# Improper $\mathcal{L}_\infty$ Optimal/Suboptimal Controllers

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***Abstract*—We consider $\mathcal{L}_\infty$-control of MIMO systems and address solvability of the problem over *all finite dimensional LTI* controllers: i.e., controllers whose transfer functions can be *proper* or *improper*. We show that improper controllers are easily dealt with using the behavioral approach, unlike the standard state-space/transfer-matrix methods, and argue that there are cases where an improper controller can outperform a proper controller. In this setting, we next formulate and prove necessary and sufficient conditions for suboptimal $\mathcal{L}_\infty$-control problem solvability and relate this to existing results about system invariant zeros. Further, we infer that in our formulation, assuming suboptimal solvability conditions on the system, an optimal controller *always exists*, possibly with an improper transfer function. In other words, the infimum $\mathcal{L}_\infty$-norm of the closed loop system is achievable when dealing with both proper and improper controller transfer functions. We illustrate these results through an example for which the optimal $\mathcal{L}_\infty$-controller has an improper transfer function.**

***Index Terms*—Dissipative systems, improper transfer function, $\mathcal{L}_\infty$-control, optimal control, polynomial matrix representations, system invariant zeros.**

## I. Introduction and Notation

While $\mathcal{H}_\infty/\mathcal{L}_\infty$ control problems have far-reaching significance in robust control and worst case scenario disturbance attenuation, one serious drawback with the current control design procedure is that controllers often end up having a large order. Consequently, the closed loop system resulting from the obtained controller has an even higher order. In our opinion, the key reason for this drawback is the inability of state-space design methodology to accommodate improper controllers in controller design. This technical note eliminates this drawback completely; we focus on $\mathcal{L}_\infty$ suboptimal/optimal control problems, without making any restrictive assumptions *a priori* that is done in state-space theory to guarantee properness of the controller transfer function. In this technical note, we do not address the *internal stability* aspect of the problem, which together with $\mathcal{L}_\infty$ control constitutes the corresponding $\mathcal{H}_\infty$ control problem. See [9] for recent work that addresses the internal stability aspect, though in a different context, namely robust stabilization, and further, without dealing with improperness of controller transfer function. An advantage of the approach in this technical note, also reflected by our main results, is that when nonproper controllers are included in $\mathcal{L}_\infty$-optimal control problem, then the optimal value is always attained, possibly by an improper controller [10]. Another of the main results in this technical note is that, even when not requiring properness of the controller/plant transfer functions, we obtain the familiar necessary and sufficient conditions for solvability of the suboptimal $\mathcal{L}_\infty$-control problem for sufficiently large $\gamma$: absence of the plant invariant zeros on the imaginary axis. See [5] for an earlier reporting of these results.

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The content of this technical note is organized as follows. The issue of controllers with improper transfer functions for the $\mathcal{L}_\infty$ suboptimal control problem is studied in Section III. Here we present necessary and sufficient conditions (Theorem 3.2) for the solvability of the sub-optimal $\mathcal{L}_\infty$-control problem under milder assumptions than made in conventional state space control theory (see [7], for example). We also relate the conditions of Theorem 3.2 with well-known system theoretic concepts of *invariant zeros*. The proof of the main result together with some auxiliary results are in Section IV. Finally, we show how our main result Theorem 3.2 can be used to infer the solvability of the $\mathcal{L}_\infty$ *optimal* control problem (Section V). Here we show that due to non-requirement of properness of the controller's transfer function, the optimal controller *always exists* under suboptimal solvability conditions. This result is our second main result of the technical note. Section II contains preliminaries of behavioral theory of dissipative dynamical systems. The rest of this section is devoted to the notation used in this technical note.

The sets $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers respectively, while $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ means the space of infinitely often differentiable maps from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. The subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ with functions having compact support is denoted by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Sometimes, when it is clear from the context, we write just $\mathfrak{C}^\infty$ and $\mathfrak{D}$. Also, in order to identify the number of components in a vector $w$, we simply use $\mathtt{w}$, for example, $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. We often require to stack vectors or matrices into a column; this is done using the 'col': $\mathrm{col}(R_1, R_2)$ denotes $[R_1^T \;\; R_2^T]^T$. Similarly, for readability purposes, we write the vector $w = (w_1, w_2, w_3)$, though $w$ is a *column* vector in equations. Finally, when defining a matrix $R$, in which the number of rows follows from matrix multiplication/addition compatibility, then we write $R(\xi) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and thus specify only the number of columns of $R$.

## II. Preliminaries

In this technical note, by a linear differential behavior $\mathfrak{B}$, we mean a subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ such that elements $w \in \mathfrak{B}$ satisfy a system of ordinary linear differential equations with constant real coefficients. This amounts to existence of a polynomial matrix $R(\xi) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) | R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}.$$

This representation is known as a *kernel representation* of $\mathfrak{B}$. We denote the set of all such linear differential behaviors with $\mathtt{w}$ number of variables by $\mathfrak{L}^{\mathtt{w}}$. Though kernel representations are not unique, the number of system inputs and system outputs do not depend on the particular kernel representation. We denote the number of inputs in the system by $\mathtt{m}(\mathfrak{B})$. The number of inputs turns out to be $\mathtt{w} - \mathrm{rank}(R)$ for a kernel representation $R(\mathrm{d}/\mathrm{d}t)w = 0$. The polynomial matrix $R$ in a kernel representation can be assumed to have full row rank without loss of generality: we assume this.

A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *controllable* if for every $w', w'' \in \mathfrak{B}$, there exists $w \in \mathfrak{B}$ and $\tau > 0$ such that

$$w(t) = \begin{cases} w'(t) & \text{for all } t \leqslant 0 \text{ and} \\ w''(t) & \text{for all } t \geqslant \tau. \end{cases}$$

We denote the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. It was shown in [6] that $\mathfrak{B} = \mathbf{ker} R(\mathrm{d}/\mathrm{d}t)$ is controllable if and only if $R(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. Controllable behaviors are precisely the behaviors that admit an *image representation*: there exists an $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ with

$$\mathfrak{B} := \left\{ w | \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\}.$$

We then write $\mathfrak{B} = \mathbf{im} M(\mathrm{d}/\mathrm{d}t)$. For the purpose of this technical note, we need the image representation to have the property that $\ell$ can be deduced from $w \in \mathfrak{B}$; this is called observability. The image representation above is said to be observable if $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. It turns out that image representations can be assumed to be observable without loss of generality: unless otherwise stated, we assume this. A detailed exposition of these concepts can be found in [6].

