The Continuous-Time Singular LQR Problem and the Riddle of Nonautonomous Hamiltonian Systems: A Behavioral Solution

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***Abstract*—In this article, we deal with the continuous-time singular linear quadratic regulator (LQR) problems, which give rise to nonautonomous Hamiltonian systems. This case arises when the system's transfer function matrix is not left-invertible. A special case of this problem can be solved using the constrained generalized continuous algebraic Riccati equation (CGCARE), when a certain condition on the input-cardinality of the Hamiltonian is satisfied. However, this condition is only a special case among many other possible cases. On the other hand, singular LQR problems with autonomous Hamiltonian systems have been well studied in the literature. In this article, we apply behavioral theoretic techniques to show that the general case of the singular LQR problem with nonautonomous Hamiltonian can be solved by a direct sum decomposition of the plant behavior, where one of the direct summands can be solved via CGCARE, while the other gives rise to an autonomous Hamiltonian system.**

***Index Terms*—Behavioral theory, constrained generalized continuous algebraic Riccati equation (CGCARE), Hamiltonian, singular linear quadratic regulator (LQR) problem.**

I. Introduction

This article deals with the most general case of the infinite horizon singular linear quadratic regulator (LQR) problem: the case when the system, along with the cost functional, may result in a *Hamiltonian system* that is nonautonomous. The infinite-horizon singular LQR problem is defined as follows:

Problem 1.1: Consider a stabilizable system with state-space dynamics $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition x_0, find an input $u(t)$ that minimizes the functional

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \text{ with } \lim_{t\to\infty} x(t) = 0 \quad (1)$$

where $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right] \geqslant 0$, and $\mathtt{rank}\, R < \mathtt{m}$.

Intimately connected to the problem is the following linear dynamical system given in singular descriptor model:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0_{\mathtt{m},\mathtt{m}} \end{bmatrix}}_{E} \frac{d}{dt} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & R \end{bmatrix}}_{H} \begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix}. \quad (2)$$

This system is known as the *Hamiltonian system* corresponding to Problem 1.1. It follows from Pontryagin's Maximum Principle that all smooth optimal trajectories must necessarily satisfy the Hamiltonian system equations [1]. Recent studies have revealed that not just the smooth optimal trajectories, but also the distributional ones pertaining to the singular LQR problem must also satisfy the Hamiltonian system equations in a distributional sense over the half-line $\mathbb{R}_+$ [2]. A hallmark of the singular LQR problem is that, unlike the regular case ($\mathtt{rank}\, R = \mathtt{m}$), the singular problem may give rise to a Hamiltonian system that is nonautonomous. This peculiarity is well-known: in the literature, this has either been ruled out by assuming left-invertibility of the system (equivalently, uniqueness of the optimal solution) [3], or been dealt with by introducing several intricate decomposition of the state space [4]. Further, a special case of the nonautonomous Hamiltonian system scenario has been pursued in [5]–[7] in connection with the constrained generalized continuous algebraic Riccati equation (CGCARE) given as follows:

$$\begin{aligned} &A^T K + KA + Q - (KB + S)R^\dagger(B^T K + S^T) = 0 \\ &\mathtt{ker}(R) \subseteq \mathtt{ker}(S + KB) \end{aligned} \quad (3)$$

where $R^\dagger$ is the Moore–Penrose pseudoinverse of R. However, it has been shown in [8] that this special case lies at an extreme end of a pretty wide gamut of special cases in the general scenario of the nonautonomy of the Hamiltonian system (we explain this phenomenon in Lemma 3.2). All in all, it is safe to say that the nonautonomy of the Hamiltonian system has been well-recognized as a problematic case in singular LQR theory. In this article, we propose an extremely simple remedy to this problematic case by employing ideas from behavioral theory and polynomial matrices [9]–[11]. We would like to note here that the germ of the ideas contained in this article can be found in [4]. Our principal contribution here is a novel proposition (see Theorem 3.7) that shows how behavioral theory naturally leads to a direct sum decomposition of the plant behavior into two subbehaviors, one of which is left-invertible, while the other admits a solution via CGCARE. Thus the singular LQR problem is made amenable to solution by these two well-established theories.

It is important to note that the abovementioned direct sum decomposition of the plant behavior is obtained here considering only $\mathfrak{C}^\infty$ trajectories. We argue that this is enough although we do not discount the possibility of distributional optimal trajectories. This can be understood by noting that the existence of distributional optimal trajectories is attributed to the singular descriptor nature of the Hamiltonian system,

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while the plant behavior is devoid of such singularities. Thus, the decomposition of the smooth plant behavior, too, will not suppress the possible distributional optimal trajectories. We explicitly show this in Section V.

II. Notation and Preliminaries

A. Notation

The symbols $\mathbb{R}, \mathbb{C}$, and $\mathbb{N}$ are used for the sets of real numbers, complex numbers, and natural numbers, respectively. The symbol $\mathbb{R}[s]$ denotes the integral domain of polynomials in s with real coefficients, while $\mathbb{R}(s)$ denotes its corresponding field of fractions. The symbols $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}, \mathbb{R}[s]^{\mathtt{n}\times\mathtt{p}}$, and $\mathbb{R}(s)^{\mathtt{n}\times\mathtt{p}}$ denote the sets of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}, \mathbb{R}[s]$, and $\mathbb{R}(s)$, respectively. We use $\bullet$ when a dimension need not be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows and an arbitrary number of columns. We use the symbol $I_{\mathtt{n}}$ for the $\mathtt{n}\times\mathtt{n}$ identity matrix and the symbol $0_{\mathtt{n},\mathtt{m}}$ for an $\mathtt{n}\times\mathtt{m}$ matrix with all entries zero. We drop these subscripts if the size of the matrix is understood from the context. Symbol $\mathtt{col}(B_1, B_2, \ldots, B_{\mathtt{n}})$ represents a matrix of the form $[B_1^T\ B_2^T\ \cdots\ B_{\mathtt{n}}^T]^T$. The symbols $\mathtt{img}\,A$ and $\mathtt{ker}\,A$ denote the image and nullspace of a matrix A, respectively. The symbols $\mathtt{rank}\,A$ and $\mathtt{nullity}(A)$ denote the rank and the dimension of the nullspace of a matrix A, respectively. The symbol $\mathtt{det}(A)$ represents the determinant of a square matrix A. The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ is represented by the symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. We use the symbol $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})|_{\mathbb{R}_+}$ to represent the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}^{\mathtt{n}}$ that are restrictions of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. By $\mathtt{col}(w_1, w_2) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}+\mathtt{q}})$ we mean that $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ and $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$. The subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ comprising of functions having compact support is denoted by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. We simply use the symbol $\mathfrak{D}$ when the domain and the co-domain of the functions in the set are clear from the context. The symbol δ represents the Dirac delta impulse function and $\delta^{(i)}$ represents its ith distributional derivative with respect to t.

B. Brief Overview of Polynomial Matrices

In this section we provide a few definitions and properties pertaining to polynomial matrices.

Definition 2.1: $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{q}\times\mathtt{q}}$ is said to be unimodular if $\mathtt{det}R(\xi) \in \mathbb{R}\setminus\{0\}$.

Definition 2.2: A matrix $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{q}}$ is said to be factor left-prime (FLP) if $\mathtt{rank}\,R(\lambda) = \mathtt{g}$ for all $\lambda \in \mathbb{C}$.

Factor right-prime (FRP) matrices can be defined analogously.

Remark 2.3: From the definition it follows that the Smith canonical form (see [9, Th. 2.5.15]) of an FLP matrix $R(\xi)$ is given by $[I_{\mathtt{g}}\ 0_{\mathtt{g},(\mathtt{q}-\mathtt{g})}]$. In other words, there exist unimodular matrices $U(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{g}}$ and $V(\xi) \in \mathbb{R}[\xi]^{\mathtt{q}\times\mathtt{q}}$ such that $R(\xi) = U(\xi)[I_{\mathtt{g}}\ 0_{\mathtt{g},(\mathtt{q}-\mathtt{g})}]V(\xi)$. Consequently, $R(\xi)$ admits completion to a unimodular matrix by the matrix $\widetilde{R}(\xi) := [0_{(\mathtt{q}-\mathtt{g}),\mathtt{g}}\ I_{(\mathtt{q}-\mathtt{g})}]V(\xi)$; i.e., $\mathtt{col}\,(R(\xi), \widetilde{R}(\xi))$ is unimodular. Notice that $\widetilde{R}(\xi)$, too, turns out to be FLP. Similarly, if $R(\xi)$ is FRP, then there exists a matrix $\widetilde{R}(\xi)$, which is FRP such that $[R(\xi)\ \widetilde{R}(\xi)]$ is unimodular. □

Definition 2.4: Let $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{q}}$ with $\ell := \mathtt{rank}\,R(\xi)$. Then, $F(\xi) \in \mathbb{R}[\xi]^{(\mathtt{g}-\ell)\times\mathtt{g}}$ is called a minimal left-annihilator (MLA) of $R(\xi)$ if both the following properties are satisfied:

1. $F(\xi)R(\xi) = 0$, and
2. $F_1(\xi)R(\xi) = 0$ for an $F_1(\xi) \in \mathbb{R}[\xi]^{\bullet\times\mathtt{g}} \Rightarrow$ there exists $F_2(\xi) \in \mathbb{R}[\xi]^{\bullet\times(\mathtt{g}-\ell)}$ such that $F_1(\xi) = F_2(\xi)F(\xi)$.

If $R(\xi)$ is full row-rank, then its MLA is the zero matrix.

The minimal right-annihilator (MRA) can be defined analogously. The following lemma related to the MLA and MRA is well-known in the literature (see [12]).

Lemma 2.5: Say $R(\xi)$ is not full row-rank (column-rank). Then, its MLA (MRA) is FLP (FRP). If $R(\xi)$ is full row-rank (column-rank), then its MLA (MRA) is a zero matrix.

C. Behavior

A linear differential behavior $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ consisting of solutions to a set of ordinary differential equations with constant coefficients, i.e.,

$$\mathfrak{B} := \mathtt{ker}R\left(\frac{d}{dt}\right) = \left\{w(t) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) | R\left(\frac{d}{dt}\right)w(t) = 0\right\}$$

where $R(\xi) \in \mathbb{R}[\xi]^{\bullet\times\mathtt{q}}$. Such a representation of a behavior $\mathfrak{B}$ is called a *kernel representation*. Every behavior $\mathfrak{B}$ always admits a full row-rank kernel representation, i.e., without loss of generality, $R(\xi)$ can be chosen to be full row-rank. Let $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{q}}$ and assume that $\widetilde{R}(\xi)$, too, provide a full row-rank representation of $\mathfrak{B}$. Then, there exists a unimodular matrix $U(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{g}}$ such that $R(\xi) = U(\xi)\widetilde{R}(\xi)$ (see [9, Chap. 2] for more on this). It also follows that $(\mathtt{q}-\mathtt{g})$ is the *input-cardinality* of the behavior $\mathfrak{B}$. This signifies that at most $(\mathtt{q}-\mathtt{g})$ components of the trajectories of $\mathfrak{B}$ can be chosen freely. Obviously, the input cardinality of a behavior is independent of the representation.

