}$$

is a map of $\mathcal{A}_1$-modules. Now, if $\mathcal{M}$ is finitely generated as an $\mathcal{A}_1$-module, then this $\mathcal{A}_1$-linear map μ_f can be represented by a square matrix. Let $\{g_1, g_2, \dots, g_n\} \subseteq \mathcal{M}$ be a finite generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. Suppose, for $1 \leqslant i \leqslant n$,

$$\mu_f(g_i) = f(\sigma)g_i = a_{i,1}(\sigma_1)g_1 + a_{i,2}(\sigma_1)g_2 + \cdots + a_{i,n}(\sigma_1)g_n. \tag{3.5}$$

We define

$$A_f(\sigma_1) := \begin{bmatrix} a_{1,1}(\sigma_1) & a_{1,2}(\sigma_1) & \cdots & a_{1,n}(\sigma_1) \\ a_{2,1}(\sigma_1) & a_{2,2}(\sigma_1) & \cdots & a_{2,n}(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(\sigma_1) & a_{n,1}(\sigma_1) & \cdots & a_{n,n}(\sigma_1) \end{bmatrix}. \tag{3.6}$$

It then follows that for any $m \in \mathcal{M}$, if m is written as an $\mathcal{A}_1$-linear combination of the generators $\{g_1, g_2, \dots, g_n\}$, say

$$\begin{aligned} m &= m_1(\sigma_1)g_1 + m_2(\sigma_1)g_2 + \cdots + m_n(\sigma_1)g_n \\ &= \begin{bmatrix} m_1(\sigma_1) & m_2(\sigma_1) & \cdots & m_n(\sigma_1) \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}, \end{aligned}$$

then $f(\sigma)m \in \mathcal{M}$ will be represented in terms of the same generating set as

$$f(\sigma)m = \begin{bmatrix} m_1(\sigma_1) & m_2(\sigma_1) & \cdots & m_n(\sigma_1) \end{bmatrix} A_f(\sigma_1) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}. \tag{3.7}$$

In Proposition 3.4 below we utilize the above-mentioned matrix representation of the multiplication map to bring out certain structural properties equivalent to $\mathcal{M}$ being finitely generated as an $\mathcal{A}_1$-module.

PROPOSITION 3.4. *Let $\mathfrak{B}$ be an autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$. Then the following conditions are equivalent:*

1. *$\mathfrak{B}$ is strongly σ_2-relevant.*
2. *The quotient module $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$.*
3. *The annihilator $\operatorname{ann}(\mathcal{M})$ contains a $p(\sigma) \in \mathcal{A}$ of the form*

$$p(\sigma) = \sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1), \tag{3.8}$$

where L is a finite positive integer, $a_0(\sigma_1), a_1(\sigma_1), \ldots, a_{L-1}(\sigma_1) \in \mathcal{A}_1$, and $a_0(\sigma_1)$ is a unit in $\mathcal{A}_1$.

4. *There exist a finite positive integer L and $a_0(\sigma_1), a_1(\sigma_1), \ldots, a_{L-1}(\sigma_1) \in \mathcal{A}_1$, with $a_0(\sigma_1)$ a unit in $\mathcal{A}_1$, such that for all standard basis vectors $\{e_i \mid 1 \leqslant i \leqslant \mathtt{w}\} \subseteq \mathcal{A}^{\mathtt{w}}$ we have*

$$\left(\sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1)\right) e_i \in \mathcal{R}. \tag{3.9}$$

5. *$\mathfrak{B}$ admits a kernel representation matrix $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$ that can be partitioned as*

$$R(\sigma) = \begin{bmatrix} R_{\text{sq}}(\sigma) \\ R'(\sigma) \end{bmatrix},$$

where $R_{\text{sq}} \in \mathcal{A}^{\mathtt{w}\times\mathtt{w}}$ is square and of the form

$$R_{\text{sq}}(\sigma) = I_{\mathtt{w}}\sigma_2^L + R_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + R_1(\sigma_1)\sigma_2 + R_0(\sigma_1) \tag{3.10}$$

with L being a finite positive integer, $R_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$ for $0 \leqslant i \leqslant L-1$, and $R_0(\sigma_1)$ being invertible (unimodular) in $\mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$.

Proof. We prove the equivalences in the following sequence: (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3), (3) $\Leftrightarrow$ (4), (4) $\Rightarrow$ (5), and (5) $\Rightarrow$ (3).

(1) $\Leftrightarrow$ (2) This is true by definition.

(2) $\Rightarrow$ (3) We assume that $\mathcal{M}$ is a finitely generated $\mathcal{A}_1$-module. We want to show that there exists a Laurent polynomial $p(\sigma) \in \operatorname{ann}(\mathcal{M})$ of the form

$$p(\sigma) = \sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1)$$

with L being a finite positive integer, $a_0(\sigma_1), a_1(\sigma_1), \ldots, a_{L-1}(\sigma_1) \in \mathcal{A}_1$, and $a_0(\sigma_1)$ being a unit in $\mathcal{A}_1$. First, we show that there exists a Laurent polynomial $p_1(\sigma) \in \operatorname{ann}(\mathcal{M})$ of the form

$$p_1(\sigma) = \sigma_2^n + b_{n-1}(\sigma_1)\sigma_2^{n-1} + \cdots + b_1(\sigma_1)\sigma_2 + b_0(\sigma_1)$$

with n being a finite positive integer and $b_0(\sigma_1), b_1(\sigma_1), \ldots, b_{n_1-1}(\sigma_1) \in \mathcal{A}_1$. However, note that we do not insist on $b_0(\sigma_1)$ being a unit in $\mathcal{A}_1$. To get this $p_1(\sigma)$, first consider a generating set $\{g_1, g_2, \ldots, g_n\}$ of $\mathcal{M}$ over $\mathcal{A}_1$, and substitute σ_2 for $f(\sigma)$ in (3.4), (3.5), and (3.6). That way, multiplication by σ_2 in $\mathcal{M}$ is represented by a square matrix, say $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$. Let ξ be any transcendental over the field of fractions of $\mathcal{A}_1$. Then $\det(\xi I_n - A(\sigma_1))$ is a monic polynomial in ξ with coefficients from $\mathcal{A}_1$, say

$$\det(\xi I_n - A(\sigma_1)) = \xi^n + b_{n-1}(\sigma_1)\xi^{n-1} + \cdots + b_1(\sigma_1)\xi + b_0(\sigma_1).$$

Define $p_1(\sigma_1, \xi) := \det(\xi I_n - A(\sigma_1))$. It follows from the Cayley–Hamilton theorem that $p_1(\sigma_1, A(\sigma_1)) \in \mathcal{A}_1^{n\times n}$ is the zero matrix. This means that for any arbitrary row-vector $r(\sigma_1) \in \mathcal{A}_1^n$ we have

$$r(\sigma_1)p_1(\sigma_1, A(\sigma_1)) = 0 \in \mathcal{A}_1^n. \tag{3.11}$$

Note that a straightforward consequence of the definition of the matrix $A(\sigma_1)$ is that the matrix $p_1(\sigma_1, A(\sigma_1))$ represents the multiplication map from $\mathcal{M}$ to itself defined by $p(\sigma_1, \sigma_2)$. It then follows from (3.7) that for any $m = m_1(\sigma_1)g_1 + m_2(\sigma_1)g_2 + \cdots + m_n(\sigma_1)g_n \in \mathcal{M}$,

$$p_1(\sigma_1, \sigma_2)m = \begin{bmatrix} m_1(\sigma_1) \; m_2(\sigma_1) \; \cdots \; m_n(\sigma_1)\end{bmatrix} p_1(\sigma_1, A(\sigma_1)) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

But the right-hand side equals zero because of (3.11). Thus, for all $m \in \mathcal{M}$, we have $p(\sigma_1, \sigma_2)m = 0$; in other words, $p_1(\sigma_1, \sigma_2) \in \operatorname{ann}(\mathcal{M})$.

Further note that multiplication by σ_2^{-1} too defines an $\mathcal{A}_1$-linear map from $\mathcal{M}$ to itself. Thus, substituting σ_2^{-1} for $f(\sigma)$ in (3.4), (3.5), and (3.6) and following exactly the same steps as above, we find that there exists $p_2(\sigma) \in \operatorname{ann}(\mathcal{M})$ of the form

$$p_2(\sigma) = \sigma_2^{-n} + c_{n-1}(\sigma_1)\sigma_2^{-n+1} + \cdots + c_1(\sigma_1)\sigma_2^{-1} + c_0(\sigma_1),$$

where $c_i(\sigma_1) \in \mathcal{A}_1$ for $0 \leqslant i \leqslant n-1$. Since $\operatorname{ann}(\mathcal{M})$ is an ideal in $\mathcal{A}$, it now follows that $p(\sigma) := \sigma_2 p_1(\sigma) + \sigma_2^n p_2(\sigma) \in \operatorname{ann}(\mathcal{M})$. Observe that $p(\sigma)$ thus defined has the form

$$\begin{aligned} p(\sigma) = \sigma_2^{n+1} &+ (b_{n-1}(\sigma_1) + c_0(\sigma_1))\,\sigma_2^n + (b_{n-2}(\sigma_1) + c_1(\sigma_1))\,\sigma_2^{n-1} + \cdots \\ &+ (b_0(\sigma_1) + c_{n-1}(\sigma_1))\,\sigma_2 + 1. \end{aligned}$$

Defining $L := n+1$, $a_i(\sigma_1) := (b_{i-1}(\sigma_1) + c_{n-i})$ for $1 \leqslant i \leqslant n$ and $a_0(\sigma_1) = 1$, we get the desired expression for $p(\sigma)$, where $a_0(\sigma_1) = 1$ is trivially a unit in $\mathcal{A}_1$.

(2) $\Leftarrow$ (3) Assuming that there exists $p(\sigma) \in \operatorname{ann}(\mathcal{M})$ of the form of (3.8), we want to show that $\mathcal{M}$ is a finitely generated $\mathcal{A}_1$-module. In order to do so it is enough to show that the ring $\mathcal{A}/\operatorname{ann}(\mathcal{M})$ is a finitely generated $\mathcal{A}_1$-module. For $\mathcal{M}$ is finitely

generated as a module over the ring $\mathcal{A}/\text{ann}(\mathcal{M})$ (see Remark 2.4), and $\mathcal{A}/\text{ann}(\mathcal{M})$ being a finitely generated module over $\mathcal{A}_1$ guarantees that $\mathcal{M}$, too, is finitely generated as a module over $\mathcal{A}_1$. Indeed, if $\{g_1, g_2, \dots, g_r\} \subseteq \mathcal{M}$ is a generating set for $\mathcal{M}$ as a module over $\mathcal{A}/\text{ann}(\mathcal{M})$, and $\{f_1, f_2, \dots, f_s\} \subseteq \mathcal{A}/\text{ann}(\mathcal{M})$ is the same for $\mathcal{A}/\text{ann}(\mathcal{M})$ as a module over $\mathcal{A}_1$, then $\{f_i g_j \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r\}$ can be shown to be a generating set for $\mathcal{M}$ as a module over $\mathcal{A}_1$.

In order to show that $\mathcal{A}/\text{ann}(\mathcal{M})$ is finitely generated as a module over $\mathcal{A}_1$, we essentially generalize the method employed in Example 3.3. First, note that every Laurent polynomial in $\mathcal{A}$ can be written as a Laurent polynomial in σ_2 with coefficients coming from $\mathcal{A}_1$; that is, any $f(\sigma) \in \mathcal{A}$ can be written as

$$f(\sigma) = \sum_{i \in \mathbb{Z}} \alpha_i(\sigma_1)\sigma_2^i, \tag{3.12}$$

where $\alpha_i(\sigma_1) \in \mathcal{A}_1$ for all i and the sum is finite, meaning $a_i(\sigma_1) \neq 0$ only for finitely many i's. We first show that the presence of $p(\sigma)$, having the form of (3.8), in $\text{ann}(\mathcal{M})$ makes every Laurent polynomial equivalent modulo $\text{ann}(\mathcal{M})$ to another Laurent polynomial with only nonnegative powers in σ_2. To see this, first observe that from (3.8) we can write

$$\sigma_2^{-1} = a_0(\sigma_1)^{-1}p(\sigma) - a_0(\sigma_1)^{-1}\left(\sigma_2^{L-1} + a_{L-1}(\sigma_1)\sigma_2^{L-2} + \cdots + a_1(\sigma_1)\right).$$

In other words,

$$\sigma_2^{-1} \equiv -a_0(\sigma_1)^{-1}\left(\sigma_2^{L-1} + a_{L-1}(\sigma_1)\sigma_2^{L-2} + \cdots + a_1(\sigma_1)\right) \text{ modulo } \text{ann}(\mathcal{M}). \tag{3.13}$$

Note that the right-hand side of (3.13) has only nonnegative degrees in σ_2. Taking higher positive powers we see that for every positive integer i, $\sigma_2^{-i} \equiv f(\sigma) \in \mathcal{A}_1[\sigma_2]$ modulo $\text{ann}(\mathcal{M})$. In mathematical terms,

$$\mathcal{A}/\text{ann}(\mathcal{M}) \cong \mathcal{A}_1[\sigma_2]/(\text{ann}(\mathcal{M}) \cap \mathcal{A}_1[\sigma_2]) \tag{3.14}$$

as rings. The next observation is yet another consequence of the presence of $p(\sigma) \in \text{ann}(\mathcal{M})$. Notice that $p(\sigma)$ is, as a polynomial in σ_2 with coefficients in $\mathcal{A}_1$, *monic.* This means, given any $f(\sigma) \in \mathcal{A}_1[\sigma_2]$, we can find $q(\sigma), r(\sigma) \in \mathcal{A}_1[\sigma_2]$, with $r(\sigma)$ having σ_2-degree less than or equal to $L-1$, such that

$$f(\sigma) = q(\sigma)p(\sigma) + r(\sigma).$$

That is, $f(\sigma) \equiv r(\sigma)$ modulo $\text{ann}(\mathcal{M})$. Since $r(\sigma) \in \mathcal{A}_1[\sigma_2]$ and has σ_2-degree less than or equal to $L-1$, it can be considered as a linear combination of $\{1, \sigma_2, \dots, \sigma_2^{L-1}\}$ with coefficients from $\mathcal{A}_1$. Therefore, it follows that every element in $\mathcal{A}_1[\sigma_2]$ is equivalent modulo $\text{ann}(\mathcal{M})$ to a linear combination of $\{1, \sigma_2, \dots, \sigma_2^{L-1}\}$ with coefficients from $\mathcal{A}_1$. In other words, $\mathcal{A}_1[\sigma_2]/\left(\text{ann}(\mathcal{M}) \cap \mathcal{A}_1[\sigma_2]\right)$ is generated as a module over $\mathcal{A}_1$ by $\{\overline{1}, \overline{\sigma_2}, \dots, \overline{\sigma_2}^{L-1}\}$. This, together with (3.14), proves that $\mathcal{A}/\text{ann}(\mathcal{M})$ is finitely generated as a module over $\mathcal{A}_1$, which is what we set out to prove.

