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ON MINIMALITY OF INITIAL DATA REQUIRED TO UNIQUELY CHARACTERIZE EVERY TRAJECTORY IN A DISCRETE n-D SYSTEM*

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Abstract. In this paper, we provide an essentially complete answer to the question of minimal initial data required to solve an overdetermined system of linear partial difference equations with real constant coefficients using the notion of characteristic sets. A characteristic set is a special subset of the domain with the defining property that for every solution trajectory of the system of equations, the knowledge of the solution trajectory restricted to this set uniquely determines the trajectory over the whole domain. We emphasize the fact that subsets which are sublattices and unions of finitely many parallel translates of such sublattices are best suited to answer the question of the minimality of initial data. We first provide an algebraic characterization of a sublattice to be a characteristic sublattice. The main result of this paper provides conditions under which a system admits a union of a sublattice and finitely many parallel translates of it as a characteristic set; an important condition is the rank of the sublattice being equal to the Krull dimension of the system. For the condition when the rank of the sublattice is strictly less than the Krull dimension of the system, we show that neither the sublattice nor a finite union of sublattices can be a characteristic set. For the case when the rank of the sublattice is strictly greater than the Krull dimension of the system, a union of the sublattice and finitely many parallel translates of it is a characteristic set. But, unlike the case when the rank of the sublattice is equal to the Krull dimension of the system, in this case a proper sublattice of the given sublattice exists which along with its finitely many parallel translates now qualify as a characteristic set. We also show that for a given overdetermined system of partial difference equations, a characteristic set of the form given by a union of a sublattice and finitely many parallel translates of it always exists.

Key words. partial difference equations, overdetermined/autonomous systems, characteristic sets, algebraic analysis, minimality, degree of autonomy

AMS subject classifications. 35Exx, 35Nxx, 35N05, 13P25, 68W30

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1. Introduction.

Characterization of minimal initial data for an overdetermined system of partial differential/difference equations has been a long-standing issue that is still largely open [28, 24]. The fundamental nature of this problem is evident from its importance in a plethora of issues concerning systems of partial differential/difference equations. Indeed, starting from the computational problem of obtaining explicit solutions of such systems of equations to theoretical questions like dissipativity [20], stability [19, 25], implementability [4], controller design [23], etc., the issue of initial data plays a crucial role in each of these problems. In this paper, we consider overdetermined systems of linear partial difference equations (pdes) with real constant coefficients having n independent variables; such systems are called *discrete n-D systems*. We approach the problem of initial data, for discrete n-D systems,

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from the perspective of "characteristic sets." Informally, a *characteristic set* is any subset $\mathcal{S}$ of the domain (here $\mathbb{Z}^n$) with the special property that for any solution trajectory, say, w, of the system of partial difference equations, the knowledge of the values w takes on $\mathcal{S}$ lets us extend w to the entire domain $\mathbb{Z}^n$, uniquely [25]. Thus a characteristic set, together with an algorithm (often iterative in nature) that enables computation of the solution from the knowledge of a trajectory on the characteristic set, formalizes the notion of initial data for a system of partial difference equations.

The next obvious question is, how large is this initial data for an overdetermined system of partial difference equations? A partial answer to this question was provided in [28, section 7.1] for systems evolving over $\mathbb{N}^n$. However, as mentioned in [28] and also in the recent paper [24], the problem of characterizing minimal initial data for systems of partial difference equations still remains open. One primary hindrance in achieving a complete answer to this problem is the fact that characteristic sets can come in any arbitrary size and shape. These sets, possibly of completely different constructs, may all have cardinalities equal to the countable infinity, $\aleph_0$ [16]. This fact can obfuscate the question of minimality. In order to overcome this issue and provide an essentially complete answer to the question of minimal initial data, in this paper, we consider a special type of set as a candidate characteristic set for overdetermined systems of partial difference equations. This special class is composed of sets that are sublattices of $\mathbb{Z}^n$ and unions of finitely many parallel translates of such sublattices. The most remarkable feature of such sets is that the issue of minimality has a natural remedy in terms of the rank of the sublattice. Incidentally, such kinds of characteristic sets are also central to various other issues concerning systems of PDEs, namely, time-relevance [3], stability [25], causality [6, 7], and so on. The choice of sublattices as candidates for the initial condition set is not new. In [28, section 7.1] the authors use co-ordinate sublattices to answer the question of largeness of initial data for systems evolving over $\mathbb{N}^n$. However, a characterization of the initial condition set was missing. In this connection, several questions which are still unanswered have been raised in a recent paper [24], the most relevant ones being whether a sublattice can qualify as a characteristic sublattice and whether the sublattice is free. We provide answers to these questions in this paper. We show in this paper that the smallest rank possible for such a characteristic sublattice (or a union of finitely many parallel translates of them) happens to be equal to the *Krull dimension* of the system. Incidentally, n minus the Krull dimension of a system (i.e., the *co-dimension* of the system) is known to be equal to the "degree of autonomy" of the system (see [28, 24, 4]). This important parameter, i.e., the degree of autonomy of an overdetermined system of PDEs, is defined as $n - \ell$, where ℓ is the largest possible rank of a co-ordinate sublattice such that the system restricted to the sublattice is underdetermined. The above-mentioned relation brings out the connection between the characteristic set, which is a finite union of parallel sublattices of minimal rank, and the degree of autonomy. We explore this connection in depth in this paper. The results in this paper are stated using the Krull dimension of the system. Using the relationship that the degree of autonomy is equal to the co-dimension of the system [24, Theorem, p. 417], the results can be interpreted using the degree of autonomy as well.

A preliminary version of a small portion of this paper, namely, results corresponding to characteristic sublattices for the scalar case, have been published in [10]. Results from this paper, for the scalar case, have been further used in [13]. The main contributions of this paper are the following.

1. A characterization for characteristic sublattices is provided (Theorem 3.7) using a variant of the well-known Malgrange's theorem for a system restricted

to a sublattice (Proposition 3.6). This characterization establishes a relationship between the rank of the sublattice and the Krull dimension of the system. In many cases such characteristic sublattices do not exist. Therefore, the notion of finitely many parallel translates of a sublattice is introduced.
2. In Theorem 4.1 we proved that when the rank of the sublattice is equal to the Krull dimension of the system, a union of the sublattice and finitely many parallel translates of it is a characteristic set for the system.
3. For the case when the rank of the sublattice is strictly less than the Krull dimension of the system, neither the sublattice nor finitely many parallel translates of it can be a characteristic set for the system (Theorem 5.1).
4. When the rank of the sublattice is strictly greater than the Krull dimension of the system, a union of the sublattice and finitely many parallel translates of it is a characteristic set (Proposition 6.2). However, the sublattice is not free with respect to the system (Lemma 6.3). It is further shown that a proper sublattice of the given sublattice exists which along with its finitely many parallel translates now qualify as a characteristic set (Theorem 6.6).
5. The existence of such a characteristic set given by a union of a sublattice and finitely many parallel translates of it is always guaranteed for a given system (Theorem 4.18).

In this paper, however, we do not dwell on the issue of freeness of the union of finitely many parallel translates of the sublattice. This is a matter of future investigation. Another important question that remains unanswered in this paper is that of the minimal *number* of parallel translates.

It is important to note here that, for an underdetermined system of PDEs, some of the dependent variables are free [14] and therefore the notion of characteristic set becomes irrelevant in such a scenario. Thus, in this paper, by a system of PDEs we always mean an overdetermined system of linear partial difference equations with real constant coefficients. It is well known that a special class of an overdetermined system of PDEs, namely, systems having Krull dimension equal to zero, admit a collection of finitely many points as a characteristic set (see [25, Lemma 2.4] for the 2-D case and [19, section 4] for the n-D case in the continuous setting). This paper extends this idea for a system having Krull dimension, say, d. In particular, we show that for a system of PDEs having Krull dimension equal to d, there exists a sublattice of rank d (satisfying some additional conditions) such that the sublattice along with finitely many parallel translates of it is a characteristic set for the system. This paper also generalizes the 2-D case [22, 16]. In particular, we generalize the idea in [16], where it was shown that every overdetermined system of PDEs in two independent variables admits a finite union of parallel lines as a characteristic set.

Organization of the paper. In section 2 we discuss the preliminaries and set the notation to be used in the rest of the paper. Section 3 characterizes initial data using characteristic sets. A necessary and sufficient algebraic condition is provided to check if a given sublattice is a characteristic sublattice for a given overdetermined system of linear PDEs with real constant coefficients. From this characterization it turns out that the rank condition, that is, the rank of the sublattice and the Krull dimension of the system, plays an important role. In section 4 we discuss when a union of a sublattice and finitely many parallel translates of it is a characteristic set for an overdetermined system of PDEs; this corresponds to the case when the Krull dimension of the system is equal to the rank of the sublattice. We also show the existence of such a characteristic set for a given system of PDEs. In sections 5 and 6 we discuss the possibilities of a characteristic set for the cases when the rank of the sublattice is strictly less than the Krull dimension of the system and when the rank is strictly greater than the Krull dimension, respectively. We conclude the paper in section 7.

2. Notation and preliminaries.

2.1. Notation. We use the symbols $\mathbb{N}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}$, and $\mathbb{R}$ to denote the set of nonnegative integers, the set of positive integers, the ring of integers, and the field of real numbers, respectively. The sets of n-tuples of nonnegative integers, n-tuples of integers, and n-tuples of real numbers are denoted by $\mathbb{N}^n$, $\mathbb{Z}^n$, and $\mathbb{R}^n$, respectively. The shorthand ξ stands for the n-tuple of indeterminates $\xi_1, \xi_2, \ldots, \xi_n$. Accordingly, we use the symbols $\mathbb{R}[\xi]$ and $\mathbb{R}[\xi, \xi^{-1}]$ to denote the rings of polynomials and Laurent polynomials, in n variables $\xi = (\xi_1, \ldots, \xi_n)$, over the field of real numbers, respectively. For brevity, we define $\mathcal{A} := \mathbb{R}[\xi, \xi^{-1}]$. We use the symbol $\mathfrak{L}^q$ to denote the set of all discrete n-D systems having q dependent variables. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. The symbol $\bullet$ is used for denoting a quantity which is unspecified. For example, $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet \times q}$ indicates that $R(\xi, \xi^{-1})$ is a matrix having entries from $\mathcal{A}$ with q columns and an unspecified number of rows.

2.2. System description. In this paper, we consider systems described by a set of partial difference equations with real constant coefficients having n independent variables; we often refer to such systems as discrete n-D systems, n standing for the number of independent variables. For such a discrete n-D system having q dependent variables, a *trajectory* is a map from the domain (here, $\mathbb{Z}^n$) to the signal space (here, $\mathbb{R}^q$). In other words, a trajectory is a multi-indexed sequence of vectors in $\mathbb{R}^q$, i.e., $w : \mathbb{Z}^n \to \mathbb{R}^q$. We use the symbol $(\mathbb{R}^q)^{\mathbb{Z}^n}$ to denote the set of all q-tuples of real-valued multi-indexed sequences. That is, $(\mathbb{R}^q)^{\mathbb{Z}^n} := \{w : \mathbb{Z}^n \to \mathbb{R}^q\}$. Following Willems [26], we define the *behavior*, $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}^n}$, of a system of partial difference equations as the collection of trajectories that satisfy the set of PDEs. We use the terms behavior and system interchangeably in this paper. By writing $\mathfrak{B} \in \mathfrak{L}^q$, we mean $\mathfrak{B}$ is the behavior of a discrete n-D system (i.e., a system of linear constant real coefficient PDEs having q dependent variables). Likewise, $\mathfrak{B} \in \mathfrak{L}^1$ denotes a discrete n-D system with one dependent variable (also called a *scalar* system).

Linear constant real coefficient PDEs, having n independent variables, are succinctly described using n shift operators $\sigma_1, \sigma_2, \ldots, \sigma_n$. The shift operator in the ith direction, σ_i, acts on a trajectory $w \in (\mathbb{R}^q)^{\mathbb{Z}^n}$ in the following manner:

$$\begin{array}{rrcl} \sigma_i : & (\mathbb{R}^q)^{\mathbb{Z}^n} & \to & (\mathbb{R}^q)^{\mathbb{Z}^n}, \\ & w(k_1, \ldots, k_n) & \mapsto & w(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n). \end{array} \tag{2.1}$$

We denote the n-tuple of shift operators by $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_n)$. Following equation (2.1), for a Laurent monomial $\xi^\nu := \xi_1^{\nu_1} \xi_2^{\nu_2} \ldots \xi_n^{\nu_n}$ in $\mathcal{A}$, the action of σ^ν on a trajectory $w \in (\mathbb{R}^q)^{\mathbb{Z}^n}$ gets defined as

$$(\sigma^\nu w)(k) := w(k_1 + \nu_1, k_2 + \nu_2, \ldots, k_n + \nu_n), \tag{2.2}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$. The action of a Laurent polynomial in the shift operators on a trajectory gets defined by extending (2.2) linearly. Indeed, for a Laurent polynomial $f(\xi, \xi^{-1}) = \sum_{\nu \in \Gamma} \alpha_\nu \xi^\nu$ in $\mathcal{A}$, where $\Gamma \subseteq \mathbb{Z}^n$ is finite and $\alpha_\nu \in \mathbb{R}$, the action of $f(\sigma, \sigma^{-1})$ on a trajectory $w \in (\mathbb{R}^q)^{\mathbb{Z}^n}$ is given by

$$f(\sigma, \sigma^{-1}) w = \sum_{\nu \in \Gamma} \alpha_\nu \sigma^\nu w. \tag{2.3}$$

Thus, a Laurent polynomial maps a trajectory w to another trajectory fw, that is, $f : (\mathbb{R}^q)^{\mathbb{Z}^n} \to (\mathbb{R}^q)^{\mathbb{Z}^n}$. A q-tuple of Laurent polynomials

$$r(\xi,\xi^{-1}) = \begin{bmatrix} r_1(\xi,\xi^{-1}) & r_2(\xi,\xi^{-1}) & \dots & r_q(\xi,\xi^{-1}) \end{bmatrix} \in \mathcal{A}^{1\times q}$$

acts on a trajectory $w \in (\mathbb{R}^q)^{\mathbb{Z}^n}$ in the following manner:

$$r(\sigma,\sigma^{-1})w = \sum_{i=1}^{q} r_i(\sigma,\sigma^{-1})w_i, \tag{2.4}$$

where $w_i \in \mathbb{R}^{\mathbb{Z}^n}$ is the ith component of w. Note that $r(\sigma,\sigma^{-1})w = 0$ corresponds to a discrete n-D system defined by just one pde. For a discrete n-D system defined by a set of, say, p, PDEs have a representation of the form

$$R(\sigma,\sigma^{-1})w = 0, \tag{2.5}$$

where $R(\xi,\xi^{-1}) \in \mathcal{A}^{p\times q}$. Thus, the behavior $\mathfrak{B}$, that is, the collection of trajectories satisfying (2.5), is equal to the kernel of the Laurent polynomial matrix $R(\sigma,\sigma^{-1})$. In other words,

$$\mathfrak{B} := \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}^n} \;\middle|\; R(\sigma,\sigma^{-1})w = 0 \right\} = \ker R(\sigma,\sigma^{-1}). \tag{2.6}$$

For obvious reasons, such a representation is called a *kernel representation* of $\mathfrak{B}$, and $R(\xi,\xi^{-1})$ is called a *kernel representation matrix*. In this paper, we analyze systems algebraically; the following subsection briefly discusses the algebraic entities associated to a discrete n-D system.

2.3. Algebraic entities associated to a system. Given a kernel representation of the system, as in (2.6), we associate with it the *equation module*, $\mathcal{R}$, which is defined as the row span over $\mathcal{A}$ of any kernel representation matrix $R(\xi,\xi^{-1})$, that is, $\mathcal{R} := \text{rowspan}_{\mathcal{A}} R(\xi,\xi^{-1})$. Note that the equation module is a submodule of the free module $\mathcal{A}^{1\times q}$. The behavior $\mathfrak{B}$, as defined in (2.6), is equivalently given by

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}^n} \;\middle|\; f(\sigma,\sigma^{-1})w = 0 \ \forall\ f(\xi,\xi^{-1}) \in \mathcal{R} \right\} =: \mathfrak{B}(\mathcal{R}). \tag{2.7}$$

It was shown in [14, equation 56, p. 34], that submodules of $\mathcal{A}^{1\times q}$ and discrete n-D systems having q dependent variables are in an inclusion reversing one-to-one correspondence with each other.