Next, we define the autonomous and controllable behaviors.

Definition 2.6: $\mathfrak{B}$ is called *autonomous* if for all $w', w'' \in \mathfrak{B}$

$$w'(t) = w''(t) \text{ for } t \leqslant 0 \Rightarrow w'(t) = w''(t) \quad \forall t \in \mathbb{R}.$$

Definition 2.7: $\mathfrak{B}$ is called *controllable* if for every $w', w'' \in \mathfrak{B}$ there exists a $w \in \mathfrak{B}$ and $\tau > 0$ such that

$$w(t) = w'(t) \quad \text{for } t \leq 0 \text{ and } w(t) = w''(t) \quad \text{for } t \geq \tau.$$

The following proposition characterizes the class of controllable behaviors (see [9, Th. 5.2.10, Th. 6.6.1]).

Proposition 2.8: Let $\mathfrak{B} := \mathtt{ker}R(\frac{d}{dt})$, where $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{q}}$ is full row-rank. Then, the following are equivalent:

1. $\mathfrak{B}$ is controllable.
2. $R(\xi)$ is FLP (see Definition 2.2).
3. There exists an $M(\xi) \in \mathbb{R}[\xi]^{\mathtt{q}\times(\mathtt{q}-\mathtt{g})}$ such that

$$\mathfrak{B} = \left\{w(t) | \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{(\mathtt{q}-\mathtt{g})}) \text{ such that } w(t) = M\left(\frac{d}{dt}\right)\ell\right\}.$$

ℓ is called a vector of *latent variables*. The representation of $\mathfrak{B}$ given in Statement 3 is called an *image representation* and $\mathfrak{B}$ is written as $\mathfrak{B} = \mathtt{img}\,M(\frac{d}{dt})$. It can be shown that $M(\xi)$ can always be chosen to be full column-rank. The following proposition establishes a relation between a kernel representation and an image representation of a controllable behavior (see [9, Th. 6.6.1] for proof).

Proposition 2.9: Assume that $\mathfrak{B} := \mathtt{ker}R(\frac{d}{dt})$ is controllable. Then, $\mathfrak{B} = \mathtt{img}M(\frac{d}{dt})$ is an image representation of $\mathfrak{B}$ if $M(\xi)$ is an MRA of $R(\xi)$ (see Definition 2.4).

Definition 2.10: An image representation $\mathfrak{B} = \mathtt{img}M(\frac{d}{dt})$ is said to be *observable* if $M(\frac{d}{dt})\ell = 0$ implies $\ell = 0$.

The following proposition gives a method to check whether an image representation is observable (see [9, Th. 5.3.3]).

Proposition 2.11: An image representation $\mathfrak{B} = \mathtt{img}M(\frac{d}{dt})$ is observable if and only if $M(\xi)$ is FRP (see Definition 2.2).

The following proposition can be obtained by combining Lemma 2.5, Proposition 2.9, and Proposition 2.11 together.

Proposition 2.12: A controllable behavior $\mathfrak{B}$ always admits an observable image representation.

Given a behavior $\mathfrak{B}$ (possibly uncontrollable), the controllable part of $\mathfrak{B}$ is defined in the following manner.

Definition 2.13: The largest controllable sub-behavior of $\mathfrak{B}$ is said to be the controllable part of a given behavior $\mathfrak{B}$.

We denote the controllable part of a behavior $\mathfrak{B}$ by $(\mathfrak{B})_{\text{cont}}$.

D. Quotient of a Behavior

Let $\widehat{\mathfrak{B}}$ be a *sub-behavior* of $\mathfrak{B}$. We define the relation $\sim$ between the trajectories of $\mathfrak{B}$ as follows: for $w_1, w_2 \in \mathfrak{B}$, we say that $w_1 \sim w_2$ if $(w_1 - w_2) \in \widehat{\mathfrak{B}}$. It can be easily verified that the relation $\sim$ is an equivalence relation. We use the symbol $[w]$ to denote the *equivalence class* of the trajectory $w \in \mathfrak{B}$, i.e., $[w]$ is the collection of all the trajectories $w_1 \in \mathfrak{B}$ such that $w_1 \sim w$. The set $\mathfrak{B}/\widehat{\mathfrak{B}}$, called a *quotient* of the behavior $\mathfrak{B}$ modulo the sub-behavior $\widehat{\mathfrak{B}}$, is the collection of all such equivalence classes. It is crucially important to note here that, when the sub-behavior $\widehat{\mathfrak{B}}$ admits a complementary sub-behavior $\widehat{\mathfrak{B}}'$ such that $\mathfrak{B} = \widehat{\mathfrak{B}} \oplus \widehat{\mathfrak{B}}'$, the quotient $\mathfrak{B}/\widehat{\mathfrak{B}}$ can then be identified with $\widehat{\mathfrak{B}}'$ as a sub-behavior of $\mathfrak{B}$. In that situation, we simply write $\mathfrak{B} = \widehat{\mathfrak{B}} \oplus \mathfrak{B}/\widehat{\mathfrak{B}}$. $\widehat{\mathfrak{B}}$ does not always admit such a complementary sub-behavior.

It is known in the literature that for the particular case when $\widehat{\mathfrak{B}}$ equals $(\mathfrak{B})_{\text{cont}}$, the behavior $\mathfrak{B}$ indeed admits $\mathfrak{B} = (\mathfrak{B})_{\text{cont}} \oplus \mathfrak{B}_{\text{aut}}$, where $\mathfrak{B}_{\text{aut}}$ is an autonomous part of $\mathfrak{B}$ (see [9, Chap. 5]). This enables the following crucial proposition derived from [9, Th. 5.2.14 and Remark 5.2.15].

Proposition 2.14: For any behavior $\mathfrak{B}$ the following hold:

1. $(\mathfrak{B})_{\text{cont}}$ is unique.
2. $\mathfrak{B}_{\text{aut}} := \mathfrak{B}/(\mathfrak{B})_{\text{cont}}$ is autonomous.
3. $\mathfrak{B} = (\mathfrak{B})_{\text{cont}} \oplus \mathfrak{B}_{\text{aut}}$.

In this article, our aim is to obtain a similar such direct sum decomposition of $\mathfrak{B}$, not with $(\mathfrak{B})_{\text{cont}}$ as a direct summand, but rather with another special sub-behavior. In order to infer the corresponding direct sum decomposition in the sequel we make use of the following theorem.

Theorem 2.15: Let $\widehat{\mathfrak{B}}$ be a controllable sub-behavior of $\mathfrak{B}$ (not necessarily the controllable part of $\mathfrak{B}$). Then, $\mathfrak{B}$ admits a direct sum decomposition $\mathfrak{B} = \widehat{\mathfrak{B}} \oplus \mathfrak{B}/\widehat{\mathfrak{B}}$.

Proof: By Definition 2.13, $\widehat{\mathfrak{B}} \subseteq (\mathfrak{B})_{\text{cont}}$ because $\widehat{\mathfrak{B}}$ has been assumed to be controllable. Further, by the direct sum decomposition of $\mathfrak{B}$ as per Proposition 2.14, it is enough to show that $(\mathfrak{B})_{\text{cont}} = \widehat{\mathfrak{B}} \oplus (\mathfrak{B})_{\text{cont}}/\widehat{\mathfrak{B}}$. Thus, we may assume without loss of generality that $\mathfrak{B} = (\mathfrak{B})_{\text{cont}}$, i.e., $\mathfrak{B}$ is controllable. Now, let $\mathfrak{B} = \texttt{ker} R(\frac{d}{dt})$ with $R(\xi) \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{q}}$. Since $\widehat{\mathfrak{B}} \subseteq \mathfrak{B}$, there exists $\widehat{R}(\xi) \in \mathbb{R}[\xi]^{\widehat{\mathtt{g}}\times\mathtt{q}}$ such that $\widehat{\mathfrak{B}} = \texttt{ker}\begin{bmatrix} R(\frac{d}{dt}) \\ \widehat{R}(\frac{d}{dt}) \end{bmatrix}$. Further, since $\widehat{\mathfrak{B}}$, too, is controllable, by Proposition 2.8, $\widehat{\mathfrak{B}} = \texttt{ker}\begin{bmatrix} R(\frac{d}{dt}) \\ \widehat{R}(\frac{d}{dt}) \end{bmatrix}$ is FLP. Thus, there exists $\widetilde{R}(\xi) \in \mathbb{R}[\xi]^{(\mathtt{q}-\mathtt{g}-\widehat{\mathtt{g}})\times\mathtt{q}}$ such that $U(\xi) := \texttt{col}(R(\xi), \widehat{R}(\xi), \widetilde{R}(\xi))$ is a unimodular matrix. We now define $\widehat{\mathfrak{B}}' := \texttt{ker}\begin{bmatrix} R(\frac{d}{dt}) \\ \widetilde{R}(\frac{d}{dt}) \end{bmatrix}$. We claim that $\mathfrak{B} = \widehat{\mathfrak{B}} \oplus \widehat{\mathfrak{B}}'$. In order to prove the claim, we first show that for any arbitrary $w \in \mathfrak{B}$ there exists $w_1 \in \widehat{\mathfrak{B}}$ and $w_2 \in \widehat{\mathfrak{B}}'$ such that $w = w_1 + w_2$. For this purpose, we first notice that, since $U(\xi)$ is unimodular, by the matrix version of the Aryabhatta–Bezout identity, there exist $A(\xi) \in \mathbb{R}[\xi]^{\mathtt{q}\times\mathtt{g}}$, $B(\xi) \in \mathbb{R}[\xi]^{\mathtt{q}\times\widehat{\mathtt{g}}}$, and $C(\xi) \in \mathbb{R}[\xi]^{\mathtt{q}\times(\mathtt{q}-\mathtt{g}-\widehat{\mathtt{g}})}$ such that $[A(\xi)\ B(\xi)\ C(\xi)]U(\xi) = A(\xi)R(\xi) + B(\xi)\widehat{R}(\xi) + C(\xi)\widetilde{R}(\xi) = I_{\mathtt{q}}$. It then follows that $A(\frac{d}{dt})R(\frac{d}{dt})w + B(\frac{d}{dt})\widehat{R}(\frac{d}{dt})w + C(\frac{d}{dt})\widetilde{R}(\frac{d}{dt})w = I_{\mathtt{q}}w = w$. Since $w \in \mathfrak{B}$, we have $A(\frac{d}{dt})R(\frac{d}{dt})w = 0$. Also, we define $w_2 := B(\frac{d}{dt})\widehat{R}(\frac{d}{dt})w$ and $w_1 := C(\frac{d}{dt})\widetilde{R}(\frac{d}{dt})w$. Note that

$$\begin{bmatrix} R(\xi) \\ \widehat{R}(\xi) \\ \widetilde{R}(\xi) \end{bmatrix} [A(\xi)\ B(\xi)\ C(\xi)] = \begin{bmatrix} R(\xi)A(\xi) & R(\xi)B(\xi) & R(\xi)C(\xi) \\ \widehat{R}(\xi)A(\xi) & \widehat{R}(\xi)B(\xi) & \widehat{R}(\xi)C(\xi) \\ \widetilde{R}(\xi)A(\xi) & \widetilde{R}(\xi)B(\xi) & \widetilde{R}(\xi)C(\xi) \end{bmatrix} = I_{\mathtt{q}}.$$