(3) $\Leftrightarrow$ (4) Statement (4) is just a restatement of the fact that $p(\sigma)$ of the form of (3.8) belongs to $\text{ann}(\mathcal{M})$, which is nothing but statement (3).

(4) $\Rightarrow$ (5) In order to show that statement (5) holds, it is enough to show that there exists $R_{\text{sq}}(\sigma) \in \mathcal{A}^{\mathtt{w} \times \mathtt{w}}$ of the form of (3.10) such that its rowspan over $\mathcal{A}$ is contained in $\mathcal{R}$. It follows from statement (4) that the rowspan of the square matrix

$$R_{\mathrm{sq}}(\sigma) := \sigma_2^L I_{\mathtt{w}} + a_{L-1}(\sigma_1)\sigma_2^{L-1} I_{\mathtt{w}} + \cdots + a_1(\sigma_1)\sigma_2 I_{\mathtt{w}} + a_0(\sigma_1) I_{\mathtt{w}}$$

over $\mathcal{A}$ is contained in the equation module $\mathcal{R}$. Defining $R_i(\sigma_1) = a_i(\sigma_1) I_{\mathtt{w}}$ for $0 \leqslant i \leqslant L-1$, we get that $\det(R_0(\sigma_1)) = a_0(\sigma_1)^{\mathtt{w}}$, which is a unit in $\mathcal{A}_1$. Therefore, $R_0(\sigma_1)$ is invertible (unimodular) in $\mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$.

(5) $\Rightarrow$ (3) We assume $\mathfrak{B}$ admits a kernel representation of the form

$$R(\sigma) = \begin{bmatrix} R_{\mathrm{sq}}(\sigma) \\ R'(\sigma) \end{bmatrix}$$

with $R_{\mathrm{sq}}(\sigma) \in \mathcal{A}^{\mathtt{w}\times\mathtt{w}}$ of the form of (3.10), that is,

$$R_{\mathrm{sq}}(\sigma) = I_{\mathtt{w}}\sigma_2^L + R_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + R_1(\sigma_1)\sigma_2 + R_0(\sigma_1),$$

L being a finite positive integer, $R_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$ for $0 \leqslant i \leqslant L-1$, and $R_0(\sigma_1)$ being invertible (unimodular) in $\mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$. We want to show that $p(\sigma)$ of the form of (3.8) belongs to $\mathrm{ann}(\mathcal{M})$. From the definition of the characteristic ideal $\mathcal{I}(\mathfrak{B})$ it follows that $\det(R_{\mathrm{sq}}(\sigma)) \in \mathcal{I}(\mathfrak{B})$. By Proposition 2.3 we have $\mathcal{I}(\mathfrak{B}) \subseteq \mathrm{ann}(\mathcal{M})$. Therefore, $\det(R_{\mathrm{sq}}(\sigma)) \in \mathrm{ann}(\mathcal{M})$. Now observe that $\det(R_{\mathrm{sq}}(\sigma))$ is of the form

$$\det(R_{\mathrm{sq}}(\sigma)) = \sigma_2^{\mathtt{w}L} + \alpha_{\mathtt{w}L-1}(\sigma_1)\sigma_2^{\mathtt{w}L-1} + \cdots + \alpha_1(\sigma_1)\sigma_2 + \alpha_0(\sigma_1).$$

Moreover, $\alpha_0(\sigma_1) = \det(R_0(\sigma_1))$. Since $R_0(\sigma_1)$ has been assumed to be unimodular, $\det(R_0(\sigma_1))$ is a unit in $\mathcal{A}_1$. Thus, defining $p(\sigma) := \det(R_0(\sigma_1))$, we get statement (3). ☐

As a corollary to the above proposition, we get the following useful result.

COROLLARY 3.5. *Suppose* $\mathfrak{a} \subseteq \mathcal{A}$ *is an ideal. Then the quotient ring* $\mathcal{A}/\mathfrak{a}$ *is a finitely generated module over* $\mathcal{A}_1$ *if and only if* $\mathfrak{a}$ *contains a Laurent polynomial* $p(\sigma)$ *of the form*

$$p(\sigma) = \sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1),$$

where L *is a finite positive integer,* $a_0(\sigma_1), a_1(\sigma_1), \ldots, a_{L-1}(\sigma_1) \in \mathcal{A}_1$, *and* $a_0(\sigma_1)$ *is a unit in* $\mathcal{A}_1$.

Proof. In Proposition 3.4 above, consider $\mathtt{w} = 1$ and take $\mathcal{R} = \mathfrak{a} \subseteq \mathcal{A}$. Then the quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$. The claim now follows from the equivalence of statements (2) and (3) by noting that in this case $\mathrm{ann}(\mathcal{M}) = \mathfrak{a}$. ☐

The defining property of strongly σ_2-relevant systems, that is, the quotient module $\mathcal{M}$ being finitely generated over $\mathcal{A}_1$, leads to a representation formula for trajectories in such systems. We present this representation formula in Theorem 3.7 below. Before we present this theorem we need some auxiliary results. Theorem 3.7 requires the following 1-variable Laurent polynomial matrices, which are guaranteed to exist once $\mathcal{M}$ is assumed to be a finitely generated module over $\mathcal{A}_1$ (i.e., $\mathfrak{B}$ is strongly σ_2-relevant). These matrices are defined in (3.15), (3.17), and (3.18) below. First, note that $\mathcal{M}$ as an $\mathcal{A}_1$-module may *not* be *free*, that is, the generators may satisfy nontrivial relations among themselves over $\mathcal{A}_1$. In that case, recalling the map $\psi : \mathcal{A}_1^n \to \mathcal{M}$ defined by (3.3), we must have $\ker(\psi)$ to be a nontrivial submodule of $\mathcal{A}_1^n$. Since $\mathcal{A}_1^n$ is a Noetherian module, this submodule $\ker(\psi)$ must be finitely generated. Let $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$ be a matrix whose rows generate $\ker(\psi)$, i.e.,

$$\mathrm{rowspan}(R_1(\sigma_1)) = \ker(\psi). \tag{3.15}$$

We call this matrix $R_1(\sigma_1)$ the matrix corresponding to the module of relations of the generating set $\{g_1, g_2, \ldots, g_n\}$, in short, the *matrix of relations* of $\{g_1, g_2, \ldots, g_n\}$. It then easily follows that

$$R_1(\sigma_1)\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = 0 \in \mathcal{M}^{n'}. \tag{3.16}$$

Next, let e_i be the standard ith basis row-vector in $\mathcal{A}^{\mathtt{w}}$. Suppose $\overline{e_i} \in \mathcal{M}$, the image of e_i under the surjection $\mathcal{A}^{\mathtt{w}} \twoheadrightarrow \mathcal{A}^{\mathtt{w}}/\mathcal{R} = \mathcal{M}$, is given by a linear combination of $\{g_1, g_2, \ldots, g_n\}$ over $\mathcal{A}_1$ as

$$\overline{e_i} = c_{i,1}(\sigma_1)g_1 + c_{1,2}(\sigma_1)g_2 + \cdots + c_{i,n}(\sigma_1)g_n.$$

Define

$$C(\sigma_1) := \begin{bmatrix} c_{1,1}(\sigma_1) & c_{1,2}(\sigma_1) & \ldots & c_{1,n}(\sigma_1) \\ c_{2,1}(\sigma_1) & c_{2,2}(\sigma_1) & \ldots & c_{2,n}(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{\mathtt{w},1}(\sigma_1) & c_{\mathtt{w},2}(\sigma_1) & \ldots & c_{\mathtt{w},n}(\sigma_1) \end{bmatrix}. \tag{3.17}$$

Finally, reconsider the map $\mu_f : \mathcal{M} \to \mathcal{M}$ defined in (3.4), with $f(\sigma) = \sigma_2$. We shall be using μ_f only for the case $f(\sigma) = \sigma_2$ in what follows, and hence, we refer to this map defined by multiplication by σ_2 as just μ. As we have mentioned before, μ is a map of $\mathcal{A}_1$-modules. We have also seen that $\mathcal{M}$ being a finitely generated $\mathcal{A}_1$-module implies that μ admits the following matrix representation:

$$A(\sigma_1) := \begin{bmatrix} a_{1,1}(\sigma_1) & a_{1,2}(\sigma_1) & \cdots & a_{1,n}(\sigma_1) \\ a_{2,1}(\sigma_1) & a_{2,2}(\sigma_1) & \cdots & a_{2,n}(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(\sigma_1) & a_{n,1}(\sigma_1) & \cdots & a_{n,n}(\sigma_1) \end{bmatrix}, \tag{3.18}$$

where, for $1 \leqslant i \leqslant n$, $\mu(g_i) = \sigma_2 g_i = a_{i,1}(\sigma_1)g_1 + a_{i,2}(\sigma_1)g_2 + \cdots + a_{i,n}(\sigma_1)g_n$. Note that this representation allows us to write

$$\sigma_2 \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} \mu(g_1) \\ \mu(g_2) \\ \vdots \\ \mu(g_n) \end{bmatrix} = A(\sigma_1) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}. \tag{3.19}$$

This matrix representation of the map μ entails the following consequences. Recall the map $\psi : \mathcal{A}_1^n \to \mathcal{M}$ of (3.3). Then, for any arbitrary $r(\sigma_1) = \left[r_1(\sigma_1)\, r_2(\sigma_1) \cdots\, r_n(\sigma_1)\right] \in \mathcal{A}_1^n$, it follows from the definition of ψ and (3.19) above that

$$\begin{aligned} \mu\left(\psi\left(r(\sigma_1)\right)\right) &= \sigma_2\left(r_1(\sigma_1)g_1 + r_2(\sigma_1)g_2 + \cdots + r_n(\sigma_1)g_n\right) \\ &= r(\sigma_1)\sigma_2 \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = r(\sigma_1)A(\sigma_1) \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = \psi(r(\sigma_1)A(\sigma_1)). \end{aligned}$$

In other words, the following diagram of maps of $\mathcal{A}_1$-modules commute:

$$\begin{array}{ccc} \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{M} \\ A(\sigma_1) \downarrow & & \downarrow \mu \\ \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{M} \end{array}, \tag{3.20}$$

where the $\mathcal{A}_1$-module morphism given by $A(\sigma_1)$ is right multiplication by $A(\sigma_1)$ to row-vectors in $\mathcal{A}_1^n$. An important consequence of the above commutative diagram is that if $R_1(\sigma) \in \mathcal{A}_1^{n' \times n}$ is a matrix of relations of $\{g_1, g_2, \dots, g_n\}$, then we have

$$R_1(\sigma_1)A(\sigma_1) = F(\sigma_1)R_1(\sigma_1) \tag{3.21}$$

for some $F(\sigma_1) \in \mathcal{A}_1^{n' \times n'}$.

At this juncture, we note that the map μ is invertible on $\mathcal{M}$, and its inverse is the map given by μ_f with $f(\sigma) = \sigma_2^{-1}$, although the matrix $A(\sigma_1)$ representing μ may not be automatically invertible (unimodular) in $\mathcal{A}_1^{n \times n}$. However, as we show below in Lemma 3.6, one can always choose a suitable generating set for $\mathcal{M}$ as a module over $\mathcal{A}_1$ such that the corresponding $A(\sigma_1)$ is indeed invertible.