An important concept required for this technical note is the notion of a quadratic differential form (QDF). (See [11] for a detailed exposition.) A QDF $Q_\Phi$ induced by a two-variable polynomial matrix $\Phi(\zeta, \eta) := \sum_{i,k} \Phi_{ik}\zeta^i\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, where $\Phi_{ik} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, is a map $Q_\Phi : \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$ defined as

$$Q_\Phi(w) := \sum_{i,k} \left(\frac{\mathrm{d}^i w}{\mathrm{d}t^i}\right)^T \Phi_{ik} \left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right).$$

When dealing with quadratic forms in $w$ and its derivatives, we assume, without loss of generality, that $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$; such a $\Phi(\zeta, \eta)$ is called *symmetric*. We often require the one-variable polynomial matrix $\Phi(-\xi, \xi)$ obtained from $\Phi(\zeta, \eta)$; we define $\partial\Phi(\xi) := \Phi(-\xi, \xi)$.

We call a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ dissipative on $\mathbb{R}$ with respect to a symmetric two-variable polynomial matrix $\Phi(\zeta, \eta)$ if $\int_{\mathbb{R}} Q_\Phi(w)\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. We will make use of the following result from [11], which relates the dissipativity of a behavior to the non-negativity of a certain polynomial matrix on the imaginary axis.

*Proposition 2.1:* Consider $\mathfrak{B} = \mathbf{im} M(\mathrm{d}/\mathrm{d}t)$ and a symmetric $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Then $\mathfrak{B}$ is $\Phi$-dissipative on $\mathbb{R}$ if and only if $M^T(-i\omega)\partial\Phi(i\omega)M(i\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$.

We require the notion of orthogonal complement of a controllable behavior in this technical note: consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ having a kernel representation $R(\mathrm{d}/\mathrm{d}t)w = 0$, the orthogonal complement $\mathfrak{B}^{\perp}$ of the behavior $\mathfrak{B}$ is defined as

$$\mathfrak{B}^{\perp} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | \int_{\mathbb{R}} w^T v \mathrm{d}t = 0 \text{ for all } v \in \mathfrak{B} \cap \mathfrak{D} \right\}.$$

## III. Suboptimal $\mathcal{L}_\infty$ Control: Problem Formulation and Main Result

In this section we address the solvability of the $\mathcal{L}_\infty$ control problem when the restrictive regularity assumptions on the "feed-through" terms of the plant are relaxed. The regularity assumptions are required in order to have the controller in the conventional observer-state-feedback structure, which is equivalent to the properness of the controller transfer function. These assumptions are restrictive in the sense that even when regularity is violated, which can make the $\mathcal{L}_\infty$ (sub)-optimal control problem unsolvable with a proper controller, an improper controller might still exist that makes the controlled system achieve the desired $\mathcal{L}_\infty$ norm[1] condition.

Our main result provides necessary and sufficient conditions for the solvability of the MIMO $\mathcal{L}_\infty$ control problem without any such assumptions. However, before we state our main result we give an example where regularity assumptions on the plant are not satisfied, but an improper transfer function controller solves the $\mathcal{L}_\infty$-control problem. We use a SISO example just for the purpose of demonstration: all the results in this technical note are for the MIMO case.

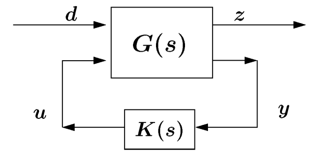


Fig. 1. Standard control problem.

*Example 3.1:* Consider the following plant:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d,$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ u \end{bmatrix} \quad \text{and} \quad y = x_1$$

where $z = (z_1, z_2)$ is the to-be-controlled variable, $u$ is the control input, $d$ is the disturbance and $y$ is the measurement. (See Fig. 1.) It can be checked that a state-space controller cannot restrict this plant to a controlled behavior whose $\mathcal{H}_\infty$ norm is at most one (of the transfer function from $d$ to $z$ in the closed loop system). However, a controller of the form $u = \dot{y}$, which is improper, solves the problem. In Section V we show that one can achieve $\mathcal{H}_\infty$ norm equal to $\sqrt{4/7}$, the optimal value, by allowing all finite dimensional LTI controllers: those with proper and improper transfer functions.

Through this observation we notice that for solvability of the $\mathcal{L}_\infty$ control problem over *all* controllers (i.e. proper and improper), more general conditions than those in state space controller design are expected. It is well-known that, in state space $\mathcal{L}_\infty$ optimal control, invariant zeros of the system play an important role in determining the solvability of the problem. It is common to assume that the system has no invariant zeros on the imaginary axis (see [2], [7]). Interestingly, our main result below (Theorem 3.2) is very much reminiscent of the invariant zeros condition (see Section III-A).

Our description of the plant is similar to that in [12]; also see Fig. 1. The system variables are partitioned into exogenous disturbance $d$, to-be-regulated output $z$ and control variable $c$. The variable $c$ includes the control inputs and the available measurements. The full-behavior of the plant is denoted here by $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}+\mathtt{c}}$. The associated plant behavior $\mathcal{P}$ is obtained by eliminating $c$ from $\mathcal{P}_{\text{full}}$. The behavior $\mathcal{P}$ is defined as

$$\mathcal{P} := \left\{ (d, z) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{z}}) | \exists c \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \text{ such that } (d, z, c) \in \mathcal{P}_{\text{full}} \right\}.$$

The control objective is to restrict this plant behavior to a sub-behavior $\mathcal{K} \in \mathcal{L}^{\mathtt{d}+\mathtt{z}}$ to meet the control specifications. In such a formulation of the control problem the controller is allowed to put in restrictions on the control variable $c$ only. In $\mathcal{H}_\infty$ control, the specification is given in terms of the dissipativity on $\mathbb{R}$ of the controlled behavior $\mathcal{K}$ with respect to a real constant matrix

$$\Sigma_\gamma := \begin{bmatrix} \gamma^2 I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{z}} \end{bmatrix} \tag{1}$$

together with internal stability, and $\mathtt{m}(\mathcal{K}) = \sigma_+(\sigma_\gamma)$ (see [12] for a detailed formulation of the problem).