A behavior $\mathfrak{B}$ defined by a kernel representation, or, equivalently, by an equation module, is closed under addition and multiplication by scalars in $\mathbb{R}$. Thus, $\mathfrak{B}$ has the structure of an $\mathbb{R}$-vector space, too. Further, $\mathfrak{B}$ is also closed under multiplication by scalars from $\mathcal{A}$, where scalar multiplication by an $f \in \mathcal{A}$ to a trajectory $w \in \mathfrak{B}$ is as defined in (2.3). Thus, $\mathfrak{B}$ also has the structure of a module over $\mathcal{A}$. Both of these structures of $\mathfrak{B}$ have been exploited crucially in this paper.

The next important algebraic entity associated to a system is the *quotient module*. Given an equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$, the quotient module, denoted by $\mathcal{M}(\mathcal{R}) := \mathcal{A}^{1\times q}/\mathcal{R}$, is defined as the set of all equivalence classes originating from the following equivalence relation: two elements $f_1(\xi,\xi^{-1}), f_2(\xi,\xi^{-1}) \in \mathcal{A}^{1\times q}$ are related if $f_1 -$

$f_2 \in \mathcal{R}$. For an element $f(\xi, \xi^{-1}) \in \mathcal{A}^{1\times q}$, the equivalence class of f is denoted by $\overline{f}$. Note that $\mathcal{M}(\mathcal{R})$ is naturally an $\mathcal{A}$-module by the operations of addition and scalar multiplication defined on $\mathcal{A}^{1\times q}$. Further, being a module over the $\mathbb{R}$-algebra $\mathcal{A}$, naturally $\mathcal{M}(\mathcal{R})$ has the structure of a vector space over $\mathbb{R}$. We often use only $\mathcal{M}$ to denote the quotient module when $\mathcal{R}$ is clear from the context.

The *canonical surjection* $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$, where every element in $\mathcal{A}^{1\times q}$ is mapped to its equivalence class in $\mathcal{M}$, plays a crucial role in this paper. The action of elements from $\mathcal{M}$ on trajectories in $\mathfrak{B}$ is defined in the following manner: for $m \in \mathcal{M}$ and $w \in \mathfrak{B}$,

$$mw := \left(\widehat{m}(\sigma, \sigma^{-1})w\right), \tag{2.8}$$

where $\widehat{m}(\sigma, \sigma^{-1})$ is a preimage of m under the canonical surjection. Note that m may have several distinct preimages in $\mathcal{A}^{1\times q}$. However, their actions on a trajectory in $\mathfrak{B}$ are the same. This is because if $\widehat{m}_1(\sigma, \sigma^{-1})$ and $\widehat{m}_2(\sigma, \sigma^{-1})$ are two distinct preimages of m, then, by the definition of $\mathcal{M}$, we have $\widehat{m}_1 - \widehat{m}_2 \in \mathcal{R}$. Since $\widehat{m}(\sigma, \sigma^{-1})w = 0$ for all $\widehat{m} \in \mathcal{R}$ it follows that $(\widehat{m}_1 - \widehat{m}_2)\, w = 0$. Thus, both of the preimages define the same action on a trajectory in $\mathfrak{B}$. That is, the action of $\mathcal{M}$ on $\mathfrak{B}$ is well-defined.

Related to a module, we have another algebraic entity called the *annihilator ideal*. For an $\mathcal{A}$-module $\mathcal{M}$, the annihilator ideal is defined as

$$\text{ann } \mathcal{M} := \{f \in \mathcal{A} \mid fm = 0 \text{ for all } m \in \mathcal{M}\}. \tag{2.9}$$

The notion of the *Krull dimension* of rings and modules plays a crucial role in this paper. An ideal $\mathfrak{p} \subseteq \mathcal{A}$ is said to be a *prime ideal* if $\mathfrak{p}$ is not equal to the full ring and for $p_1p_2 \in \mathfrak{p}$ either $p_1 \in \mathfrak{p}$ or $p_2 \in \mathfrak{p}$. The Krull dimension of a ring $\mathcal{A}$ is defined to be the supremum of the lengths of chains of prime ideals in $\mathcal{A}$, where a chain of prime ideals of the form $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_\ell$ is said to be of length ℓ. The Krull dimension of an $\mathcal{A}$-module $\mathcal{M}$ is defined to be the Krull dimension of the quotient ring $\mathcal{A}/\text{ann } \mathcal{M}$ [5, Chapter 9]. That is,

$$\text{Krull dimension } \mathcal{M} := \text{ Krull dimension } (\mathcal{A}/\text{ann } \mathcal{M}). \tag{2.10}$$

The *co-dimension* of a ring or a module is defined to be the global dimension (here, n) minus the Krull dimension of the ring or the module. In this paper, for a behavior $\mathfrak{B} \in \mathfrak{L}^q$ with corresponding quotient module $\mathcal{M}$, we often say that the Krull dimension of $\mathfrak{B}$ is d, by which we mean the Krull dimension of $\mathcal{M}$ is d. On several occasions we use the fact that the Krull dimension remains invariant under isomorphism of rings and modules [5].

2.4. Overdetermined/autonomous systems. In this paper, we consider only overdetermined systems of linear PDEs. By this we mean that we consider those systems that have no *free* variables (see [19] for the definition of free variables). Such systems are also called *autonomous* in the literature. Autonomous/overdetermined systems have been characterized using various equivalent conditions in the literature [21, 19, 25, 29, 27]. We summarize in Proposition 2.1 some important characterizations of autonomous/overdetermined discrete n-D systems.

PROPOSITION 2.1. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D system. Then the following are equivalent:*

1. $\mathfrak{B}$ *is autonomous/overdetermined.*
2. $\mathfrak{B} = \ker\, R(\sigma, \sigma^{-1})$*, where $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$ has full column rank over $\mathcal{A}$ (see* [29, Theorem 2], [27, Theorem 4] *among others for a proof).*

3. *The corresponding quotient module* $\mathcal{M}$ *is a* torsion module[1] *(see* [19, section 4], [27, Theorem 4] *among others for a proof).*
4. *The annihilator ideal,* ann $\mathcal{M}$*, is nonzero (see* [29, Theorem 2], [27, Theorem 4] *among others for a proof).*

3. Characterization of initial data: Characteristic sets. A characteristic set is one of the ways to formalize the notion of initial conditions required to solve an overdetermined system of linear PDEs with real constant coefficients. Informally, these sets are subsets of the domain such that the knowledge of a trajectory in this subset enables one to uniquely extend the trajectory over the whole domain. In the literature, a characteristic set is formally defined using the notion of restriction of trajectories.

Definition 3.1. *Given a trajectory* $w \in (\mathbb{R}^q)^{\mathbb{Z}^n}$ *and a subset* $\mathcal{S} \subseteq \mathbb{Z}^n$*, the* restriction of w to $\mathcal{S}$*, denoted by* $w|_{\mathcal{S}}$*, is defined as*

$$\begin{array}{rccl} w|_{\mathcal{S}}: & \mathcal{S} & \to & \mathbb{R}^q, \\ w|_{\mathcal{S}}(k) & := & w(k) & \text{for all } k \in \mathcal{S}. \end{array} \tag{3.1}$$

Applying Definition 3.1 to every trajectory in $\mathfrak{B}$, we obtain the restriction of $\mathfrak{B}$ to $\mathcal{S}$, denoted as $\mathfrak{B}|_{\mathcal{S}}$. That is,

$$\mathfrak{B}|_{\mathcal{S}} := \{w|_{\mathcal{S}} \text{ such that } w \in \mathfrak{B}\}. \tag{3.2}$$

In the subsequent parts of this paper, $\mathfrak{B}|_{\mathcal{S}}$ plays a crucial role. A characteristic set is defined using the notion of restriction as follows [25].

Definition 3.2. *Given a behavior* $\mathfrak{B} \in \mathfrak{L}^q$*, a subset* $\mathcal{S} \subseteq \mathbb{Z}^n$ *is said to be a* characteristic set *for* $\mathfrak{B}$ *if for every* $w, w' \in \mathfrak{B}$*,*

$$w|_{\mathcal{S}} = w'|_{\mathcal{S}} \Longrightarrow w = w'.$$

In other words, $\mathcal{S} \subseteq \mathbb{Z}^n$ is a characteristic set if and only if for every trajectory $w \in \mathfrak{B}$, the knowledge of $w|_{\mathcal{S}}$ allows us to *uniquely* determine $w|_{\mathbb{Z}^n \setminus \mathcal{S}}$.

It is known that a system admits a proper subset of the domain, $\mathbb{Z}^n$, as a characteristic set if and only if it is an overdetermined/autonomous system [21, 7, 29, 25]. In this paper, we consider a special class of subsets of $\mathbb{Z}^n$ as candidate characteristic sets: sublattices of $\mathbb{Z}^n$ (defined below in Definition 3.3) and unions of finitely many parallel translates of such sublattices. As mentioned in the introduction, the reason for considering these sets is that with these sets we have a natural answer to the question of minimality in terms of the rank of the sublattice (see Theorem 4.1).

In this section, we derive a necessary and sufficient algebraic criterion for a given sublattice to be a characteristic set. A necessary and sufficient condition for a *cone* in $\mathbb{Z}^n$ to be a characteristic cone can be found in [12, Theorem 4]. In what follows, we call a sublattice a *characteristic sublattice* if it is a characteristic set.

3.1. Sublattice and sublattice algebra. In this paper, by a *sublattice* we mean a subset $\mathcal{S} \subseteq \mathbb{Z}^n$ that has the structure of a submodule of the $\mathbb{Z}$-module $\mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a Noetherian module, every sublattice $\mathcal{S}$ is finitely generated as a $\mathbb{Z}$-module. Also, $\mathbb{Z}$ being a principal ideal domain, it follows that a sublattice actually is freely generated as a $\mathbb{Z}$-module [8, Chapter 3, section 7]. In other words, every sublat-

[1]An $\mathcal{A}$-module $\mathcal{M}$ is said to be a torsion module if for every element $m \in \mathcal{M}$, there exists a nonzero element $f \in \mathcal{A}$ such that $fm = 0 \in \mathcal{M}$.

tice is generated by finitely many elements from $\mathbb{Z}^n$, which are linearly independent over $\mathbb{Z}$. The cardinality of a linearly independent generating set is called the *rank* of the sublattice. Thus, we have the following definition.

DEFINITION 3.3. *A subset $\mathcal{S} \subseteq \mathbb{Z}^n$ is called a* sublattice of rank $r \leqslant n$ *if there exists a set $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{Z}^n$, of cardinality r, linearly independent over $\mathbb{Z}$, that generates $\mathcal{S}$ as a $\mathbb{Z}$-module:*

$$\mathcal{S} = \{\lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_r s_r \mid \lambda_1, \ldots, \lambda_r \in \mathbb{Z}\}. \tag{3.3}$$

Remark 3.4. It would be worthwhile to note here that this definition of a sublattice is more general than some existing notions of sublattices in the literature. For example, in [4] sublattices have been defined to be those which are more precisely called *co-ordinate sublattices*. These sublattices are defined in the following manner: let $\Gamma \subseteq \{1, 2, \ldots, n\}$, then define sublattice

$$\mathcal{S}_\Gamma := \{(\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n \mid \nu_k = 0 \text{ for all } k \in \Gamma\}.$$

Clearly, such a sublattice is a special case of the ones defined above in Definition 3.3.

A subalgebra, denoted here by $\mathbb{R}[\mathcal{S}]$, of $\mathcal{A}$ can be naturally associated with a given sublattice $\mathcal{S}$ of $\mathbb{Z}^n$ in the following manner:

$$\mathbb{R}[\mathcal{S}] := \left\{ \sum_{\nu \in \mathcal{S}_1} \alpha_\nu \xi^\nu \;\middle|\; \mathcal{S}_1 \subseteq \mathcal{S},\ |\mathcal{S}_1| < \infty,\ \alpha_\nu \in \mathbb{R} \right\}. \tag{3.4}$$

We call this ring $\mathbb{R}[\mathcal{S}]$, the *sublattice algebra* corresponding to $\mathcal{S}$—this ring plays a crucial role throughout this paper.

3.2. Algebraic characterization of characteristic sublattices. We present the first main result of this paper, Theorem 3.7, in this section. This theorem provides a necessary and sufficient algebraic condition for a given sublattice to be a characteristic sublattice for a given autonomous behavior. In order to get to Theorem 3.7 (and also for developments later in the paper), the following algebraic construction is required.

Given a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, we have the sublattice algebra $\mathbb{R}[\mathcal{S}]$, as defined in (3.4). Consider the free module $\mathbb{R}[\mathcal{S}]^{1\times q}$ over the ring $\mathbb{R}[\mathcal{S}]$. Note that the free module $\mathcal{A}^{1\times q}$ has the structure of an $\mathbb{R}[\mathcal{S}]$-module via the injection $\mathbb{R}[\mathcal{S}] \hookrightarrow \mathcal{A}$. Thus, the natural inclusion map $\psi : \mathbb{R}[\mathcal{S}]^{1\times q} \hookrightarrow \mathcal{A}^{1\times q}$ is an $\mathbb{R}[\mathcal{S}]$-module homomorphism. Now, consider the composite homomorphism $\widetilde{\Psi}$ of $\mathbb{R}[\mathcal{S}]$-modules obtained by the composition of ψ with the canonical surjection $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$, i.e.,

$$\begin{array}{cccccccc} \widetilde{\Psi}: & \mathbb{R}[\mathcal{S}]^{1\times q} & \hookrightarrow & \mathcal{A}^{1\times q} & \twoheadrightarrow & \mathcal{M}, & & \\ & p & \mapsto & p & \mapsto & \overline{p} & = p + \mathcal{R} & =: \widetilde{\Psi}(p). \end{array} \tag{3.5}$$

Since $\widetilde{\Psi}$ is an $\mathbb{R}[\mathcal{S}]$-module homomorphism, $\ker \widetilde{\Psi}$ is a submodule of $\mathbb{R}[\mathcal{S}]^{1\times q}$, and it is easy to verify that $\ker \widetilde{\Psi} = \mathcal{R} \cap \mathbb{R}[\mathcal{S}]^{1\times q}$. Define the quotient module

$$\mathcal{Q} := \frac{\mathbb{R}[\mathcal{S}]^{1\times q}}{\ker \widetilde{\Psi}} = \frac{\mathbb{R}[\mathcal{S}]^{1\times q}}{\mathcal{R} \cap \mathbb{R}[\mathcal{S}]^{1\times q}}. \tag{3.6}$$

Then, the $\mathbb{R}[\mathcal{S}]$-module homomorphism $\Psi : \mathcal{Q} \to \mathcal{M}$, induced by $\widetilde{\Psi}$, is defined in the following manner: for any $f \in \mathcal{Q}$, let $\widehat{f}$ be a preimage of f in $\mathbb{R}[\mathcal{S}]^{1\times q}$ under the surjective $\mathbb{R}[\mathcal{S}]$-module homomorphism $\mathbb{R}[\mathcal{S}]^{1\times q} \twoheadrightarrow \mathcal{Q}$. Then define

$$\Psi : \begin{array}{ccc} \mathcal{Q} & \to & \mathcal{M}, \\ f & \mapsto & \widetilde{\Psi}(\widehat{f}). \end{array} \tag{3.7}$$

It can be easily checked that Ψ is well defined. While it easily follows from the definition that Ψ is injective, Theorem 3.7 below states that in order for $\mathcal{S}$ to be a characteristic sublattice, it is necessary and sufficient that the homomorphism Ψ be *surjective* as well. This gives us the desired algebraic characterization of characteristic sublattices. In order to prove this we need the following two important results: Propositions 3.5 and 3.6. Proposition 3.5 is a variant of the well-known Malgrange's theorem. Proposition 3.6 is a derivative of Malgrange's theorem that applies to the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$ when $\mathcal{S} \subseteq \mathbb{Z}^n$ is a sublattice.

PROPOSITION 3.5. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D system with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. Let $\mathcal{M}$ denote the corresponding quotient module $\mathcal{A}^{1\times q}/\mathcal{R}$. Define the $\mathcal{A}$-module homomorphism $\Phi : \mathfrak{B} \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ as $(\Phi(w))\,(m) := (mw)(0)$ for $m \in \mathcal{M}$. Then Φ is an isomorphism.*

Proof. See the proof of [12, Proposition 6]. ∎

PROPOSITION 3.6. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D system with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and the corresponding quotient module $\mathcal{M}$. Further, let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice. Define the $\mathbb{R}[\mathcal{S}]$-module $\mathcal{Q}$ as in (3.6). Then the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$ is isomorphic to $\mathrm{Hom}_{\mathbb{R}}(\mathcal{Q}, \mathbb{R})$ as $\mathbb{R}[\mathcal{S}]$-modules.*

Proof. We prove this by setting up an $\mathbb{R}[\mathcal{S}]$-linear map $\varphi : \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q}, \mathbb{R}) \to \mathfrak{B}|_{\mathcal{S}}$ in the following manner: let $A \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q}, \mathbb{R})$ be arbitrary. Since $\Psi : \mathcal{Q} \to \mathcal{M}$ is injective, the corresponding dual map $\Psi^* : \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q}, \mathbb{R})$ is surjective.[2] Thus, there exists $\hat{A} \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ such that $\Psi^*(\hat{A}) = A$. According to Proposition 3.5 there exists a unique $w^{\hat{A}} \in \mathfrak{B}$ corresponding to this $\hat{A}$ that satisfies $(\Phi(w^{\hat{A}}))(m) = (mw^{\hat{A}})(0) = \hat{A}(m)$ for all $m \in \mathcal{M}$. Define $\varphi(A) := w^{\hat{A}}|_{\mathcal{S}}$.