Lemma 3.6. *Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule such that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Then there exists a finite generating set $\{g_1, g_2, \dots, g_n\}$ of $\mathcal{M}$ as a module over $\mathcal{A}_1$ such that the corresponding matrix $A(\sigma_1) \in \mathcal{A}_1^{n \times n}$, as defined in* (3.18)*, is invertible (unimodular) in $\mathcal{A}_1^{n \times n}$.*

Proof. In order to show that $A(\sigma_1)$ is invertible it is enough to show that $\det(A(\sigma_1))$ is a unit in $\mathcal{A}_1$. By assumption, $\mathcal{M}$ is a finitely generated module over $\mathcal{A}_1$. It then follows from Proposition 3.4 that there exists a Laurent polynomial $p(\sigma) \in \mathcal{A}$ of the form

$$\sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1),$$

where L is a finite positive integer and $a_0(\sigma_1), a_1(\sigma_1), \dots, a_{L-1}(\sigma_1) \in \mathcal{A}_1$, with $a_0(\sigma_1)$ a unit in $\mathcal{A}_1$ such that for all standard basis vectors $\{e_i \mid 1 \leqslant i \leqslant \mathtt{w}\} \subseteq \mathcal{A}^{\mathtt{w}}$ we have

$$p(\sigma)e_i \in \mathcal{R}.$$

We claim that the images of the elements

$$\{\sigma_2^j e_i \mid 1 \leqslant i \leqslant \mathtt{w}, 0 \leqslant j \leqslant L-1\}$$

in $\mathcal{M}$ form a generating set for $\mathcal{M}$ as a module over $\mathcal{A}_1$. In order to see this, first note that $p(\sigma)e_i \in \mathcal{R}$ for all $1 \leqslant i \leqslant \mathtt{w}$ implies that for two row-vectors

$$r_1(\sigma) = \begin{bmatrix} r_{11}(\sigma) & r_{12}(\sigma) & \cdots & r_{1\mathtt{w}}(\sigma) \end{bmatrix} \in \mathcal{A}^{\mathtt{w}}$$

and

$$r_2(\sigma) = \begin{bmatrix} r_{21}(\sigma) & r_{22}(\sigma) & \cdots & r_{2\mathtt{w}}(\sigma) \end{bmatrix} \in \mathcal{A}^{\mathtt{w}}$$

we have

$$r_{1i}(\sigma) \equiv r_{2i}(\sigma) \text{ modulo } \langle p(\sigma) \rangle \text{ for all } 1 \leqslant i \leqslant \mathtt{w} \;\Rightarrow\; r_1(\sigma) \equiv r_2(\sigma) \text{ modulo } \mathcal{M}. \tag{3.22}$$

However, the structure of the polynomial $p(\sigma)$ implies first that every Laurent polynomial is equivalent to a Laurent polynomial with only nonnegative powers in σ_2

modulo $\langle p(\sigma)\rangle$ (this is because the constant term $a_0(\sigma_1)$ is a unit in $\mathcal{A}_1$; see (2) $\Leftarrow$ (3) in the proof of Proposition 3.4). Second, every Laurent polynomial with only non-negative powers in σ_2 is equivalent to a linear combination of $\{1, \sigma_2, \dots, \sigma_2^{L-1}\}$ with coefficients from $\mathcal{A}_1$ (this is because $p(\sigma)$ is monic with σ_2-degree L). This, together with (3.22), implies that for an arbitrary $m(\sigma) \in \mathcal{A}^{\mathtt{w}}$, there exist $\{\alpha_{i,j}(\sigma_1) \in \mathcal{A}_1 \mid 1 \leqslant i \leqslant \mathtt{w}, 0 \leqslant j \leqslant L-1\}$ such that

$$m(\sigma) \equiv \begin{bmatrix} r_1(\sigma) & r_2(\sigma) & \cdots & r_{\mathtt{w}}(\sigma)\end{bmatrix} \text{ modulo } \mathcal{R},$$

where $r_i(\sigma) = \alpha_{i,0}(\sigma_1) + \alpha_{i,1}(\sigma_1)\sigma_2 + \cdots + \alpha_{i,L-1}(\sigma_1)\sigma_2^{L-1}$. That is,

$$m(\sigma) \equiv \sum_{i=1}^{\mathtt{w}} \sum_{j=0}^{L-1} \alpha_{i,j}(\sigma_1)\sigma_2^j e_i \text{ modulo } \mathcal{R}.$$

In other words, $\mathcal{M}$ is generated as an $\mathcal{A}_1$-module by $\{\overline{\sigma_2^j e_i} \mid 1 \leqslant i \leqslant \mathtt{w}, 0 \leqslant j \leqslant L-1\}$.

Let us now order this generating set as $\{\overline{e_1}, \overline{\sigma_2 e_1}, \dots, \overline{\sigma_2^{L-1} e_1}, \overline{e_2}, \overline{\sigma_2 e_2}, \dots\}$ and define

$$\overline{\sigma_2^j e_i} =: g_{L(i-1)+(j+1)}.$$

With this generating set $\{g_1, g_2, \dots, g_{L\mathtt{w}}\}$ it is straightforward to check that μ admits a matrix representation of the form

$$A(\sigma_1) = \begin{bmatrix} A_{\text{block}}(\sigma_1) & 0 & \cdots & 0 \\ 0 & A_{\text{block}}(\sigma_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\text{block}}(\sigma_1) \end{bmatrix} \in \mathcal{A}_1^{L\mathtt{w} \times L\mathtt{w}},$$

where the $A_{\text{block}}(\sigma_1) \in \mathcal{A}_1^{L \times L}$ is given by

$$A_{\text{block}}(\sigma_1) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0(\sigma_1) & -a_1(\sigma_1) & -a_2(\sigma_1) & \cdots & -a_{L-1}(\sigma_1) \end{bmatrix},$$

where the $a_i(\sigma_1)$'s are the coefficients of $p(\sigma)$. Now, clearly $\det(A(\sigma_1)) = \det(A_{\text{block}}(\sigma_1))^{\mathtt{w}}$. Since $A_{\text{block}}(\sigma_1)$ is in companion form, $\det(A_{\text{block}}(\sigma_1)) = a_0(\sigma_1)$. Therefore, $\det(A(\sigma_1)) = a_0(\sigma_1)^{\mathtt{w}}$, which is clearly a unit because $a_0(\sigma_1)$ is. ∎

Keeping Lemma 3.6 in mind, in what follows we always assume that $A(\sigma_1)$ is invertible in $\mathcal{A}_1^{n\times n}$. The inverse of $A(\sigma_1)$ clearly represents the map defined by multiplication by σ_2^{-1}. This enables us to extend the commutative diagram of (3.20) to the case when μ is replaced by any integer power of it. That is, for any $i \in \mathbb{Z}$, define $\mu^i : \mathcal{M} \ni \overline{m} \mapsto \overline{\sigma_2^i m} \in \mathcal{M}$; then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{M} \\ A(\sigma_1)^i \downarrow & & \downarrow \mu^i \\ \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{M} \end{array}, \tag{3.23}$$

where $A(\sigma_1)^i : \mathcal{A}_1^n \ni r(\sigma_1) \mapsto r(\sigma_1)A(\sigma_1)^i \in \mathcal{A}_1^n$. Consequently, for all $i \in \mathbb{Z}$ we have $F_i(\sigma_1) \in \mathcal{A}_1^{n' \times n'}$ such that

$$R_1(\sigma_1)A(\sigma_1)^i = F_i(\sigma_1)R_1(\sigma_1). \tag{3.24}$$

We now present the representation formula for strongly σ_2-relevant autonomous systems. It is important to note at this point that elements from $\mathcal{A}_1^n$ act on n-tuples of 1D trajectories. As in (2.2), in this case we have that for $r(\sigma_1) = \begin{bmatrix} r_1(\sigma_1) & r_2(\sigma_1) & \cdots & r_n(\sigma_1) \end{bmatrix} \in \mathcal{A}_1^n$ and $x = \mathrm{col}(x_1, x_2, \ldots, x_n) \in (\mathbb{R}^n)^{\mathbb{Z}}$ the action of $r(\sigma_1)$ on x is defined as

$$r(\sigma_1)x = r_1(\sigma_1)x_1 + r_2(\sigma_1)x_2 + \cdots + r_n(\sigma_1)x_n \in \mathbb{R}^{\mathbb{Z}}. \tag{3.25}$$

THEOREM 3.7. *Let $\mathfrak{B}$ be an autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$. Suppose $\mathfrak{B}$ is strongly σ_2-relevant; that is, $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Let $\{g_1, g_2, \ldots, g_n\} \subseteq \mathcal{M}$ be a set of generators of $\mathcal{M}$ as an $\mathcal{A}_1$-module, and consider the $\mathcal{A}_1$-module map $\psi : \mathcal{A}_1^n \to \mathcal{M}$ as in* (3.3)*. Further, let $R_1(\sigma_1) \in \mathcal{A}_1^{n' \times n}$, $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w} \times n}$, and $A(\sigma_1) \in \mathcal{A}_1^{n \times n}$ be as defined in* (3.15), (3.17)*, and* (3.18)*, respectively, with $A(\sigma_1)$ invertible in $\mathcal{A}_1^{n \times n}$. Then $w \in \mathfrak{B}$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ satisfying*

$$R_1(\sigma_1)x = 0 \tag{3.26}$$

such that for all $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$w(\nu) = \left(C(\sigma_1)A(\sigma_1)^{\nu_2}x\right)(\nu_1). \tag{3.27}$$

In order to prove Theorem 3.7 we first need Lemma 3.8. The content of Lemma 3.8 is that given any arbitrary $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$, the lemma provides a candidate row-vector in $\mathcal{A}_1^n$ such that the image of this row-vector under ψ is equal to $\overline{r(\sigma)} \in \mathcal{M}$. As a consequence of this observation, we get a necessary and sufficient condition, in terms of $R(\sigma_1)$, $A(\sigma_1)$, and $C(\sigma_1)$, for $r(\sigma)$ to be in the equation module $\mathcal{R}$. First, in Lemma 3.8 and in the proof of Theorem 3.7 we shall require writing row-vectors in $\mathcal{A}^{\mathtt{w}}$ in an alternative form. For this we first define the following finite subset of $\mathbb{Z}^2$: suppose a row-vector $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ is given by

$$r(\sigma) = \sum_{\nu \in \mathbb{Z}^2} \alpha_\nu \sigma^\nu, \tag{3.28}$$

where $\alpha_\nu \in \mathbb{R}^{\mathtt{w}}_{\mathrm{row}}$ are row-vectors of real numbers and only finitely many of α_ν's are nonzero; then define

$$\mathrm{supp}(r) := \{\nu \in \mathbb{Z}^2 \mid \alpha_\nu \neq 0 \in \mathbb{R}^{\mathtt{w}}_{\mathrm{row}}\}.$$

Further, for any subset Γ of $\mathbb{Z}^2$ we define

$$\pi_2(\Gamma) := \{i \in \mathbb{Z} \mid \exists\, j \in \mathbb{Z} \text{ such that } (j, i) \in \Gamma\}.$$

Then (3.28) can be rewritten in powers of σ_2 with coefficients from $\mathcal{A}_1^{\mathtt{w}}$ as

$$r(\sigma) = \sum_{i \in \pi_2(\mathrm{supp}(r))} \beta_i(\sigma_1)\sigma_2^i, \tag{3.29}$$

where $\beta_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}}$.

Next, observe that it follows from the definition of $C(\sigma_1)$ that $\psi(C(\sigma_1)) = \overline{I_{\mathtt{w}}}$. Then for any $\beta(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}} \subseteq \mathcal{A}^{\mathtt{w}}$ we have

$$\psi(\beta(\sigma_1)C(\sigma_1)) = \overline{\beta(\sigma_1)} \in \mathcal{M}. \tag{3.30}$$

With this we are now in a position to state and prove Lemma 3.8.

LEMMA 3.8. *Suppose $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ is a submodule such that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Let $\{g_1, g_2, \ldots, g_n\} \subseteq \mathcal{M}$ be a set of generators of $\mathcal{M}$ as an $\mathcal{A}_1$-module, and consider the $\mathcal{A}_1$-module map $\psi : \mathcal{A}_1^n \to \mathcal{M}$ as in* (3.3)*. Further, let $R_1(\sigma_1) \in \mathcal{A}_1^{n' \times n}$, $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w} \times n}$, and $A(\sigma_1) \in \mathcal{A}_1^{n \times n}$ be as defined in* (3.15), (3.17), *and* (3.18)*, respectively, with $A(\sigma_1)$ invertible in $\mathcal{A}_1^{n \times n}$. Suppose $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ is given by* (3.29)*. Then we have*

$$\overline{r(\sigma)} = \psi\left(\sum_{i \in \pi_2(\mathrm{supp}(r))} \beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i\right). \tag{3.31}$$

Therefore, $r(\sigma) \in \mathcal{R}$ if and only if

$$\sum_{i \in \pi_2(\mathrm{supp}(r))} \beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i \in \mathrm{rowspan}(R_1(\sigma_1)). \tag{3.32}$$

Proof. First note that

$$\begin{aligned} \overline{r(\sigma)} &= \sum_{i \in \pi_2(\mathrm{supp}(r))} \overline{\beta_i(\sigma_1)}\,\overline{\sigma_2^i} \\ &= \sum_{i \in \pi_2(\mathrm{supp}(r))} \mu^i\left(\overline{\beta_i(\sigma_1)}\right). \end{aligned} \tag{3.33}$$

Since $\beta_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}}$ it follows from (3.30) that $\overline{\beta_i(\sigma_1)} = \psi(\beta_i(\sigma_1)C(\sigma_1))$. So the right-hand side of (3.33) can be written as

$$\overline{r(\sigma)} = \sum_{i \in \pi_2(\mathrm{supp}(r))} \mu^i\left(\psi(\beta_i(\sigma_1)C(\sigma_1))\right). \tag{3.34}$$

Now by the commutative diagram of (3.23) we get

$$\begin{aligned} \overline{r(\sigma)} &= \sum_{i \in \pi_2(\mathrm{supp}(r))} \psi\left(\beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i\right) \\ &= \psi\left(\sum_{i \in \pi_2(\mathrm{supp}(r))} \beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i\right). \end{aligned}$$

This proves the first claim of the lemma, that is, (3.31).

The second claim, (3.32), easily follows from here, because $\ker(\psi) = \mathrm{rowspan}(R_1(\sigma_1))$ and $r(\sigma) \in \mathcal{R}$ if and only if $\overline{r(\sigma)} = 0 \in \mathcal{M}$. ◻

We now prove Theorem 3.7.

Proof of Theorem 3.7. *(Only if)* We assume that $w \in \mathfrak{B}$ and show existence of $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ satisfying (3.26) and (3.27), that is,

$$R_1(\sigma_1)x = 0 \text{ and for all } \nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2, \quad w(\nu) = \left(C(\sigma_1)A(\sigma_1)^{\nu_2}x\right)(\nu_1).$$

We claim that an x with the desired properties can be obtained as follows: recall the definition of the action of elements in $\mathcal{M}$ on $\mathfrak{B}$ shown in (2.6); then, for all $\nu \in \mathbb{Z}$, define

$$x(\nu) := \left(\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} w \right) (\nu, 0).$$

We first show that x, as defined above, satisfies $R_1(\sigma_1)x = 0$. This follows from the following equations:

$$R_1(\sigma_1)x = R_1(\sigma_1) \left(\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} w \right) (\bullet, 0) = 0,$$

where the last equality follows from (3.16).