It was shown in [12] that a controlled behavior $\mathcal{K}$, with the controller putting restrictions *only* on the control variables, exists if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$, where $\mathcal{N}$, called the "hidden behavior" is defined as

$$\mathcal{N} := \left\{ (d, z) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{d}+\mathtt{z}}) | (d, z, 0) \in \mathcal{P}_{\text{full}} \right\}. \tag{2}$$

[1]The $\mathcal{L}_\infty$-norm $\|G\|_{\mathcal{L}_\infty}$ of a transfer matrix $G$ is defined as $\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(i\omega))$, where $\sigma_{\max}$ denotes the maximum singular value.

The $\mathcal{L}_\infty$-control problem is called solvable for a plant if there exist a controlled behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{d+z}}_{\text{cont}}$ and $\gamma > 0$ such that

1) $\mathcal{K}$ is $\Sigma_\gamma$ dissipative on $\mathbb{R}$ ($\mathcal{L}_\infty$-control specification);
2) $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ (controller acting only on the $c$-variable);
3) $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma_\gamma)$ (liveness).

Note that condition 3 in the above problem is a maximality requirement: this is in view of [12, Proposition 2, Part I] that for a $\Sigma_\gamma$-dissipative behavior $\mathfrak{B}$, the number of inputs $\mathtt{m}(\mathfrak{B})$ cannot exceed $\sigma_+(\Sigma_\gamma)$. This condition ensures that in the controlled system, $d$ is an input; this allows speaking about the transfer function from $d$ to $z$. Conditions 1 and 3 together mean that the transfer matrix from $d$ to $z$ (see Fig. 1) has $\mathcal{L}_\infty$ norm at most $\gamma$.

We are now in a position to state our main result Theorem 3.2, which provides necessary and sufficient conditions for the $\mathcal{L}_\infty$ suboptimal control problem to be solvable. In the sequel, we assume that the full plant behavior $\mathcal{P}_{\text{full}}$ is given by the following kernel representation:

$$\mathcal{P}_{\text{full}} := \left\{(d, z, c) | R_{\text{d}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)d + R_{\text{z}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z + R_{\text{c}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c = 0\right\}. \quad (3)$$

After elimination, we get a kernel representation of the plant behavior $\mathcal{P} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{d+z}})$ associated with $\mathcal{P}_{\text{full}}$ as

$$\mathcal{P} := \left\{(d, z) | R_{\text{de}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)d + R_{\text{ze}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z = 0\right\}. \quad (4)$$

*Theorem 3.2:* Consider the kernel representation of the full plant behavior as in (3) and assume the associated plant behavior $\mathcal{P}$ is given by (4). Suppose the hidden behavior $\mathcal{N}$ and the plant behavior $\mathcal{P}$ are controllable. Then the $\mathcal{L}_\infty$ control problem is solvable if and only if the following four conditions are satisfied.

1) $R_{\text{z}}(\lambda)$ has full column rank for all $\lambda \in i\mathbb{R}$.
2) There exists a partition of $d$ into $(d_1, d_2)$ such that $d_1$ is input and $(d_2, z)$ is output for $\mathcal{N}$ and the corresponding transfer function from $d_1$ to $(d_2, z)$ is proper.
3) $R_{\text{ze}}(\lambda)$ is full row rank for every $\lambda \in i\mathbb{R}$.
4) There exists a partition of $z$ into $(z_1, z_2)$ such that $(d, z_1)$ is input and $z_2$ is output for $\mathcal{P}$ and the corresponding transfer function from $(d, z_1)$ to $z_2$ is proper.

It is noteworthy that none of the four conditions involve the parameter $\gamma$: they only assure the existence of a finite $\gamma$ such that the $\mathcal{L}_\infty$-control problem is solvable. For a particular $\gamma$, whether the suboptimal control problem is solvable is the problem addressed (in more generality) in [12] (see Proposition 4.1 below). The crucial fact of independence from $\gamma$ suggests that instead of iterating over different $\gamma$ for obtaining $\mathcal{L}_\infty$-solvability, one ought to first verify if the four 'system level properties' to conclude the existence of a sufficiently large $\gamma$. The procedure to calculate the least $\gamma$ is the subject of Section V where we deal with optimal $\mathcal{L}_\infty$ control problem.

### A. Relation With System Invariant Zeros

In this subsection we relate Conditions 1 and 3 (i.e. no loss of rank on the imaginary axis of certain polynomial matrices) in Theorem 3.2 to the notion of 'invariant' zeros of a system. Traditionally, invariance refers to invariance under feedback, i.e. these complex numbers are closed loop poles no matter which feedback controller is used. It is well-known in the state space literature that a system pole which is either uncontrollable or unobservable is invariant with respect to feedback. Note that Conditions 1 and 3 of Theorem 3.2 are nothing but $i\mathbb{R}$-detectability and $i\mathbb{R}$-stabilizability of certain auxiliary behaviors derived from $\mathcal{P}_{\text{full}}$, as defined below. We now elaborate how these two conditions are, in fact, about system invariant zeros on the imaginary axis. A similar relation in the context of state-space/polynomial methods for suboptimal $\mathcal{H}_\infty$-control has been studied in [4], [7].

The following auxiliary behaviors, derived from $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d+z+c}}$, play a role for this purpose:

- $\mathcal{P}_{\text{full,unforced}} := \{(z, c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{z+c}}) | (0, z, c) \in \mathcal{P}_{\text{full}}\}$;
- $\mathcal{K}_{\text{unforced}} := \{z \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{z}}) | (0, z) \in \mathcal{K}\}$.