So, we have the identities $R(\xi)B(\xi) = 0$, $\widetilde{R}(\xi)B(\xi) = 0$, $R(\xi)C(\xi) = 0$, $\widehat{R}(\xi)C(\xi) = 0$. It then easily follows that $\begin{bmatrix} R(\frac{d}{dt}) \\ \widehat{R}(\frac{d}{dt}) \end{bmatrix} w_1 = 0$, and $\begin{bmatrix} R(\frac{d}{dt}) \\ \widetilde{R}(\frac{d}{dt}) \end{bmatrix} w_2 = 0$. That is, $w_1 \in \widehat{\mathfrak{B}}$ and $w_2 \in \widehat{\mathfrak{B}}'$. Hence, $\mathfrak{B} = \widehat{\mathfrak{B}} + \widehat{\mathfrak{B}}'$. In order to show that the sum is a direct sum, we have to prove that $\widehat{\mathfrak{B}} \cap \widehat{\mathfrak{B}}' = \{0\}$. Let $w \in \widehat{\mathfrak{B}} \cap \widehat{\mathfrak{B}}'$. Then this w must satisfy $\begin{bmatrix} R(\frac{d}{dt}) \\ \widehat{R}(\frac{d}{dt}) \end{bmatrix} w = 0$, and also $\begin{bmatrix} R(\frac{d}{dt}) \\ \widetilde{R}(\frac{d}{dt}) \end{bmatrix} w = 0$. Thus, combining the above two equations we get that $U(\frac{d}{dt})w = 0$. Since $U(\xi)$ is unimodular, it follows that $w = 0$. Identifying $\mathfrak{B}/\widehat{\mathfrak{B}}$ with $\widehat{\mathfrak{B}}'$ we get the desired result. □

E. Projection of a Behavior

Let $\mathfrak{B}_{\texttt{full}} := \{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{g}+\mathtt{q}}) | [R_1(\frac{d}{dt})\ R_2(\frac{d}{dt})] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0\}$, where $R_1(\xi) \in \mathbb{R}[\xi]^{\bullet\times\mathtt{g}}$ and $R_2(\xi) \in \mathbb{R}[\xi]^{\bullet\times\mathtt{q}}$. A projection of the behavior $\mathfrak{B}_{\texttt{full}}$ on the variable w_2 is defined as $\Pi_{w_2}(\mathfrak{B}_{\texttt{full}}) := \mathfrak{B} := \{w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) | \exists w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$ such that $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathfrak{B}_{\texttt{full}}\}$. The following lemma can be derived from [13, Lemma 2.9.5]. It shows that elimination of output variables does not change the input-cardinality.

Lemma 2.16: Let $[R_1(\xi)\ R_2(\xi)] \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{q}}$ and $\mathfrak{B} := \{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) | [R_1(\frac{d}{dt})\ R_2(\frac{d}{dt})] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0\}$. The partitioning in $\texttt{col}(w_1, w_2)$ conforms with the partitioning in $[R_1(\xi)\ R_2(\xi)]$. Assume that $R_1(\xi)$ is full column-rank. Then, the input-cardinality of $\mathfrak{B}$ = the input-cardinality of $\Pi_{w_2}(\mathfrak{B})$.

III. Main Results

Since the cost matrix $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$, it admits a factorization $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} =: \begin{bmatrix} C^T \\ D^T \end{bmatrix} [C\ D]$, where $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ with $\mathtt{p}$ being the rank of the cost matrix. Therefore, using this factorization, Problem 1.1 can be equivalently rewritten as:

Problem 3.1: Consider a stabilizable system

$$\Sigma : \frac{d}{dt}x(t) = Ax(t) + Bu(t) \text{ and } y(t) = Cx(t) + Du(t) \tag{4}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$. Then, for every initial condition x_0, find an input $u(t)$ that minimizes

$$J(x_0, u) := \int_0^\infty y(t)^T y(t) dt \text{ with } \lim_{t\to\infty} x(t) = 0. \tag{5}$$

The problem is a singular LQR problem if $\texttt{rank}\, D < \mathtt{m}$. We formulate this problem in the behavioral theory set up. Therefore, we first define the behavior of the system Σ as

$$\mathfrak{B}_{\text{full}} := \left\{ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{m}+\mathtt{p}}) | x, u, \text{ and } y \text{ satisfy equation (4)} \right\}.$$

The objective function defined in equation (5) consists of the output $y(t)$ of Σ only. Therefore, we restrict ourselves to the input trajectory $u(t)$ and the output trajectory $y(t)$. Define the behavior (following the notation used in Section II-E)

$$\mathfrak{B} := \Pi_{\begin{pmatrix} u \\ y \end{pmatrix}}(\mathfrak{B}_{\rm full}). \tag{6}$$

The dual $\mathfrak{B}^{\perp}$ of the behavior $\mathfrak{B}$ is defined as

$$\mathfrak{B}^{\perp} := \left\{ \begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}+\mathtt{p}}) | L\left(\begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix} \right) = 0 \forall \begin{pmatrix} u \\ y \end{pmatrix} \in \mathfrak{B} \cap \mathfrak{D} \right\}$$

where $$L\left(\begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix} \right) := \int_{-\infty}^{\infty} \hat{y}^T y dt. \tag{7}$$

By Pontryagin's maximum principle $\mathtt{col}(u^*, y^*)$ is an optimal trajectory of the behavior $\mathfrak{B}$ only if $\mathtt{col}(u^*, y^*) \in \mathfrak{B}_{\rm Ham}$, where $\mathfrak{B}_{\rm Ham}$ is the Hamiltonian of $\mathfrak{B}$ defined as

$$\mathfrak{B}_{\rm Ham} := \mathfrak{B} \cap \mathfrak{B}^{\perp}. \tag{8}$$

It is evident that the behavior $\mathfrak{B}_{\rm Ham}$ is the projection $\Pi_{\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)}(\mathfrak{B}_{\rm Hfull})$, where $\mathfrak{B}_{\rm Hfull}$ is defined as

$$\mathfrak{B}_{\rm Hfull} := \left\{ \begin{pmatrix} x \\ z \\ u \\ y \end{pmatrix} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{n}+\mathtt{m}+\mathtt{p}}) | \begin{pmatrix} x \\ z \\ u \end{pmatrix} \text{ satisfies equation (2) and } y = Cx + Du \right\}. \tag{9}$$

Recall that $Q = C^T C, S = C^T D$, and $R = D^T D$. The following lemma derived from [8, Th. 1] presents a necessary and sufficient condition for CGCARE solvability.

Lemma 3.2: Consider the LQR Problem 3.1 and the Hamiltonian behavior $\mathfrak{B}_{\rm Ham}$ defined in equation (8). Then, the CGCARE given by equation (3) (where $Q = C^T C, S = C^T D$, and $R = D^T D$) admits a solution if and only if the input-cardinality of $\mathfrak{B}_{\rm Ham}$ is equal to $\mathtt{nullity}(D)$.

Proof: From equation (2) and equation (9), it follows that $\mathfrak{B}_{\rm Hfull}$ is given by $\mathfrak{B}_{\rm Hfull} = \mathtt{ker} \widehat{R}(\frac{d}{dt})$, where $\widehat{R}(\xi) := \left[\begin{smallmatrix} \xi E - H & 0 \\ -F & I_{\mathtt{p}} \end{smallmatrix}\right]$, $F := [C \; 0_{\mathtt{p},\mathtt{n}} \; D]$. Next, by [8, Th. 1], it follows that CGCARE is solvable $\Leftrightarrow \mathtt{rank}\,(\xi E - H) = 2\mathtt{n} + \mathtt{rank}\, R$. But, clearly $\mathtt{rank}\,(\xi E - H) = 2\mathtt{n} + \mathtt{rank}\, R \Leftrightarrow \mathtt{rank}\, \widehat{R}(\xi) = 2\mathtt{n} + \mathtt{p} + \mathtt{rank}\, R$. Therefore, CGCARE is solvable $\Leftrightarrow \mathtt{rank}\, \widehat{R}(\xi) = 2\mathtt{n} + \mathtt{p} + \mathtt{rank}\, R$. Now, define $\hat{\mathtt{m}} :=$ the input-cardinality of $\mathfrak{B}_{\rm Hfull}$. Then, recall from Section II-C that $\hat{\mathtt{m}} = 2\mathtt{n} + \mathtt{m} + \mathtt{p} - \mathtt{rank}\, \widehat{R}(\xi)$. Thus, CGCARE is solvable $\Leftrightarrow \hat{\mathtt{m}} = \mathtt{m} - \mathtt{rank}\, R = \mathtt{nullity}(R)$. Since $\mathfrak{B}_{\rm Ham}$ is a projection of the behavior $\mathfrak{B}_{\rm Hfull}$, which is obtained by eliminating the output variables $\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right)$, by Lemma 2.16 it follows that input-cardinality of $\mathfrak{B}_{\rm Hfull} =$ input-cardinality of $\mathfrak{B}_{\rm Ham}$. Also, since $R = D^T D$, we have $\mathtt{nullity}(R) = \mathtt{nullity}(D)$. Hence, we conclude that CGCARE is solvable if and only if the input-cardinality of $\mathfrak{B}_{\rm Ham}$ is equal to $\mathtt{nullity}(D)$. □

By using the idea of Schur complement, it can be shown that $\mathtt{rank}\, \widehat{R}(\xi) \geqslant 2\mathtt{n} + \mathtt{p} + \mathtt{rank}\, D$. The maximum rank that $\widehat{R}(\xi)$ can attain is, of course, $2\mathtt{n} + \mathtt{m} + \mathtt{p}$. Therefore, the following inequality always holds:

$$0 \leqslant \text{ input-cardinality of } \mathfrak{B}_{\rm Ham} \leqslant \mathtt{nullity}(D). \tag{10}$$

Two boundary cases of inequality (10) are the input-cardinality of $\mathfrak{B}_{\rm Ham} = 0$ and the input-cardinality of $\mathfrak{B}_{\rm Ham} = \mathtt{nullity}(D)$. The former corresponds to $\mathfrak{B}_{\rm Ham}$ being autonomous and thus [3] becomes applicable. The latter corresponds to CGCARE being solvable and thus [5], [6] can be employed. However, to the best of our knowledge, the cases when inequality (10) does not lie in any of the boundaries have remained unsolved. In this article, we deal with the cases when inequality (10) is strict. The following corollary of Lemma 3.2 is a case when inequality (10) attains its upper bound.

Corollary 3.3: $\mathfrak{B}$ admits a solvable CGCARE if $\mathfrak{B}_{\rm Ham} = \mathfrak{B}$.