Next we show that x, as defined above, also satisfies $w(\nu) = (C(\sigma_1)A(\sigma_1)^{\nu_2}x)(\nu_1)$ for all $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$. Let $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ be arbitrary. It follows from the definition of the $\mathcal{A}_1$-module map ψ that

$$\begin{aligned} (C(\sigma_1)A(\sigma_1)^{\nu_2}x)(\nu_1) &= \left(C(\sigma_1)A(\sigma_1)^{\nu_2} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} w \right) (\nu_1, 0) \\ &= (\psi(C(\sigma_1)A(\sigma_1)^{\nu_2})\, w)(\nu_1, 0). \end{aligned} \tag{3.35}$$

Now utilizing the commutative diagram of (3.21) it follows from (3.35) that

$$\begin{aligned} (C(\sigma_1)A(\sigma_1)^{\nu_2}x)(\nu_1) &= (\psi(C(\sigma_1)A(\sigma_1)^{\nu_2})\, w)(\nu_1, 0) \\ &= (\mu^{\nu_2}(\psi(C(\sigma_1)))\, w)(\nu_1, 0) \\ &= \left(\overline{\sigma_2}^{\nu_2}\overline{I_{\mathtt{w}}}w\right)(\nu_1, 0) = w(\nu_1, \nu_2), \end{aligned}$$

where we have used the fact that $\psi(C(\sigma_1)) = \overline{I_{\mathtt{w}}}$.

(If) We show that if $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ satisfies $R(\sigma_1)x = 0$, then w, given by

$$w(\nu_1, \nu_2) := (C(\sigma_1)A(\sigma_1)^{\nu_2}x)(\nu_1),$$

is a trajectory in $\mathfrak{B}$. Note that, since $\ker(\psi)$ is a proper submodule of $\mathcal{A}_1^n$, $\ker(R_1(\sigma_1)) \neq \emptyset$. Therefore, there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ satisfying $R_1(\sigma_1)x = 0$. In order to show that w as defined above is a trajectory in $\mathfrak{B}$, it is enough to show that $r(\sigma)w = 0$ for all $r(\sigma) \in \mathcal{R}$. Suppose $r(\sigma) \in \mathcal{R}$ is written in powers of σ_2 as in (3.29):

$$r(\sigma) = \sum_{i \in \pi_2(\mathrm{supp}(r))} \beta_i(\sigma_1)\sigma_2^i,$$

where $\beta_i(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}}$. For arbitrary $(\nu_1, \nu_2) \in \mathbb{Z}^2$, by making this $r(\sigma)$ act on w we get

$$
\begin{aligned}
(r(\sigma)w)(\nu_1,\nu_2) &= \sum_{i\in\pi_2(\mathrm{supp}(r))} \left(\beta_i(\sigma_1)\sigma_2^i w\right)(\nu_1,\nu_2)\\
&= \sum_{i\in\pi_2(\mathrm{supp}(r))} \left(\beta_i(\sigma_1)w\right)(\nu_1,\nu_2+i)\\
&= \sum_{i\in\pi_2(\mathrm{supp}(r))} \left(\beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^{\nu_2+i}x\right)(\nu_1) \text{ (by definition of } w)\\
&= \left(\left(\sum_{i\in\pi_2(\mathrm{supp}(r))} \beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i\right)A(\sigma_1)^{\nu_2}x\right)(\nu_1). \qquad (3.36)
\end{aligned}
$$

Since $r(\sigma)\in\mathcal{R}$, by Lemma 3.8 $\sum_{i\in\pi_2(\mathrm{supp}(r))}\beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i \in \mathrm{rowspan}(R_1(\sigma_1))$; that is, there exists $f(\sigma_1)\in\mathcal{A}_1^{1\times n'}$ such that

$$\sum_{i\in\pi_2(\mathrm{supp}(r))}\beta_i(\sigma_1)C(\sigma_1)A(\sigma_1)^i = f(\sigma_1)R_1(\sigma_1).$$

Putting this in (3.36), we get

$$
\begin{aligned}
(r(\sigma)w)(\nu_1,\nu_2) &= \left(f(\sigma_1)R_1(\sigma_1)A(\sigma_1)^{\nu_2}x\right)(\nu_1)\\
&= \left(f(\sigma_1)F_{\nu_2}(\sigma_1)R_1(\sigma_1)x\right)(\nu_1) \quad \text{(by (3.24))}\\
&= 0,
\end{aligned}
$$

where $F_{\nu_2}(\sigma_1)\in\mathcal{A}_1^{n'\times n'}$. Since $(\nu_1,\nu_2)\in\mathbb{Z}^2$ was arbitrary, it follows that $r(\sigma)w = 0\in\mathbb{R}^{\mathbb{Z}^2}$. Thus $w\in\mathfrak{B}$. ∎

The result of Theorem 3.7 is illustrated in the following two examples involving scalar behaviors.

Example 3.9. Consider the scalar ($\mathtt{w}=1$) strongly σ_2-relevant behavior of Example 3.3,

$$\mathfrak{B} = \ker\begin{bmatrix}\sigma_2^2-2\sigma_2+1\\ \sigma_1\sigma_2-\sigma_1-\sigma_2+1\end{bmatrix}.$$

Here the equation module is the ideal $\mathfrak{a}=\langle\sigma_2^2-2\sigma_2+1, \sigma_1\sigma_2-\sigma_1-\sigma_2+1\rangle$, and consequently, the quotient module $\mathcal{M}=\mathcal{A}/\mathfrak{a}$. As we have already seen, here $\mathcal{M}$ is a finitely generated module over $\mathcal{A}_1$; $\{\overline{1},\overline{\sigma_2}\}$ generate $\mathcal{M}$ as an $\mathcal{A}_1$-module. In this case, we have $n=2$ and the 1-variable Laurent polynomial matrices $R_1(\sigma_1)$, $C(\sigma_1)$, and $A(\sigma_1)$ are given by

1. $R_1(\sigma_1)=\begin{bmatrix}(\sigma_1-1) & -(\sigma_1-1)\end{bmatrix}$,
2. $C(\sigma_1)=\begin{bmatrix}1 & 0\end{bmatrix}$,
3. $A(\sigma_1)=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}$.

So every solution in $\mathfrak{B}$ is of the form

$$w(\nu_1,\nu_2)=\left(\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}^{\nu_2}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)(\nu_1),$$

where $\mathrm{col}(x_1,x_2)\in(\mathbb{R}^2)^{\mathbb{Z}}$ satisfies

$$\begin{bmatrix}(\sigma_1-1) & -(\sigma_1-1)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0.$$

Example 3.10. Consider the following kernel representation:

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2^3 - (\sigma_1+3)\sigma_2^2 + (\sigma_1+4)\sigma_2 - 2, \\ (\sigma_1 - 1)\sigma_2^2 - (\sigma_1^2 + \sigma_1 - 2)\sigma_2 + 2\sigma_1 - 2 \end{bmatrix}.$$

The equation ideal is $\mathfrak{a} = \langle \sigma_2^3 - (\sigma_1+3)\sigma_2^2 + (\sigma_1+4)\sigma_2 - 2, (\sigma_1-1)\sigma_2^2 - (\sigma_1^2+\sigma_1-2)\sigma_2 + 2\sigma_1 - 2\rangle$. By Proposition 3.4, the presence of the polynomial $\sigma_2^3 - (\sigma_1+3)\sigma_2^2 + (\sigma_1 + 4)\sigma_2 - 2$ in the kernel representation matrix implies that the quotient ring $\mathcal{M} = \mathcal{A}/\mathfrak{a}$ is finitely generated as a module over $\mathcal{A}_1$. In fact, $\{\overline{1}, \overline{\sigma_2}, \overline{\sigma_2}^2\}$ is a possible generating set. The corresponding 1-variable matrices turn out to be

1. $R_1(\sigma_1) = \begin{bmatrix} 2(\sigma_1-1) & -(\sigma_1^2+\sigma_1-2) & (\sigma_1-1) \end{bmatrix}$,
2. $C(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$,
3. $A(\sigma_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -(\sigma_1+4) & (\sigma_1+3) \end{bmatrix}$.

It can be checked that every trajectory in $\mathfrak{B}$ can be obtained by

$$w(\nu_1, \nu_2) = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -(\sigma_1+4) & (\sigma_1+3) \end{bmatrix}^{\nu_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)(\nu_1),$$

where $\mathrm{col}(x_1, x_2, x_3) \in (\mathbb{R}^3)^{\mathbb{Z}}$ satisfies

$$\begin{bmatrix} 2(\sigma_1 - 1) & -(\sigma_1^2 + \sigma_1 - 2) & (\sigma_1 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Remark 3.11. The 1D trajectory x appearing in Theorem 3.7 plays the role of *initial conditions*. Note that $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ must satisfy the equation $R_1(\sigma_1)x = 0$ in order to qualify as a valid initial condition. We denote this set of valid initial conditions by $\mathscr{X}$:

$$\mathscr{X} := \{x \in (\mathbb{R}^n)^{\mathbb{Z}} \mid R_1(\sigma_1)x = 0\}. \tag{3.37}$$

This $\mathscr{X}$ can in fact be treated as a 1D behavior with kernel representation given by $R_1(\sigma_1)$. Thus, $\mathscr{X}$ turns out to be a subspace of $(\mathbb{R}^n)^{\mathbb{Z}}$. It is interesting to note that under the operator $A(\sigma_1) : (\mathbb{R}^n)^{\mathbb{Z}} \to (\mathbb{R}^n)^{\mathbb{Z}}$, $\mathscr{X}$ is an *invariant subspace*. This follows from (3.21). As a result, restriction of $A(\sigma_1)$ to $\mathscr{X}$ is a well defined endomorphism of $\mathscr{X}$. It then makes sense to define a 1D first order dynamical system over $\mathscr{X}$ given by (3.38) below. For this purpose we define $\mathbf{x} : \mathbb{Z} \to \mathscr{X}$ as a 1D trajectory taking values in $\mathscr{X}$. That is, for $\nu \in \mathbb{Z}$ we have $\mathbf{x}(\nu) \in \mathscr{X}$. Now consider the first order dynamical system on $\mathscr{X}$ is defined by

$$\mathbf{x}(\nu + 1) = A(\sigma_1)\mathbf{x}(\nu) \tag{3.38}$$

for all $\nu \in \mathbb{Z}$. Then the solution to this first order equation is given by

$$\mathbf{x}(\nu) = A(\sigma_1)^{\nu}\mathbf{x}(0). \tag{3.39}$$

By identifying $\mathbf{x}(0) = x \in \mathscr{X}$ as the initial condition, and $C\mathbf{x}(\nu) =: w(\bullet, \nu)$ it follows that (3.39) is just another way of rewriting (3.27) of Theorem 3.7. Thus, Theorem 3.7 in fact shows that the first order system over $\mathscr{X}$ defined by (3.38) is equivalent to the behavior $\mathfrak{B}$. This way, $\mathscr{X}$ is like a state-space for $\mathfrak{B}$. The state-variables get defined as

$$\mathbf{x}(\nu) = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} w(\bullet, \nu),$$

where $\{g_1, g_2, \ldots, g_n\} \subseteq \mathcal{M}$ is a suitable generating set for $\mathcal{M}$ as a module over $\mathcal{A}_1$.

The assumption of $\mathfrak{B}$ being strongly σ_2-relevant in Theorem 3.7 is very restrictive. There are many systems which do not satisfy this requirement. For example, consider a scalar behavior given by a single equation $\mathfrak{B} = \ker(f(\sigma))$, where $f(\sigma) \in \mathcal{A}$ is of the form

$$f(\sigma) = \sigma_2^n + \alpha_{n-1}(\sigma_1)\sigma_2^{n-1} + \cdots + \alpha_1(\sigma_1)\sigma_2 + \alpha_0(\sigma_1),$$

where n is a positive integer, with $\alpha_i(\sigma_1) \in \mathcal{A}_1$ for $0 \leqslant i \leqslant n-1$. Suppose that $\alpha_0(\sigma_1)$ is *not* a unit in $\mathcal{A}_1$. It can be shown, in that case, that $\mathfrak{B}$ *cannot* be strongly σ_2-relevant. Indeed, supposing that $\mathfrak{B}$ is σ_2-relevant, it follows from Proposition 3.4 (or, equivalently, Corollary 3.5) that the ideal $\langle f(\sigma)\rangle$ must contain a Laurent polynomial of the form $p(\sigma) = \sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1)$, where $a_0(\sigma_1)$ is a unit in $\mathcal{A}_1$. This means there exists $g(\sigma) \in \mathcal{A}$ such that $p(\sigma) = f(\sigma)g(\sigma)$. Let $g(\sigma)$ be written as

$$g(\sigma) = \sum_{i=r_1}^{r_2} \beta_i(\sigma_1)\sigma_2^i,$$

where $\beta_i(\sigma_1) \in \mathcal{A}_1$ for $r_1 \leqslant i \leqslant r_2$. It follows from the equation $p(\sigma) = f(\sigma)g(\sigma)$, by equating various powers of σ_2 in $p(\sigma)$, that $\beta_i(\sigma_1) = 0$ for $r_1 \leqslant i < 0$. Furthermore, it also follows that $a_0(\sigma_1) = \alpha_0(\sigma_1)\beta_0(\sigma_1)$. Now observe that $a_0(\sigma_1)$ being a unit in $\mathcal{A}_1$ forces $\alpha_0(\sigma_1)$ to be a unit too in $\mathcal{A}_1$; this is clearly a contradiction. Therefore, the scalar behavior $\mathfrak{B} = \ker(f(\sigma))$ cannot be strongly σ_2-relevant. A concrete example of such an $f(\sigma)$ is $f(\sigma) = \sigma_2 - \sigma_1 - 1$. Another example of a scalar behavior that is not strongly σ_2-relevant would be $\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)$. In sections 4 and 5 following we overcome this drawback of Theorem 3.7 and present a representation formula for general autonomous systems in Theorem 5.3. The main idea behind this is that every autonomous system can be converted to a strongly σ_2-relevant system by a suitable change of coordinates. In section 4 we show how to achieve this transformation for ideals; we call this result the *discrete version of Noether's normalization lemma*. Then in section 5 we first extend the normalization process to submodules and then use this result to give the general representation formula (Theorem 5.3).