The 'unforced' here refers to the condition that the external disturbance $d$ is zero in these two behaviors. Further, due to the condition $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$ (which is equal to $\mathtt{d}$) in the problem formulation (see text after (2) above), it turns out that $\mathcal{K}_{\text{unforced}}$ is autonomous, i.e. the input cardinality of $\mathcal{K}_{\text{unforced}}$ is zero. For an autonomous behavior, its poles are defined as the column zeros of any kernel representation matrix. It is known (see [6], for example) that the poles of the autonomous behavior $\mathcal{K}_{\text{unforced}}$ are also the poles of the transfer function from $d$ to $z$ in the behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. Since this transfer function is required to have a bounded $\mathcal{L}_\infty$-norm, neither this transfer function nor $\mathcal{K}_{\text{unforced}}$ are allowed to have any imaginary axis poles. We now come to the relation with system invariant zeros.

Of course, different controlled behaviors $\mathcal{K} \in \mathfrak{L}^{\mathtt{d+z}}$ can be obtained from a given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d+z+c}}$ by attaching different feedback[2] controllers $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ on the control variables $c$. Consider again the equations defining $\mathcal{P}_{\text{full}}$ and $\mathcal{P}$, namely (3) and (4)

$$\mathcal{P}_{\text{full}} := \left\{(d, z, c) | R_{\text{d}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)d + R_{\text{z}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z + R_{\text{c}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c = 0\right\}$$

and

$$\mathcal{P} := \left\{(d, z) | R_{\text{de}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)d + R_{\text{ze}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z = 0\right\}.$$

It turns out that the column zeros of both $R_{\text{z}}$ and $R_{\text{ze}}^T$ are amongst the poles of $\mathcal{K}_{\text{unforced}}$ no matter which feedback controller $\mathcal{C}$ is attached on the control variables: this can be inferred as follows for the case of $R_{\text{ze}}^T$, the case of $R_{\text{z}}$ being analogous and 'dual'.

Consider the minimal kernel representation $R_{\text{de}}(\mathrm{d}/\mathrm{d}t)d + R_{\text{ze}}(\mathrm{d}/\mathrm{d}t)z = 0$ of the plant $\mathcal{P}$. Notice that the control that is possible by a controller $\mathcal{C}_1 \in \mathfrak{L}^{\mathtt{c}}$ that acts on the control variables $c$ to influence the to-be-controlled variables $(d, z)$ cannot[3] be better than that possible by a controller $\mathcal{C}_2 \in \mathfrak{L}^{\mathtt{d+z}}$ that acts on the variables $(d, z)$ directly. Let $\mathcal{C}_2$ have a minimal kernel representation $C_d(\mathrm{d}/\mathrm{d}t)d + C_z(\mathrm{d}/\mathrm{d}t)z = 0$. Due to $\mathcal{C}_2$ being a feedback controller, the matrix $\mathrm{col}(R_{\text{ze}}, C_z)$ is square and nonsingular, further with the roots of its determinant being the poles of $\mathcal{K}_{\text{unforced}}$, and hence the poles of the transfer function from $d$ to $z$ in the controlled system. Notice that the column zeros of $R_{\text{ze}}^T$ are amongst these poles independent of the polynomial matrix $C_z$ and hence independent of the controller $\mathcal{C}_2$. This explains why column zeros of $R_{\text{ze}}^T$ are invariant with respect to every feedback controller that leaves the disturbance free in the closed loop system.

Thus conditions 1 and 3 of Theorem 3.2, in fact, imply that there are no invariant zeros of $\mathcal{P}_{\text{full,unforced}}$ on the imaginary axis. Further, these invariant zeros are unaffected by elementary row operations on the system equations, and hence these are properties of the system, and not of the particular representation we used.

[2]A feedback controller is one which allows its variables to be partitioned such that controller inputs and outputs are respectively plant outputs and inputs. This has been shown to be equivalent to 'regularity' of the controller interconnection in terms of output cardinalities of the plant and controller behaviors adding up to that of the controlled behavior (see [6, Section 10.8.2]). Since this is not the focus of this technical note, we don't dwell further on this topic.

[3]One of the reasons that the influence cannot be the same is because $R_{\mathbf{z}}$ can have column zeros; these column zeros are then inevitable in $\mathcal{K}_{\text{unforced}}$ due to the control action on the $(d, z)$ variables being transmitted through the $c$-variables; this is the dual and analogous situation which we have skipped. These invariant zeros reflect the limitation of having to use the control variables to influence the $(d, z)$ variables.

## IV. Proof of Theorem 3.2

For proof of the first main result of this technical note, Theorem 3.2, we need a few results which we state/prove in this section.

Below is a result about synthesis of dissipative systems, without internal stability, when a supply rate $\Sigma_\gamma$ is given (see [1, Theorem 7.2.1]).

*Proposition 4.1:* Given a $\gamma > 0$, the $\mathcal{L}_\infty$ control problem is solvable if and only if:

- $\mathcal{N}$ is $\Sigma_\gamma$ dissipative on $\mathbb{R}$;
- $\mathcal{P}^\perp$ is $-\Sigma_\gamma^{-1}$ dissipative on $\mathbb{R}$.

We also need a property of polynomial matrices and its column zeros. For a polynomial matrix $R(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ we define the column zeros as follows:

$$\text{colzeros}\,(R(\xi)) := \{\lambda \in \mathbb{C} |\text{ there exists } v \in \mathbb{C}^{\mathtt{q}} \setminus 0 \text{ such that } R(\lambda)v = 0\}.$$

In case $R(\xi)$ is not full column rank, $\text{colzeros}(R(\xi))$ turns out to be the whole of $\mathbb{C}$. Otherwise, it is a finite set. The following lemma (whose proof we skip due to space constraints) relates column zeros of a polynomial matrix and that of its maximal right annihilator[4].