(φ is well-defined) Let $\hat{A}_1$ and $\hat{A}_2$ be two distinct elements from $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ such that $\Psi^*(\hat{A}_1) = \Psi^*(\hat{A}_2) = A$. Since Ψ^* is the dual of Ψ, it follows that for all $f \in \mathcal{Q}$, $\hat{A}_1(\Psi(f)) = \hat{A}_2(\Psi(f)) = A(f)$. Further, for any $\nu \in \mathbb{Z}^n$, $i \in \{1, 2, \ldots, q\}$, and $j \in \{1, 2\}$, we have

$$w_i^{\hat{A}_j}(\nu) = \left(\Phi(w^{\hat{A}_j})\right)\left(\overline{\sigma^\nu e_i^T}\right) = \hat{A}_j\left(\overline{\sigma^\nu e_i^T}\right).$$

It then follows that for every $\nu \in \mathcal{S}$, $i \in \{1, 2, \ldots, q\}$, and $j \in \{1, 2\}$,

$$w_i^{\hat{A}_j}(\nu) = \hat{A}_j\left(\overline{\sigma^\nu e_i^T}\right) = A\left(\overline{\sigma^\nu e_i^T}\right),$$

where the last equality follows from the fact that $\overline{\sigma^\nu e_i^T} \in \mathcal{Q}$. Thus $w^{\hat{A}_1}|_{\mathcal{S}} = w^{\hat{A}_2}|_{\mathcal{S}}$, and hence φ is well-defined.

(φ is $\mathbb{R}[\mathcal{S}]$-linear) This is straightforward.

(φ is injective) Let $A \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q}, \mathbb{R})$ be such that $\varphi(A) = w^{\hat{A}}|_{\mathcal{S}} = 0$. This means $w_i^{\hat{A}}(\nu) = 0$ for all $\nu \in \mathcal{S}$ and $i \in \{1, 2, \ldots, q\}$. From the discussion above, it follows that

$$w_i^{\hat{A}}(\nu) = A\left(\overline{\sigma^\nu e_i^T}\right) = 0$$

[2]Since $\mathbb{R}$ is a field, and $\mathcal{M}$ and $\mathcal{Q}$ are vector spaces over $\mathbb{R}$, the functor $\mathrm{Hom}_{\mathbb{R}}(\bullet, \mathbb{R})$ is exact.

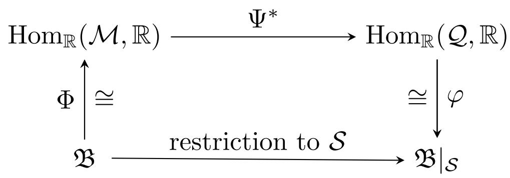

FIG. 1. *Complete commutative diagram.*

for all $\nu \in \mathcal{S}$ and $i \in \{1,2,\ldots,q\}$. But this means $A(f) = 0$ for all $f \in \mathcal{Q}$ because every element in $\mathcal{Q}$ is a finite $\mathbb{R}$-linear combination of monomials of the form $\overline{\sigma^\nu e_i^T}$ with $\nu \in \mathcal{S}$ and $i \in \{1,2,\ldots,q\}$. Thus $A = 0 \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q},\mathbb{R})$. This proves that φ is injective.

(φ is surjective) Note that an arbitrary element from $\mathfrak{B}|_{\mathcal{S}}$ is of the form $w|_{\mathcal{S}}$, where $w \in \mathfrak{B}$. For such an arbitrary element $w|_{\mathcal{S}} \in \mathfrak{B}|_{\mathcal{S}}$ define $A := \Psi^*(\Phi(w)) \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q},\mathbb{R})$, where Φ is as defined in Proposition 3.5 above. We claim that $\varphi(A) = w|_{\mathcal{S}}$. Note that for any $\nu \in \mathcal{S}$ and $i \in \{1,2,\ldots,q\}$

$$w_i|_{\mathcal{S}}(\nu) = w_i(\nu) = \left(\overline{\sigma^\nu e_i^T} w\right)(0) = (\Phi(w))\left(\overline{\sigma^\nu e_i^T}\right).$$

However, since $\overline{\sigma^\nu e_i^T} \in \mathcal{Q}$, we must have

$$w_i|_{\mathcal{S}}(\nu) = (\Phi(w))\left(\overline{\sigma^\nu e_i^T}\right) = \left(\Psi^*(\Phi(w))\right)\left(\overline{\sigma^\nu e_i^T}\right) = A\left(\overline{\sigma^\nu e_i^T}\right).$$

Hence $\varphi(A) = w|_{\mathcal{S}}$. ∎

The results of Propositions 3.5 and 3.6 can be summarized in the commutative diagram (Figure 1); all the maps involved in it are $\mathbb{R}[\mathcal{S}]$-module homomorphisms.

We are now in a position to prove the first main result of this paper.

THEOREM 3.7. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D autonomous system with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Then a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ is a characteristic sublattice for $\mathfrak{B}$ if and only if the homomorphism of $\mathbb{R}[\mathcal{S}]$-modules $\Psi : \mathcal{Q} \to \mathcal{M}$ is surjective.*

Proof. (**If**) Since $\Psi : \mathcal{Q} \to \mathcal{M}$ is surjective, the corresponding dual map $\Psi^* : \mathrm{Hom}_{\mathbb{R}}(\mathcal{M},\mathbb{R}) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q},\mathbb{R})$ is injective. Using the isomorphisms in Propositions 3.5 and 3.6 it follows that $\mathfrak{B} \to \mathfrak{B}|_{\mathcal{S}}$ is injective. By the definition of injectivity, it follows that $w|_{\mathcal{S}} = 0$ implies $w \equiv 0$. Thus $\mathcal{S}$ is a characteristic sublattice for $\mathfrak{B}$.

(**Only If**) Suppose $\mathcal{S}$ is a characteristic sublattice for $\mathfrak{B}$. We need to show that $\Psi : \mathcal{Q} \to \mathcal{M}$ is surjective. This is equivalent to showing that the corresponding dual map $\Psi^* : \mathrm{Hom}_{\mathbb{R}}(\mathcal{M},\mathbb{R}) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{Q},\mathbb{R})$ is injective. Now $\mathcal{S}$ being a characteristic sublattice implies that if $w|_{\mathcal{S}} = 0$, then $w \equiv 0$ (see [25, Lemma 2.3]). Using the isomorphisms in Propositions 3.5 and 3.6, it follows that Ψ^* is injective when $\mathcal{S}$ is a characteristic sublattice. ∎

We illustrate the result of Theorem 3.7 with the help of Example 3.8 below.

Example 3.8. Consider the scalar 3-D autonomous system with kernel representation

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_1^5\sigma_3 + \sigma_2^6 + \sigma_3^7\sigma_2 \\ \sigma_3\sigma_1^{-1} - 1 \end{bmatrix}.$$

The equation ideal is $\mathfrak{a} = \langle \xi_1^5\xi_3 + \xi_2^6 + \xi_3^7\xi_2, \xi_3\xi_1^{-1} - 1\rangle \subseteq \mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}]$. The sublattice $\mathcal{S}$ generated by $\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$ and $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$ is a characteristic sublattice. This is because the monomials ξ_2, ξ_3, ξ_2^{-1}, and ξ_3^{-1} already belong to the sublattice. Simple calculations show that $\xi_1 \equiv \xi_3 \mod \mathfrak{a}$ and $\xi_1^{-1} \equiv \xi_3^{-1} \mod \mathfrak{a}$, where ξ_3 and ξ_3^{-1} belong to $\mathbb{R}[\mathcal{S}]$. Thus the monomials $\{\overline{\xi_1}, \overline{\xi_2}, \overline{\xi_3}, \overline{\xi_1}^{-1}, \overline{\xi_2}^{-1}, \overline{\xi_3}^{-1}\} \subseteq \mathcal{M}$ that generate $\mathcal{M}$ as an $\mathbb{R}$-algebra are in the image of $\Psi : \mathcal{Q} \to \mathcal{M}$. Therefore, Ψ is surjective.

3.3. Nonautonomy of $\mathfrak{B}|_{\mathcal{S}}$. Since $\mathbb{Z}^n$ is a Noetherian module over a principal ideal domain $\mathbb{Z}$, every sublattice $\mathcal{S}$ of $\mathbb{Z}^n$ is a finitely generated free module over $\mathbb{Z}$. This means the sublattice algebra $\mathbb{R}[\mathcal{S}]$ is isomorphic to the r-variable Laurent polynomial ring over $\mathbb{R}$, where r is the rank of $\mathcal{S}$. Indeed, let $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$ be a free generating set of $\mathcal{S}$ and define $\eta_i := \xi^{s_i}$ for $i = 1, 2, \dots, r$. It can be easily checked that the sublattice algebra $\mathbb{R}[\mathcal{S}]$ is then the $\mathbb{R}$-algebra generated by $\{\eta_1, \eta_1^{-1}, \eta_2, \eta_2^{-1}, \dots, \eta_r, \eta_r^{-1}\}$. Note that by virtue of being a free generating set, $\{s_1, s_2, \dots, s_r\}$ is linearly independent over $\mathbb{Z}$, and hence $\{\eta_1, \eta_2, \dots, \eta_r\}$ is algebraically independent over $\mathbb{R}$. Therefore, $\mathbb{R}[\mathcal{S}] = \mathbb{R}[\eta_1, \eta_1^{-1}, \eta_2, \eta_2^{-1}, \dots, \eta_r, \eta_r^{-1}]$ is isomorphic to the r-variable Laurent polynomial ring. It then follows from Propositions 3.5 and 3.6 that the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$ is isomorphic to an r-D behavior. Note that $\mathcal{S}$ being a characteristic sublattice for $\mathfrak{B}$ is equivalent to saying that this r-D behavior $\mathfrak{B}|_{\mathcal{S}}$ is in one-to-one correspondence with the original n-D behavior $\mathfrak{B}$. Clearly, in this scenario, it is desirable that $\mathfrak{B}|_{\mathcal{S}}$ be nonautonomous as an r-D behavior. For if $\mathfrak{B}|_{\mathcal{S}}$ is autonomous, then $\mathfrak{B}|_{\mathcal{S}}$ would admit a proper subset of its domain, i.e., $\mathcal{S}$, as a characteristic set, and by transitivity, that proper subset of $\mathcal{S}$ would be a characteristic set for $\mathfrak{B}$, too. From the perspective of minimality of a characteristic set, it is therefore desirable to have $\mathfrak{B}|_{\mathcal{S}}$ be a nonautonomous r-D behavior.

Definition 3.9. *Given a discrete n-D behavior $\mathfrak{B} \in \mathfrak{L}^q$, with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$, and a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$ is said to be* nonautonomous *if the annihilator ideal of the quotient module $\mathcal{Q} := \mathbb{R}[\mathcal{S}]^{1\times q}/\mathcal{R} \cap \mathbb{R}[\mathcal{S}]^{1\times q}$ as an $\mathbb{R}[\mathcal{S}]$-module is zero, i.e.,*

$$\text{(3.8)} \qquad \operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q} := \{f \in \mathbb{R}[\mathcal{S}] \mid f(\sigma, \sigma^{-1})e_i^T \in \mathcal{R} \text{ for all } 1 \leqslant i \leqslant q\} = \{0\}.$$

Proposition 3.10 below characterizes the property of $\mathfrak{B}|_{\mathcal{S}}$ being nonautonomous in terms of the algebraic entities associated with the original behavior $\mathfrak{B}$.

Proposition 3.10. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and quotient module $\mathcal{M} := \mathcal{A}^{1\times q}/\mathcal{R}$. Further, let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice of rank r. Define the $\mathbb{R}[\mathcal{S}]$-module $\mathcal{Q}$ as in* (3.6). *Then the following are true.*

1. $\operatorname{ann} \mathcal{M} \cap \mathbb{R}[\mathcal{S}] = \operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q}$.
2. *The restricted r-D behavior $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous if and only if* $\operatorname{ann} \mathcal{M} \cap \mathbb{R}[\mathcal{S}] = \{0\}$.

Proof. (1) We first prove $\operatorname{ann} \mathcal{M} \cap \mathbb{R}[\mathcal{S}] \subseteq \operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q}$. Let $f \in \operatorname{ann} \mathcal{M} \cap \mathbb{R}[\mathcal{S}]$. In particular, $f \in \operatorname{ann} \mathcal{M}$, which implies that for any $r \in \mathcal{A}^{1\times q}$, $fr \in \mathcal{R}$. That is, for all $i \in \{1, 2, \dots, q\}$, $fe_i^T \in \mathcal{R}$. Therefore, the row span over $\mathcal{A}$ of the $q \times q$ matrix fI_q is contained in $\mathcal{R}$. Since f also belongs to $\mathbb{R}[\mathcal{S}]$, the row span of fI_q over $\mathbb{R}[\mathcal{S}]$ is contained in $\mathcal{R} \cap \mathbb{R}[\mathcal{S}]^{1\times q}$. This implies $f \in \operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q}$.

To show $\operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q} \subseteq \operatorname{ann} \mathcal{M} \cap \mathbb{R}[\mathcal{S}]$, let $f \in \operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q}$. By the definition of $\operatorname{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{Q}$, for all $i \in \{1, 2, \dots, q\}$, $fe_i^T \in \mathcal{R}$. Therefore, the row span of fI_q over

$\mathcal{A}$ is contained in $\mathcal{R}$. This implies $f \in \text{ann } \mathcal{M}$. By assumption $f \in \mathbb{R}[\mathcal{S}]$ as well. Therefore, $f \in \text{ann } \mathcal{M} \cap \mathbb{R}[\mathcal{S}]$.

(2) Follows from Definition 3.9 and the fact that ann $\mathcal{M} \cap \mathbb{R}[\mathcal{S}] = \text{ann }_{\mathbb{R}[\mathcal{S}]} \mathcal{Q}$. ∎

As noted earlier, characteristic sets formalize the notion of initial data required for solving an overdetermined system of PDEs. Albeit in addition to a characteristic set, one must also be equipped with an algorithm for the extension of a trajectory from its restriction on the characteristic set to the entire domain. Note that Theorem 3.7 above not only gives an algebraic criterion for checking characteristic sublattices, but it also provides a method for doing this extension. Algorithm 3.14 below is adapted from [12, Algorithm 17], [11] to carry out this job.

3.4. Algorithms. In this section, we provide algorithms, based on Gröbner bases, for computing solutions for an overdetermined system of PDEs. This requires us to first specify initial data for a given system of PDEs. We do this using characteristic sets. In particular, for a given system of PDEs we first check if a sublattice is a characteristic sublattice. Once this is done, trajectories restricted to that sublattice form the initial data for the given system of PDEs. Then computing the solution of the given system of PDEs at an arbitrary point in the domain is possible. The algorithms require several important algebraic reductions, such as reducing the Laurent polynomial ring for using Gröbner bases theory (applicable for polynomial rings) and converting Theorem 3.7 to another equivalent algebraic condition (Proposition 3.11) for implementation. We state the constructions briefly here, without proof, as they can be worked out from [12, section 6]. Proofs of the correctness of algorithms are provided that borrow heavily from [12, section 6].

3.4.1. Auxiliary equivalent criterion of the one in Theorem 3.7. In order to obtain implementable algorithms using the theory of Gröbner bases, we convert the algebraic condition of Theorem 3.7 to another equivalent algebraic condition in Proposition 3.11. Note that the theory of Gröbner bases is applicable for polynomial rings. Therefore, it is essential to convert the Laurent polynomial ring to a polynomial ring. The other important algebraic reduction involves the representation of the sublattice algebra as an image of a ring homomorphism. These two reductions play an important role in converting the algebraic condition of Theorem 3.7 to the equivalent condition in Proposition 3.11 below.