4. Discrete version of Noether's normalization. Recall that given a unimodular $T \in \mathbb{Z}^{2\times 2}$, it defines an automorphism of $\mathcal{A}$ as

$$\begin{array}{rccc} \varphi_T : & \mathcal{A} & \to & \mathcal{A}, \\ & \sigma^\nu & \mapsto & \sigma^{T\nu}. \end{array} \tag{4.1}$$

In this section, we show that, given a nonzero ideal $\mathfrak{a} \subseteq \mathcal{A}$, either $\mathcal{A}/\mathfrak{a}$ is a finite dimensional vector space over $\mathbb{R}$, or there exists a unimodular $T \in \mathbb{Z}^{2\times 2}$ such that under the corresponding φ_T the quotient ring $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated *faithful* module over $\mathcal{A}_1$. This observation constitutes the main theorem, Theorem 4.3, of this section. However, before we state and prove this theorem, we first prove following Lemma 4.1, which is a precursor to Theorem 4.3. The lemma shows that given a 2D Laurent polynomial, there exists a unimodular T such that under φ_T the given Laurent polynomial is mapped to a Laurent polynomial with a special structure: when written as a Laurent polynomial in σ_2 with coefficients from $\mathcal{A}_1$, these coefficients are all *units* in $\mathcal{A}_1$. A similar result can be found in [9], where the result has been used in a different context, namely design of inverse 2D filters.

LEMMA 4.1. *Let $0 \neq f(\sigma) \in \mathcal{A}$ be given by*

$$f(\sigma) = \sum_{\nu \in \mathbb{Z}^2} \alpha_\nu \sigma^\nu, \quad \alpha_\nu \in \mathbb{R},$$

with only finitely many $\alpha_\nu \neq 0$. Then there exists a unimodular $T \in \mathbb{Z}^{2\times 2}$ such that under the corresponding automorphism φ_T given by (4.1)*, we have*

$$\varphi_T(f(\sigma)) = \left(\sum_{k=0}^{\delta} u_k(\sigma_1)\sigma_2^k\right) u(\sigma_2), \tag{4.2}$$

where $\{u_0(\sigma_1), u_1(\sigma_1), \ldots, u_\delta(\sigma_1)\} \subseteq \mathcal{A}_1$, and $u(\sigma_2) \in \mathbb{R}[\sigma_2^{\pm 1}]$ are all units in $\mathcal{A}$, and δ is some finite positive integer.

Proof. There are, in general, many T's which will render $f(\sigma)$ in the form of (4.2); we construct one particular T for which the monomials of $f(\sigma)$ are mapped to monomials having different σ_2 degrees. To this end let us define $T \in \mathbb{Z}^{2\times 2}$ as

$$T := \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},$$

where $t \in \mathbb{Z}$. Clearly, T is unimodular. It follows from the structure of T that the ring map φ_T maps an arbitrary monomial σ^ν with $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$ to

$$\varphi_T(\sigma^\nu) = \sigma^{T\nu} = \sigma_1^{\nu_1}\sigma_2^{t\nu_1+\nu_2}.$$

We claim that $t \in \mathbb{Z}$ can be chosen such that the σ_2-degrees of $\varphi_T(\sigma^\nu)$, for the monomials σ^ν having nonzero coefficient α_ν in $f(\sigma)$, are all different from each other. For easy referencing we define the following finite subset of $\mathbb{Z}^2$:

$$\mathrm{supp}(f) := \{\nu \in \mathbb{Z}^2 \mid \alpha_\nu \neq 0\}.$$

Indeed, a $t \in \mathbb{Z}$ fails to achieve this if and only if there exist $(\nu_1, \nu_2), (\nu_1', \nu_2') \in \mathrm{supp}(f)$, $(\nu_1, \nu_2) \neq (\nu_1', \nu_2')$ such that

$$t(\nu_1 - \nu_1') = (\nu_2' - \nu_2). \tag{4.3}$$

Since, $(\nu_1, \nu_2) \neq (\nu_1', \nu_2')$, both $(\nu_1 - \nu_1')$ and $(\nu_2' - \nu_2)$ cannot be zero simultaneously. Therefore, the set

$$\mathcal{S} := \{t \in \mathbb{Z} \mid t(\nu_1 - \nu_1') = (\nu_2' - \nu_2) \text{ for } (\nu_1, \nu_2), (\nu_1', \nu_2') \in \mathrm{supp}(f), (\nu_1, \nu_2) \neq (\nu_1', \nu_2')\}$$

is either empty or finite. Choosing t from the nonempty set $\mathbb{Z} \setminus \mathcal{S}$ ensures that

$$(t\nu_1 + \nu_2) \neq (t\nu_1' + \nu_2') \text{ for all } (\nu_1, \nu_2), (\nu_1', \nu_2') \in \mathrm{supp}(f), (\nu_1, \nu_2) \neq (\nu_1', \nu_2').$$

This means the monomials appearing in $\varphi_T(f(\sigma))$ have all distinct powers of σ_2. Therefore, $\varphi_T(f(\sigma))$ can be written as

$$\varphi_T(f(\sigma)) = \sum_{\nu \in \operatorname{supp}(f)} \alpha_\nu \sigma_1^{\nu_1} \sigma_2^{t\nu_1+\nu_2}.$$

Note that $\alpha_\nu \sigma_1^{\nu_1}$ are units in $\mathcal{A}_1$ for all $\nu \in \operatorname{supp}(f)$. Rearranged in ascending order of powers of σ_2, we get $u_k(\sigma_1)$'s from $\alpha_\nu \sigma_1^{\nu_1}$s. Further, taking out the smallest power of σ_2 and calling it $u(\sigma_2)$ we get the desired expression of (4.2). ◻

Before we get to Theorem 4.3 we require one more important lemma.

LEMMA 4.2. *Let $\mathfrak{a} \subseteq \mathcal{A}$ be a nonzero ideal. Then there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that under the corresponding automorphism $\varphi_T : \mathcal{A} \to \mathcal{A}$ we have that $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1/(\varphi_T(\mathfrak{a}) \cap \mathcal{A}_1)$.*

Proof. Let $0 \neq f(\sigma) \in \mathfrak{a}$. By Lemma 4.1 above, there exists a unimodular matrix T such that $\varphi_T(f(\sigma))$ has the form of (4.2). Let us define $\mathfrak{b} := \varphi_T(\mathfrak{a})$. As we have mentioned earlier, since φ_T is an automorphism, $\mathfrak{b}$ is an ideal. Note that $\varphi_T(f(\sigma)) \in \mathfrak{b}$. Since $\varphi_T(f(\sigma)) \in \mathfrak{b}$, and $u(\sigma_2), u_\delta(\sigma_1)$ in the expression of $\varphi_T(f(\sigma))$ are units in $\mathcal{A}$, we also have $g(\sigma) := u(\sigma_2)^{-1} u_\delta(\sigma_1)^{-1} \varphi_T(f(\sigma)) \in \mathfrak{b}$. Now note that $g(\sigma)$ is of the following form:

$$g(\sigma) = \sigma_2^\delta + u_\delta(\sigma_1)^{-1} u_{\delta-1}(\sigma_1) \sigma_2^{\delta-1} + \cdots + u_\delta(\sigma_1)^{-1} u_0(\sigma_1).$$

Thus, $g(\sigma_2)$ is a monic polynomial in σ_2 with coefficients from $\mathcal{A}_1$ such that the constant term is $u_\delta(\sigma_1)^{-1} u_0(\sigma_1)$, which is a unit in $\mathcal{A}_1$. It then follows from Corollary 3.5 that $\mathcal{A}/\mathfrak{b}$ is a finitely generated module over $\mathcal{A}_1$.

Further, the annihilator of $\mathcal{A}/\mathfrak{b}$ as a module over $\mathcal{A}_1$ is clearly the intersection ideal $\mathfrak{b} \cap \mathcal{A}_1$. Hence, by Remark 2.4, $\mathcal{A}/\mathfrak{b}$ is a finitely generated module over $\mathcal{A}_1/\,(\mathfrak{b} \cap \mathcal{A}_1)$ as claimed in the statement of the lemma. ◻

We now state and prove the discrete version of Noether's normalization lemma.

THEOREM 4.3. *Suppose $\{0\} \neq \mathfrak{a} \subseteq \mathcal{A}$ is an ideal. Then exactly one of the following statements is true:*

1. *$\mathcal{A}/\mathfrak{a}$ is a finite-dimensional vector space over $\mathbb{R}$.*
2. *There exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that under the corresponding ring automorphism $\varphi_T : \mathcal{A} \to \mathcal{A}$, the quotient ring $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated faithful module over $\mathcal{A}_1$.*

Proof. By Lemma 4.2 there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1/(\varphi_T(\mathfrak{a}) \cap \mathcal{A}_1)$. Once again, we define $\mathfrak{b} := \varphi_T(\mathfrak{a})$. Now looking at $\mathfrak{b} \cap \mathcal{A}_1$, there are two possible situations:

(a) $\mathfrak{b} \cap \mathcal{A}_1 = 0$,
(b) $\mathfrak{b} \cap \mathcal{A}_1 \neq 0$.

For the first case we have $\mathcal{A}_1/(\mathfrak{b} \cap \mathcal{A}_1) = \mathcal{A}_1$. It then follows from Lemma 4.2 that $\mathcal{A}/\mathfrak{b}$ is a finitely generated module over the 1-variable Laurent polynomial ring $\mathcal{A}_1$. Also, since $\mathfrak{b} \cap \mathcal{A}_1$ is equal to the kernel of the ring map $\mathcal{A}_1 \to \mathcal{A}/\mathfrak{b}$, we have $\mathcal{A}_1$ contained in $\mathcal{A}/\mathfrak{b}$ as a subring. That is, $\mathcal{A}/\mathfrak{b}$ is faithful as an $\mathcal{A}_1$-module. Hence in this case, statement (2) of the theorem holds.

On the other hand, when $\mathfrak{b} \cap \mathcal{A}_1 \neq 0$, the quotient ring $\mathcal{A}_1/(\mathfrak{b} \cap \mathcal{A}_1)$ turns out to be a finite-dimensional vector space over $\mathbb{R}$. This is true essentially because $\mathcal{A}_1$, being the 1-variable Laurent polynomial ring, is a principal ideal domain (PID). This finite dimensionality of $\mathcal{A}_1/(\mathfrak{b} \cap \mathcal{A}_1)$ as a vector space over $\mathbb{R}$, together with the fact that $\mathcal{A}/\mathfrak{b}$ is finitely generated as a module over $\mathcal{A}_1/(\mathfrak{b} \cap \mathcal{A}_1)$, implies that $\mathcal{A}/\mathfrak{b}$ too is a finite-dimensional vector space over $\mathbb{R}$. This is equivalent to $\mathcal{A}/\mathfrak{b}$ being an Artinian ring (see footnote 1). Since φ_T is an automorphism of $\mathcal{A}$ it follows that $\mathcal{A}/\mathfrak{a}$ too is Artinian (see [3]) and thus a finite-dimensional vector space over $\mathbb{R}$. This is nothing but statement 1 of the theorem.

Thus, since exactly one of the statements (a) and (b) is true, we consequently have that exactly one of the statements 1 and 2 holds. This completes the proof. $\square$

Remark 4.4. Incidentally, statement 1 of Theorem 4.3 corresponds to $\mathcal{A}/\mathfrak{a}$ having *Krull dimension*[3] equal to 0. And, statement 2 corresponds to Krull dimension of $\mathcal{A}/\mathfrak{a}$ being 1. Since the global dimension of $\mathcal{A}$ is 2, only these two situations are possible. Thus, depending upon the Krull dimension of $\mathcal{A}/\mathfrak{a}$, one of the two statements of Theorem 4.3 holds.

When statement 1 of Theorem 4.3 above does not hold, the process of obtaining a unimodular $T \in \mathbb{Z}^{2\times 2}$ to get the automorphism $\varphi_T : \mathcal{A} \to \mathcal{A}$ so that statement 2 holds will be referred to in what follows as *Noether's normalization*.

5. Representation formula for general autonomous systems. In this section, we utilize the discrete version of Noether's normalization lemma to obtain a representation formula for a general 2D autonomous system. This is stated as Theorem 5.3 below. In order to make use of Noether's normalization, we need first to extend Theorem 4.3 to the module case. We do this in Theorem 5.2. For the proof of this extension, we need the following technical lemma, which relates annihilators of two quotient modules after a coordinate change, as in the Noether's normalization process, is done. Recall from (2.8) how a unimodular $T \in \mathbb{Z}^{2\times 2}$ induces a map $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$.

LEMMA 5.1. *Let $T \in \mathbb{Z}^{2\times 2}$ be unimodular, and let $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$ be the corresponding map of $\mathcal{A}$-modules via the ring map $\varphi_T : \mathcal{A} \to \mathcal{A}$. Then*

$$\varphi_T(\mathrm{ann}(\mathcal{M})) = \mathrm{ann}(\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})).$$

Proof. Since $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$ is a bijection, $f(\sigma) \in \mathrm{ann}(\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R}))$ if and only if for all $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ we have $f(\sigma)\widehat{\varphi}_T(r(\sigma)) \in \widehat{\varphi}_T(\mathcal{R})$. Recall that φ_T is an automorphism of $\mathcal{A}$. Therefore, we have $g(\sigma) := \varphi_T^{-1}(f(\sigma))$. Then we can write

$$f(\sigma)\widehat{\varphi}_T(r(\sigma)) = \varphi_T(g(\sigma))\widehat{\varphi}_T(r(\sigma)) = \widehat{\varphi}_T(g(\sigma)r(\sigma)) \in \widehat{\varphi}_T(\mathcal{R}) \;\Leftrightarrow\; g(\sigma) \in \mathrm{ann}(\mathcal{M}).$$

But $g(\sigma) \in \mathrm{ann}(\mathcal{M})$ if and only if $f(\sigma) \in \varphi_T(\mathrm{ann}(\mathcal{M}))$. $\square$

Lemma 5.1 above, together with Remark 2.4, leads to Theorem 5.2 below.

THEOREM 5.2. *Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule such that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a torsion module. Then exactly one of the following statements is true:*

1. *$\mathcal{M}$ is a finite-dimensional vector space over $\mathbb{R}$.*
2. *There exists $T \in \mathbb{Z}^{2\times 2}$ such that under the corresponding module map $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$, the quotient module $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated faithful module over $\mathcal{A}_1$.*

Proof. From the fact that $\mathcal{M}$ is a finitely generated faithful module over $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ (Remark 2.4) it follows that $\mathcal{M}$ is a finite-dimensional $\mathbb{R}$-vector space if and only if $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ too is a finite-dimensional $\mathbb{R}$-vector space (see [3]). Now applying Theorem 4.3 to the ideal $\mathrm{ann}(\mathcal{M})$ we get that exactly one of the following statements is true:

[3]The Krull dimension of $\mathcal{A}/\mathfrak{a}$ is the length of a maximal proper chain of prime ideals in $\mathcal{A}/\mathfrak{a}$. See [3] for a detailed exposition of this concept.