*Lemma 4.2:* Consider $R(\xi) := [R_1(\xi)\ R_2(\xi)]$, with $R_1(\xi) \in \mathbb{R}^{(\mathtt{d}+\mathtt{z}-\ell)\times\mathtt{d}}[\xi]$ and $R_2(\xi) \in \mathbb{R}^{(\mathtt{d}+\mathtt{z}-\ell)\times\mathtt{z}}[\xi]$. Let $M(\xi) := \left[\begin{smallmatrix} M_1(\xi) \\ M_2(\xi)\end{smallmatrix}\right]$, with $M_1(\xi) \in \mathbb{R}^{\mathtt{d}\times\ell}[\xi]$ and $M_2(\xi) \in \mathbb{R}^{\mathtt{z}\times\ell}[\xi]$ be such that $R(\xi)M(\xi) = 0$ and $M(\lambda)$ full column rank for all $\lambda \in \mathbb{C}$. Then,

1) $\text{colzeros}(M_1(\xi)) \subseteq \text{colzeros}(R_2(\xi))$.
2) If $R_2(\xi)$ is full column rank then so is $M_1(\xi)$.
3) If $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, then $\text{colzeros}(R_2(\xi)) = \text{colzeros}(M_1(\xi))$.

*Proof of Theorem 3.2:* "**Only if**": We first assume that the $\mathcal{L}_\infty$ problem is solvable and we show that each of the four conditions listed in Theorem 3.2 are satisfied.

1) The hidden plant behavior is given by $\mathcal{N} = \mathbf{ker}[R_{\mathrm{d}}(\mathrm{d}/\mathrm{d}t)\quad R_{\mathrm{z}}(\mathrm{d}/\mathrm{d}t)]$. Since $\mathcal{N}$ is controllable, it allows an observable image representation $\mathcal{N} = \mathbf{im}[\text{col}(D_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t), Z_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t))]$; $D_{\mathcal{N}}(\xi) \in \mathbb{R}^{\mathtt{d}\times\ell}[\xi], Z_{\mathcal{N}}(\xi) \in \mathbb{R}^{\mathtt{z}\times\ell}[\xi]$. The input cardinality of the hidden behavior is taken to be $\ell$, which is at most $\mathtt{d}$. Let $R_{\mathrm{z}}(\xi)$ lose its rank at $i\omega$, where $\omega \in \mathbb{R}$. Then from Lemma 4.2, in the image representation of $\mathcal{N}$, the matrix $D_{\mathcal{N}}(\xi)$ also loses its rank at $i\omega$. Let $0 \neq v \in \mathbf{ker} D_{\mathcal{N}}(i\omega)$. Then there exists a nonzero periodic trajectory $w = \left[\begin{smallmatrix} 0 \\ Z_{\mathcal{N}}(i\omega)v\end{smallmatrix}\right]e^{i\omega t} \in \mathcal{N}$. Thus the integral
$$\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} w^T\Sigma_\gamma w \mathrm{d}t = \frac{-2\pi\gamma^2 v^* Z_{\mathcal{N}}^T(-i\omega)Z_{\mathcal{N}}(i\omega)v}{\omega} < 0.$$
This implies that $\mathcal{N}$ is not $\Sigma_\gamma$ dissipative on $\mathbb{R}$, which contradicts solvability (see Proposition 4.1). This proves that condition 1 is satisfied.
2) The hidden behavior satisfies $\mathcal{N} \subseteq \mathcal{K}$, which implies that its input cardinality is at most the positive signature of $\Sigma_\gamma$. Therefore in the above image representation $D_{\mathcal{N}}(\xi)$ has at least as many rows as its columns. The fact that $\mathcal{N}$ is $\Sigma_\gamma$ dissipative implies that there exist polynomial matrices $D_{\mathcal{C}}(\xi) \in \mathbb{R}^{\mathtt{d}\times(\mathtt{d}-\ell)}[\xi]$ and $Z_{\mathcal{C}}(\xi) \in \mathbb{R}^{\mathtt{z}\times(\mathtt{d}-\ell)}[\xi]$ such that the behavior given by
$$\mathcal{K} := \text{im}\begin{bmatrix} D_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & D_{\mathcal{C}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ Z_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & Z_{\mathcal{C}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\end{bmatrix}$$
is $\Sigma_\gamma$ dissipative and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma_\gamma)$. This in turn means that the transfer matrix
$$[Z_{\mathcal{N}}(\xi) \quad Z_{\mathcal{C}}(\xi)]\,[D_{\mathcal{N}}(\xi) \quad D_{\mathcal{C}}(\xi)]^{-1}$$
is proper (see [11, Theorem 5.7]). Without loss of generality, we can assume that $[D_{\mathcal{N}}(\xi) \quad D_{\mathcal{C}}(\xi)]$ is column-reduced.[5] Then from [3, Lemma 6.3–11] we conclude that properness of
$$[Z_{\mathcal{N}}(\xi) \quad Z_{\mathcal{C}}(\xi)]\,[D_{\mathcal{N}}(\xi) \quad D_{\mathcal{C}}(\xi)]^{-1}$$
implies that each column of $[Z_{\mathcal{N}}(\xi)\ Z_{\mathcal{C}}(\xi)]$ has degree at most the degree of the corresponding column of $[D_{\mathcal{N}}(\xi)\ D_{\mathcal{C}}(\xi)]$. In particular degree of each column of $Z_{\mathcal{N}}(\xi)$ is at most the degree of the corresponding column of $D_{\mathcal{N}}(\xi)$. This implies that there exists a square block $D_1(\xi)$ in $D_{\mathcal{N}}(\xi)$ such that $\deg\det D_1(\xi)$ is at least the degrees of the determinant of any square block in $\left[\begin{smallmatrix} D_{\mathcal{N}}(\xi) \\ Z_{\mathcal{N}}(\xi)\end{smallmatrix}\right]$. Partitioning $d$ corresponding to this partition of $D_{\mathcal{N}}(\xi)$ meets the requirement in condition 2, thus proving necessity of condition 2.
3) Due to the similar nature of the proofs of necessities of conditions 1 and 3, we only outline the proof for 3). The given kernel representation in (4) is used to obtain an image representation of $\mathcal{P}^\perp$, the orthogonal complement of $\mathcal{P}$. Loss of row rank of $R_{\mathrm{ze}}$ on the imaginary axis results eventually in a nonzero periodic trajectory in $\mathcal{P}^\perp$ that leads to a contradiction to $(-\Sigma_\gamma)^{-1}$-dissipativity for every $\gamma$ of $\mathcal{P}^\perp$ and hence to a contradiction to solvability of the $\mathcal{L}_\infty$ control problem through Proposition 4.1.
4) Using [12, Theorem 5, Part I] and [1, Theorem 3], we first note that $\mathcal{P}^\perp$ is $-(\Sigma_\gamma)^{-1}$ dissipative, and hence this implies $\mathtt{m}(\mathcal{P}^\perp) \leqslant \sigma_+(-\Sigma_\gamma^{-1})$. Following the same line of arguments as in 2) it can be shown that a maximal minor of maximal determinantal degree amongst all maximal minors of $R_{\mathrm{ze}}^T(-\xi)$ is also maximal determinantal degree in $[R_{\mathrm{ze}}^T(-\xi)\ \ R_{\mathrm{ze}}^T(-\xi)]$. Using the partition corresponding to this maximal minor, say $z_1$, leads to a proper transfer function from $(z_2, d)$ to $z_1$. Like done in proof of necessity of Condition 2, this $z_1$ satisfies the requirement. This completes the 'only if' part of the proof of Theorem 3.2, i.e. solvability of the $\mathcal{L}_\infty$ suboptimal control problem for some $\gamma$ implies each of the conditions 1 to 4 are satisfied.