Define the $2n$-variable polynomial ring $\mathbb{R}[x, y]$ and the $\mathbb{R}$-algebra homomorphism $\pi : \mathbb{R}[x, y] \twoheadrightarrow \mathcal{A}$ as follows: for $i \in \{1, 2, \ldots, n\}$,

$$\begin{array}{rccc} \pi : & \mathbb{R}[x, y] & \twoheadrightarrow & \mathcal{A}, \\ & x_i & \mapsto & \xi_i, \\ & y_i & \mapsto & \xi_i^{-1}. \end{array} \tag{3.9}$$

Note that ker $\pi = \langle x_1 y_1 - 1, x_2 y_2 - 1, \ldots, x_n y_n - 1 \rangle$. It follows from the first isomorphism theorem [2] that $\mathcal{A} \cong \mathbb{R}[x, y]/\text{ker } \pi$.

To extend $\pi : \mathbb{R}[x, y] \twoheadrightarrow \mathcal{A}$ to corresponding modules, construct the homomorphism of $\mathbb{R}$-algebra modules $\Pi : \mathbb{R}[x, y]^{1\times q} \twoheadrightarrow \mathcal{A}^{1\times q}$ induced[3] by the $\mathbb{R}$-algebra homo-

[3] The scalar multiplication property of the homomorphism obeys the following relation: for $\alpha \in \mathbb{R}[x, y]$, $t \in \mathbb{R}[x, y]^{1\times q}$

$$\begin{array}{rccc} \Pi : & \mathbb{R}[x, y]^{1\times q} & \twoheadrightarrow & \mathcal{A}^{1\times q}, \\ & \Pi(\alpha t) & = & \pi(\alpha)\Pi(t). \end{array} \tag{3.10}$$

morphism $\pi : \mathbb{R}[x,y] \twoheadrightarrow \mathcal{A}$ as follows: for $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,\dots,q\}$,

$$\begin{array}{rccc} \Pi: & \mathbb{R}[x,y]^{1\times q} & \twoheadrightarrow & \mathcal{A}^{1\times q}, \\ & x_i e_j^T & \mapsto & \xi_i e_j^T, \\ & y_i e_j^T & \mapsto & \xi_i^{-1} e_j^T. \end{array} \tag{3.11}$$

The kernel of Π is a submodule of $\mathbb{R}[x,y]^{1\times q}$ and is given by the rowspan of the following matrix:

$$P = \operatorname{diag}\left(\underbrace{\begin{bmatrix} x_1y_1 - 1 \\ \vdots \\ x_ny_n - 1 \end{bmatrix}, \begin{bmatrix} x_1y_1 - 1 \\ \vdots \\ x_ny_n - 1 \end{bmatrix}, \dots, \begin{bmatrix} x_1y_1 - 1 \\ \vdots \\ x_ny_n - 1 \end{bmatrix}}_{q \text{ entries}}\right).$$

That is, $\ker \Pi = \operatorname{rowspan}_{\mathbb{R}[x,y]} P$.

Consider a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ of rank r, generated by $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$. Define the $2r$-variable polynomial ring $\mathbb{R}[t,u]$ and the $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[t,u] \to \mathcal{A}$ as follows: for $i \in \{1,2,\dots,r\}$,

$$\begin{array}{rccc} \Phi: & \mathbb{R}[t,u] & \to & \mathcal{A}, \\ & t_i & \mapsto & \xi^{s_i}, \\ & u_i & \mapsto & \xi^{-s_i}. \end{array} \tag{3.12}$$

It can be shown that the sublattice algebra is given by im Φ. That is, $\mathbb{R}[\mathcal{S}] = \operatorname{im} \Phi$. Let $\Phi^\star : \mathbb{R}[t,u]^{1\times q} \to \mathcal{A}^{1\times q}$ be the homomorphism of $\mathbb{R}$-algebra modules, induced by the $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[t,u] \to \mathcal{A}$. Then, it follows from the scalar counterpart that $\mathbb{R}[\mathcal{S}]^{1\times q} = \operatorname{im} \Phi^\star$. Using the first isomorphism theorem, it then follows that

$$\mathbb{R}[\mathcal{S}]^{1\times q} = \operatorname{im} \Phi^\star \cong \frac{\mathbb{R}[t,u]^{1\times q}}{\ker \Phi^\star},$$

where the isomorphism is between $\mathbb{R}$-algebra modules induced by the $\mathbb{R}$-algebra homomorphism $\Phi : \mathbb{R}[t,u] \to \mathcal{A}$.

Recall that the sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ is generated by $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$. Let $s_{i+} \in \mathbb{N}^n$ denote the n-tuple of nonnegative integers that contains the nonnegative components of s_i with the negative components replaced by zero. Similarly, $s_{i-} \in \mathbb{N}^n$ represents the n-tuple of nonnegative integers that contains the negative of the negative components of s_i with the positive components replaced by zero. For $i \in \{1,2,\dots,r\}$, define $m_i(x,y) := x^{s_{i+}} y^{s_{i-}}$ and $n_i(x,y) := y^{s_{i+}} x^{s_{i-}}$. Using this construction we define the $\mathbb{R}$-algebra homomorphism, $\widehat{\Phi} : \mathbb{R}[t,u] \to \mathbb{R}[x,y]$, as follows: for $i \in \{1,2,\dots,r\}$,

$$\begin{array}{rccc} \widehat{\Phi}: & \mathbb{R}[t,u] & \to & \mathbb{R}[x,y], \\ & t_i & \mapsto & m_i(x,y), \\ & u_i & \mapsto & n_i(x,y). \end{array} \tag{3.13}$$

The vector version of $\widehat{\Phi}$, that is, $\widehat{\Phi}^\star : \mathbb{R}[t,u]^{1\times q} \to \mathbb{R}[x,y]^{1\times q}$, is defined accordingly. The complete commutative diagram is shown in Figure 2.

The algebraic equivalent of Theorem 3.7 is stated in Proposition 3.11. The proof can be worked out from the proof of [12, Proposition 14]. Hence we provide only a brief sketch of the proof here.

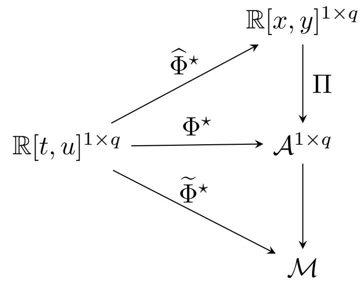

FIG. 2. *Complete commutative diagram showing the reductions.*

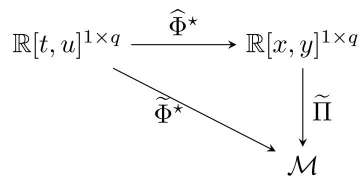

FIG. 3. *Commutative diagram in terms of polynomial rings.*

PROPOSITION 3.11. *Consider the $\mathbb{R}[\mathcal{S}]$-module homomorphism $\Psi : \mathcal{Q} \to \mathcal{M}$ as defined in* (3.7) *and $\widetilde{\Phi}^\star : \mathbb{R}[t,u]^{1\times q} \to \mathcal{M}$ as shown in the commutative diagram of Figure* 2*. Then, Ψ is surjective if and only if $\widetilde{\Phi}^\star$ is surjective.*

Sketch of proof. Let $m \in \mathcal{M}$. Note that by the commutativity of the diagram of Figure 2, $m \in \operatorname{im} \widetilde{\Phi}^\star$ if and only if there exists a preimage of m in $\mathcal{A}^{1\times q}$, denoted by $\widehat{m}$, such that $\widehat{m} \in \operatorname{im} \Phi^\star$. However, note that $\operatorname{im} \Phi^\star = \mathbb{R}[\mathcal{S}]^{1\times q}$. Therefore, $m \in \operatorname{im} \widetilde{\Phi}^\star$ if and only if $\widehat{m} \in \mathbb{R}[\mathcal{S}]^{1\times q}$. Recall from (3.6) and (3.7) that $\widehat{m} \in \mathbb{R}[\mathcal{S}]^{1\times q}$ is equivalent to $\widetilde{\Psi}(\widehat{m})$, i.e., the image of $\widehat{m}$ under the canonical surjection $\widetilde{\Psi} : \mathbb{R}[\mathcal{S}]^{1\times q} \twoheadrightarrow \mathcal{Q}$, satisfying $\Psi(\widetilde{\Psi}(\widehat{m})) = m$. Thus, $m \in \operatorname{im} \widetilde{\Phi}^\star$ if and only if $m \in \operatorname{im} \Psi$. The equivalence of the surjectivity of the two maps, Ψ and $\widetilde{\Phi}^\star$, hence follows immediately. ∎

3.4.2. Details of algorithms and their proofs of correctness. In this section, we provide algorithms to first check whether a given sublattice is a characteristic sublattice. Then using the knowledge of trajectories restricted to a characteristic sublattice we provide an algorithm that computes explicit solutions of a given system of PDEs at an arbitrary point in the domain.

To algorithmically test if a given sublattice is a characteristic sublattice, we use the constructions in subsection 3.4.1. Using Proposition 3.11, checking if a sublattice is a characteristic sublattice is equivalent to checking surjectivity of $\widetilde{\Phi}^\star$. To do this check, using algorithms based on Gröbner bases, we consider the commutative diagram in Figure 3, derived from the commutative diagram in Figure 2. Note that the commutative diagram in Figure 3 circumvents the use of Laurent polynomial rings by using suitable polynomial rings, thereby allowing the use of the theory of Gröbner bases.

The first step in checking whether a given sublattice is a characteristic sublattice for a given system of PDEs is computing a Gröbner basis of the submodule $\mathcal{K}$, as defined in (3.14) below, using an elimination term ordering. This is outlined in Algorithm 3.12.

ALGORITHM 3.12. *Algorithm for computing a Gröbner basis for a given system of PDEs and a given sublattice.*

Input:

1. *The system of equations given in kernel representation as* $\mathfrak{B} = \ker R$*, where* $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$*, forming the equation module* $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$*.*
2. *A sublattice* $\mathcal{S}$ *generated by* $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$*.*

Output: *A Gröbner basis* $\mathcal{G}$ *of the submodule* $\mathcal{K}$ *defined in* (3.14) *below.*

Computation:

1. *Define the* $(2n+2r)$*-variable polynomial ring* $\mathbb{R}[x, y, t, u]$*.*
2. *Define the free module* $\mathbb{R}[x, y, t, u]^{1\times q}$ *and the sub-module* $\mathcal{K} \subseteq \mathbb{R}[x, y, t, u]^{1\times q}$ *as*

$$\mathcal{K} := \widetilde{\mathcal{R}} + \mathcal{T}, \tag{3.14}$$

where $\widetilde{\mathcal{R}} = \mathrm{rowspan}_{\mathbb{R}[x,y,t,u]}\widetilde{R} + \mathrm{rowspan}_{\mathbb{R}[x,y,t,u]}P$*,* $\mathcal{T} = \mathrm{rowspan}_{\mathbb{R}[x,y,t,u]}T$*, with* $\widetilde{R}$*,* P*, and* T *defined as*

$$\widetilde{R} = R(x, y) \in \mathbb{R}[x, y]^{\bullet\times q} \text{ (substituting } \xi \text{ by } x \text{ and } \xi^{-1} \text{ by } y \text{ in } R(\xi, \xi^{-1})),$$

$$P = \mathrm{diag}\left(\underbrace{\begin{bmatrix} x_1y_1 - 1 \\ \vdots \\ x_ny_n - 1 \end{bmatrix}, \begin{bmatrix} x_1y_1 - 1 \\ \vdots \\ x_ny_n - 1 \end{bmatrix}, \dots, \begin{bmatrix} x_1y_1 - 1 \\ \vdots \\ x_ny_n - 1 \end{bmatrix}}_{q \text{ entries}}\right) \in \mathbb{R}[x, y]^{nq\times q},$$

$$T = \mathrm{diag}\left(\underbrace{\begin{bmatrix} t_1 - m_1 \\ \vdots \\ t_r - m_r \end{bmatrix}, \dots, \begin{bmatrix} t_1 - m_1 \\ \vdots \\ t_r - m_r \end{bmatrix}}_{q \text{ entries}}\right) + \mathrm{diag}\left(\underbrace{\begin{bmatrix} u_1 - n_1 \\ \vdots \\ u_r - n_r \end{bmatrix}, \dots, \begin{bmatrix} u_1 - n_1 \\ \vdots \\ u_r - n_r \end{bmatrix}}_{q \text{ entries}}\right).$$

Note that $T \in \mathbb{R}[x, y, t, u]^{rq\times q}$*, and* $m_i := x^{s_{i+}}y^{s_{i-}}$*,* $n_i = y^{s_{i+}}x^{s_{i-}}$ *for* $i \in \{1, 2 \dots, r\}$*.*
3. *Calculate a Gröbner basis*[4] $\mathcal{G} = \{g_1, \dots, g_s\}$ *of* $\mathcal{K}$ *with elimination term ordering* $x \succ y \succ t \succ u$ *and corresponding elimination module term ordering* $\succ_{TOP}$*.*

Proof of correctness. The proof follows from standard results in Gröbner basis theory [1, Chapter 3]. □

ALGORITHM 3.13. *Algorithm for checking if a given sublattice is a characteristic sublattice.*

Input:

1. *The system of equations given in kernel representation as* $\mathfrak{B} = \ker R$*, where* $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$*, forming the equation module* $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$*.*
2. *A sublattice* $\mathcal{S}$ *generated by* $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$*.*

[4] An algorithm for calculating the Gröbner basis of a module can be found in [1, Algorithm 3.5.2].

Output:

1. *Yes, if $\mathcal{S}$ is a characteristic sublattice for $\mathfrak{B}$.*
2. *No, if $\mathcal{S}$ is not a characteristic sublattice.*

Computation:

1. *Use Algorithm* 3.12 *to compute a Gröbner basis $\mathcal{G}$.*
2. *For all $j \in \{1, 2, \ldots q\}$, calculate the remainders of $x_1 e_j^T, \ldots, x_n e_j^T, y_1 e_j^T, \ldots, y_n e_j^T$, where e_j is the jth standard basis (column) vector of $\mathbb{R}^q$, by division with $\mathcal{G}$.*
3. *If $\overline{x_1 e_j^T}^{\mathcal{G}}, \ldots, \overline{x_n e_j^T}^{\mathcal{G}}, \overline{y_1 e_j^T}^{\mathcal{G}}, \ldots, \overline{y_n e_j^T}^{\mathcal{G}} \in \mathbb{R}[t, u]^{1\times q}$ for all $j \in \{1, 2, \ldots q\}$, then $\mathcal{S}$ is a characteristic sublattice for $\mathfrak{B}$.*
4. *If not, then $\mathcal{S}$ is not a characteristic sublattice for $\mathfrak{B}$.*

Proof of correctness. To check if a given sublattice is a characteristic sublattice, Proposition 3.11 is used. The surjectivity of $\widetilde{\Phi}^*$ is checked using a Gröbner basis for $\mathcal{K}$, computed using the elimination term ordering, and the corresponding division-with-remainder algorithm. That is, let $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$ be a Gröbner basis of $\mathcal{K}$ with respect to the elimination module ordering $\succ_{\text{TOP}}$ having the property that $x \succ y \succ t \succ u$. For an element $f(x, y) \in \mathbb{R}[x, y]^{1\times q}$, let $\overline{f}^{\mathcal{G}}$ denote the remainder of f obtained after dividing it by elements of $\mathcal{G}$. Then, $\overline{f}$, the image of f under the canonical surjection $\mathbb{R}[x, y]^{1\times q} \twoheadrightarrow \mathcal{M}$, belongs to im $\widetilde{\Phi}^\star$ if and only if $\overline{f}^{\mathcal{G}} \in \mathbb{R}[t, u]^{1\times q}$ [12, Lemma 15]. Therefore, it follows that $\mathcal{S} \subseteq \mathbb{Z}^n$ is a characteristic sublattice for $\mathfrak{B}$ if and only if the remainders of $x_1 e_j^T, \ldots, x_n e_j^T, y_1 e_j^T, \ldots, y_n e_j^T$ for all $j \in \{1, 2, \ldots q\}$ contain elements only in $\mathbb{R}[t, u]^{1\times q}$ [12, Theorem 16]. ◻

ALGORITHM 3.14. *The algorithm is for computing the solution at an arbitrary point in the domain.*

Input:

1. *The system of equations given in kernel representation as $\mathfrak{B} = \ker R$, where $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$, forming the equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$.*
2. *A sublattice $\mathcal{S}$ generated by $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{Z}^n$.*
3. *The value the trajectory takes on the sublattice, that is, $w|_{\mathcal{S}}$.*
4. *A point $\nu \in \mathbb{Z}^n \setminus \mathcal{S}$ where the value of the trajectory needs to be computed.*

Output: *The trajectory w evaluated at ν, that is, $w(\nu)$.*

Computation:

1. *Use Algorithm* 3.13 *to check if the given sublattice $\mathcal{S}$ is a characteristic sublattice.*
   (a) *If yes, proceed further.*
   (b) *If no, choose a different sublattice and try again.*
2. *Use Algorithm* 3.12 *to compute a Gröbner basis $\mathcal{G}$.*
3. *For the given $\nu \in \mathbb{Z}^n \setminus \mathcal{S}$, write $\nu = \nu_+ - \nu_-$, where $\nu_+, \nu_- \in \mathbb{N}^n$. Calculate the remainders $(x^{\nu_+} y^{\nu_-}) e_j^T$ for all $j \in \{1, 2, \ldots q\}$ by division with $\mathcal{G}$.*
4. *The trajectory evaluated at ν is*

$$w(\nu) = w_1(\nu)e_1 + w_2(\nu)e_2 + \ldots + w_q(\nu)e_q = \sum_{j=1}^{q} w_j(\nu)e_j, \tag{3.15}$$

   where

$$w_j(\nu) = \left(\left(\overline{\sigma^\nu e_j^T}\right) w\right)(0).$$

5. *Let $\overline{(x^{\nu_+} y^{\nu_-}) e_j^T}^{\mathcal{G}} = \begin{bmatrix} f_{j_1} & f_{j_2} & \ldots & f_{j_q} \end{bmatrix}$, where $f_{j_i} \in \mathbb{R}[t, u]$. That is, f_{j_i} is of the form $\sum_{\gamma_1, \gamma_2 \in \Gamma} \alpha_\gamma t^{\gamma_1} u^{\gamma_2}$, where $\alpha_\gamma \in \mathbb{R}$, $\gamma_1, \gamma_2 \in \mathbb{N}^r$, and $|\Gamma| < \infty$.*

6. *Now,* $t^{\gamma_1} = \xi^{\sum_{i=1}^{r} \gamma_{1i} s_i}$ *and* $u^{\gamma_2} = \xi^{-\sum_{i=1}^{r} \gamma_{2i} s_i}$. *Define* $w_{\gamma_{ji}} := \left(\left(f_{j_i} e_i^T\right) w\right)(0)$. *Thus* $w_{\gamma_j} = \begin{bmatrix} w_{\gamma_{j1}} & w_{\gamma_{j2}} & \dots & w_{\gamma_{jq}} \end{bmatrix}^T$.
7. *Using* (3.15) *the trajectory evaluated at* ν *is* $w(\nu) = \sum_{j=1}^{q} w_{\gamma_j} e_j$.