Proof: Clearly, $\mathtt{nullity}(D) \leqslant \mathtt{m} =$ the input-cardinality of $\mathfrak{B} =$ the input-cardinality of $\mathfrak{B}_{\rm Ham}$ (since $\mathfrak{B}_{\rm Ham} = \mathfrak{B}$). Thus, by inequality (10), it is evident that input-cardinality of $\mathfrak{B}_{\rm Ham} = \mathtt{nullity}(D)$. Hence, the result follows. □

Our main result shows that we can always write the behavior $\mathfrak{B}$ as a direct sum of two behaviors such that one of the behaviors admits a solvable CGCARE, while the Hamiltonian of the other behavior is autonomous. We further show that the Hamiltonians of these direct summands are direct summands of the Hamiltonian behavior $\mathfrak{B}_{\rm Ham}$ as well. To achieve this goal, we first define the *output-nulling* behavior $\mathfrak{B}_{\rm null}$ as

$$\mathfrak{B}_{\rm null} := \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}+\mathtt{p}}) | \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathfrak{B} \right\} . \tag{11}$$

The controllable parts of $\mathfrak{B}_{\rm null}$ and $\mathfrak{B}_{\rm Ham}$ are defined as

$$\mathfrak{B}_{\rm nc} := (\mathfrak{B}_{\rm null})_{\rm cont} \text{ and } \mathfrak{B}_{\rm Hc} := (\mathfrak{B}_{\rm Ham})_{\rm cont}. \tag{12}$$

Since $\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm null} \subseteq \mathfrak{B}$, we define the following quotient:

$$\mathfrak{B}_{\rm LI} := \mathfrak{B}/\mathfrak{B}_{\rm nc}. \tag{13}$$

Since $\mathfrak{B}_{\rm nc}$ is controllable, Theorem 2.15 ensures that $\mathfrak{B}_{\rm LI}$ can be embedded into $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}+\mathtt{p}})$ as a sub-behavior of $\mathfrak{B}$.

The following lemma plays a very important role in finding the direct summands of the behavior $\mathfrak{B}$.

Lemma 3.4: Consider $\mathfrak{B}_{\rm nc}$ and $\mathfrak{B}_{\rm Hc}$ as defined in equation (12). Then, $\mathfrak{B}_{\rm nc} = \mathfrak{B}_{\rm Hc}$.

Proof: $\underline{\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm Hc}}$: We start by showing that $\mathfrak{B}_{\rm null} \subseteq \mathfrak{B}_{\rm Ham}$.

Clearly, $\mathfrak{B}_{\rm null} \subseteq \mathfrak{B}$ by construction. Further, by equation (7), it is evident that $\mathfrak{B}_{\rm null} \subseteq \mathfrak{B}^{\perp}$. Hence, $\mathfrak{B}_{\rm null} \subseteq \mathfrak{B} \cap \mathfrak{B}^{\perp} = \mathfrak{B}_{\rm Ham} \Rightarrow \mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm Ham}$. But, since $\mathfrak{B}_{\rm nc}$ is controllable, by Definition 2.13 we further have $\mathfrak{B}_{\rm nc} \subseteq (\mathfrak{B}_{\rm Ham})_{\rm cont} = \mathfrak{B}_{\rm Hc}$.

$\underline{\mathfrak{B}_{\rm Hc} \subseteq \mathfrak{B}_{\rm nc}}$: We first show that $\mathfrak{B}_{\rm Hc} \cap \mathfrak{D} \subseteq \mathfrak{B}_{\rm null}$. Let $\left(\begin{smallmatrix} \hat{u} \\ \hat{y} \end{smallmatrix}\right) \in \mathfrak{B}_{\rm Hc} \cap \mathfrak{D}$ be arbitrary. Then, $\left(\begin{smallmatrix} \hat{u} \\ \hat{y} \end{smallmatrix}\right) \in \mathfrak{B} \cap \mathfrak{D}$ and $\left(\begin{smallmatrix} \hat{u} \\ \hat{y} \end{smallmatrix}\right) \in \mathfrak{B}^{\perp} \cap \mathfrak{D}$. Now, since $\left(\begin{smallmatrix} \hat{u} \\ \hat{y} \end{smallmatrix}\right) \in \mathfrak{B}^{\perp}$, it follows that $\int_{-\infty}^{\infty} \hat{y}^T y dt = 0$ for all $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \mathfrak{B} \cap \mathfrak{D}$. Since $\left(\begin{smallmatrix} \hat{u} \\ \hat{y} \end{smallmatrix}\right) \in \mathfrak{B} \cap \mathfrak{D}$, we must have $\int_{-\infty}^{\infty} \hat{y}^T \hat{y} dt = 0 \Leftrightarrow \hat{y} = 0$. Therefore, $\mathfrak{B}_{\rm Hc} \cap \mathfrak{D} \subseteq \mathfrak{B}_{\rm null}$. By taking closure in $\mathfrak{C}^{\infty}$-topology, we further have $\mathfrak{B}_{\rm Hc} \subseteq \mathfrak{B}_{\rm null}$ (see proof of [14, Th. 4] for instance). But, since $\mathfrak{B}_{\rm Hc}$ is controllable, it further follows that $\mathfrak{B}_{\rm Hc} \subseteq (\mathfrak{B}_{\rm null})_{\rm cont} = \mathfrak{B}_{\rm nc}$. Hence, $\mathfrak{B}_{\rm nc} = \mathfrak{B}_{\rm Hc}$. □

Remark 3.5: Recall that, $\mathfrak{B}_{\rm LI}$ [defined in equation (13)] is a quotient behavior. So, the elements in $\mathfrak{B}_{\rm LI}$ are equivalence classes of trajectories from $\mathfrak{B}$ modulo the behavior $\mathfrak{B}_{\rm nc}$. Say, $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right) \in \mathfrak{B}$, then we denote the equivalence class of this trajectory by the notation $[\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)]$. As discussed before, $\mathfrak{B}_{\rm LI}$ can be represented as a sub-behavior of $\mathfrak{B}$. But, being a quotient behavior, this representation is not unique. Different representations of $\mathfrak{B}_{\rm LI}$ signify different choices of the representatives of the equivalence classes. Therefore, in order to define the Hamiltonian of $\mathfrak{B}_{\rm LI}$ with respect to the cost function (5), the natural questions that one encounters are the following:

1) Is the function $L([\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)], \cdot)$ in equation (7) well-defined for all $[\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)] \in \mathfrak{B}_{\rm LI}$?
2) Is the cost function $J(x_0, u)$ in equation (5) well-defined for the quotient behavior $\mathfrak{B}_{\rm LI}$ as well?

The answer to both these questions is "yes." To show this, first consider $[\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)] \in \mathfrak{B}_{\rm LI}$. Note that a trajectory $\left(\begin{smallmatrix} \hat{u} \\ \hat{y} \end{smallmatrix}\right) \in \mathfrak{B}$ belongs to the equivalence class $[\binom{u}{y}]$ if and only if $\binom{\hat{u}}{\hat{y}} - \binom{u}{y} \in \mathfrak{B}_{\rm nc}$, which in turn implies $\hat{y} = y$. Now, say $\binom{\tilde{u}}{\tilde{y}} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}+\mathtt{p}})$ is arbitrary, then $L(\binom{u}{y}, \binom{\tilde{u}}{\tilde{y}}) = \int_{-\infty}^{\infty} y^T \tilde{y} dt = \int_{-\infty}^{\infty} \hat{y}^T \tilde{y} dt = L(\binom{\hat{u}}{\hat{y}}, \binom{\tilde{u}}{\tilde{y}})$. Thus, $L([\binom{u}{y}], \cdot)$ is independent of the representative of $[\binom{u}{y}]$. Hence, $L([\binom{u}{y}], \cdot)$ is well-defined for $\mathfrak{B}_{\rm LI}$. Similar argument shows that $J(x_0, u)$, too, is well-defined.

As a consequence of Remark 3.5, we can define $\mathfrak{B}_{\rm LI}^\perp$ with respect to the same cost function $J(x_0, u)$ as $\mathfrak{B}_{\rm LI}^\perp := \{\binom{\hat{u}}{\hat{y}} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}+\mathtt{p}}) | L(\binom{\hat{u}}{\hat{y}}, [\binom{u}{y}]) = 0 \; \forall \; [\binom{u}{y}] \in \mathfrak{B}_{\rm LI} \cap \mathfrak{D}\}$. Therefore, the Hamiltonian of $\mathfrak{B}_{\rm LI}$ can be defined as

$$\mathfrak{B}_{\rm LHam} := \mathfrak{B}_{\rm LI} \cap \mathfrak{B}_{\rm LI}^\perp. \tag{14}$$

The following lemma is pivotal in showing that $\mathfrak{B}_{\rm LHam}$ is autonomous.

Lemma 3.6: Consider $\mathfrak{B}_{\rm Ham}, \mathfrak{B}_{\rm nc}, \mathfrak{B}_{\rm LHam}$ given in equation (8), equation (12), and equation (14), respectively. Then, $\mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc} = \mathfrak{B}_{\rm LHam}$.

Proof: $\underline{\mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm LHam}}$: Recall that $\mathfrak{B}_{\rm LHam} = \mathfrak{B}_{\rm LI} \cap \mathfrak{B}_{\rm LI}^\perp = (\mathfrak{B}/\mathfrak{B}_{\rm nc}) \cap (\mathfrak{B}/\mathfrak{B}_{\rm nc})^\perp$ [see equation (13) and equation (14)]. Now, $\mathfrak{B}_{\rm Ham} \subseteq \mathfrak{B} \Rightarrow \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}/\mathfrak{B}_{\rm nc}$ by natural inclusion. Thus, it suffices to show that $\mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc} \subseteq (\mathfrak{B}_{\rm LI})^\perp$.

Let $[\binom{\hat{u}}{\hat{y}}] \in \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc}$ be arbitrary $\Rightarrow \binom{\hat{u}}{\hat{y}} \in \mathfrak{B}_{\rm Ham}$. Thus, $\binom{\hat{u}}{\hat{y}} \in \mathfrak{B}^\perp$. Therefore, by equation (7)

$$L\left(\begin{pmatrix}\hat{u}\\ \hat{y}\end{pmatrix}, \begin{pmatrix}u\\ y\end{pmatrix}\right) = 0 \text{ for all } \begin{pmatrix}u\\ y\end{pmatrix} \in \mathfrak{B} \cap \mathfrak{D}$$

$$\Rightarrow L\left(\begin{pmatrix}\hat{u}\\ \hat{y}\end{pmatrix}, \left[\begin{pmatrix}u\\ y\end{pmatrix}\right]\right) = 0 \text{ for all } \left[\begin{pmatrix}u\\ y\end{pmatrix}\right] \in \mathfrak{B}_{\rm LI} \cap \mathfrak{D}, \text{ (as } \mathfrak{B}_{\rm LI} \subseteq \mathfrak{B})$$

$$\Rightarrow L\left(\left[\begin{pmatrix}\hat{u}\\ \hat{y}\end{pmatrix}\right], \left[\begin{pmatrix}u\\ y\end{pmatrix}\right]\right) = 0 \text{ for all } \left[\begin{pmatrix}u\\ y\end{pmatrix}\right] \in \mathfrak{B}_{\rm LI} \cap \mathfrak{D}. \tag{15}$$

Note that, the last two implications in the above equation follow from Remark 3.5. From equation (15), it directly follows that $[\binom{\hat{u}}{\hat{y}}] \in (\mathfrak{B}_{\rm LI})^\perp$. Hence, $\mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm LHam}$.