(a) $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ is a finite-dimensional vector space over $\mathbb{R}$.

(b) There exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that under the corresponding ring automorphism $\varphi_T : \mathcal{A} \to \mathcal{A}$, the quotient ring $\mathcal{A}/\varphi_T(\mathrm{ann}(\mathcal{M}))$ is a finitely generated faithful module over $\mathcal{A}_1$.

As mentioned at the beginning of this proof, statement (a) here is equivalent to statement 1 of the theorem. On the other hand, using the unimodular $T \in \mathbb{Z}^{2\times 2}$ of statement (b) to define $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$, we get by Lemma 5.1 that $\varphi_T(\mathrm{ann}(\mathcal{M})) = \mathrm{ann}(\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R}))$. Therefore, statement (b) is equivalent to $\mathcal{A}/\mathrm{ann}(\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R}))$ being a finitely generated faithful module over $\mathcal{A}_1$. Once again using the fact that $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated faithful module over $\mathcal{A}/\mathrm{ann}(\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R}))$, we get that $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ too is a finitely generated faithful module over $\mathcal{A}_1$. This is nothing but statement 2 of the theorem. Since exactly one of the statements (a) and (b) is true, it follows that exactly one of the statements 1 and 2 of the theorem is true. □

It is well known that statement 1 of Theorem 5.2 above corresponds to $\mathfrak{B}(\mathcal{R})$ being strongly autonomous. Since such behaviors are already known to have a representation formula given by (2.5), in what follows we shall concentrate only on autonomous systems which are not strongly autonomous. Recall that strongly autonomous systems are always strongly σ_2-relevant. As a consequence of Theorem 5.2 above and Theorem 2.6 it follows that for every 2D autonomous system $\mathfrak{B}$ there exists a coordinate transformation T such that $\mathfrak{B}$ is related with a strongly σ_2-relevant behavior, say $\mathfrak{B}'$, by $\mathfrak{B} = \Phi_T(\mathfrak{B}')$. This is the key idea behind the general representation formula stated in Theorem 5.3 below.

THEOREM 5.3. *Suppose $\mathfrak{B}$ is an autonomous behavior whose equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ is such that the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is not a finite-dimensional vector space over $\mathbb{R}$. Then there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular, two positive integers n, n', and the following* 1*-variable Laurent polynomial matrices:*

- $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$,
- $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$,
- $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$,

with $A(\sigma_1)$ invertible in $\mathcal{A}_1^{n\times n}$, such that $w \in \mathfrak{B}$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$, which satisfies

$$R_1(\sigma_1)x = 0,$$

and for all $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$w(\nu) = \left(C(\sigma_1)A(\sigma_1)^{(T\nu)_2}x\right)((T\nu)_1),$$

where $T\nu = \mathrm{col}((T\nu)_1, (T\nu)_2)$.

Proof. Since $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is not a finite-dimensional $\mathbb{R}$-vector space, by Theorem 5.2, there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that under the corresponding map $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$ we have $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ as a finitely generated module over $\mathcal{A}_1$. It then follows from Theorem 3.7 that there exist two positive integers n and n' and the following 1-variable Laurent polynomial matrices:

- $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$,
- $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$,
- $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$

such that $v \in \mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$, which satisfies

$$R_1(\sigma_1)x = 0,$$

and for all $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$v(\nu) = (C(\sigma_1)A(\sigma_1)^{\nu_2}x)\,(\nu_1). \tag{5.1}$$

Now, by Theorem 2.6, $\mathfrak{B} = \Phi_T(\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R})))$. Hence $w \in \mathfrak{B}$ if and only if there exists $v \in \mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ such that for all $\nu \in \mathbb{Z}^2$,

$$w(\nu) = v(T\nu).$$

It then follows from (5.1) that $w \in \mathfrak{B}$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ satisfying $R_1(\sigma_1)x = 0$ and

$$w(\nu) = \left(C(\sigma_1)A(\sigma_1)^{(T\nu)_2}x\right)((T\nu)_1),$$

where $T\nu = \mathrm{col}((T\nu)_1, (T\nu)_2)$. $\square$

Example 5.4. Consider the scalar behavior

$$\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1).$$

The equation module is the principal ideal $\mathfrak{a} = \langle \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1\rangle$. The quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$ is clearly not a finitely generated module over $\mathcal{A}_1$. Therefore, $\mathfrak{B}$ is not strongly σ_2-relevant. However, under the coordinate transformation $T = \left[\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right]$ the transformed ideal $\varphi_T(\mathfrak{a})$ turns out to be

$$\varphi_T(\mathfrak{a}) = \langle \varphi_T(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)\rangle = \langle \sigma_2^3 - \sigma_2^2 - \sigma_1^{-1}\sigma_2 + \sigma_1^{-1}\rangle.$$

Clearly, $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1$. Generators can be chosen to be $\{\overline{1}, \overline{\sigma_2}, \overline{\sigma_2}^2\}$. In fact, these generators freely generate $\mathcal{A}/\varphi_T(\mathfrak{a})$ as an $\mathcal{A}_1$-module. Here, $n = 3$ and

- $R_1(\sigma_1) = 0$,
- $A(\sigma_1) = \left[\begin{smallmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\sigma_1^{-1} & \sigma_1^{-1} & 1\end{smallmatrix}\right]$,
- $C(\sigma_1) = \left[\begin{smallmatrix}1 & 0 & 0\end{smallmatrix}\right]$.

Hence, solutions in $\mathfrak{B}$ are given by

$$w(\nu_1, \nu_2) = \left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\sigma_1^{-1} & \sigma_1^{-1} & 1\end{bmatrix}^{2\nu_1+\nu_2} x\right)(\nu_1),$$

where $x \in (\mathbb{R}^3)^{\mathbb{Z}}$ is arbitrary.

6. The set of valid initial conditions and alternative representation formulae. As mentioned in Remark 3.11, the variable x can be thought of as initial conditions. Unlike the situation in 1D systems where x can always be constructed so that the initial conditions can be chosen freely (see [20]), here x must satisfy certain constraints to qualify as a valid initial condition and, consequently, may not always be free. We have already seen that the set of valid initial conditions is given by $\mathscr{X} := \ker(R_1(\sigma_1))$. Note that a different choice of generators for $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ results in a different $\mathscr{X}$; however, these different $\mathscr{X}$'s are *isomorphic* to each other as 1D behaviors in the sense of [11]. In this section, we first show in Theorem 6.5 how a better choice of generators leads to an alternative representation formula. Later, in

Theorem 6.10 we resolve another issue related to $\mathscr{X}$: when is $\mathscr{X}$ *free*, when is $\mathscr{X}$ *free*, i.e., when is $\mathscr{X} = (\mathbb{R}^n)^{\mathbb{Z}}$ for some n?

Remark 6.1. Note that, if $\mathfrak{B}$ is assumed to be *not* strongly autonomous, then, by Theorem 5.2, $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a faithful $\mathcal{A}_1$-module. Therefore, $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ as an $\mathcal{A}_1$-module has the trivial ideal $\{0\}$ as its annihilator. In other words, $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$, as an $\mathcal{A}_1$-module, cannot be torsion. This means $\mathscr{X}$, when viewed as a 1D behavior, *cannot* be autonomous. Hence $\mathscr{X}$, the set of valid initial conditions, is *not* a finite-dimensional vector space over $\mathbb{R}$. Also, $\mathscr{X}$ not being autonomous means that some of the x variables can be treated as *inputs* and some as *outputs* (see [21]). Those of x which are treated as inputs can be any 1D trajectory. We call these variables *free variables*. In general, not all of the x variables are free, but it is desirable to have all of x free. Theorem 6.10 gives conditions equivalent to this freeness. Recall that if $\mathfrak{B}$ is strongly autonomous, then it is already strongly σ_2-relevant. In that case, although $\mathcal{M}$ will be a finitely generated module over $\mathcal{A}_1$, it will not be faithful. However, if we carry out the construction of $\mathscr{X}$ in this case we will find that $\mathscr{X}$ is an *autonomous* 1D behavior. This is the other extreme case where all of the x variables will be outputs.

In order to resolve the issue of free initial conditions, it will be useful to look at an alternative representation formula. In this alternative formula, we give a more precise description of the initial conditions. We first prove in Lemma 6.3 below how a clever choice of the generators results in a better structure of the matrix of relations $R_1(\sigma_1)$. Note that we can make $R_1(\sigma_1)$ *full row-rank* over the field of fractions $\mathrm{qt}(\mathcal{A}_1)$. This is true because $\mathcal{A}_1$ is a PID.[4] Lemma 6.3 below utilizes another consequence of $\mathcal{A}_1$ being a PID: $R_1(\sigma_1)$ admits a *Smith form*. For our purpose, the Smith canonical form in full generality is not required; a weaker version suffices. We state this result as Proposition 6.2 below. See [12] for a proof.

PROPOSITION 6.2. *Let $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$ be a full row-rank matrix. Then there exist square matrices $U(\sigma_1) \in \mathcal{A}_1^{n'\times n'}$ and $V(\sigma_1) \in \mathcal{A}_1^{n\times n}$, with the property that* $\det(U(\sigma_1))$ *and* $\det(V(\sigma_1))$ *are units in $\mathcal{A}_1$, such that*

$$U(\sigma_1)R_1(\sigma_1)V(\sigma_1) = \begin{bmatrix} D(\sigma_1) & 0 \end{bmatrix},$$

where $D(\sigma_1) \in \mathcal{A}_1^{n'\times n'}$ is square with nonzero determinant.

LEMMA 6.3. *Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule such that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Then there exists a set of generators of $\mathcal{M}$ as an $\mathcal{A}_1$-module, which admits a matrix of relations $R_1(\sigma_1)$ of the form*

$$R_1(\sigma_1) = \begin{bmatrix} D(\sigma_1) & 0 \end{bmatrix}, \tag{6.1}$$

where $D(\sigma_1)$ is a square matrix with nonzero determinant.

Proof. Let $\{\overline{g_1'(\sigma)}, \overline{g_2'(\sigma)}, \ldots, \overline{g_n'(\sigma)}\}$ be an arbitrary set of generators for $\mathcal{M}$ as an $\mathcal{A}_1$-module, and let $R_1'(\sigma_1) \in \mathcal{A}_1^{n'\times n}$ be its matrix of relations. As mentioned earlier, $R_1'(\sigma_1)$ can be assumed to be full row-rank. Then by Proposition 6.2 there exist square matrices $U(\sigma_1) \in \mathcal{A}_1^{n'\times n}$ and $V(\sigma_1) \in \mathcal{A}_1^{n\times n}$, both having units for determinants, such that

$$U(\sigma_1)R_1'(\sigma_1)V(\sigma_1) = \begin{bmatrix} D(\sigma_1) & 0 \end{bmatrix}, \tag{6.2}$$

[4] $\mathcal{A}_1$ being a PID implies that the submodule $\ker(\psi)$ of the free module $\mathcal{A}_1^n$ is free, and hence there exists a full row-rank matrix $R_1(\sigma_1)$ whose rows will generate the free module $\ker(\psi)$ over $\mathcal{A}_1$.

where $D(\sigma_1) \in \mathcal{A}_1^{n'\times n'}$ with nonzero determinant. Since $\det(V(\sigma_1))$ is a unit in $\mathcal{A}_1$, it follows that $V(\sigma_1)$ has an inverse in $\mathcal{A}_1^{n\times n}$. Define

$$\begin{bmatrix} g_1(\sigma) \\ g_2(\sigma) \\ \vdots \\ g_n(\sigma) \end{bmatrix} := V(\sigma_1)^{-1} \begin{bmatrix} g_1'(\sigma) \\ g_2'(\sigma) \\ \vdots \\ g_n'(\sigma) \end{bmatrix}. \tag{6.3}$$

Clearly, $\mathcal{G} := \{\overline{g_1(\sigma)}, \overline{g_2(\sigma)}, \dots, \overline{g_n(\sigma)}\}$ is a generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. It then follows that a matrix of relations for this new set of generators is given by

$$R_1(\sigma_1) := U(\sigma_1)R_1'(\sigma_1)V(\sigma_1).$$

Indeed, $R_1'(\sigma_1)V(\sigma_1)$ is clearly a matrix of relations for $\mathcal{G}$. Since $\det(U(\sigma_1))$ is a unit in $\mathcal{A}_1$, it also has an inverse in $\mathcal{A}_1^{n'\times n'}$. It then follows that the rowspan of $R_1'(\sigma_1)V(\sigma_1)$ is the same as that of $U(\sigma_1)R_1'(\sigma_1)V(\sigma_1)$. Therefore, $R_1(\sigma_1) := U(\sigma_1)R_1'(\sigma_1)V(\sigma_1)$ is a matrix of relations for $\mathcal{G}$. The statement of the lemma then follows from (6.2). ∎

The purpose of obtaining $R_1(\sigma_1)$ in the form of (6.1) is that now the set of valid initial conditions $\mathscr{X} := \ker(R_1(\sigma_1))$ has a nice description. Suppose $R_1(\sigma_1) = \begin{bmatrix} D(\sigma_1) & 0 \end{bmatrix} \in \mathcal{A}_1^{n'\times n}$, with $D(\sigma_1) \in \mathcal{A}_1^{n'\times n'}$ having nonzero determinant. Now partition $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ as $x = (x_1, x_2)$, where $x_1 \in (\mathbb{R}^{n'})^{\mathbb{Z}}$ and $x_2 \in (\mathbb{R}^{n-n'})^{\mathbb{Z}}$. Then $x \in \ker(R_1(\sigma_1))$ if and only if

$$D(\sigma_1)x_1 = 0,$$

and x_2 is free. Now, since $D(\sigma_1)$ is square with nonzero determinant, it follows that $\ker(D(\sigma_1))$ is a finite-dimensional vector space over $\mathbb{R}$. In other words, there exists a fixed set of finitely many 1D trajectories $\{z_1, z_2, \dots, z_r\} \subseteq (\mathbb{R}^{n'})^{\mathbb{Z}}$ such that $x \in \mathscr{X}$ if and only if it is of the form

$$x = \begin{bmatrix} a_1 z_1 + a_2 z_2 + \cdots + a_r z_r \\ x_2 \end{bmatrix}, \tag{6.4}$$

where $\{a_1, a_2, \dots, a_r\} \subseteq \mathbb{R}$ and $x_2 \in (\mathbb{R}^{n-n'})^{\mathbb{Z}}$. This leads to the following alternative representation formula.