Proof of correctness. The trajectory evaluated at any point in the domain is given by (3.15). This applies to a point $\nu \in \mathbb{Z}^n \setminus \mathcal{S}$ as well. From (3.15), it is clear that to compute $w(\nu)$, we need to calculate

$$w_j(\nu) = \left(\left(\overline{\sigma^{\nu} e_j^T}\right) w\right)(0)$$

for all $j \in \{1, 2, \dots, q\}$. In order to do so, we first write $\nu = \nu_+ - \nu_-$, where $\nu_+, \nu_- \in \mathbb{N}^n$. Then, for all $j \in \{1, 2, \dots q\}$, calculating the remainders of $(x^{\nu_+} y^{\nu_-}) e_j^T$ by division with $\mathcal{G}$, we have $\overline{(x^{\nu_+} y^{\nu_-}) e_j^T}^{\mathcal{G}} = \begin{bmatrix} f_{j_1} & f_{j_2} & \dots & f_{j_q} \end{bmatrix}$, where $f_{j_i} \in \mathbb{R}[t, u]$. This follows from the fact that $\mathcal{S}$ is a characteristic sublattice. Since $\mathcal{S}$ is a characteristic sublattice, it is assumed that $w|_{\mathcal{S}}$ is known. In particular, for all $i \in \{1, 2, \dots, q\}$ and for a fixed $j \in \{1, 2, \dots q\}$ the action of f_{j_i} on a trajectory w can be computed using the knowledge of $w|_{\mathcal{S}}$. We denote this by w_{γ_j}. Repeating this for all $j \in \{1, 2, \dots q\}$, we evaluate the trajectory at ν using $w(\nu) = \sum_{j=1}^{q} w_{\gamma_j} e_j$. ☐

The isomorphism between $\mathcal{Q}$ and $\mathcal{M}$, which is the necessary and sufficient condition proved in Theorem 3.7, also reveals that if $\mathcal{S}$ is to be a characteristic set, then the Krull dimensions of $\mathcal{Q}$ and $\mathcal{M}$ must match. Thus, a necessary condition for $\mathcal{S}$ to be a characteristic sublattice is that the Krull dimension of $\mathcal{Q}$ is the same as that of $\mathcal{M}$. This, however, is not sufficient, as shown in Example 3.15 below.

Example 3.15. Consider the scalar 3-D autonomous system given by

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_1^2\sigma_2^3 + \sigma_2^2 + \sigma_1^{-1} \\ \sigma_3^4 + \sigma_1^{-2}\sigma_3^3 + \sigma_1^2\sigma_3 + 5 \end{bmatrix},$$

where $\mathfrak{a} = \langle \xi_1^2\xi_2^3 + \xi_2^2 + \xi_1^{-1}, \xi_3^4 + \xi_1^{-2}\xi_3^3 + \xi_1^2\xi_3 + 5 \rangle \subseteq \mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}]$ is the equation ideal. It can be checked that the Krull dimension of the quotient ring $\mathcal{M} = \mathcal{A}/\mathfrak{a}$ is equal to one. Consider the sublattice $\mathcal{S}$ of rank 1 given by

$$\mathcal{S} = \operatorname{span}_{\mathbb{Z}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus, the rank of the sublattice is equal to the Krull dimension of $\mathcal{M}$. However, using Algorithm 3.13 it follows that $\mathcal{S}$ is not a characteristic sublattice.

As Example 3.15 has shown, a sublattice often turns out not to be a characteristic set. When a given sublattice is indeed not a characteristic set, we propose in this paper that a natural choice then would be to look at the union of that sublattice with a finitely many parallel translates of the same and ask whether this union of affine sublattices is a characteristic set or not. In the next few sections we answer this question by concentrating on three distinct cases that exhaust all possible cases. These cases are based on the rank of a given sublattice $\mathcal{S}$.

4. Rank of $\mathcal{S}$ is equal to the Krull dimension: Finite union of $\mathcal{S}$ as a characteristic set. This section provides a complete answer to the question posed in [28] regarding how large the initial data is for a discrete autonomous n-D system. In short, we prove Theorem 4.1, which is the second main result of this paper. Recall from Definition 3.1 the notion of restriction of a behavior $\mathfrak{B}$ to a sublattice $\mathcal{S}$ denoted by $\mathfrak{B}|_{\mathcal{S}}$. Another notion that plays a crucial role in Theorem 4.1 is that of a sublattice $\mathcal{S}$ of $\mathbb{Z}^n$ being a *direct summand* of $\mathbb{Z}^n$. A sublattice $\mathcal{S}$ is called a direct summand of $\mathbb{Z}^n$ if there exists another sublattice of $\mathbb{Z}^n$, say, $\mathcal{S}'$, such that $\mathbb{Z}^n = \mathcal{S} \oplus \mathcal{S}'$.

THEOREM 4.1 (vector version). *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that $\mathcal{S}$ is a direct summand of $\mathbb{Z}^n$ and $\mathfrak{B}|_{\mathcal{S}}$ is a nonautonomous system. Further, let the Krull dimension of $\mathcal{M}$ be equal to the rank of $\mathcal{S}$. Then, a union of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ is a characteristic set for $\mathfrak{B}$.*

This result is proved in two stages. We first prove the scalar version of this result and extend it to the general vector case.

4.1. Scalar system. Some modifications are in order to work with scalar autonomous systems $\mathfrak{B} \in \mathfrak{L}^1$. The role of the equation module is played by the *equation ideal* (denoted by $\mathfrak{a}$), while the quotient module $\mathcal{M}$ now becomes the quotient ring $\mathcal{A}/\mathfrak{a}$, that is, $\mathcal{M} = \mathcal{A}/\mathfrak{a}$. For a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, note that $\mathfrak{B}|_{\mathcal{S}}$ is a scalar r-D behavior, where $r = \mathrm{rank}(\mathcal{S})$. Further, it follows from [4, Theorem 6] that, when $\mathcal{S}$ is a direct-summand of $\mathbb{Z}^n$, then the equation ideal of the restricted scalar behavior $\mathfrak{B}|_{\mathcal{S}}$ is $\mathbb{R}[\mathcal{S}] \cap \mathfrak{a}$. It then follows from Proposition 3.10 that the scalar behavior $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous if and only if $\mathbb{R}[\mathcal{S}] \cap \mathfrak{a} = \{0\} \subseteq \mathbb{R}[\mathcal{S}]$. However, note that that means $\mathfrak{B}|_{\mathcal{S}} = \mathbb{R}^{\mathcal{S}}$; in this case, we call $\mathcal{S}$ to be *free* with respect to $\mathfrak{B}$. Further recall that the sublattice $\mathcal{S}$ must also satisfy the rank condition. That is, the rank of $\mathcal{S}$ must be equal to the Krull dimension of $\mathcal{A}/\mathfrak{a}$. Following these observations, we define the notion of a rank-maximally free sublattice with respect to a discrete scalar autonomous n-D system.

DEFINITION 4.2. *A sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ is said to be* rank-maximally free *with respect to a scalar autonomous n-D behavior $\mathfrak{B} \in \mathfrak{L}^1$ if $\mathcal{S}$ is free with respect to $\mathfrak{B}$ (i.e., $\mathfrak{a} \cap \mathbb{R}[\mathcal{S}] = \{0\}$) and* $\mathrm{rank}(\mathcal{S}) = \text{Krull dimension}(\mathcal{A}/\mathfrak{a})$.

Remark 4.3. Note that if a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, which is also a direct summand of $\mathbb{Z}^n$, is rank-maximally free, then for any other sublattice $\widetilde{\mathcal{S}} \subseteq \mathbb{Z}^n$ such that $\mathcal{S} \subseteq \widetilde{\mathcal{S}}$ and $\mathrm{rank}(\mathcal{S}) < \mathrm{rank}(\widetilde{\mathcal{S}})$, we must have $\widetilde{\mathcal{S}}$ to be not free with respect to $\mathfrak{B}$ (see [10, Corollary 12]).

Proposition 4.4 below gives a characterization of rank-maximally free sublattices that is crucially used in this paper.

PROPOSITION 4.4. *Consider a scalar autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^1$ given by equation ideal $\mathfrak{a} \subseteq \mathcal{A}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice. Then $\mathcal{S}$ is rank-maximally free with respect to $\mathfrak{B}$ if and only if $\mathfrak{a} \cap \mathbb{R}[\mathcal{S}] = \{0\}$ and* $\mathrm{rank}(\mathcal{S}) = \text{Krull dimension}(\mathcal{A}/\mathfrak{a})$.

Proof. The proof is straightforward. ∎

Let us now state the scalar version of Theorem 4.1 below.

THEOREM 4.1 (scalar version). *Let $\mathfrak{B} \in \mathfrak{L}^1$ be a discrete scalar autonomous n-D behavior with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$ and corresponding quotient ring $\mathcal{A}/\mathfrak{a}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice that is a direct summand of $\mathbb{Z}^n$. Further, let $\mathcal{S}$ be rank-maximally free with respect to $\mathfrak{B}$. Then, a union of $\mathcal{S}$ and finitely many parallel translates of it is a characteristic set for $\mathfrak{B}$.*

In order to prove the scalar version of Theorem 4.1, we first show that when $\mathcal{S}$ is rank-maximally free with respect to $\mathfrak{B} \in \mathfrak{L}^1$, then the quotient ring $\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful[5] module over the sublattice algebra $\mathbb{R}[\mathcal{S}]$. Then, under the mild assumption that $\mathcal{S}$ is a direct summand of $\mathbb{Z}^n$, we show that $\mathcal{A}/\mathfrak{a}$ being a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$ ensures that a union of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ is a characteristic set for $\mathfrak{B}$. The proofs crucially use the concept of *integrality* and *integral ring extension*. We discuss these ideas briefly here; for a detailed exposition, please refer to [2].

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be rings such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ as a subring. Then an element $\alpha \in \mathcal{A}_2$ is said to be *integral* over $\mathcal{A}_1$ if α satisfies a monic polynomial equation with coefficients from $\mathcal{A}_1$. In other words, there exists a monic $f \in \mathcal{A}_1[x]$ such that $f(\alpha) = 0$. When $\mathcal{A}_1 \subseteq \mathcal{A}_2$ as a subring, $\mathcal{A}_2$ is said to be an *integral extension* of $\mathcal{A}_1$ if every element of $\mathcal{A}_2$ is integral over $\mathcal{A}_1$. The following proposition summarizes the results on integral ring extension required for this paper. (For details and proofs please see [2, Chapter 5].)

PROPOSITION 4.5. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be rings. Further, let $\mathcal{A}_2$ be a finitely generated algebra over $\mathcal{A}_1$, that is, $\mathcal{A}_2 = \mathcal{A}_1[\alpha_1, \alpha_2, \ldots, \alpha_p]$ with $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathcal{A}_2$. Then the following are equivalent.*

1. *$\mathcal{A}_2$ is integral over $\mathcal{A}_1$.*
2. *The elements $\alpha_1, \alpha_2, \ldots, \alpha_p$ are integral over $\mathcal{A}_1$.*
3. *$\mathcal{A}_2$ is a finitely generated module over $\mathcal{A}_1$.*

We show in Theorem 4.7 that when $\mathcal{S}$ is a direct summand and rank-maximally free with respect to $\mathfrak{B} \in \mathfrak{L}^1$, then the quotient ring $\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$. In order to prove this theorem, we need the following auxiliary lemma.

LEMMA 4.6. *Let $\mathfrak{B} \in \mathfrak{L}^1$ be a scalar discrete n-D autonomous behavior with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$ and corresponding quotient ring $\mathcal{A}/\mathfrak{a}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice that is a direct summand of $\mathbb{Z}^n$ and is rank-maximally free with respect to $\mathfrak{B}$. Then, the canonical $\mathbb{R}$-algebra map $\psi^\star : \mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathfrak{a}$ is injective and integral.*

Proof. Injectivity follows by noting that $\mathcal{S}$ is free with respect to $\mathfrak{B}$ if and only if $\mathfrak{a} \cap \mathbb{R}[\mathcal{S}] = \{0\}$ and $\ker \psi^\star = \mathfrak{a} \cap \mathbb{R}[\mathcal{S}]$.

We prove integrality by contradiction. Suppose $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathfrak{a}$ is not integral. Then there exists an element $\xi \in \mathcal{A}/\mathfrak{a}$ transcendental over $\mathbb{R}[\mathcal{S}]$. Therefore, we have the following chain of ring extensions: $\mathbb{R}[\mathcal{S}] \subsetneq \mathbb{R}[\mathcal{S}][\xi] \subseteq \mathcal{A}/\mathfrak{a}$. Since ξ is transcendental over $\mathbb{R}[\mathcal{S}]$, it follows from the dimension theory of rings [5, Corollary 10.13b] that the Krull dimension of $\mathbb{R}[\mathcal{S}][\xi]$ is one more than the Krull dimension of $\mathbb{R}[\mathcal{S}]$. On the other hand, since $\mathbb{R}[\mathcal{S}][\xi] \subseteq \mathcal{A}/\mathfrak{a}$, we have the Krull dimension of $\mathcal{A}/\mathfrak{a}$ to be more than or equal to the Krull dimension of $\mathbb{R}[\mathcal{S}][\xi]$ [5, Corollary 13.5]. Thus

$$\operatorname{rank}(\mathcal{S}) = \text{Krull dim}(\mathbb{R}[\mathcal{S}]) < \text{Krull dim}(\mathbb{R}[\mathcal{S}][\xi]) \leqslant \text{Krull dim}(\mathcal{A}/\mathfrak{a}).$$

This is a contradiction to the assumption that $\operatorname{rank}(\mathcal{S})$ is equal to the Krull dimension of $\mathcal{A}/\mathfrak{a}$. ∎

We now prove, using Lemma 4.6, that the quotient ring $\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$ when $\mathcal{S}$ is a direct summand and is rank-maximally free with respect to a scalar behavior $\mathfrak{B}$.

THEOREM 4.7. *Let $\mathfrak{B} \in \mathfrak{L}^1$ be a scalar discrete n-D autonomous behavior with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$ and corresponding quotient ring $\mathcal{A}/\mathfrak{a}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice*

[5]An $\mathcal{A}$-module $\mathcal{M}$ is said to be a faithful module if $\operatorname{ann} \mathcal{M} = \{0\}$.

that is a direct summand of $\mathbb{Z}^n$ and is rank-maximally free with respect to $\mathfrak{B}$. Then, the quotient ring $\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$.