$\underline{\mathfrak{B}_{\rm LHam} \subseteq \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc}}$: Let $[\binom{\hat{u}}{\hat{y}}] \in \mathfrak{B}_{\rm LHam} = \mathfrak{B}_{\rm LI} \cap (\mathfrak{B}_{\rm LI})^\perp$ be arbitrary. Then, $[\binom{\hat{u}}{\hat{y}}] \in (\mathfrak{B}_{\rm LI})^\perp$; which implies that $L([\binom{\hat{u}}{\hat{y}}], [\binom{u}{y}]) = 0$ for all $[\binom{u}{y}] \in \mathfrak{B}_{\rm LI} \cap \mathfrak{D}$. But, due to Remark 3.5, this further implies that $L(\binom{\hat{u}}{\hat{y}}, \binom{u}{y}) = 0$ for all $\binom{u}{y} \in \mathfrak{B} \cap \mathfrak{D}$. Therefore, $\binom{\hat{u}}{\hat{y}} \in \mathfrak{B}^\perp$. But, by our assumption, $[\binom{\hat{u}}{\hat{y}}] \in \mathfrak{B}_{\rm LHam} \subseteq \mathfrak{B}_{\rm LI} \subseteq \mathfrak{B}$. So, $\binom{\hat{u}}{\hat{y}} \in \mathfrak{B}$ and hence $\binom{\hat{u}}{\hat{y}} \in \mathfrak{B}_{\rm Ham} \Rightarrow [\binom{\hat{u}}{\hat{y}}] \in \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc}$. Thus, $\mathfrak{B}_{\rm LHam} \subseteq \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc}$. □

Now, we prove the main result of this article. This result shows that the behavior $\mathfrak{B}$ can be written as $\mathfrak{B} = \mathfrak{B}_{\rm nc} \oplus \mathfrak{B}_{\rm LI}$, where $\mathfrak{B}_{\rm nc}$ admits a CGCARE which is solvable, while $\mathfrak{B}_{\rm LI}$ gives rise to an autonomous Hamiltonian.

Theorem 3.7: Consider the behaviors $\mathfrak{B}$, $\mathfrak{B}_{\rm nc}$, and $\mathfrak{B}_{\rm LI}$ as defined in equation (6), equation (12), and equation (13), respectively. Define $\mathfrak{B}_{\rm ncHam} := \mathfrak{B}_{\rm nc} \cap \mathfrak{B}_{\rm nc}^\perp$. Then

1. $\mathfrak{B} = \mathfrak{B}_{\rm nc} \oplus \mathfrak{B}_{\rm LI}$.
2. CGCARE is solvable for the behavior $\mathfrak{B}_{\rm nc}$.
3. The Hamiltonian $\mathfrak{B}_{\rm LHam}$ of $\mathfrak{B}_{\rm LI}$ is autonomous.
4. $\mathfrak{B}_{\rm Ham} = \mathfrak{B}_{\rm ncHam} \oplus \mathfrak{B}_{\rm LHam}$.

Proof: 1. Since $\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm null} \subseteq \mathfrak{B}$, $\mathfrak{B}_{\rm LI} = \mathfrak{B}/\mathfrak{B}_{\rm nc}$, and $\mathfrak{B}_{\rm nc}$ is controllable, by Theorem 2.15, $\mathfrak{B} = \mathfrak{B}_{\rm nc} \oplus \mathfrak{B}_{\rm LI}$.

2. The dual of the behavior $\mathfrak{B}_{\rm nc}$ with respect to the cost function (5) is given by $\mathfrak{B}_{\rm nc}^\perp := \{\binom{\hat{u}}{\hat{y}} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}+\mathtt{p}}) | L(\binom{\hat{u}}{\hat{y}}, \binom{u}{y}) = 0 \forall \binom{u}{y} \in \mathfrak{B}_{\rm nc} \cap \mathfrak{D}\}$, where $L(\binom{\hat{u}}{\hat{y}}, \binom{u}{y})$ is as defined in equation (7). Now, from the construction of $\mathfrak{B}_{\rm nc}$ it follows that for $\mathtt{col}(u, y) \in \mathfrak{B}_{\rm nc}$, where $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$, we must have that $y \equiv 0$. Thus, $\mathfrak{B}_{\rm nc} \subseteq \mathfrak{B}_{\rm nc}^\perp \Rightarrow \mathfrak{B}_{\rm ncHam} = \mathfrak{B}_{\rm nc}$. Thus, using Corollary 3.3, we conclude that $\mathfrak{B}_{\rm nc}$ admits a CGCARE, which is solvable.

3. From Lemma 3.6, $\mathfrak{B}_{\rm LHam} = \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc}$. But, by Lemma 3.4, we also know that $\mathfrak{B}_{\rm nc} = \mathfrak{B}_{\rm Hc}$, where $\mathfrak{B}_{\rm Hc} = (\mathfrak{B}_{\rm Ham})_{\rm cont}$. Hence, by Proposition 2.14, $\mathfrak{B}_{\rm LHam}$ is autonomous.

4. $\mathfrak{B}_{\rm LHam} = \mathfrak{B}_{\rm Ham}/\mathfrak{B}_{\rm nc} \Rightarrow \mathfrak{B}_{\rm Ham} = \mathfrak{B}_{\rm nc} \oplus \mathfrak{B}_{\rm LHam}$. Also, $\mathfrak{B}_{\rm ncHam} = \mathfrak{B}_{\rm nc}$. Hence, $\mathfrak{B}_{\rm Ham} = \mathfrak{B}_{\rm ncHam} \oplus \mathfrak{B}_{\rm LHam}$. □

Remark 3.8: As mentioned earlier, we have considered only smooth trajectories for defining the behaviors involved in this article, because our primary aim is to write the behavior $\mathfrak{B}$ as $\mathfrak{B} = \mathfrak{B}_{\rm nc} \oplus \mathfrak{B}_{\rm LI}$. This direct sum decomposition enables us to solve the singular LQR problems for the behaviors $\mathfrak{B}_{\rm nc}$ and $\mathfrak{B}_{\rm LI}$ separately. While solving the problems for these behaviors, we extend the trajectory space to allow the impulsive trajectories as well (see Section V for more on impulsive trajectories). Let $\mathtt{col}(u_1^*, y_1^*)$ and $\mathtt{col}(u_2^*, y_2^*)$ be optimal trajectories for the behavior $\mathfrak{B}_{\rm nc}$ and $\mathfrak{B}_{\rm LI}$, respectively. Then, it is obvious that $\mathtt{col}(u^*, y^*) := \mathtt{col}(u_1^*, y_1^*) + \mathtt{col}(u_2^*, y_2^*)$ is an optimal trajectory for $\mathfrak{B}$. Because, from the structure of $\mathfrak{B}_{\rm nc}$ it is clear that the optimal cost for the behavior $\mathfrak{B}_{\rm nc}$ is zero. Thus, optimal cost for the behavior $\mathfrak{B}$ is the same as the optimal cost for the behavior $\mathfrak{B}_{\rm LI}$. Further, $\mathfrak{B}_{\rm nc}$ can be solved using a proportional state-feedback, because it admits a solvable CGCARE [6]. On the other hand, $\mathfrak{B}_{\rm LI}$ can be solved using a proportional–derivative (P-D) state-feedback. Hence, a controller that solves the original problem is of a P–D feedback nature. It also follows that $\mathfrak{B}_{\rm nc}$ does not have a unique optimal trajectory. This results in the nonuniqueness of an optimal trajectory for $\mathfrak{B}$ as well.

IV. Method to Obtain the Direct Summands $\mathfrak{B}_{\rm nc}$ and $\mathfrak{B}_{\rm LI}$ of $\mathfrak{B}$

In this section, we obtain the representations of the behaviors $\mathfrak{B}_{\rm nc}$ and $\mathfrak{B}_{\rm LI}$. Recall from Proposition 2.14 that $\mathfrak{B}$ admits a direct sum decomposition given by $\mathfrak{B} = (\mathfrak{B})_{\rm cont} \oplus \mathfrak{B}_{\rm aut}$, where $\mathfrak{B}_{\rm aut} := \mathfrak{B}/(\mathfrak{B})_{\rm cont}$ is autonomous. Since $\mathfrak{B}$ is stabilizable, $\mathfrak{B}_{\rm aut}$ must be asymptotically stable; i.e., if $\mathfrak{B}_{\rm aut} =: \mathtt{ker} R_{\rm aut}(\frac{d}{dt})$, then all the roots of $\mathtt{det} R_{\rm aut}(\xi)$ lie in the open left-half of the complex plane. Such a matrix $R_{\rm aut}(\xi)$ can be obtained from the kernel representation of $\mathfrak{B}$ (see [9, Th. 5.2.14]). Similarly, a kernel representation of $(\mathfrak{B})_{\rm cont}$ can also be found. Furthermore, $(\mathfrak{B})_{\rm cont}$ being controllable admits an observable image representation given by

$$(\mathfrak{B})_{\rm cont} =: \mathtt{img} \begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix}, U(\xi) \in \mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{m}}, Y(\xi) \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{m}}. \tag{16}$$

Obtaining the behavior $\mathfrak{B}_{\rm nc}$: In the following lemma, we obtain an observable image representation of $\mathfrak{B}_{\rm nc}$ from the image representation of $(\mathfrak{B})_{\rm cont}$.

Lemma 4.1: Consider the behaviors $\mathfrak{B}$ and $\mathfrak{B}_{\rm nc}$ as defined in equation (6) and equation (12), respectively. Recall the matrices $U(\xi)$ and $Y(\xi)$ from equation (16). Let $M(\xi) \in \mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{q}}$ be an MRA of $Y(\xi)$, where $\mathtt{q} := \mathtt{m} - \mathtt{rank}\, Y(\xi)$. Then, $\mathfrak{B}_{\rm nc}$ is given by the observable image representation: $\mathfrak{B}_{\rm nc} = \mathtt{img} \begin{bmatrix} U(\frac{d}{dt})M(\frac{d}{dt}) \\ 0_{\mathtt{p},\mathtt{q}} \end{bmatrix}$.