"**If**": Assuming all the four conditions are satisfied we now show that the $\mathcal{L}_\infty$ control problem is solvable. In order to conclude that $\mathcal{L}_\infty$ problem is solvable, we will show that the first two conditions together imply that $\mathcal{N}$ is $\Sigma_\gamma$ dissipative and the last two imply the dissipativity of $\mathcal{P}^\perp$ with respect to $-\Sigma_\gamma^{-1}$.

As done in the proof of necessity of condition 2), we partition $d$ is into $(d_1, d_2)$ such that the transfer function from $d_1$ to $(d_2, z)$ is proper. Corresponding to this partitioning $D_{\mathcal{N}}(\xi)$ can be partitioned as $D_{\mathcal{N}}(\xi) = \left[\begin{smallmatrix} D_1(\xi) \\ D_2(\xi)\end{smallmatrix}\right]$ after a possible permutation of the rows of $D_{\mathcal{N}}(\xi)$ if required. Since condition 1) implies $R_{\mathrm{z}}(i\omega)$ has full column rank for all $\omega \in \mathbb{R}$, from Lemma 4.2 $\det D_1(\xi)$ has no roots on the imaginary axis. Therefore the transfer function from $d_1$ to $(d_2, z)$, namely, $G_{d_1 \to (d_2,z)} := \left[\begin{smallmatrix} D_2(\xi) \\ Z_{\mathcal{N}}(\xi)\end{smallmatrix}\right] D_1^{-1}(\xi)$ is proper and has no poles on the

[4]The Maximal Right Annihilator (MRA) of a full row rank polynomial matrix $R \in \mathbb{R}^{n\times(n+p)}[\xi]$ is a polynomial matrix $M \in \mathbb{R}^{(n+p)\times p}[\xi]$ such that $M(\lambda)$ has rank $p$ for every $\lambda \in \mathbb{C}$.

[5]A square nonsingular matrix is said to be *column-reduced* if the degree of its determinant is equal to the sum of the maximum degrees of each column (see [3] for a detailed exposition). A polynomial matrix, possibly not square, is said to be column-reduced if this matrix forms the columns of a column-reduced square and nonsingular matrix. Column-reduced matrices are also called *column-proper*. We use the property that elementary column operations can be used on a given polynomial matrix to obtain a column-reduced polynomial matrix. See [3, Section 6.3].

imaginary axis. This means it has bounded $\mathcal{L}_\infty$ norm. Define the symmetric nonsingular matrix $\Sigma' := \mathrm{diag}(\gamma_1^2 I_{\mathtt{d}_1}, -I_{\mathtt{d}_2}, -I_{\mathtt{z}})$, where $\gamma_1 \geqslant \|G_{d_1\to(d_2,z)}\|_{\mathcal{L}_\infty}$, then

$$\begin{bmatrix} D_{\mathcal{N}}(-i\omega) \\ Z_{\mathcal{N}}(-i\omega)\end{bmatrix}^T \Sigma' \begin{bmatrix} D_{\mathcal{N}}(i\omega) \\ Z_{\mathcal{N}}(i\omega)\end{bmatrix} \geqslant 0$$

for all $\omega \in \mathbb{R}$. Since $\Sigma_{\gamma_1} \geqslant \Sigma'$, we obtain that $\mathcal{N}$ is dissipative with respect to $\Sigma_{\gamma_1}$ also.

With exactly the same line of arguments conditions 3) and 4) imply that there exists $\gamma_2 > 0$ such that $\mathcal{P}^\perp$ is $-\Sigma_{\gamma_2}^{-1}$ dissipative. Thus by taking $\gamma = \max(\gamma_1, \gamma_2)$, and utilizing Proposition 4.1 we conclude that all the four conditions together imply the solvability of the $\mathcal{L}_\infty$ control problem. This completes the proof of Theorem 3.2. □

## V. $\mathcal{L}_\infty$ Optimal Control

In this section we address the problem of solving the $\mathcal{L}_\infty$ *optimal* control problem, i.e., finding a controller that minimizes the $\mathcal{L}_\infty$ norm of the closed loop system in the configuration of Fig. 1. We first note that none of the four necessary and sufficient conditions in Theorem 3.2 above for solvability of the $\mathcal{L}_\infty$-control problem involve the parameter $\gamma$ explicitly. It tells us that the four conditions are equivalent to existence of a positive real $\gamma$ for which the sub-optimal problem is solvable in the sense of Proposition 4.1. We will see in this section that if the sub-optimal $\mathcal{L}_\infty$ control problem is solvable for some $\gamma$ (equivalently, if the four conditions in Theorem 3.2 are satisfied) then, in fact, the optimal control problem too is solvable. This is in contrast with the results for the state space case (see [8]). The reason behind the difference is that we optimize over the class of proper and *improper* controllers, unlike the state space case where the controller could lose properness at optimality.