Proof. Note that $\mathcal{A}/\mathfrak{a}$ is a finitely generated $\mathbb{R}$-algebra. It then trivially follows that $\mathcal{A}/\mathfrak{a}$ is a finitely generated $\mathbb{R}[\mathcal{S}]$-algebra, too. Proposition 4.5 then applies to this situation, whence it follows that $\mathcal{A}/\mathfrak{a}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$ due to integrality of $\mathcal{A}/\mathfrak{a}$ over $\mathbb{R}[\mathcal{S}]$, as shown in Lemma 4.6 above.

Moreover, since $\mathbb{R}[\mathcal{S}]$ injects into $\mathcal{A}/\mathfrak{a}$, the annihilator ideal of $\mathcal{A}/\mathfrak{a}$ as a module over $\mathbb{R}[\mathcal{S}]$ is the zero ideal. Thus $\mathcal{A}/\mathfrak{a}$ is a faithful module over $\mathbb{R}[\mathcal{S}]$. ∎

Recall that since $\mathcal{S}$ is a direct-summand of $\mathbb{Z}^n$, there exists a sublattice $\mathcal{S}' \subseteq \mathbb{Z}^n$ such that $\mathbb{Z}^n = \mathcal{S} \oplus \mathcal{S}'$. This sublattice $\mathcal{S}'$ is called a *complementary* sublattice to $\mathcal{S}$. Since $\mathbb{Z}$ is a principal ideal domain, $\mathcal{S}'$ is also free. Assuming $\operatorname{rank}(\mathcal{S}) = r$, let $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$ be a free generating set for $\mathcal{S}$. Further, let $\mathcal{S}'$ be freely generated by $\{t_1, t_2, \dots, t_{n-r}\} \subseteq \mathbb{Z}^n$. It then follows that the matrix

$$T := \begin{bmatrix} s_1 & s_2 & \dots & s_r & t_1 & t_2 & \dots & t_{n-r} \end{bmatrix} \in \mathbb{Z}^{n\times n}$$

is *unimodular*[6] over $\mathbb{Z}$. Lemma 4.8 is an easy consequence of this observation.

LEMMA 4.8. *Consider $\mathcal{S}, \mathcal{S}' \subseteq \mathbb{Z}^n$, $T \in \mathbb{Z}^{n\times n}$ as defined above. Define $\zeta := \{\zeta_1, \dots, \zeta_r\}$ and $\eta := \{\eta_1, \dots, \eta_{n-r}\}$ in the following manner:*

$$\zeta_i := \xi^{s_i}, \qquad \eta_j := \xi^{t_j}$$

for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n-r$. Then

$$\mathcal{A} = \mathbb{R}[\zeta_1, \zeta_1^{-1}, \dots, \zeta_r, \zeta_r^{-1}, \eta_1, \eta_1^{-1}, \dots, \eta_{n-r}, \eta_{n-r}^{-1}] =: \mathbb{R}[\zeta, \zeta^{-1}, \eta, \eta^{-1}].$$

Proof. To show $\mathcal{A} = \mathbb{R}[\zeta, \zeta^{-1}, \eta, \eta^{-1}]$ it suffices to show that $\mathcal{A} \subseteq \mathbb{R}[\zeta, \zeta^{-1}, \eta, \eta^{-1}]$. For this purpose, it is enough to show that any $\xi^\nu \in \mathcal{A}$, for arbitrary $\nu \in \mathbb{Z}^n$, can be expressed as a monomial in terms of ζ_is and η_js and their inverses. Recall the matrix $T \in \mathbb{Z}^{n\times n}$ defined above, and note that T is unimodular. It then follows that there exist $\kappa := (\kappa_1, \kappa_2, \dots, \kappa_n) \in \mathbb{Z}^n$ such that $\nu = T\kappa$. Thus

$$\begin{aligned} \xi^\nu = \xi^{T\kappa} &= \xi^{\sum_{i=1}^r \kappa_i s_i + \sum_{j=1}^{n-r} \kappa_{r+j} t_j} \\ &= \xi^{\sum_{i=1}^r \kappa_i s_i} \xi^{\sum_{j=1}^{n-r} \kappa_{r+j} t_j} \\ &= \prod_{i=1}^r \zeta_i^{\kappa_i} \prod_{j=i}^{n-r} \eta_j^{\kappa_{r+j}}. \end{aligned}$$ ∎

We now state Corollary 4.9, which follows directly from Theorem 4.7 and Lemma 4.8. For discrete 2-D autonomous systems, an analogous result was proved in [17, Proposition 3.4].

COROLLARY 4.9. *Let $\mathfrak{B} \in \mathfrak{L}^1$ be a discrete scalar autonomous n-D system with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$ and quotient ring $\mathcal{A}/\mathfrak{a}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice, which is rank-maximally free with respect to $\mathfrak{B}$, and also is a direct summand of $\mathbb{Z}^n$. Let $\mathcal{S}' \subseteq \mathbb{Z}^n$ be a complementary sublattice of $\mathcal{S}$ such that $\mathbb{Z}^n = \mathcal{S} \oplus \mathcal{S}'$. Let $\{s_1, s_2, \dots, s_r\} \subseteq \mathbb{Z}^n$ and $\{t_1, t_2, \dots, t_{n-r}\} \subseteq \mathbb{Z}^n$ be free generating sets for $\mathcal{S}$, and $\mathcal{S}'$, respectively, over $\mathbb{Z}$. Define ζ, η as done in Lemma 4.8. Then the following are true:*

[6] An integer matrix with determinant ± 1 is known as a *unimodular* matrix.

1. *$\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$.*
2. *For every $1 \leqslant i \leqslant n-r$ there exists $d_i \in \mathbb{Z}_{>0}$, such that*

$$p_i(\zeta,\eta_i) = \eta_i^{d_i} + \alpha_{i,d_i-1}(\zeta)\eta_i^{d_i-1} + \ldots + \alpha_{i,1}(\zeta)\eta_i + \alpha_{i,0}(\zeta) \in \mathfrak{a}, \tag{4.1}$$

where $\alpha_{i,j}(\zeta) \in \mathbb{R}[\mathcal{S}]$ for every $i \in \{1,\ldots,n-r\}$ and $j \in \{1,\ldots,d_i-1\}$, with $\alpha_{i,0}(\zeta)$ a unit for all $i \in \{1,\ldots,n-r\}$.

Proof. (1) This is restatement of Theorem 4.7.

(2) Note that it follows from the definition of ζ that $\mathbb{R}[\mathcal{S}] = \mathbb{R}[\zeta,\zeta^{-1}]$. Now, from Proposition 4.5, we have that $\mathcal{A}/\mathfrak{a}$ being a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$ implies that the $\mathbb{R}[\mathcal{S}]$-algebra homomorphism $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathfrak{a}$ is injective and integral. Using Lemma 4.8 and the fact that $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathfrak{a}$ is integral, it follows that for all $1 \leqslant i \leqslant n-r$, every η_i and η_i^{-1} satisfy monic polynomial equations modulo $\mathfrak{a}$ with coefficients from $\mathbb{R}[\mathcal{S}]$. In other words, for all $1 \leqslant i \leqslant n-r$, there exists $\ell_i \in \mathbb{Z}_{>0}$, such that

$$q_{1_i}(\zeta,\eta_i) = \eta_i^{\ell_i} + \beta_{i,\ell_i-1}(\zeta)\eta_i^{\ell_i-1} + \cdots + \beta_{i,1}(\zeta)\eta_i + \beta_{i,0}(\zeta) \in \mathfrak{a} \quad \text{and}$$
$$q_{2_i}(\zeta,\eta_i) = \eta_i^{-\ell_i} + \rho_{i,\ell_i-1}(\zeta)\eta_i^{-\ell_i+1} + \cdots + \rho_{i,1}(\zeta)\eta_i^{-1} + \rho_{i,0}(\zeta) \in \mathfrak{a},$$

where $\beta_{i,j}(\zeta) \in \mathbb{R}[\mathcal{S}]$ and $\rho_{i,j}(\zeta) \in \mathbb{R}[\mathcal{S}]$ for every $i \in \{1,\ldots,n-r\}$ and $j \in \{0,\ldots,\ell_i-1\}$. Note that $\beta_{i,0}(\zeta)$ and $\rho_{i,0}(\zeta)$ may not be units in $\mathbb{R}[\mathcal{S}]$. To get to (4.1), define $p_i(\zeta,\eta_i) := \eta_i q_{1_i}(\zeta,\eta_i) + \eta_i^{\ell_i} q_{2_i}(\zeta,\eta_i)$. Since $\mathfrak{a}$ is an ideal $p_i(\zeta,\eta_i) \in \mathfrak{a}$. Note that $d_i := \ell_i + 1$, $\alpha_{i,j}(\zeta) = \beta_{i,j-1}(\zeta) + \rho_{i,l_i-j}(\zeta)$, where $1 \leqslant j \leqslant \ell_i$ and $\alpha_{i,0} = 1$, which is a unit in $\mathbb{R}[\mathcal{S}]$. ∎

Using Corollary 4.9, the following lemma gives an explicit list of generators for $\mathcal{A}/\mathfrak{a}$ as a module over $\mathbb{R}[\mathcal{S}]$.

LEMMA 4.10. *Consider a discrete scalar autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^1$ with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$ and corresponding quotient ring $\mathcal{A}/\mathfrak{a}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice of rank r, which is rank-maximally free with respect to $\mathfrak{B}$ and is a direct summand of $\mathbb{Z}^n$. Then, there exist $t_1,\ldots,t_{n-r} \in \mathbb{Z}^n$, and $d_1,\ldots,d_{n-r} \in \mathbb{Z}_{>0}$ such that the following finite subset of $\mathcal{A}/\mathfrak{a}$,*

$$\mathcal{G} := \left\{ \overline{\xi^\nu} \mid \nu \in \Gamma \right\}, \tag{4.2}$$

where

$$\Gamma := \left\{ \sum_{i=1}^{n-r} \pi_i t_i \;\middle|\; 0 \leqslant \pi_i \leqslant d_i - 1 \right\}, \tag{4.3}$$

is a generating set of $\mathcal{A}/\mathfrak{a}$ as a module over $\mathbb{R}[\mathcal{S}]$.

Proof. Since $\mathcal{S}$ is rank-maximally free, it follows from Corollary 4.9 that $\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful module over the sublattice algebra $\mathbb{R}[\mathcal{S}]$. Further, since $\mathcal{S}$ is assumed to be direct summand of $\mathbb{Z}^n$, it follows that there exists a complementary sublattice $\mathcal{S}'$ such that $\mathcal{S} \oplus \mathcal{S}' = \mathbb{Z}^n$. As done in Lemma 4.8, let $\{s_1,\ldots,s_r\} \subseteq \mathbb{Z}^n$ and $\{t_1,\ldots,t_{n-r}\} \subseteq \mathbb{Z}^n$ be free generating sets for $\mathcal{S}$ and $\mathcal{S}'$, respectively, as $\mathbb{Z}$-modules. Note that Lemma 4.8 applies in this case, and hence $\mathcal{A} = \mathbb{R}[\zeta,\zeta^{-1},\eta,\eta^{-1}]$, where the r-tuple of monomials $\zeta = \{\zeta_1,\ldots,\zeta_r\}$ and the $(n-r)$-tuple of monomials $\eta = \{\eta_1,\ldots,\eta_{n-r}\}$ are as defined in Lemma 4.8. Therefore, every Laurent polynomial in $\mathcal{A}$ can be rewritten as a finite linear combination of monomials in ζ,η; these monomials look like $\zeta^\mu\eta^\pi$, where $\mu \in \mathbb{Z}^r$ and $\pi \in \mathbb{Z}^{n-r}$. We prove the statement of the lemma by showing that the image under the canonical surjection $\mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{a}$ of every monomial of the form $\zeta^\mu\eta^\pi$, where $\mu \in \mathbb{Z}^r$ and $\pi \in \mathbb{Z}^{n-r}$, is a finite linear combination of elements from the set $\mathcal{G}$ (defined in (4.2) above) with coefficients coming from $\mathbb{R}[\mathcal{S}]$. We do this by categorizing these monomials into three classes.

Class 1 (monomials of the form ζ^μ with $\mu \in \mathbb{Z}^r$). Such a monomial is already an element from $\mathbb{R}[\mathcal{S}]$. Therefore, these monomials are trivially expressible as linear combinations of elements in $\mathcal{G}$, namely, $\overline{1}$, over $\mathbb{R}[\mathcal{S}]$.

Class 2 (monomials of the form $\eta_i^{\pi_i}$ with $\pi_i \in \mathbb{Z}_{>0}$ for any $i = 1, 2, \ldots, n-r$). Since $\mathcal{S}$ is rank-maximally free with respect to $\mathfrak{B}$, it follows that $\mathcal{A}/\mathfrak{a}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$. Therefore, according to Corollary 4.9, for all $1 \leqslant i \leqslant n-r$ there exists $d_i \in \mathbb{Z}_{>0}$ such that

$$p_i(\zeta, \eta_i) := \eta_i^{d_i} + \alpha_{i,d_i-1}(\zeta)\eta_i^{d_i-1} + \cdots + \alpha_{i,1}(\zeta)\eta_i + \alpha_{i,0}(\zeta) \in \mathfrak{a}, \tag{4.4}$$

where $\alpha_{i,j}(\zeta) \in \mathbb{R}[\mathcal{S}]$ for every $i \in \{1, \ldots, n-r\}$ and $j \in \{1, \ldots, d_i - 1\}$. Now, given any monomial of the form $\eta_i^{\pi_i}$ with $\pi_i \in \mathbb{Z}_{>0}$, it follows that one can carry out Euclidean division on $\eta_i^{\pi_i}$ by $p_i(\zeta, \eta_i)$ because $p_i(\zeta, \eta_i)$ is a monic polynomial in $\mathbb{R}[\mathcal{S}][\eta_i]$. The result of this Euclidean division is as follows:

$$\eta_i^{\pi_i} - \rho_{i,\pi_i}(\zeta, \eta_i) = q_i(\zeta, \eta_i)p_i(\zeta, \eta_i) \in \mathfrak{a}, \tag{4.5}$$

where the remainder $\rho_{i,\pi_i}(\zeta, \eta_i)$ is an $\mathbb{R}[\mathcal{S}]$-linear combination of $\{1, \eta_i, \eta_i^2, \ldots, \eta_i^{d_i-1}\}$. Therefore, going modulo $\mathfrak{a}$ we find that

$$\overline{\eta_i^{\pi_i}} = \overline{\rho_{i,\pi_i}(\zeta, \eta_i)} \in \mathcal{A}/\mathfrak{a}. \tag{4.6}$$

Thus, for $1 \leqslant i \leqslant n-r$, every monomial of the form $\overline{\eta_i^{\pi_i}}$ with $\pi_i \in \mathbb{Z}_{>0}$ is equal to an $\mathbb{R}[\mathcal{S}]$-linear combination of $\{\overline{1}, \overline{\eta_i}, \overline{\eta_i^2}, \ldots, \overline{\eta_i^{d_i-1}}\}$.

Class 3 (monomials of the form $\eta_i^{-\pi_i}$ with $\pi_i \in \mathbb{Z}_{>0}$ for any $i = 1, 2, \ldots, n-r$). Arguing like above till (4.4), we now note from Corollary 4.9 that $\alpha_{i,0}(\zeta)$ is a unit in $\mathbb{R}[\mathcal{S}]$. Hence, multiplying both sides of (4.4) by $\alpha_{i,0}(\zeta)^{-1}\eta_i^{-1}$ we get that

$$\alpha_{i,0}(\zeta)^{-1}\eta_i^{-1}p_i(\zeta, \eta_i) = \frac{\eta_i^{d_i-1}}{\alpha_{i,0}(\zeta)} + \frac{\alpha_{i,d_i-1}(\zeta)\eta_i^{d_i-2}}{\alpha_{i,0}(\zeta)} + \cdots + \frac{\alpha_{i,1}(\zeta)}{\alpha_{i,0}(\zeta)} + \eta_i^{-1} \in \mathfrak{a}, \tag{4.7}$$

in other words, $\overline{\eta_i^{-1}}$ is equal, in $\mathcal{A}/\mathfrak{a}$, to a finite linear combination of monomials from Class 2 above. By raising both sides of (4.7) to higher positive powers, it follows that every monomial of the form $\overline{\eta_i^{-\pi_i}}$ is equal, in $\mathcal{A}/\mathfrak{a}$, to a finite linear combination of monomials from Class 2 above. Hence, from the conclusion of the analysis of monomials in Class 2, we infer that every monomial of the form $\overline{\eta_i^{-\pi_i}}$ is equal to an $\mathbb{R}[\mathcal{S}]$-linear combination of $\{\overline{1}, \overline{\eta_i}, \overline{\eta_i^2}, \ldots, \overline{\eta_i^{d_i-1}}\}$.