Proof: Since $\mathfrak{B}_{\rm nc}$ is controllable, it admits an image representation. Also, by Definition 2.13, we have that $\mathfrak{B}_{\rm nc} \subseteq (\mathfrak{B})_{\rm cont}$. Thus, there exists $M_1(\xi) \in \mathbb{R}[\xi]^{\mathtt{m}\times\bullet}$ such that $\mathfrak{B}_{\rm nc} = \mathtt{img} \begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix} M_1(\frac{d}{dt})$. From the definition of $\mathfrak{B}_{\rm nc}$, we must have $Y(\frac{d}{dt})M_1(\frac{d}{dt}) = 0$. Thus, $M_1(\xi)$ is a right-annihilator of $Y(\xi)$. Therefore, $\mathtt{img} M_1(\frac{d}{dt}) \subseteq \mathtt{img} M(\frac{d}{dt})$. So, $\mathfrak{B}_{\rm nc} = \mathtt{img} \begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix} M_1(\frac{d}{dt}) \subseteq \mathtt{img} \begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix} M(\frac{d}{dt}) = \mathtt{img} \begin{bmatrix} U(\frac{d}{dt})M(\frac{d}{dt}) \\ 0_{\mathtt{p},\mathtt{q}} \end{bmatrix}$. But, clearly $\mathtt{img} \begin{bmatrix} U(\frac{d}{dt})M(\frac{d}{dt}) \\ 0_{\mathtt{p},\mathtt{q}} \end{bmatrix} \subseteq \mathfrak{B}_{\rm null}$. Also, the behavior

$\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})M(\frac{d}{dt})\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}$ is controllable. Therefore, by Definition 2.13, $\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})M(\frac{d}{dt})\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}\subseteq\mathfrak{B}_{\mathrm{nc}}$. Hence, $\mathfrak{B}_{\mathrm{nc}}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})M(\frac{d}{dt})\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}$.

Next, since $\begin{bmatrix}U(\xi)M(\xi)\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}=\begin{bmatrix}U(\xi)\\ Y(\xi)\end{bmatrix}M(\xi)$ and both $\begin{bmatrix}U(\xi)\\ Y(\xi)\end{bmatrix}$ and $M(\xi)$ are FRP, we have $\begin{bmatrix}U(\xi)M(\xi)\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}$ is FRP. Hence, $\mathfrak{B}_{\mathrm{nc}}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})M(\frac{d}{dt})\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}$ is an observable image representation. □

Obtaining the behavior $\mathfrak{B}_{\mathrm{LI}}$: Recall that $\mathfrak{B}_{\mathrm{LI}}=\mathfrak{B}/\mathfrak{B}_{\mathrm{nc}}$. Being a quotient behavior, $\mathfrak{B}_{\mathrm{LI}}$ is not unique. The next lemma provides a representation for $\mathfrak{B}_{\mathrm{LI}}$ as a sub-behavior of $\mathfrak{B}$.

Lemma 4.2: Recall behavior $\mathfrak{B}_{\mathrm{aut}}=\mathfrak{B}/(\mathfrak{B})_{\mathrm{cont}}$ and the matrices $\begin{bmatrix}U(\xi)\\ Y(\xi)\end{bmatrix}$ and $M(\xi)$ from equation (16) and Lemma 4.1, respectively. Then, the following are true

1. There exists a matrix $N(\xi)\in\mathbb{R}[\xi]^{\mathtt{m}\times(\mathtt{m}-\mathtt{q})}$, which is FRP such that $[M(\xi)\ N(\xi)]$ is unimodular.
2. $\widehat{\mathfrak{B}}_{\mathrm{LI}}:=(\mathfrak{B})_{\mathrm{cont}}/\mathfrak{B}_{\mathrm{nc}}$ is given by an observable image representation $\widehat{\mathfrak{B}}_{\mathrm{LI}}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})$.
3. $\mathfrak{B}_{\mathrm{LI}}=\mathfrak{B}_{\mathrm{aut}}\oplus\widehat{\mathfrak{B}}_{\mathrm{LI}}$.

Proof: 1. Follows from Remark 2.3.

2. Since $[M(\xi)\ N(\xi)]$ is unimodular, $\mathtt{img}[M(\frac{d}{dt})\ N(\frac{d}{dt})]=\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})$. Consequently, using Lemma 4.1, it is evident that

$$\begin{aligned}(\mathfrak{B})_{\mathrm{cont}}&=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}\left[M(\tfrac{d}{dt})\ N(\tfrac{d}{dt})\right]\\ &=\mathfrak{B}_{\mathrm{nc}}+\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N\left(\frac{d}{dt}\right).\end{aligned}\tag{17}$$

Now, let $\mathtt{col}\,(u(t),y(t))\in\mathfrak{B}_{\mathrm{nc}}\cap\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})$. Then, there exist $\ell_1\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ and $\ell_2\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{(\mathtt{m}-\mathtt{q})})$ such that $\begin{bmatrix}u(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}U(\frac{d}{dt})M(\frac{d}{dt})\\ 0_{\mathtt{p},\mathtt{q}}\end{bmatrix}\ell_1=\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})\ell_2\Leftrightarrow\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}M(\frac{d}{dt})\ell_1=\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})\ell_2\Leftrightarrow\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}[M(\frac{d}{dt})\ N(\frac{d}{dt})]\begin{bmatrix}\ell_1\\ -\ell_2\end{bmatrix}=0$. Since $(\mathfrak{B})_{\mathrm{cont}}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}$ is observable, by Definition 2.10, we infer that $[M(\frac{d}{dt})\ N(\frac{d}{dt})]\begin{bmatrix}\ell_1\\ -\ell_2\end{bmatrix}=0$. But, $[M(\frac{d}{dt})\ N(\frac{d}{dt})]$ being unimodular further implies that $\mathtt{col}\,(\ell_1,-\ell_2)=0$. Therefore, $\mathtt{col}\,(u(t),y(t))=0$. Thus, $\mathfrak{B}_{\mathrm{nc}}\cap\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})=\{0\}$. Hence, from equation (17), we infer that $(\mathfrak{B})_{\mathrm{cont}}=\mathfrak{B}_{\mathrm{nc}}\oplus\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})$. Therefore, $\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})=(\mathfrak{B})_{\mathrm{cont}}/\mathfrak{B}_{\mathrm{nc}}=\widehat{\mathfrak{B}}_{\mathrm{LI}}$.

Since $\mathtt{col}\,(U(\xi),Y(\xi))$ and $N(\xi)$ are FRP, $\widehat{\mathfrak{B}}_{\mathrm{LI}}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}N(\frac{d}{dt})$ is an observable image representation.

3. By definition $\mathfrak{B}_{\mathrm{LI}}=\mathfrak{B}/\mathfrak{B}_{\mathrm{nc}}=[\mathfrak{B}_{\mathrm{aut}}\oplus(\mathfrak{B})_{\mathrm{cont}}]/\mathfrak{B}_{\mathrm{nc}}$. Now, $\mathfrak{B}_{\mathrm{nc}}\subseteq(\mathfrak{B})_{\mathrm{cont}}$ implies that $\mathfrak{B}_{\mathrm{aut}}\cap\mathfrak{B}_{\mathrm{nc}}=\{0\}$. Therefore, $\mathfrak{B}_{\mathrm{LI}}=\mathfrak{B}_{\mathrm{aut}}\oplus[(\mathfrak{B})_{\mathrm{cont}}/\mathfrak{B}_{\mathrm{nc}}]=\mathfrak{B}_{\mathrm{aut}}\oplus\widehat{\mathfrak{B}}_{\mathrm{LI}}$.

Remark 4.3: Here, we summarize the relations between the behaviors encountered in this article.

1) The plant behavior $\mathfrak{B}$ is written as a direct sum decomposition $\mathfrak{B}=\mathfrak{B}_{\mathrm{nc}}\oplus\mathfrak{B}_{\mathrm{LI}}$, where $\mathfrak{B}_{\mathrm{nc}}$ is the controllable part of the output-nulling behavior $\mathfrak{B}_{\mathrm{null}}$ [see equation (11)]. Hence, $\mathfrak{B}_{\mathrm{nc}}\subseteq(\mathfrak{B})_{\mathrm{cont}}$.
2) On the other hand, $\mathfrak{B}_{\mathrm{aut}}\subseteq\mathfrak{B}_{\mathrm{LI}}$. As mentioned before, $\mathfrak{B}_{\mathrm{aut}}$ is asymptotically stable. So, by Lemma 4.2, we can infer that $\mathfrak{B}_{\mathrm{LI}}$ is stabilizable.
3) $\mathfrak{B}_{\mathrm{Ham}}$ is the Hamiltonian behavior of the plant behavior $\mathfrak{B}$. We showed that $\mathfrak{B}_{\mathrm{Ham}}$ admits the decomposition $\mathfrak{B}_{\mathrm{Ham}}=\mathfrak{B}_{\mathrm{ncHam}}\oplus\mathfrak{B}_{\mathrm{LHam}}$, where $\mathfrak{B}_{\mathrm{ncHam}}$ and $\mathfrak{B}_{\mathrm{LHam}}$ are the Hamiltonian behaviors corresponding to the behaviors $\mathfrak{B}_{\mathrm{nc}}$ and $\mathfrak{B}_{\mathrm{LI}}$, respectively.
4) Further, due to the structure of $\mathfrak{B}_{\mathrm{nc}}$, it follows that $\mathfrak{B}_{\mathrm{ncHam}}=\mathfrak{B}_{\mathrm{nc}}$ while $\mathfrak{B}_{\mathrm{LHam}}$ is autonomous.

V. On the Distributional Optimal Trajectories

Singular LQR problems, in general, exhibit optimal trajectories, which are impulsive in nature. But, while defining the behaviors, we have considered only smooth trajectories. This may lead one to believe that the approach presented in this article works only for the smooth optimal trajectories. In this section, we show that this apparent impression is not true. As discussed before (see Remark 3.8), smooth trajectories have been considered only to get a direct sum decomposition of the plant behavior, which is our initial aim. While solving the problems for the direct summand behaviors individually, we extend our trajectory space to allow the impulsive trajectories as well. We show that the method presented in this article can obtain the impulsive optimal trajectories, too. We first define the set of impulsive-smooth distributions.

Definition 5.1: The set of impulsive-smooth distributions $\mathfrak{C}^{\mathtt{w}}_{\mathrm{imp}}$ is defined as

$$\mathfrak{C}^{\mathtt{w}}_{\mathrm{imp}}:=\{\,f=f_{\mathrm{reg}}+f_{\mathrm{imp}}|f_{\mathrm{reg}}\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})|_{\mathbb{R}_+}\text{ and }$$
$$f_{\mathrm{imp}}=\sum_{i=0}^{k}a_i\delta^{(i)},\ \text{with } a_i\in\mathbb{R}^{\mathtt{w}}\text{ and }k\in\mathbb{N}\,\}.$$

Next, we formally define the set of allowable impulsive trajectories for a system with transfer function matrix $G(s)$.

Definition 5.2: Consider a system with the transfer function matrix $G(s)\in\mathbb{R}(s)^{\mathtt{p}\times\mathtt{m}}$. Assume that the system produces the output $y(t)$ on application of the input $u(t)$. Then, $\left(\begin{smallmatrix}u\\ y\end{smallmatrix}\right)$ is said to be an *allowable impulsive trajectory*, if $u(t)\in\mathfrak{C}^{\mathtt{m}}_{\mathrm{imp}}$ and $y(t)\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{p}})|_{\mathbb{R}_+}$ (i.e., the output is regular).