Recall the definition of $\Sigma_\gamma$ from (1). In this section $\gamma$ is viewed as a parameter which is to be minimized with the condition that the two key behaviors concerned in Proposition 4.1 are dissipative. We consider the minimum $\gamma$ such that a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is dissipative with respect to $\Sigma_\gamma$. Define

$$\gamma_{\mathcal{N}} := \inf\nolimits_{\gamma\in\mathbb{R}_+}\{\mathcal{N} \text{ is } \Sigma_\gamma \text{ dissipative}\} \tag{5}$$

i.e. the infimum $\gamma$ for which $\mathcal{N}$ is $\Sigma_\gamma$ dissipative. Define $\gamma_{\mathcal{P}}$ as the infimum $\gamma$ such that $\mathcal{P}^\perp$ is $-(\Sigma_\gamma)^{-1}$ dissipative

$$\gamma_{\mathcal{P}} := \inf\nolimits_{\gamma\in\mathbb{R}_+}\left\{\mathcal{P}^\perp \text{ is } -(\Sigma_\gamma)^{-1} \text{ dissipative}\right\}. \tag{6}$$

Note that fulfilment of the four conditions in Theorem 3.2 guarantees that the above mentioned two sets, namely $\{\gamma \in \mathbb{R}_+ | \mathcal{N} \text{ is } \Sigma_\gamma \text{ dissipative}\}$ and $\{\gamma \in \mathbb{R}_+ | \mathcal{P}^\perp \text{ is } -(\Sigma_\gamma)^{-1} \text{ dissipative}\}$ are non-empty. Moreover, except when the infimum is zero, these two sets are closed[6] subsets of $\mathbb{R}_+$. This is because the sets can be thought of as solution sets of nonstrict polynomial inequalities parametrized by $\omega \in \mathbb{R}$, and thus, as (infinite) intersections of closed sets. Therefore, the infima, $\gamma_{\mathcal{N}}$ and $\gamma_{\mathcal{P}}$, exist and are within the above mentioned sets. In other words, whenever the four conditions in Theorem 3.2 are satisfied, $\gamma_{\mathrm{opt}} := \max\{\gamma_{\mathcal{N}}, \gamma_{\mathcal{P}}\}$ turns out to be the smallest $\gamma$ for which the two behaviors $\mathcal{N}$ and $\mathcal{P}^\perp$ are $\Sigma_\gamma$ and $-(\Sigma_\gamma)^{-1}$ dissipative, respectively.

The above discussion and Theorem 3.2 of the previous section, in fact, shows solvability of the optimal $\mathcal{L}_\infty$ problem whenever the four conditions in Theorem 3.2 are satisfied; this leads to the following theorem, which is our second main result of the technical note.

*Theorem 5.1:* Consider $\mathcal{N}$ and $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, the hidden and the plant behaviors of a system. Suppose the $\mathcal{L}_\infty$ control problem is solvable for some $\gamma > 0$, equivalently, the necessary and sufficient conditions listed in Theorem 3.2 are satisfied. Define $\gamma_{\mathcal{N}}$ and $\gamma_{\mathcal{P}}$ as in (5) and (6). Then the $\mathcal{L}_\infty$-optimal control problem is also solvable. The optimal $\gamma$ value is $\gamma_{\mathrm{opt}} = \max(\gamma_{\mathcal{N}}, \gamma_{\mathcal{P}})$

We demonstrate the utility of the above theorem in an example below where we use a $J$-spectral factorization to compute the optimal $\mathcal{L}_\infty$ controller, in addition to calculating $\gamma_{\mathrm{opt}}$. We have chosen a SISO example only for the convenience of demonstration: the results hold for MIMO plants/controllers too. The controller turns out to be what-can-be-called a $PD^2$ controller, an improper transfer function controller.

*Example 5.2:* Consider again the state-space description of the plant in Example 3.1. Consider the problem of finding a controller that takes input $y$, gives output $u$ and minimizes the $\mathcal{L}_\infty$ norm of the transfer function from $d$ to $z$. We demonstrate the procedure for finding the optimal value and obtain equations of the optimal controller for this example.

Elimination (see [6]) of $u$, $y$, $x_1$ and $x_2$ from the above set of equations to obtain an image representation of the plant $\mathcal{P}$ in just the variables $w := (d, z_1, z_2)$ gives $w = M_P(\mathrm{d}/\mathrm{d}t)\ell$ with $M_P(\xi) = \begin{bmatrix} -1 & 0 \\ 1 & \xi^2+\xi+1 \\ 0 & -1\end{bmatrix}$. In order to compute $\gamma_{\mathcal{P}}$, we obtain an image representation of $\mathcal{P}^\perp$ and find the minimum $\gamma$ such that $\mathcal{P}^\perp$ is $-(\Sigma_\gamma)^{-1}$ dissipative (see (6) above). This procedure yields $\gamma_{\mathcal{P}} = \sqrt{4/7}$.

The other candidate for $\gamma_{\mathrm{opt}}$ comes from the hidden behavior $\mathcal{N}$. For this example, we get $\mathcal{N} = 0$, which is dissipative with respect to every supply rate: hence the infimum $\gamma_{\mathcal{N}} = 0$. Thus $\gamma_{\mathrm{opt}}$, the maximum of $\gamma_{\mathcal{P}}$ and $\gamma_{\mathcal{N}}$ is $\sqrt{4/7}$.