As mentioned at the beginning of this proof, every monomial in $\mathcal{A}$ is a finite product of monomials from the above-mentioned three classes. It then follows from the analysis presented above that the image of a typical monomial in $\mathcal{A}$ under the canonical surjection $\mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{a}$ is an $\mathbb{R}[\mathcal{S}]$-linear combination of monomials that themselves are products of monomials $\{\overline{1}, \overline{\eta_i}, \overline{\eta_i^2}, \ldots, \overline{\eta_i^{d_i-1}}\}$, in other words, $\mathbb{R}[\mathcal{S}]$-linear combinations of monomials from the set

$$\mathcal{G}_\eta := \left\{ \overline{\prod_{i=1}^{n-r} \eta_i^{\pi_i}} \mid 0 \leqslant \pi_i \leqslant d_i - 1 \right\}. \tag{4.8}$$

From the definition of ζ, η, however, it follows that

$$\mathcal{G}_\eta = \mathcal{G}.$$

Therefore, the image of a typical monomial in $\mathcal{A}$ under the canonical surjection $\mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{a}$ is an $\mathbb{R}[\mathcal{S}]$-linear combination of monomials from $\mathcal{G}$. Since every element in $\mathcal{A}/\mathfrak{a}$ is a finite $\mathbb{R}$-linear combination of images of monomials in $\mathcal{A}$, it follows that every element in $\mathcal{A}/\mathfrak{a}$ is an $\mathbb{R}[\mathcal{S}]$-linear combination of monomials from $\mathcal{G}$. ∎

Lemma 4.10 gives an explicit list of generators for $\mathcal{A}/\mathfrak{a}$ as a module over $\mathbb{R}[\mathcal{S}]$. This forms an integral part of the proof of Theorem 4.1, which we now provide.

THEOREM 4.1 (scalar version). *Let $\mathfrak{B} \in \mathfrak{L}^1$ be a discrete scalar autonomous n-D behavior with equation ideal $\mathfrak{a} \subseteq \mathcal{A}$ and corresponding quotient ring $\mathcal{A}/\mathfrak{a}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice that is a direct summand of $\mathbb{Z}^n$. Further, let $\mathcal{S}$ be rank-maximally free with respect to $\mathfrak{B}$. Then, a union of $\mathcal{S}$ and finitely many parallel translates of it is a characteristic set for $\mathfrak{B}$.*

Proof. By Lemma 4.10 there exist $\{t_1, \ldots, t_{n-r}\} \subseteq \mathbb{Z}^n$ with $r = \text{rank}(\mathcal{S})$ and $d_i \in \mathbb{Z}_{>0}$ such that $\mathcal{A}/\mathfrak{a}$ is finitely generated as an $\mathbb{R}[\mathcal{S}]$-module by the elements from the following set:

$$\mathcal{G} = \left\{ \overline{\xi^\nu} \mid \nu \in \Gamma \right\},$$

where

$$\Gamma = \left\{ \sum_{i=1}^{n-r} \pi_i t_i \,\middle|\, 0 \leqslant \pi_i \leqslant d_i - 1 \right\}.$$

Given $\mathcal{S} \subseteq \mathbb{Z}^n$, we define a parallel translate of $\mathcal{S}$ by a $\nu \in \mathbb{Z}^n$, and denote it by $\mathcal{S}_\nu$, as follows:

$$\mathcal{S}_\nu := \nu + \mathcal{S}. \tag{4.9}$$

Now define

$$\mathcal{C} := \bigcup_{\nu \in \Gamma} \mathcal{S}_\nu.$$

Note that since Γ is a finite set, the set $\mathcal{C}$ as defined above is a finite union of parallel translates of $\mathcal{S}$. Also note that since $0 \in \Gamma$, we must have $\mathcal{S} \subseteq \mathcal{C}$. We claim that $\mathcal{C}$ is a characteristic set of $\mathfrak{B}$.

In order to prove the claim, note that it is sufficient that we prove for any $w \in \mathfrak{B}$, $w|_\mathcal{C} \equiv 0$ implies that $w \equiv 0$ [25, Lemma 2.3]. For this purpose, let $\kappa \in \mathbb{Z}^n$ be arbitrary. We want to show that $w|_\mathcal{C} \equiv 0$ implies that $w(\kappa) = 0$. Recall that for any $w \in \mathfrak{B}$, we must have $w(\kappa) = \left(\overline{\sigma^\kappa} w\right)(0)$. From Lemma 4.10, however, we can write that

$$\overline{\xi^\kappa} = \sum_{\nu \in \Gamma} \alpha_\nu(\xi) \overline{\xi^\nu},$$

where $\alpha_\nu(\xi) \in \mathbb{R}[\mathcal{S}]$. Therefore,

$$w(\kappa) = \left(\overline{\sigma^\kappa} w\right)(0) = \sum_{\nu \in \Gamma} \alpha_\nu(\sigma) \left(\overline{\sigma^\nu} w\right)(0). \tag{4.10}$$

Since $\alpha_\nu(\xi) \in \mathbb{R}[\mathcal{S}]$, it must be a finite $\mathbb{R}$-linear combination of monomials of the form $\xi^{\widetilde{\nu}}$, where $\widetilde{\nu} \in \mathcal{S}$. It then follows that the right-most expression in (4.10) can be written as

$$\begin{aligned} w(\kappa) = \sum_{\nu\in\Gamma} \alpha_\nu(\sigma)\left(\overline{\sigma^\nu} w\right)(0) &= \sum_{\nu\in\Gamma}\sum_{\widetilde{\nu}\in\mathcal{S}} \beta_{\widetilde{\nu}} \left(\overline{\sigma^{\widetilde{\nu}+\nu}} w\right)(0) \text{ with } \beta_{\widetilde{\nu}} \in \mathbb{R} \\ &= \sum_{\nu\in\mathcal{C}} \widetilde{\beta}_\nu \left(\overline{\sigma^\nu} w\right)(0) \text{ with } \widetilde{\beta}_\nu \in \mathbb{R} \\ &= \sum_{\nu\in\mathcal{C}} \widetilde{\beta}_\nu w(\nu). \end{aligned}$$

But, $w|_{\mathcal{C}} \equiv 0$ means $w(\nu) = 0$ for all $\nu \in \mathcal{C}$. Therefore, $w(\kappa) = \sum_{\nu\in\mathcal{C}} \widetilde{\beta}_\nu w(\nu) = 0$. This proves that $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$. ∎

In the following subsection we show that the result obtained for the scalar case is sufficient to conclude the result for the general vector case.

4.2. The vector case. Given a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$, define the associated scalar behavior $\mathfrak{B}_{\mathrm{sc}} \in \mathfrak{L}^1$ as

$$\mathfrak{B}_{\mathrm{sc}} := \mathfrak{B}(\mathrm{ann}\ \mathcal{M}) \in \mathfrak{L}^1. \tag{4.11}$$

This alternate description of the behavior, using the equation ideal, following (2.7), plays an important role. The scalar behavior, $\mathfrak{B}_{\mathrm{sc}}$, has some special significance, which we state and prove in Proposition 4.11 below.

PROPOSITION 4.11. *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Define $\mathfrak{B}_{\mathrm{sc}} \in \mathfrak{L}^1$ as in* (4.11). *Then the following are true:*

1. *If $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$, then $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$, too.*
2. *If a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ is such that $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous and the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, then $\mathcal{S}$ is rank-maximally free with respect to $\mathfrak{B}_{\mathrm{sc}}$.*

Proof. (1) It follows from [16, Lemmas 22, 23] that if a set $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$, then $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$, too.

(2) It follows from statement 2 of Proposition 3.10 that if $\mathcal{S} \subseteq \mathbb{Z}^n$ is such that $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous, then $\mathcal{S}$ is free with respect to $\mathfrak{B}_{\mathrm{sc}}$. Also, the Krull dimension of $\mathcal{M}$ is equal to the Krull dimension of $\mathcal{A}/\mathfrak{a}$, by definition. Therefore, if $\mathcal{S} \subseteq \mathbb{Z}^n$ is a sublattice such that $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous and the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, then $\mathcal{S}$ is rank-maximally free with respect to $\mathfrak{B}_{\mathrm{sc}}$. ∎

Recall Theorem 4.1. We prove the result here.

THEOREM 4.1 (vector version). *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that $\mathcal{S}$ is a direct summand of $\mathbb{Z}^n$ and $\mathfrak{B}|_{\mathcal{S}}$ is a nonautonomous system. Further, let the Krull dimension of $\mathcal{M}$ be equal to the rank of $\mathcal{S}$. Then, a union of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ is a characteristic set for $\mathfrak{B}$.*

Proof. By statement 2 of Proposition 4.11, $\mathcal{S}$ is rank-maximally free with respect to $\mathfrak{B}_{\mathrm{sc}}$. It follows from the scalar version of Theorem 4.1 that the union, say, $\mathcal{C}$, of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ is a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$. By statement 1 of Proposition 4.11, the set $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$. ∎

Remark 4.12. It is important to note that when a sublattice and finitely many parallel translates of the sublattice is a characteristic set for a system, the *number* of parallel translates of the sublattice is not an *invariant* property of the system. In fact, the number of parallel translates depends on the choice of the generators for $\mathcal{S}'$. The minimum number of these translates required for a given system (for a given sublattice) is a matter of future investigation.

Remark 4.13. When a finite union of sublattices is a characteristic set for an overdetermined system of PDEs, that is, trajectories restricted to this set serve as initial data, Algorithm 3.14 can be suitably modified to compute the explicit solution of the overdetermined system of PDEs.

We now investigate the question of existence of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ for a given system $\mathfrak{B} \in \mathfrak{L}^q$ that satisfies the rank condition and is such that $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous. The analysis so far has used the assumption that for a given autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$, a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ satisfying the desired specifications is also available to us. However, it is also imperative to show that such a sublattice can indeed be found. This is precisely what we do in section 4.3 below: we show that for a given system one can always find a sublattice which satisfies the desired specifications, namely, the sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ is a direct summand of $\mathbb{Z}^n$, the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, and $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous.

4.3. Existence of a sublattice with the desired specifications for a given system. For a given n-D autonomous system $\mathfrak{B} \in \mathfrak{L}^q$, the existence of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, such that $\mathcal{S}$ is a direct summand, $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous, and the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, can be shown by invoking the discrete Noether's normalization lemma (DNNL) [18, 15].

4.3.1. Discrete Noether's normalization lemma. Before stating the DNNL we briefly discuss co-ordinate transformations on $\mathbb{Z}^n$ and their effects, which play an important role in DNNL. By a co-ordinate transformation, we mean a change of basis in the domain, here $\mathbb{Z}^n$. A transformation T on $\mathbb{Z}^n$ is defined as

$$\begin{array}{rccc} T: & \mathbb{Z}^n & \to & \mathbb{Z}^n, \\ & \nu & \mapsto & T\nu, \end{array} \tag{4.12}$$

such that T is $\mathbb{Z}$-linear and bijective. Since the transformation T is bijective and $\mathbb{Z}$-linear, it follows that a representation of the transformation is given by a square unimodular matrix (see footnote 6 for a definition of unimodular matrix). With a slight abuse of notation we denote the matrix representation also by T, that is, $T \in \mathbb{Z}^{n\times n}$ with $\det T = \pm 1$. This coordinate transformation induces two maps—the *push-forward* map, T_*, and the *pull-back* map, T^*.

The push-forward map, T_*, is an automorphism of the $\mathbb{R}$-algebra $\mathcal{A}$. For $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\nu \in \mathbb{Z}^n$, the action of T_* on a monomial is defined as

$$\begin{array}{rccc} T_*: & \mathcal{A} & \to & \mathcal{A}, \\ & \xi^\nu & \mapsto & \xi^{T\nu} \end{array} \tag{4.13}$$

and is extended to Laurent polynomials by $\mathbb{R}$-linearity. The bijectivity of T ensures the bijectivity of T_* as well. The push-forward map is extended to $\mathcal{A}^{1\times q}$ componentwise.

That is, we have the $\mathcal{A}$-module homomorphism $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$ induced[7] by T_* as follows:

$$\begin{array}{lccc} \widetilde{T}_* : & \mathcal{A}^{1\times q} & \to & \mathcal{A}^{1\times q} \\ & \begin{bmatrix} f_1(\xi) & f_2(\xi) & \dots & f_q(\xi) \end{bmatrix} & \mapsto & \begin{bmatrix} T_*(f_1(\xi)) & T_*(f_2(\xi)) & \dots & T_*(f_q(\xi)) \end{bmatrix}. \end{array} \tag{4.14}$$

Therefore, under $\widetilde{T}_*$ a submodule $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ is mapped to another submodule because $\widetilde{T}_*$ is a bijective $\mathbb{R}$-linear homomorphism.

The pull-back map, T^*, is an automorphism of the $\mathbb{R}$-vector space $(\mathbb{R}^q)^{\mathbb{Z}^n}$ defined as follows: for $w \in (\mathbb{R}^q)^{\mathbb{Z}^n}$ and $\nu \in \mathbb{Z}^n$,

$$\begin{array}{cccc} T^* : & (\mathbb{R}^q)^{\mathbb{Z}^n} & \to & (\mathbb{R}^q)^{\mathbb{Z}^n}, \\ & w(\nu) & \mapsto & w(T\nu). \end{array} \tag{4.15}$$

The pull-back map is also $\mathbb{R}$-linear and bijective as T is bijective.

Under this transformation of co-ordinates, the following result relates the original behavior and the transformed behavior. The details of the proof can be found in [18, Theorem 3.1] for a general n-D system.

PROPOSITION 4.14. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be given by equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. Let $T \in \mathbb{Z}^{n\times n}$ be a unimodular matrix representing the co-ordinate transformation on $\mathbb{Z}^n$. Let $T_* : \mathcal{A} \to \mathcal{A}$, $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$, and $T^* : (\mathbb{R}^q)^{\mathbb{Z}^n} \to (\mathbb{R}^q)^{\mathbb{Z}^n}$, as defined in* (4.13), (4.14)*, and* (4.15)*, respectively, be induced by T. Then*

$$\mathfrak{B}(\mathcal{R}) = T^*(\mathfrak{B}(\widetilde{T}_*(\mathcal{R}))). \tag{4.16}$$

We now state the vector version of the discrete Noether's normalization lemma in Proposition 4.15 below. The details of the proof for the 2-D vector case can be found in [17, Theorem 5.2]. The scalar version of the general n-D case is proved in [15, Theorem 7.7]. For the general n-D vector case see [18, Lemma 3.2]. We use the symbol $\mathcal{A}_d$ to denote the subring of the Laurent polynomial ring $\mathcal{A}$ generated by the first d indeterminates $(\xi_1, \xi_2 \dots, \xi_d)$. That is, $\mathcal{A}_d := \mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \dots, \xi_d, \xi_d^{-1}]$. Also recall the definition of a faithful module from footnote 5.

PROPOSITION 4.15. *Suppose $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ is a proper submodule such that the quotient module $\mathcal{M}$ is a torsion module. Then there exist a nonnegative integer $d < n$ and a unimodular matrix $T \in \mathbb{Z}^{n\times n}$, inducing the maps $T_* : \mathcal{A} \to \mathcal{A}$ and $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$, as defined in* (4.13) *and* (4.14)*, respectively, such that the quotient module $\mathcal{A}^{1\times q}/\widetilde{T}_*(\mathcal{R})$ is a finitely generated faithful module over $\mathcal{A}_d$.*

Using these results, we show the existence of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ such that for a given discrete n-D autonomous system $\mathfrak{B} \in \mathfrak{L}^q$, $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous, the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, and $\mathcal{S}$ is a direct summand. Therefore, a characteristic set for the given system can be constructed using this sublattice and finitely many parallel translates of it.

4.3.2. Existence of a sublattice with the desired specifications. We first prove some auxiliary results using the construction and results of the last subsection.

[7] The scalar multiplication property of the homomorphism obeys the following relation: for $f \in \mathcal{A}$ and $r \in \mathcal{A}^{1\times q}$,

$$\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q},$$
$$\widetilde{T}_*(fr) = T_*(f)\widetilde{T}_*(r).$$

LEMMA 4.16. *Let $T \in \mathbb{Z}^{n\times n}$ be a co-ordinate transformation on the domain $\mathbb{Z}^n$. Let $T_* : \mathcal{A} \to \mathcal{A}$ and $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$ be the T-induced maps as defined in* (4.13), *and* (4.14), *respectively. Let $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ be a submodule. Define the behaviors $\mathfrak{B}_1 := \mathfrak{B}(\mathcal{R})$ and $\mathfrak{B}_2 := \mathfrak{B}(\widetilde{T}_*(\mathcal{R}))$ in $\mathfrak{L}^q$ with the corresponding quotient modules $\mathcal{M}_1$ and $\mathcal{M}_2$. Further, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sublattices in $\mathbb{Z}^n$ such that $\mathcal{S}_2 := T(\mathcal{S}_1)$. Then*

1. *the rank of $\mathcal{S}_1$ is equal to the rank of $\mathcal{S}_2$,*
2. *the Krull dimension of $\mathfrak{B}_1$ is equal to the Krull dimension of $\mathfrak{B}_2$, and*
3. *the restricted behavior $\mathfrak{B}_1|_{\mathcal{S}_1}$ is nonautonomous if and only if the restricted behavior $\mathfrak{B}_2|_{\mathcal{S}_2}$ is non-autonomous.*

To prove Lemma 4.16 we require a small result, which we prove below.