Remark 5.3: Let $u(t)=u_{\mathrm{reg}}+u_{\mathrm{imp}}$, where $u_{\mathrm{reg}}\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}$ and $u_{\mathrm{imp}}=\sum_{i=0}^{k}a_i\delta^{(i)}$, with $a_i\in\mathbb{R}^{\mathtt{m}}$ and $k\in\mathbb{N}$. Define $\breve{u}(s):=\breve{u}_{\mathrm{reg}}(s)+\breve{u}_{\mathrm{imp}}(s)$, where $\breve{u}_{\mathrm{reg}}(s)$ is the Laplace transform of u_{reg} and $\breve{u}_{\mathrm{imp}}(s):=\sum_{i=0}^{k}a_is^i$. Then, it can be shown that the output $y(t)$ corresponding to the input $u(t)$ is regular if and only if $G(s)\breve{u}(s)$ is strictly proper.

Next, we characterize the allowable impulsive trajectories for the behavior $(\mathfrak{B})_{\mathrm{cont}}$ in terms of its image representation.

Lemma 5.4: Recall that $(\mathfrak{B})_{\mathrm{cont}}=\mathtt{img}\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}$, where $Y(s)\in\mathbb{R}[s]^{\mathtt{p}\times\mathtt{m}}$, and $U(s)\in\mathbb{R}[s]^{\mathtt{m}\times\mathtt{m}}$ is nonsingular. Then, $\left(\begin{smallmatrix}u\\ y\end{smallmatrix}\right)\in\mathfrak{C}^{\mathtt{m}+\mathtt{p}}_{\mathrm{imp}}$ is an allowable impulsive trajectory of $(\mathfrak{B})_{\mathrm{cont}}$, if and only if there exists $\ell\in\mathfrak{C}^{\mathtt{m}}_{\mathrm{imp}}$ such that $\left(\begin{smallmatrix}u\\ y\end{smallmatrix}\right)=\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}\ell$, where $u=U(\frac{d}{dt})\ell\in\mathfrak{C}^{\mathtt{m}}_{\mathrm{imp}}$ and $y=Y(\frac{d}{dt})\ell\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{p}})|_{\mathbb{R}_+}$.

Proof if: This direction follows directly.

only if: The transfer function matrix of the behavior $(\mathfrak{B})_{\mathrm{cont}}$ is given by $G(s)=Y(s)U(s)^{-1}$. Now, since $\left(\begin{smallmatrix}u\\ y\end{smallmatrix}\right)$ is an allowable impulsive trajectory, $u(t)\in\mathfrak{C}^{\mathtt{m}}_{\mathrm{imp}}$ and $y(t)\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{p}})|_{\mathbb{R}_+}$. Moreover, $\breve{y}(s)=Y(s)U(s)^{-1}\breve{u}(s)$, where $\breve{u}(s)$ is as defined in Remark 5.3 and $\breve{y}(s)$ is the Laplace transform of $y(t)$. Now, define $\breve{\ell}(s)=U(s)^{-1}\breve{u}(s)$. Then, $\breve{\ell}(s)$ can be written as $\breve{\ell}(s)=\breve{\ell}_1(s)+\breve{\ell}_2(s)$, where $\breve{\ell}_1(s)$ is strictly proper and $\breve{\ell}_2(s)=\sum_{i=0}^{j}b_is^i$, for some $b_i\in\mathbb{R}^{\mathtt{m}}$ and $j\in\mathbb{N}$. Define $\ell(t)=\ell_1(t)+\ell_2(t)$, where $\ell_1(t)$ is the Laplace inverse of $\breve{\ell}_1(s)$ and $\ell_2(t):=\sum_{i=0}^{j}b_i\delta^{(i)}$. Then, evidently, $\left(\begin{smallmatrix}u\\ y\end{smallmatrix}\right)=\begin{bmatrix}U(\frac{d}{dt})\\ Y(\frac{d}{dt})\end{bmatrix}\ell$. □

In Theorem 3.7, we have shown that the plant behavior admits a decomposition $\mathfrak{B}=\mathfrak{B}_{\mathrm{nc}}\oplus\mathfrak{B}_{\mathrm{LI}}$. But, while doing so, we have considered smooth trajectories only. The following theorem shows that this decomposition works for the impulsive trajectories as well. Therefore, if $\binom{u}{y}$ is an allowable impulsive trajectory of $\mathfrak{B}$, then $\binom{u}{y}$ can be written as $\binom{u}{y} = \binom{\widehat{u}}{0} + \binom{\widetilde{u}}{y}$, where $\binom{\widehat{u}}{0} \in \mathfrak{B}_{\rm nc}$ and $\binom{\widetilde{u}}{y} \in \mathfrak{B}_{\rm LI}$. Furthermore, $\binom{u}{y} \in \mathfrak{B}_{\rm LI}$ and $\binom{\widetilde{u}}{y} \in \mathfrak{B}_{\rm LI}$ incur the same cost. Thus, even though we have used only smooth trajectories to get a direct sum decomposition of the plant behavior, the impulsive optimal trajectories are not lost when we solve the singular LQR problems for the behaviors $\mathfrak{B}_{\rm nc}$ and $\mathfrak{B}_{\rm LI}$ individually.

Theorem 5.5: Let $\binom{u}{y} \in \mathfrak{B}$ be an impulsive optimal trajectory that incurs the cost J_0. Then, there exist $\widehat{u}, \widetilde{u} \in \mathfrak{C}^{\mathtt{m}}_{\rm imp}$ such that $\binom{u}{y} = \binom{\widehat{u}}{0} + \binom{\widetilde{u}}{y}$ with $\binom{\widehat{u}}{0} \in \mathfrak{B}_{\rm nc}$ and $\binom{\widetilde{u}}{y} \in \mathfrak{B}_{\rm LI}$. Furthermore, $\binom{\widetilde{u}}{y}$ is an impulsive optimal trajectory for $\mathfrak{B}$ that incurs the cost J_0.

Proof: Recall that $\mathfrak{B} = \mathfrak{B}_{\rm aut} \oplus (\mathfrak{B})_{\rm cont}$. $\mathfrak{B}_{\rm aut}$ being an autonomous behavior does not contain any impulsive optimal trajectory. Thus, all the impulsive trajectories of $\mathfrak{B}$ are contained in the behavior $(\mathfrak{B})_{\rm cont}$. Therefore, without loss of generality, we may assume that $\mathfrak{B} = (\mathfrak{B})_{\rm cont}$. So, $\mathfrak{B} = \mathtt{img}\begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix}$, where $U(\xi) \in \mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{m}}$ and $Y(\xi) \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{m}}$. Assume that $M(\xi) \in \mathbb{R}[\xi]^{\mathtt{m}\times\mathtt{q}}$ is an MRA of $Y(\xi)$ and $N(\xi) \in \mathbb{R}[\xi]^{\mathtt{m}\times(\mathtt{m}-\mathtt{q})}$ is such that $[M(\xi)\ N(\xi)]$ is unimodular, where $\mathtt{q} := \mathtt{m} - \mathtt{rank}\, Y(\xi)$. Then, from Lemmas 4.1 and 4.2 it follows that $\mathfrak{B}$ is given by $\mathfrak{B} = \mathtt{img}\begin{bmatrix} U_1(\frac{d}{dt}) & U_2(\frac{d}{dt}) \\ 0 & Y_2(\frac{d}{dt}) \end{bmatrix}$, where $U_1(\frac{d}{dt}) := U(\frac{d}{dt})M(\frac{d}{dt}), U_2(\frac{d}{dt}) := U(\frac{d}{dt})N(\frac{d}{dt})$, and $Y_2(\frac{d}{dt}) := Y(\frac{d}{dt})N(\frac{d}{dt})$. Since $\binom{u}{y} \in \mathfrak{B}$, by Lemma 5.4, there exist $\ell_1 \in \mathfrak{C}^{\mathtt{q}}_{\rm imp}$ and $\ell_2 \in \mathfrak{C}^{(\mathtt{m}-\mathtt{q})}_{\rm imp}$ such that $\binom{u}{y} = \begin{bmatrix} U_1(\frac{d}{dt}) & U_2(\frac{d}{dt}) \\ 0 & Y_2(\frac{d}{dt}) \end{bmatrix}\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$. Define $\widehat{u} := U_1(\frac{d}{dt})\ell_1$ and $\widetilde{u} := U_2(\frac{d}{dt})\ell_2$. Then, clearly $\binom{u}{y} = \binom{\widehat{u}}{0} + \binom{\widetilde{u}}{y}$ with $\binom{\widehat{u}}{0} \in \mathtt{img}\begin{bmatrix} U_1(\frac{d}{dt}) \\ 0 \end{bmatrix} = \mathfrak{B}_{\rm nc}$ and $\binom{\widetilde{u}}{y} \in \mathtt{img}\begin{bmatrix} U_2(\frac{d}{dt}) \\ Y_2(\frac{d}{dt}) \end{bmatrix} = \mathfrak{B}_{\rm LI}$.

Furthermore, by equation (5), it follows that $\binom{u}{y}$ and $\binom{\widetilde{u}}{y}$ incur the same cost J_0. Since $\mathfrak{B}_{\rm LI} \subseteq \mathfrak{B}$, we have that $\binom{\widetilde{u}}{y}$, too, is an impulsive optimal trajectory of $\mathfrak{B}$, which incurs the cost J_0. □

VI. Illustrative Example

Consider the singular LQR Problem 3.1, where the matrices of the system Σ are given as $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, C = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the transfer function matrix $G(s)$ of Σ is given by

$$G(s) = C(sI_3 - A)^{-1}B + D = \frac{1}{3}\begin{bmatrix} -\frac{(5s-1)}{(s-1)^2} & \frac{(4s-2)}{(s-1)^2} & \frac{(2s+2)}{(s-1)^2} \\ -2 & -2 & -1 \\ \frac{2}{(s-1)} & \frac{2}{(s-1)} & \frac{1}{(s-1)} \end{bmatrix}.$$

It can be verified that $\mathtt{nrank}\, G(s) = 2$, where $\mathtt{nrank}\, G(s) := \mathtt{max}\{\mathtt{rank}\, G(\lambda) | \lambda \in \mathbb{C}$ and $G(s)$ is analytic at $\lambda\}$ is the *normal rank* of $G(s)$. But, since $\mathtt{rank}\,(D) = 1$, by [8, Th. 1], we conclude that CGCARE is *not* solvable.

Again, it can be easily verified that $\mathtt{det}(sE - H) \equiv 0$, where E, H are as defined in equation (2) with $Q = C^TC, S = C^TD$, and $R = D^TD$. Hence, the Hamiltonian system is nonautonomous. Therefore, the problem cannot be solved by applying either [3], or [5], [6]. Next, we solve this problem by using the behavioral formulation presented in this article.