Remark 6.4. It is important to note that when a new set of generators, say $\{\overline{g_1'(\sigma)}, \overline{g_2'(\sigma)}, \dots, \overline{g_n'(\sigma)}\}$, is obtained from an old one, say $\{\overline{g_1(\sigma)}, \overline{g_2(\sigma)}, \dots, \overline{g_n(\sigma)}\}$, by (6.3) in Lemma 6.3, the corresponding matrix representations of the map μ turn out to obey the following equation:

$$A'(\sigma_1) = V(\sigma_1)A(\sigma_1)V(\sigma_1)^{-1}.$$

This is analogous to a similarity transformation done on the state-space in 1D systems. Observe that $A'(\sigma_1)$ is invertible if and only if $A(\sigma_1)$ is.

THEOREM 6.5. *Suppose $\mathfrak{B}$ is an autonomous behavior whose equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ is such that the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is not a finite-dimensional vector space over $\mathbb{R}$. Then there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular, two positive integers n, n', the* 1*-variable Laurent polynomial matrices*

- $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$,
- $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$ *invertible,*

and a fixed set of finitely many 1*D trajectories* $\{z_1, z_2, \ldots, z_r\} \subseteq (\mathbb{R}^{n'})^{\mathbb{Z}}$ *such that* $w \in \mathfrak{B}$ *if and only if for all* $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$
\begin{aligned}
w(\nu) = &\left(C(\sigma_1) A(\sigma_1)^{(T\nu)_2} \begin{bmatrix} a_1 z_1 + a_2 z_2 + \cdots + a_r z_r \\ 0 \end{bmatrix} \right) ((T\nu)_1) \\
&+ \left(C(\sigma_1) A(\sigma_1)^{(T\nu)_2} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right) ((T\nu)_1),
\end{aligned}
\tag{6.5}
$$

where $\{a_1, a_2, \ldots, a_r\} \subseteq \mathbb{R}$ *and* $x_2 \in (\mathbb{R}^{n-n'})^{\mathbb{Z}}$.

Proof. By Theorem 5.2 there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated module over $\mathcal{A}_1$. By Lemma 6.3 there exists a generating set for $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ over $\mathcal{A}_1$ such that the matrix of relations, say $R_1(\sigma_1)$, for this generating set is of the form $\begin{bmatrix} D(\sigma_1) & 0 \end{bmatrix}$ with $D(\sigma_1)$ square and having nonzero determinant. Note that by Remark 6.4 we can guarantee that the matrix $A(\sigma_1)$ in this generating set is invertible in $\mathcal{A}_1^{n\times n}$. By (6.4) it follows that in this case there exists a fixed set of finitely many 1D trajectories $\{z_1, z_2, \ldots, z_n\} \subseteq (\mathbb{R}^{n'})^{\mathbb{Z}}$ such that all the 1D trajectories in $\mathscr{X} := \ker(R_1(\sigma_1))$ are given by

$$
x = \begin{bmatrix} a_1 z_1 + a_2 z_2 + \cdots + a_n z_n \\ x_2 \end{bmatrix},
$$

where $\{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}$ and $x_2 \in (\mathbb{R}^{n-n'})^{\mathbb{Z}}$. From Theorem 5.3 we get our desired result that in this case $w(\nu) \in \mathfrak{B}$ is given by

$$
\begin{aligned}
w(\nu) = &\left(C(\sigma_1) A(\sigma_1)^{(T\nu)_2} \begin{bmatrix} a_1 z_1 + a_2 z_2 + \cdots + a_n z_n \\ 0 \end{bmatrix} \right) ((T\nu)_1) \\
&+ \left(C(\sigma_1) A(\sigma_1)^{(T\nu)_2} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right) ((T\nu)_1). \qquad \square
\end{aligned}
$$

Remark 6.6. The representation formula in Theorem 6.5 explicitly brings out how free the initial conditions are; the $x_2 \in (\mathbb{R}^{n-n'})^{\mathbb{Z}}$ trajectories constitute the free part. Once again, viewing $\mathscr{X}$ as a 1D system, x_2 can be thought of as input. In [21] this number $(n - n')$ has been called the *input cardinality* of the concerned 1D behavior.

Remark 6.7. Note that a variant of (6.5) holds even for the case when $\mathfrak{B}$ is strongly autonomous. In that case, as we have already mentioned in Remark 6.1, the set of valid initial conditions $\mathscr{X}$ turns out to be a finite-dimensional vector space over $\mathbb{R}$. Viewed as a 1D system, then, $\mathscr{X}$ turns out to have input cardinality equal to zero. So, all the initial conditions are of the form

$$
x = a_1 z_1 + a_2 z_2 + \cdots + a_r z_r,
$$

where $\{z_1, z_2, \ldots, z_r\} \subseteq (\mathbb{R}^n)^{\mathbb{Z}}$ are a fixed finite set of 1D trajectories. Moreover, since strongly autonomous implies strongly σ_2-relevant, no explicit coordinate change is required in this case. With these considerations then, we get for strongly autonomous systems the following representation formula, which is a variant of (6.5):

$$
w(\nu) = \left(C(\sigma_1) A(\sigma_1)^{\nu_2} \begin{bmatrix} a_1 z_1 + a_2 z_2 + \cdots + a_r z_r \end{bmatrix} \right) (\nu_1),
$$

where $a_1, a_2, \ldots, a_r \in \mathbb{R}$ are arbitrary.

In light of Theorem 6.5, a set of valid initial conditions will be *free* if the determinant of $D(\sigma_1)$ is a unit in $\mathcal{A}_1$. This means $\ker(D(\sigma_1)) = 0$. In that case, only the free x_2 part constitutes all the valid initial conditions. Now $D(\sigma_1)$ having a unit for determinant is equivalent to $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ being a free module over $\mathcal{A}_1$ because $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R}) \cong \mathcal{A}_1^n/\mathrm{rowspan}(R_1(\sigma_1))$ as $\mathcal{A}_1$-modules. In Theorem 6.10 below we provide equivalent conditions for $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ to be a free $\mathcal{A}_1$-module. For the proof of this result we make use of the following crucial lemma, Lemma 6.8. The proof of Lemma 6.8 utilizes various intricate algebraic ideas, which we do not make use of elsewhere in the paper. Due to this technicality, we provide the proof of Lemma 6.8 in Appendix A.

LEMMA 6.8. *Let $R(\sigma) \in \mathcal{A}^{\mathtt{w}\times\mathtt{w}}$ have a nonzero determinant $g(\sigma)$. Further, let*

$$g(\sigma) = u(\sigma)p_1(\sigma)^{n_1}p_2(\sigma)^{n_2}\cdots p_k(\sigma)^{n_k}$$

be a prime factorization of $g(\sigma)$, with $u(\sigma)$ a unit and $\{p_i(\sigma)\}$ irreducible elements in $\mathcal{A}$ and $p_i(\sigma) \neq p_j(\sigma)$ for $i \neq j$. If $f(\sigma) \in \mathcal{A}$ is a zerodivisor on $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathrm{rowspan}(R(\sigma))$ then $f(\sigma)$ must be divisible by one of the $p_i(\sigma)$s.

We require one more technical result before we proceed to Theorem 6.10. Lemma 6.9 below is a consequence of Proposition 3.4. It says that if $R(\sigma) \in \mathcal{A}^{\mathtt{w}\times\mathtt{w}}$ is a square matrix whose determinant has a special structure (similar to the right-hand side of (4.2) in Lemma 4.1), then the quotient module $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathrm{rowspan}(R(\sigma))$ will be finitely generated as a module over $\mathcal{A}_1$.

LEMMA 6.9. *Let $R(\sigma) \in \mathcal{A}^{\mathtt{w}\times\mathtt{w}}$ be such that*

$$\det(R(\sigma)) = u(\sigma)\left(\sum_{i=0}^{\delta-1} u_i(\sigma_1)\sigma_2^i + \sigma_2^\delta\right),$$

where $u(\sigma) \in \mathcal{A}$ is a unit and $\{u_0(\sigma_1), u_1(\sigma_1), \ldots, u_{\delta-1}(\sigma_1)\} \subseteq \mathcal{A}_1$ are also units. Then $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathrm{rowspan}(R(\sigma))$ is a finitely generated module over $\mathcal{A}_1$.

Proof. Let us define $f(\sigma) := u(\sigma)^{-1}\det(R(\sigma))$. Since $u(\sigma)$ is a unit in $\mathcal{A}$, it follows that $f(\sigma)$ thus defined belongs to $\mathcal{A}$. Note that $f(\sigma)$ can also be viewed as a polynomial in σ_2 with coefficients from $\mathcal{A}_1$. That way, $f(\sigma)$ is a monic polynomial in σ_2 with the constant term being a unit in $\mathcal{A}$. Now, recall that the characteristic ideal of a behavior $\mathfrak{B}$ is defined to be the ideal generated by all $(\mathtt{w}\times\mathtt{w})$ minors of its kernel representation matrix. Let us define $\mathfrak{B} = \ker(R(\sigma))$. Then, in this case, the characteristic ideal turns out to be

$$\mathcal{I}(\mathfrak{B}) = \langle \det(R(\sigma))\rangle.$$

Clearly, then $f(\sigma) \in \mathcal{I}(\mathfrak{B})$. However, by Proposition 2.3, $\mathcal{I}(\mathfrak{B}) \subseteq \mathrm{ann}(\mathcal{M})$. Thus, $f(\sigma) \in \mathrm{ann}(\mathcal{M})$. Now, we have already seen that $f(\sigma)$, as a polynomial in σ_2 with coefficients from $\mathcal{A}_1$, is monic and has as constant term a unit in $\mathcal{A}$. Therefore, by equivalence of statements 2 and 3 of Proposition 3.4, we get that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathrm{rowspan}(R(\sigma))$ is a finitely generated module over $\mathcal{A}_1$. $\square$

We are now in a position to state and prove the second main result of this section: Theorem 6.10. This theorem shows that initial conditions can be chosen freely if and only if the behavior has a square kernel representation matrix with nonzero determinant. As a consequence of free initial conditions, the first order equation over $\mathscr{X}$ defined in Remark 3.11 becomes a first order equation over $(\mathbb{R}^n)^{\mathbb{Z}}$. This results in a first order representation of $\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ as given in statement 3 of Theorem 6.10 below.

THEOREM 6.10. *Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule and $\mathfrak{B}$ its behavior. Then the following statements are equivalent:*

1. *There exists a square matrix $R(\sigma) \in \mathcal{A}_1^{\mathtt{w}\times\mathtt{w}}$ with $\det(R(\sigma)) \neq 0$ such that $\mathfrak{B} = \ker(R(\sigma))$.*
2. *There exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated free $\mathcal{A}_1$-module.*
3. *There exist $T \in \mathbb{Z}^{2\times 2}$ unimodular, a positive integer n, and 1-variable Laurent polynomial matrices $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$ and $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$ such that*

$$\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R})) = \left\{ v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \,\middle|\, \exists\, \ell \in (\mathbb{R}^n)^{\mathbb{Z}^2} \ s.t. \begin{bmatrix} \sigma_2 I_n - A(\sigma_1) & 0 \\ C(\sigma_1) & -I_{\mathtt{w}} \end{bmatrix} \begin{bmatrix} \ell \\ v \end{bmatrix} = 0 \right\}.$$

Proof. (1) $\Rightarrow$ (2) Suppose $g(\sigma) = \det(R(\sigma))$. By assumption, $g(\sigma) \neq 0$. Using Lemma 4.1 we get a unimodular $T \in \mathbb{Z}^{2\times 2}$ such that $\varphi_T(g(\sigma))$ has the form

$$\varphi_T(g(\sigma)) = u(\sigma) \left(\sum_{i=0}^{\delta-1} u_i(\sigma_1)\sigma_2^i + \sigma_2^\delta \right), \tag{6.6}$$

where $u(\sigma) \in \mathcal{A}$ is a unit and $\{u_0(\sigma_1), u_1(\sigma_1), \dots, u_{\delta-1}(\sigma)\} \subseteq \mathcal{A}_1$ are also units. Let $\varphi_T(R(\sigma))$ denote the matrix obtained by applying φ_T to each entry in $R(\sigma)$. Then clearly, $\det(\varphi_T(R(\sigma))) = \varphi_T(g(\sigma))$. Note that $\mathrm{rowspan}(\varphi_T(R(\sigma))) = \widehat{\varphi}_T(\mathcal{R})$. By Lemma 6.9 $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated $\mathcal{A}_1$-module. We want to show that $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is also free as an $\mathcal{A}_1$-module. In order to show this, it is enough that we show $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is torsion-free as an $\mathcal{A}_1$-module. This is true because $\mathcal{A}_1$ is a PID, and therefore, torsion-free is equivalent to free. Suppose $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is not torsion-free as an $\mathcal{A}_1$-module. This would imply that there exists $0 \neq f(\sigma_1) \in \mathcal{A}_1$ and $0 \neq \overline{m(\sigma)} \in \mathcal{A}_1^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ such that $\overline{f(\sigma_1)m(\sigma)} = 0 \in \mathcal{A}_1^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$. This means $f(\sigma_1)$ is a zerodivisor on $\mathcal{A}_1^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$. Then by Lemma 6.8 $f(\sigma_1)$ must be divisible by one of the irreducible factors of $\det(\varphi_T(R(\sigma))) = \varphi_T(g(\sigma))$. However, the right-hand side of (6.6) is the product of a unit and a monic polynomial in σ_2 with coefficients in $\mathcal{A}_1$. Therefore, no irreducible factor of $\varphi_T(g(\sigma))$ can be purely in $\mathcal{A}_1$. This means $f(\sigma_1)$ must be zero, that is, $\mathcal{A}_1^n/\mathrm{rowspan}(R_1(\sigma_1))$ is a torsion-free $\mathcal{A}_1$-module.