The next step is to find a controller that results in the controlled behavior $\mathcal{K}$ being dissipative with respect to this $\gamma$ value. This is obtained using the procedure described in [1]. Obtain a $J$-spectral factorization of $M_P(-\xi)^T\Sigma_\gamma M_P(\xi) =$

$$\begin{bmatrix} -3 & -7\xi^2-7\xi-7 \\ -7\xi^2+7\xi-7 & -7\xi^4-7\xi^2-14\end{bmatrix} = F^T(-\xi)\begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{7}{3}\end{bmatrix}F(\xi) \tag{7}$$

with $F(\xi) = \begin{bmatrix} 3 & 7\xi^2+7\xi+7 \\ 0 & 2\xi^2+1\end{bmatrix}$. Note that the optimality has caused $\det(F)$ to have some imaginary axis roots $\pm j\omega_0$. Define the top row of $F$ as $F_-$; the required controller is

$$w = M_P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \quad \text{with } \ell \text{ satisfying} \quad F_-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0. \tag{8}$$

Dissipativity of $\mathcal{K}$ follows from the $J$-spectral factorization and the controller ensuring that $F_-(\mathrm{d}/\mathrm{d}t)\ell = 0$. We now simplify (8): $F_-(\mathrm{d}/\mathrm{d}t)M_P^\dagger(\mathrm{d}/\mathrm{d}t)w = 0$, where $M_P^\dagger(\xi)$ is a left-inverse of $M_P(\xi)$. Of course, this controller acts on $z_1$, $z_2$ and $d$. However, note that $z_1 = y$ and $z_2 = u$. A left-inverse $M_P^\dagger$ can be chosen to have its last column zero, thus resulting in a 'feedback controller', i.e. a controller whose equation involves only $y$ and $u$, and further, $y$ its input and $u$ its output. Such a left inverse $M_P^\dagger$ can be chosen as

$$M_P^\dagger(\xi) = \begin{bmatrix} 0 & 1 & \xi^2+\xi+1 \\ 0 & 0 & -1\end{bmatrix}.$$

[6]The 'closed' aspect of the interval in which $\gamma$ takes its values is true except when the infimum is zero. This can happen only when the behavior is such that for all its trajectories those variables corresponding to negative signature in the supply rate are identically zero. The zero behavior and the behavior with transfer function 0 are two such extreme examples. The situation that $\gamma = 0$ causes $\Sigma_\gamma$ to be singular, say rank $r$, and this just means that the supply rate $w^T\Sigma_\gamma w$ in $w$ (now, a singular supply rate) penalizes only an $r$-dimensional subspace of linear combinations of the variables $w$: after a projection onto this subspace, one reconsiders the supply rate on a new behavior, with only $r$-variables, for which we again have nonsingularity of the supply rate. This is straightforward and hence not pursued.

The calculation of $F_{-}(\mathrm{d}/\mathrm{d}t)M_P^{\dagger}(\mathrm{d}/\mathrm{d}t)w$ gives $3u = 4((\mathrm{d}^2/\mathrm{d}t^2) + (\mathrm{d}/\mathrm{d}t) + 1)y$ as the corresponding controller.

The significance of the results of this section is that if the measurements are noise-free, then there is no harm, in fact, it can be helpful, to differentiate the measurements and achieve better disturbance attenuation.

## VI. Conclusion

We proved an alternative and easily verifiable set of necessary and sufficient conditions for solvability of the MIMO $\mathcal{L}_\infty$-control problem (Theorem 3.2). One of the prime features of this result is that it brings out the relation of $\mathcal{L}_\infty$ problem solvability with system invariant zeros. Further, the result importantly relaxes properness requirements on the controller's transfer function.

Another feature of Theorem 3.2 is that when these conditions are satisfied, then the theorem states existence of a $\gamma$ sufficiently large for which the suboptimal control problem is solvable. The obvious next step: to compute the minimum $\gamma$ is the one we dealt in Section V. An important conclusion there was that the optimal $\mathcal{L}_\infty$-control problem admits a solution whenever the suboptimal case admits one, in other words, when the four necessary and sufficient conditions are satisfied. We demonstrated the procedure to determine the optimal $\gamma$ value for $\mathcal{L}_\infty$-control and the calculation of a controller using an example: note that controller design packages like Matlab point that inbuilt algorithms cannot work due to violation of regularity assumptions.

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# $H_\infty$ Control of Switched Nonlinear Systems in $p$-Normal Form Using Multiple Lyapunov Functions

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***Abstract*—The problem of $H_\infty$ control of switched nonlinear systems in $p$-normal form is investigated in this technical note where the solvability of the $H_\infty$ control problem for individual subsystems is unnecessary. Using the generalized multiple Lyapunov functions method and the adding a power integrator technique, we design a switching law and construct continuous state feedback controllers of subsystems explicitly by a recursive design algorithm to produce global asymptotical stability and a prescribed $H_\infty$ performance level. Multiple Lyapunov functions are exploited to reduce the conservativeness caused by adoption of a common Lyapunov function for all subsystems, which is usually required when applying the backstepping-like recursive design scheme. An example is provided to demonstrate the effectiveness of the proposed design method.**

***Index Terms*—$H_\infty$ control, multiple Lyapunov functions, $p$-normal form, power integrator, switched systems.**

## I. Introduction

A switched system is a hybrid system which consists of a family of subsystems, either continuous-time or discrete-time subsystems, and a switching law, which defines a specific subsystem that is active at each instant of time. In the last decade, switched systems have received a great amount of attention because of their importance from both theoretical and practical points of view (see, e.g., [7], [8], [12], [17], [18] and the references therein). The motivation for studying switched systems comes partly from the fact that many practical systems are inherently multimodel in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors [6], [9], and many complex nonlinear continuous or discrete systems that are not stabilizable by a single continuous or discrete controller can be stabilized by switching between finitely many controllers [2]–[5]. Meanwhile, several methods, such as common Lyapunov function (CLF), single Lyapunov function, multiple Lyapunov functions (MLFs), and so forth, have been proposed in the study of switched systems.

Turning to non-switched nonlinear systems, the strict-feedback form is a typical system structure for which several effective design approaches are available to solve the stabilization problem (see, [19], [20]). Further, as a generalization of strict-feedback structure, the $p$-normal form has also been extensively studied (see, [24]). However, in switched nonlinear systems, only a few studies have appeared on the strict-feedback form. In [13]–[15], global stabilization for strict-feedback switched nonlinear systems under arbitrary switchings is achieved by constructing a CLF; An adaptive control scheme for strict-feedback switched nonlinear systems with switching jumps and

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