To obtain a behavioral representation of the system Σ, we first obtain a left co-prime factorization of $G(s)$ as $G(s) = \widetilde{Y}(s)^{-1}\widetilde{U}(s)$, where $\widetilde{Y}(s) := \begin{bmatrix} (s-1)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (s-1) \end{bmatrix}$ and $\widetilde{U}(s) := \frac{1}{3}\begin{bmatrix} 1-5s & 4s-2 & 2s+2 \\ -2 & -2 & -1 \\ 2 & 2 & 1 \end{bmatrix}$. Then, the behavior $\mathfrak{B}$ of the system Σ is given by $\mathfrak{B} = \mathtt{ker} R(\frac{d}{dt})$, where $R(\xi) = [-\widetilde{U}(\xi)\ \widetilde{Y}(\xi)]$. Since, $\mathfrak{B}$ is controllable, by Proposition 2.9, we can obtain an observable image representation $\mathfrak{B} = \mathtt{img}\begin{bmatrix} U(\xi) \\ Y(\xi) \end{bmatrix}$ by computing an MRA $\begin{bmatrix} U(\xi) \\ Y(\xi) \end{bmatrix}$ of $R(\xi)$. These matrices can be found out to be

$$U(\xi) := \begin{bmatrix} -1 & 0 & \frac{(3\xi-3)}{2} \\ \frac{3\xi+1}{2} & -\frac{(\xi-1)^2}{2} & -\frac{5\xi^2-6\xi+1}{4} \\ 1-3\xi & (\xi-1)^2 & \frac{5\xi^2-6\xi+1}{2} \end{bmatrix} \text{ and } Y(\xi) := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1-\xi \\ 0 & 0 & 1 \end{bmatrix}.$$

We apply Lemma 4.1 to obtain the behavior $\mathfrak{B}_{\rm nc}$. Thus, we find a matrix $M(\xi)$ which is an MRA of $Y(\xi)$. We find this matrix to be $M(\xi) = [1\ 0\ 0]^T$. Then

$$\mathfrak{B}_{\rm nc} = \mathtt{img}\underbrace{\begin{bmatrix} U(\frac{d}{dt})M(\frac{d}{dt}) \\ 0_{3,1} \end{bmatrix}}_{M_{\rm nc}(\frac{d}{dt})} = \mathtt{img}\begin{bmatrix} -1 \\ \frac{1}{2}(3\frac{d}{dt}+1) \\ 1-3\frac{d}{dt} \\ 0_{3,1} \end{bmatrix}. \quad (18)$$

To obtain the behavior $\mathfrak{B}_{\rm LI}$, we apply Lemma 4.2. Thus, we find a matrix $N(\xi)$ such that $\left[M(\xi)\ N(\xi)\right]$ is unimodular. Notice that, $N(\xi) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies this condition. Thus, an observable image representation of $\mathfrak{B}_{\rm LI}$ is given as

$$\mathfrak{B}_{\rm LI} = \mathtt{img}\underbrace{\begin{bmatrix} U(\frac{d}{dt}) \\ Y(\frac{d}{dt}) \end{bmatrix} N(\frac{d}{dt})}_{M_{\rm LI}(\frac{d}{dt})} = \mathtt{img}\begin{bmatrix} 0 & \frac{3\xi-3}{2} \\ -\frac{(\xi-1)^2}{2} & -\frac{5\xi^2-6\xi+1}{4} \\ (\xi-1)^2 & \frac{5\xi^2-6\xi+1}{2} \\ 1 & 0 \\ 0 & 1-\xi \\ 0 & 1 \end{bmatrix}. \quad (19)$$

Notice that $\mathfrak{B} = \mathfrak{B}_{\rm nc} \oplus \mathfrak{B}_{\rm LI}$.

Say, $\mathtt{col}(u_1(t), u_2(t), u_3(t), y_1(t), y_2(t), y_3(t)) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^6)$ is an arbitrary trajectory of $\mathfrak{B}_{\rm nc}$, then from equation (18) it is evident that $\mathtt{col}(y_1(t), y_2(t), y_3(t)) \equiv 0$. Therefore, the Hamiltonian of $\mathfrak{B}_{\rm nc}$ with respect to the cost function (5) is same as the behavior $\mathfrak{B}_{\rm nc}$. Hence, by Corollary 3.3, CGCARE is solvable for the behavior $\mathfrak{B}_{\rm nc}$.

Next, $\mathtt{rank}\, M_{\rm LI}(\xi) = 2$. Hence, the input-cardinality of $\mathfrak{B}_{\rm LI} = 2$. Let $\mathtt{col}(u_1(t), u_2(t), u_3(t), y_1(t), y_2(t), y_3(t)) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^6)$ be an arbitrary trajectory of $\mathfrak{B}_{\rm LI}$. To obtain a proper input–output partition, we choose $\mathtt{col}(u_1(t), u_3(t))$ as the input. The variable $u_2(t)$ is ignored, because it does not influence the objective function. Now, we obtain the input-state-output (i/s/o) representation of the behavior $\mathfrak{B}_{\rm LI}$ from the input $\tilde{u} := \mathtt{col}(u_1(t), u_3(t))$ to the output $y = \mathtt{col}(y_1(t), y_2(t), y_3(t))$ (see [9, Ch. 6] for details about obtaining the i/s/o form). This representation is found to be

$$\frac{d}{dt}\tilde{x}(t) = \widetilde{A}\tilde{x}(t) + \widetilde{B}\tilde{u}(t) \text{ and } y(t) = \widetilde{C}\tilde{x}(t) + \widetilde{D}\tilde{u}(t) \quad (20)$$

where $\widetilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{3} & 0 & -1 \\ -\frac{4}{3} & 1 & 2 \end{bmatrix}, \widetilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \widetilde{C} = \begin{bmatrix} -\frac{5}{3} & 0 & 1 \\ 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \end{bmatrix}$, and $\widetilde{D} = \begin{bmatrix} 0 & 0 \\ -\frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}$. For this system, the Hamiltonian matrix pair (E, H) [defined in equation (2)] is found to be $E = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & 0_{2,2} \end{bmatrix}$ and $H = \begin{bmatrix} \widetilde{A} & 0 & \widetilde{B} \\ -\widetilde{C}^T\widetilde{C} & -\widetilde{A}^T & -\widetilde{C}^T\widetilde{D} \\ \widetilde{D}^T\widetilde{C} & \widetilde{B}^T & \widetilde{D}^T\widetilde{D} \end{bmatrix}$. It can be verified that $\mathtt{det}(sE - H) = \frac{4}{9}(s^2 - 2)$. Thus, the Hamiltonian system is autonomous. By solving the problem for the state-space system given by equation (20), we get that the optimal state and the optimal input are given by $\tilde{x}^* = 0.5 \begin{bmatrix} 3 \\ -(5\sqrt{2}+6) \\ 5 \end{bmatrix} e^{-\sqrt{2}t}x_{03} + \begin{bmatrix} 0 \\ x_{01}+x_{02} \\ 0 \end{bmatrix} \delta$ and $\widetilde{u}^* = 0.5 \begin{bmatrix} -3(\sqrt{2}+1) \\ 6\sqrt{2}+11 \end{bmatrix} e^{-\sqrt{2}t}x_{03} - \begin{bmatrix} 0 \\ x_{01}-x_{02}+\frac{5}{2}(\sqrt{2}+1)x_{03} \end{bmatrix} \delta + \begin{bmatrix} 0 \\ x_{01}+x_{02} \end{bmatrix} \delta^{(1)}$, respectively, where $x_0 = \mathtt{col}\,(x_{01}, x_{02}, x_{03})$ is the initial condition of the given plant Σ. Consequently, the optimal output trajectory of $\mathfrak{B}_{\mathrm{LI}}$ (and also of $\mathfrak{B}$) is given by $y^* = \begin{bmatrix} 0 \\ \sqrt{2}+1 \\ 1 \end{bmatrix} e^{-\sqrt{2}t}x_{03}$. Evidently, an optimal input for the given plant Σ is given by $u^* = 0.5 \begin{bmatrix} -3(\sqrt{2}+1) \\ -\frac{6\sqrt{2}+11}{2} \\ 6\sqrt{2}+11 \end{bmatrix} e^{-\sqrt{2}t}x_{03} - \begin{bmatrix} 0 \\ -\frac{1}{2}\{x_{01}-x_{02}+\frac{5}{2}(\sqrt{2}+1)x_{03}\} \\ x_{01}-x_{02}+\frac{5}{2}(\sqrt{2}+1)x_{03} \end{bmatrix} \delta + \begin{bmatrix} 0 \\ -\frac{1}{2}(x_{01}+x_{02}) \\ x_{01}+x_{02} \end{bmatrix} \delta^{(1)} =: u_r e^{-\sqrt{2}t} + r_0\delta + r_1\delta^{(1)}$. This optimal input produces the optimal state trajectory of Σ given by $x^* = \frac{1}{4} \begin{bmatrix} -5(\sqrt{2}+1) \\ 5(\sqrt{2}+1) \\ 4 \end{bmatrix} e^{-\sqrt{2}t}x_{03} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} (x_{01}+x_{02})\delta = x_r e^{-\sqrt{2}t} + x_d\delta$. So, $\frac{d}{dt}x^* = -\sqrt{2}x_r e^{-\sqrt{2}t} - x_0\delta + x_d\delta^{(1)}$ (in distributional sense). To obtain a P–D feedback law, we solve the equation $u^* = F_p x^* + F_d \frac{d}{dt}x^*$ for F_p and F_d. Equivalently, $(F_p - \sqrt{2}F_d)x_r = u_r$, $(F_p x_d - F_d x_0) = r_0$, and $F_d x_d = r_1$. $F_p = \begin{bmatrix} 0 & 0 & -\frac{3}{2}(\sqrt{2}+1) \\ 0 & 0 & -\frac{1}{4}(6\sqrt{2}+11) \\ 0 & 0 & \frac{1}{2}(6\sqrt{2}+11) \end{bmatrix}$ and $F_d = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{5}{4}(\sqrt{2}+1) \\ 1 & -1 & \frac{5}{2}(\sqrt{2}+1) \end{bmatrix}$ satisfy the set of equations. Hence, the feedback law $u(t) = F_p x(t) + F_d \frac{d}{dt}x(t)$ solves the given singular LQR problem.

VII. Conclusion

A singular LQR problem with nonautonomous Hamiltonian for which the corresponding CGCARE is not solvable has been dealt with in this article. We formulated the problem in a behavioral theoretic setting. Then, we showed that the original problem can be divided into two subproblems such that one problem admits a solvable CGCARE, while the other admits an autonomous Hamiltonian. We achieve this by obtaining a direct sum decomposition of the original behavior. Finally, we show that the method described in this article can be used to obtain both the smooth and the distributional optimal trajectories. In Section VI, we demonstrated the theory presented in this article through an illustrative example. We also provided a closed-loop solution for this particular example. However, a closed-loop solution for a general problem with nonautonomous Hamiltonian has not been provided here. We plan to pursue this elsewhere in the future.

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