(2) $\Rightarrow$ (3) We assume that there exists $T \in \mathbb{Z}^{2\times 2}$ such that $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated free $\mathcal{A}_1$-module. This means we can fix a free basis for $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$. Since this basis is free the matrix of relations $R_1(\sigma_1)$ turns out to be zero. Therefore, it follows from Theorem 3.7 that there exist

- a positive integer n,
- a 1-variable Laurent polynomial matrix $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$, and
- another 1-variable Laurent polynomial matrix $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$

such that $v \in \mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$, and for all $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$w(\nu) = (C(\sigma_1)A(\sigma_1)^{\nu_2}x)(\nu_1). \tag{6.7}$$

Define the following 2D trajectory: for all $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$\ell(\nu) := (A(\sigma_1)^{\nu_2}x)(\nu_1).$$

Then by this definition ℓ satisfies

$$\sigma_2\ell = A(\sigma_1)\ell, \tag{6.8}$$

and (6.7) can be rewritten as

$$w = C(\sigma_1)\ell. \tag{6.9}$$

Combining (6.8) and (6.9), we get that $v \in \mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ if and only if there exists $\ell \in (\mathbb{R}^n)^{\mathbb{Z}^2}$ such that

$$\begin{bmatrix} \sigma_2 I_n - A(\sigma_1) & 0 \\ C(\sigma_1) & -I_{\mathtt{w}} \end{bmatrix} \begin{bmatrix} \ell \\ v \end{bmatrix} = 0.$$

(3) $\Rightarrow$ (1) First, note that it is enough to prove $\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ admits a square kernel representation. Indeed, by applying $\varphi_{T^{-1}}$ elementwise we can obtain a square kernel representation for $\mathfrak{B}$.

In order to show that $\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))$ admits a square kernel representation we make use of the elimination theorem from [8]. Let $E_1(\sigma) \in \mathcal{A}^{\mathtt{g}\times n}$ and $E_2(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$ be such that the matrix $E(\sigma) := \left[\begin{smallmatrix} E_1(\sigma) & E_2(\sigma)\end{smallmatrix}\right]$ is a maximal left annihilator[5] of $\left[\begin{smallmatrix} \sigma_2 I_n - A(\sigma_1) \\ C(\sigma_1)\end{smallmatrix}\right]$. Then according to the elimination theorem,

$$\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R})) = \ker(-E_2(\sigma)).$$

We first show that $E_2(\sigma)$ can be chosen to be square. Consider the 2D behavior

$$\mathfrak{B}_{\text{aux}} := \ker(E(\sigma)) \subseteq (\mathbb{R}^{n+\mathtt{w}})^{\mathbb{Z}^2}.$$

Since $E(\sigma)$ is a maximal left annihilator matrix, it follows that $\mathfrak{B}_{\text{aux}}$ is controllable (see [10]). By Corollary 4 in [10], we get that rowspan$(E(\sigma))$ is a free submodule of $\mathcal{A}^{n+\mathtt{w}}$, which means $E(\sigma)$ can be chosen to be full row-rank. But since rank$\left[\begin{smallmatrix} \sigma_2 I_n - A(\sigma_1) \\ C(\sigma_1)\end{smallmatrix}\right] = n$, and $E(\sigma)$ is its maximal left annihilator, it follows that rank$(E(\sigma)) = \mathtt{w}$. Therefore, $E(\sigma)$ can be chosen to have full row-rank with $\mathtt{w}$ number of rows, thus making $E_2(\sigma)$ square.

We now prove that $\det(-E_2(\sigma)) \neq 0$. For this purpose, it is enough that we show $\mathcal{A}^{\mathtt{w}}/\text{rowspan}(-E_2(\sigma))$ is a torsion module. Let $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and consider $\begin{bmatrix} 0 & r(\sigma)\end{bmatrix} \in \mathcal{A}^{n+\mathtt{w}}$. Define

$$R'(\sigma) := \begin{bmatrix} \sigma_2 I_n - A(\sigma_1) & 0 \\ C(\sigma_1) & -I_{\mathtt{w}} \end{bmatrix}.$$

Since $\det(R'(\sigma)) \neq 0$, it follows that $\mathcal{A}^{n+\mathtt{w}}/\text{rowspan}(R'(\sigma))$ is a torsion module. Hence there exists $f(\sigma) \in \mathcal{A}$ such that

$$f(\sigma)\begin{bmatrix} 0 & r(\sigma)\end{bmatrix} = \begin{bmatrix} 0 & f(\sigma)r(\sigma)\end{bmatrix} \in \text{rowspan}(R'(\sigma)).$$

In other words, there exist $r_1(\sigma) \in \mathcal{A}^n$ and $r_2(\sigma) \in \mathcal{A}^{\mathtt{w}}$ such that

$$\begin{bmatrix} 0 & f(\sigma)r(\sigma)\end{bmatrix} = \begin{bmatrix} r_1(\sigma) & r_2(\sigma)\end{bmatrix} R'(\sigma). \tag{6.10}$$

This means, $\begin{bmatrix} r_1(\sigma) & r_2(\sigma)\end{bmatrix}\left[\begin{smallmatrix} \sigma_2 I_n - A(\sigma_1) \\ C(\sigma_1)\end{smallmatrix}\right] = 0$. Since $E(\sigma) \in \mathcal{A}^{\mathtt{w}\times(n+\mathtt{w})}$ is a maximal left annihilator of $\left[\begin{smallmatrix} \sigma_2 I_n - A(\sigma_1) \\ C(\sigma_1)\end{smallmatrix}\right]$ it follows that there exists $r_3(\sigma) \in \mathcal{A}^{\mathtt{w}}$ such that

$$\begin{bmatrix} r_1(\sigma) & r_2(\sigma)\end{bmatrix} = r_3(\sigma)\begin{bmatrix} E_1(\sigma) & E_2(\sigma)\end{bmatrix}.$$

[5]By *maximal left annihilator* of a polynomial matrix $M(\sigma)$ with entries in $\mathcal{A}$ we mean a matrix, say $N(\sigma)$, with entries in $\mathcal{A}$ such that the rows of $N(\sigma)$ generate the module of relations of the rows of $M(\sigma)$.

Putting this in (6.10), we get

$$\begin{aligned}\begin{bmatrix}0 & f(\sigma)r(\sigma)\end{bmatrix} &= r_3(\sigma)E(\sigma)R'(\sigma)\\ &= r_3(\sigma)\begin{bmatrix}E_1(\sigma) & E_2(\sigma)\end{bmatrix}\begin{bmatrix}\sigma_2 I_n - A(\sigma_1) & 0\\ C(\sigma_1) & -I_{\mathtt{w}}\end{bmatrix}\\ &= r_3(\sigma)\begin{bmatrix}0 & E_2(\sigma)\end{bmatrix}\\ \Rightarrow f(\sigma)r(\sigma) &= r_3(\sigma)E_2(\sigma) \in \text{rowspan}(-E_2(\sigma)).\end{aligned}$$

Since $r(\sigma)$ was arbitrary, this proves that $\mathcal{A}^{\mathtt{w}}/\text{rowspan}(E_2(\sigma))$ is a torsion module. ∎

7. Concluding remarks. In this paper, we looked into representation formulae for discrete 2D autonomous systems. These representation formulae generalize the solution formula for 1D autonomous systems given by a flow acting on initial conditions. The crucial difference in the 2D case is that here the initial conditions are given by 1D trajectories as opposed to real vectors in the 1D case. Moreover, instead of a constant matrix, here in the 2D case the flow operator is a 1-variable Laurent polynomial matrix. We first looked at systems whose corresponding quotient modules are finitely generated as modules over $\mathbb{R}[\sigma_1^{\pm 1}]$. We showed that these systems admit representation formulae of the above-mentioned type. Then we used a discrete version of Noether's normalization to obtain representation formulae for general 2D autonomous systems. A crucial step in the normalization process is finding a suitable coordinate transformation in $\mathbb{Z}^n$.

There are a number of issues related to the results presented in this paper that have not been addressed here, for example, the question of how to get minimal size of the 1-variable Laurent polynomial matrix $A(\sigma_1)$, or algorithms for computing the matrix. The extension of the formulae to nonautonomous systems is also another important unresolved issue.

Appendix A. Proof of Lemma 6.8. We require the following notions for the proof. A prime ideal $\mathfrak{p} \subseteq \mathcal{A}$ is said to be an *associated prime of* $\mathcal{M}$ if there exists $0 \neq m \in \mathcal{M}$ such that

$$\mathfrak{p} = \{h(\sigma) \in \mathcal{A} \mid h(\sigma)m = 0 \in \mathcal{M}\}.$$

The set of all associated primes of $\mathcal{M}$ is denoted by $\text{Ass}(\mathcal{M})$. Let $\mathcal{Z}(\mathcal{M})$ denote the set of zerodivisors on $\mathcal{M}$, i.e.,

$$\mathcal{Z}(\mathcal{M}) := \{h(\sigma) \in \mathcal{A} \mid \exists \;\; 0 \neq m \in \mathcal{M} \text{ such that } h(\sigma)m = 0 \in \mathcal{M}\}.$$

In order to prove the claim of the lemma it is enough that we show

$$\mathcal{Z}(\mathcal{M}) \subseteq \langle p_1(\sigma)\rangle \cup \langle p_2(\sigma)\rangle \cup \cdots \cup \langle p_k(\sigma)\rangle. \tag{A.1}$$

Since $\mathcal{A}$ is Noetherian and $\mathcal{M}$ is finitely generated over $\mathcal{A}$, we have

$$\mathcal{Z}(\mathcal{M}) = \bigcup_{\mathfrak{p} \in \text{Ass}(\mathcal{M})} \mathfrak{p}$$

(see [17, Corollary 7.1.12]). Therefore, our claim of (A.1) will follow if we show that

$$\text{Ass}(\mathcal{M}) \subseteq \{\langle p_i(\sigma)\rangle \mid 1 \leqslant i \leqslant k\}. \tag{A.2}$$

Before we go ahead and prove (A.2), we point out the following important notions from commutative algebra, which will be used below for showing that (A.2) holds. Detailed expositions about these notions can be found in textbooks on commutative algebra, e.g., [3, 17].

- Localization of a ring $\mathcal{A}$ and an $\mathcal{A}$-module $\mathcal{M}$ at a prime ideal $\mathfrak{p}$, denoted here by $\mathcal{A}_\mathfrak{p}$ and $\mathcal{M}_\mathfrak{p}$, respectively.
- *Projective dimension* of $\mathcal{M}_\mathfrak{p}$, written as proj $\dim(\mathcal{M}_\mathfrak{p})$ here.
- *Depth* of $\mathcal{M}_\mathfrak{p}$, written as $\mathrm{depth}(\mathcal{M}_\mathfrak{p})$ here.
- *Height* of a prime ideal $\mathfrak{p}$, written as $\mathrm{ht}(\mathfrak{p})$ here.
- *Krull dimension* of $\mathcal{A}_\mathfrak{p}$, denoted by $\dim(\mathcal{A}_\mathfrak{p})$ here.
- Auslander–Buchsbaum formula.

In order to show (A.2), we first show that $\mathrm{Ass}(\mathcal{M})$ contains only principal ideals. Let $\mathfrak{p} \in \mathrm{Ass}(\mathcal{M})$. Localizing at $\mathfrak{p}$ we get that $\mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(\mathcal{M}_\mathfrak{p})$. Therefore, $\mathrm{depth}(\mathcal{M}_\mathfrak{p}) = 0$. Also, $\mathcal{M}_\mathfrak{p}$ is torsion, so it is not free, but $\mathcal{M}_\mathfrak{p} = \mathcal{A}_\mathfrak{p}^{\mathtt{w}}/\mathrm{rowspan}(R(\sigma))$, and $\mathrm{rowspan}(R(\sigma))$ is free, because $\det(R(\sigma)) \neq 0$. Therefore, proj $\dim(\mathcal{M}_\mathfrak{p}) = 1$. Hence by the Auslander–Buchsbaum formula,

$$\dim(\mathcal{A}_\mathfrak{p}) = \text{proj } \dim(\mathcal{M}_\mathfrak{p}) + \mathrm{depth}(\mathcal{M}_\mathfrak{p}) = 1 + 0 = 1.$$

But $\dim(\mathcal{A}_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$, so $\mathrm{ht}(\mathfrak{p}) = 1$. By Corollary 10.6 of [3], $\mathfrak{p}$ is a principal ideal because $\mathcal{A}$ is a unique factorization domain (UFD). Let $p(\sigma) \in \mathcal{A}$ be a generator for $\mathfrak{p}$, that is, $\mathfrak{p} = \langle p(\sigma) \rangle$. Now since $\mathfrak{p} \in \mathrm{Ass}(\mathcal{M})$, we have $\mathrm{ann}(\mathcal{M}) \subseteq \mathfrak{p}$. Note that if we define a behavior $\mathfrak{B} = \ker(R(\sigma))$, then the characteristic ideal of this behavior is $\mathcal{I}(\mathfrak{B}) = \langle g(\sigma) \rangle$, where $g(\sigma) = \det(R(\sigma))$. By Proposition 2.3 $\langle g(\sigma) \rangle = \mathcal{I}(\mathfrak{B}) \subseteq \mathrm{ann}(\mathcal{M})$. Therefore, $\langle g(\sigma) \rangle \subseteq \mathfrak{p} = \langle p(\sigma) \rangle$. Since $\mathfrak{p}$ is prime, it follows from the prime factorization of $g(\sigma)$ that for some $1 \leqslant i \leqslant k$ we have $p_i(\sigma) \in \mathfrak{p}$. That is, there exists $f(\sigma) \in \mathcal{A}$ such that $p_i(\sigma) = f(\sigma)p(\sigma)$. But $p_i(\sigma)$ has been assumed to be irreducible. So $f(\sigma)$ must be a unit, which means $\langle p(\sigma) \rangle = \langle p_i(\sigma) \rangle$. This proves (A.2), and hence our claim of (A.1) follows.

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