LEMMA 4.17. *Let $T \in \mathbb{Z}^{n\times n}$ be a co-ordinate transformation on the domain $\mathbb{Z}^n$. Let $T_* : \mathcal{A} \to \mathcal{A}$ and $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$ be the T-induced maps as defined in* (4.13), *and* (4.14), *respectively. Let $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ be a submodule. Define the behaviors $\mathfrak{B}_1 := \mathfrak{B}(\mathcal{R})$ and $\mathfrak{B}_2 := \mathfrak{B}(\widetilde{T}_*(\mathcal{R}))$ in $\mathfrak{L}^q$ with the corresponding quotient modules $\mathcal{M}_1$ and $\mathcal{M}_2$. Then $T_*(\text{ann } \mathcal{M}_1) = \text{ann } \mathcal{M}_2$.*

Proof. ($\subseteq$): Let $f \in \text{ann } \mathcal{M}_1$. Then for any $r \in \mathcal{A}^{1\times q}$, $fr \in \mathcal{R}$. Since $\widetilde{T}_*$ is an automorphism this implies $\widetilde{T}_*(fr) \in \widetilde{T}_*(\mathcal{R})$, which in turn implies $T_*(f)\widetilde{T}_*(r) \in \widetilde{T}_*(\mathcal{R})$. As $\widetilde{T}_*$ is an automorphism for every $r \in \mathcal{A}^{1\times q}$, there exists a unique $r_1 \in \mathcal{A}^{1\times q}$ such that $r_1 = \widetilde{T}_*(r)$. Thus, for $r_1 \in \mathcal{A}^{1\times q}$, $T_*(f)r_1 \in \widetilde{T}_*(\mathcal{R})$. Therefore, $T_*(f) \in \text{ann } \mathcal{M}_2$.

($\supseteq$): Let $f \in \text{ann } \mathcal{M}_2$. Then for any $r \in \mathcal{A}^{1\times q}$, $fr \in \widetilde{T}_*(\mathcal{R})$. Since T_* is an automorphism, for every $f \in \mathcal{A}$, there exists a unique $f_1 \in \mathcal{A}$ such that $T_*(f_1) = f$. Similarly, since $\widetilde{T}_*$ is an automorphism, for every $r \in \mathcal{A}^{1\times q}$, there exists a unique $r_1 \in \mathcal{A}^{1\times q}$ such that $\widetilde{T}_*(r_1) = r$. Thus, $T_*(f_1)\widetilde{T}_*(r_1) \in \widetilde{T}_*(\mathcal{R})$, which in turn implies $\widetilde{T}_*(f_1 r_1) \in \widetilde{T}_*(\mathcal{R})$. Therefore, $f_1 r_1 \in \mathcal{R}$ and $f_1 \in \text{ann } \mathcal{M}_1$. Hence, $f \in T_*(\text{ann } \mathcal{M}_1)$. ∎

We now prove Lemma 4.16.

Proof of Lemma 4.16. (1) Since T is an isomorphism of $\mathbb{Z}$-modules, the rank of $\mathcal{S}_1$ is equal to the rank of $\mathcal{S}_2$.

(2) Note that the Krull dimension of $\mathfrak{B}_1$ is equal to the Krull dimension of the quotient module $\mathcal{A}^{1\times q}/\mathcal{R}$. Since $\widetilde{T}_*$ is an automorphism induced by T_*, the Krull dimension of the quotient module $\mathcal{A}^{1\times q}/\widetilde{T}_*(\mathcal{R})$ is equal to the Krull dimension of $\mathcal{A}^{1\times q}/\mathcal{R}$. Hence, the Krull dimension of $\mathfrak{B}_1$ is equal to the Krull dimension $\mathfrak{B}_2$.

(3) Recall that $\mathfrak{B}_1|_{\mathcal{S}_1}$ is a d-D behavior, where rank of $\mathcal{S}_1$ is equal to d. Since the rank of $\mathcal{S}_2$ is equal to the rank of $\mathcal{S}_1$ it, therefore, follows that $\mathfrak{B}_2|_{\mathcal{S}_2}$ is also a d-D behavior. Let $\mathcal{M}_{\mathcal{S}_1} := \mathbb{R}[\mathcal{S}_1]^{1\times q}/\mathcal{R}\cap\mathbb{R}[\mathcal{S}_1]^{1\times q}$ and $\mathcal{M}_{\mathcal{S}_2} := \mathbb{R}[\mathcal{S}_2]^{1\times q}/\widetilde{T}_*(\mathcal{R})\cap\mathbb{R}[\mathcal{S}_2]^{1\times q}$ be the corresponding quotient modules for $\mathfrak{B}_1|_{\mathcal{S}_1}$, and $\mathfrak{B}_2|_{\mathcal{S}_2}$, respectively. Suppose $\mathfrak{B}_1|_{\mathcal{S}_1}$ is nonautonomous. Then, from Proposition 2.1, it follows that $\text{ann } \mathcal{M}_{\mathcal{S}_1} = \{0\}$. Note that $T_*(\mathbb{R}[\mathcal{S}_1]) = \mathbb{R}[\mathcal{S}_2]$. Now,

$$
\begin{aligned}
T_*(\text{ann } \mathcal{M}_{\mathcal{S}_1}) = \{0\} &\quad \text{since } T_* \text{is an automorphism}\\
\Rightarrow T_*(\text{ann } \mathcal{M}_1 \cap \mathbb{R}[\mathcal{S}_1]) = \{0\} &\quad \text{follows from Proposition 3.10}\\
\Rightarrow T_*(\text{ann } \mathcal{M}_1) \cap T_*(\mathbb{R}[\mathcal{S}_1]) = \{0\} &\\
\Rightarrow T_*(\text{ann } \mathcal{M}_1) \cap \mathbb{R}[\mathcal{S}_2] = \{0\} &\quad \text{since } T_*(\mathbb{R}[\mathcal{S}_1]) = \mathbb{R}[\mathcal{S}_2]\\
\Rightarrow \text{ann } \mathcal{M}_2 \cap \mathbb{R}[\mathcal{S}_2] = \{0\} &\quad \text{follows from Lemma 4.17}\\
\Rightarrow \text{ann } \mathcal{M}_{\mathcal{S}_2} = \{0\} &\quad \text{follows from Proposition 3.10.}
\end{aligned}
$$

Therefore, $\mathfrak{B}_2|_{\mathcal{S}_2}$ is nonautonomous. Using the same line of arguments, it can be shown that $\mathfrak{B}_1|_{\mathcal{S}_1}$ is nonautonomous when $\mathfrak{B}_2|_{\mathcal{S}_2}$ is nonautonomous. ◻

Using Lemma 4.16 we now prove Theorem 4.18.

THEOREM 4.18. *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$. Let the Krull dimension of $\mathcal{M}$ be equal to d. Then, there exists a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ such that $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous, the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, and $\mathcal{S}$ is a direct summand.*

Proof. From Proposition 4.15, it follows that there exists a unimodular matrix $T \in \mathbb{Z}^{n\times n}$ and the T-induced maps $T_* : \mathcal{A} \to \mathcal{A}$ and $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$ such that $\mathcal{A}^{1\times q}/\widetilde{T}_*(\mathcal{R})$ is a finitely generated faithful module over $\mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \dots, \xi_d, \xi_d^{-1}]$. For $i \in \{1, 2, \dots, d\}$, let e_i be the standard basis (column) vectors of $\mathbb{Z}^n$. Define $\mathcal{S}_2 := \operatorname{span}_{\mathbb{Z}}\{e_1, e_2, \dots, e_d\}$ and $\mathfrak{B}_2 := \mathfrak{B}(\widetilde{T}_*(\mathcal{R}))$. Note that $\mathcal{S}_2$ is a direct summand and rank of $\mathcal{S}_2$ is equal to d. Also, the quotient module $\mathcal{M}_2 := \mathcal{A}^{1\times q}/\widetilde{T}_*(\mathcal{R})$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}_2]$. We claim that $\mathcal{S}_1 := T^{-1}(\mathcal{S}_2)$ is the required sublattice with the desired specifications for $\mathfrak{B}_1 = \mathfrak{B}(\mathcal{R})$. Clearly, $\mathcal{S}_1$ is a direct summand as T is unimodular. From Lemma 4.16 it follows that the rank of $\mathcal{S}_1$ is equal to d. Also, the Krull dimension of $\mathfrak{B}_1$ is equal to the Krull dimension of $\mathfrak{B}_2$, which is equal to the rank of $\mathcal{S}$. It remains to show that $\mathfrak{B}_1|_{\mathcal{S}_1}$ is nonautonomous. It follows from statement 3 of Lemma 4.16 that $\mathfrak{B}_1|_{\mathcal{S}_1}$ is nonautonomous if and only if $\mathfrak{B}_2|_{\mathcal{S}_2}$ is nonautonomous. Note that $\mathfrak{B}_2|_{\mathcal{S}_2}$ is a d-D behavior with quotient module $\mathcal{M}_{\mathcal{S}_2} := \mathbb{R}[\mathcal{S}_2]^{1\times q}/\widetilde{T}_*(\mathcal{R}) \cap \mathbb{R}[\mathcal{S}_2]^{1\times q}$. Now, the quotient module $\mathcal{M}_2$ being a faithful module over $\mathbb{R}[\mathcal{S}_2]$ implies that $\operatorname{ann} \mathcal{M}_2 \cap \mathbb{R}[\mathcal{S}_2] = \{0\}$. Using Proposition 3.10, it follows that $\operatorname{ann} \mathcal{M}_2 \cap \mathbb{R}[\mathcal{S}_2] = \operatorname{ann} \mathcal{M}_{\mathcal{S}_2} = \{0\}$. This implies that $\mathfrak{B}_2|_{\mathcal{S}_2}$ is nonautonomous and thus $\mathfrak{B}_1|_{\mathcal{S}_1}$ is also nonautonomous. ◻

As previously mentioned, the existence of a sublattice with the desired specifications is crucial because it allows us to *construct* a characteristic set, given by a union of the sublattice and finitely many parallel translates of it, for a given discrete autonomous n-D system. This important consequence of Theorem 4.18 is stated in Corollary 4.19.

COROLLARY 4.19. *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ having a Krull dimension equal to d. Then there exists a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ of dimension d, satisfying that $\mathfrak{B}|_{\mathcal{S}}$ is nonautonomous and $\mathcal{S}$ is a direct summand, such that a characteristic set for $\mathfrak{B}$ given by a union of a sublattice and finitely many parallel traslates of it can be constructed using the sublattice $\mathcal{S}$.*

Proof. Theorem 4.18 establishes the existence of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ satisfying the desired specifications. Applying Theorem 4.1, a characteristic set for $\mathfrak{B}$ is given by a union of $\mathcal{S}$ and finitely many parallel translates of it. ◻

Corollary 4.19 shows that *every* discrete autonomous n-D system admits a characteristic set given by a union of a sublattice and finitely many parallel translates of it. In the following section, we elaborate on the construction of a characteristic set for a given discrete autonomous n-D system from scratch.

4.4. Explicit construction of a characteristic set for a given discrete autonomous n-D system. Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ having a Krull dimension equal to d. In this section, we outline the procedure for constructing a characteristic set for $\mathfrak{B}$ given by a union of a sublattice and finitely many parallel translates of it.

It is clear from the proof of Theorem 4.18 that the discrete Noether's normalization lemma plays a crucial role in establishing existence and subsequently in the construction of a characteristic set. In fact, the existence of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, with the desired specifications, is guaranteed by the existence of a transformation matrix $T \in \mathbb{Z}^{n\times n}$ representing a co-ordinate transformation on $\mathbb{Z}^n$. Therefore, the question of constructing a characteristic set for $\mathfrak{B}$ is intrinsically related to constructing a transformation matrix $T \in \mathbb{Z}^{n\times n}$ for a given system $\mathfrak{B}$. One method of constructing a unimodular transformation matrix $T \in \mathbb{Z}^{n\times n}$ for DNNL has been discussed in [15, Lemma 7.3].

Therefore, to construct a characteristic set for a given discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$, having Krull dimension equal to d, we do the following. The steps follow from the proof of Theorem 4.18.

1. Construct a unimodular transformation matrix $T \in \mathbb{Z}^{n\times n}$ for performing the discrete Noether's normalization. That is, construct a $T \in \mathbb{Z}^{n\times n}$ and the T-induced maps $T_* : \mathcal{A} \to \mathcal{A}$ and $\widetilde{T}_* : \mathcal{A}^{1\times q} \to \mathcal{A}^{1\times q}$ such that $\mathcal{A}^{1\times q}/\widetilde{T}_*(\mathcal{R})$ is a finitely generated faithful module over $\mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \ldots, \xi_d, \xi_d^{-1}]$.
2. Define the behavior in the transformed domain as $\mathfrak{B}_2 := \mathfrak{B}(\widetilde{T}_*(\mathcal{R}))$. Define the sublattice $\mathcal{S}_2 := \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, \ldots, e_d\} \subseteq \mathbb{Z}^n$, where e_i is the ith standard basis of $\mathbb{Z}^n$. It has been shown in the proof of Theorem 4.18 that $\mathcal{S}_2$ satisfies all the desired properties, namely, $\mathcal{S}_2$ is a direct summand, rank of $\mathcal{S}_2$ is equal to d, and $\mathfrak{B}_2|_{\mathcal{S}_2}$ is nonautonomous. Therefore, a union of $\mathcal{S}_2$ and finitely many parallel translates of $\mathcal{S}_2$ is a characteristic set for $\mathfrak{B}_2$.
3. Transforming back to the original domain, we obtain a characteristic set for $\mathfrak{B}$. More precisely, defining $\mathcal{S} := T^{-1}(\mathcal{S}_2)$, it has been shown in the proof of Theorem 4.18 that $\mathcal{S} \subseteq \mathbb{Z}^n$ satisfies the desired properties with respect to $\mathfrak{B}$. Hence, a characteristic set for $\mathfrak{B}$ given by a union of $\mathcal{S}$ and finitely many parallel translates of it is obtained.

In this section, we have shown when a system of overdetermined PDEs admits a characteristic set given by a finite union of sublattices. In other words, a characteristic set given by a union of a sublattice and finitely many parallel translates of it is a characteristic set for a given system of overdetermined PDEs with real constant coefficients if the Krull dimension of the system is equal to the rank of the sublattice, the system restricted to the sublattice is underdetermined, and the sublattice is a direct summand. We have also shown that for a given system of overdetermined PDEs, a sublattice satisfying the above criteria always exits. Thus, for a given system having Krull dimension equal to d, a characteristic set given by a finite union of sublattices having rank d can always be constructed. The following section discusses the possibilities of a characteristic set for the case when the rank of the sublattice is strictly less than the Krull dimension of the system.

5. Rank of $\mathcal{S}$ is less than the Krull dimension of the system. Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D autonomous system with Krull dimension d. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice of rank r. Supposing $r < d$, the question we now ask is the following: can the sublattice along with finitely many parallel translates of it be a characteristic set for $\mathfrak{B}$? The answer to this question is negative. We show in this section that finitely many parallel translates along with the sublattice $\mathcal{S}$ does not qualify as a characteristic set for $\mathfrak{B}$. In short, we prove Theorem 5.1.

THEOREM 5.1. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D autonomous system with Krull dimension equal to d. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice of rank r. Supposing $r < d$, then a union of $\mathcal{S}$ and finitely many parallel translates of it cannot be a characteristic set for $\mathfrak{B}$.*

Remark 5.2. Note that for a shift-invariant system, a union of a sublattice and finitely many parallel translates of it is a characteristic set if and only if the union shifted away from the origin is a characteristic set [21]. Thus, Theorem 5.1 also implies that a finite union of parallel translates of a sublattice $\mathcal{S}$, which may not contain $\mathcal{S}$, cannot be a characteristic set if rank($\mathcal{S}$) is strictly less than the Krull dimension of the behavior. In other words, the assumption in Theorem 5.1 that the finite union must contain $\mathcal{S}$ can be relaxed.

To prove Theorem 5.1, we require some auxiliary results, which we first discuss. Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D autonomous system and let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice. Suppose $\Gamma := \{\nu_1, \nu_2, \dots, \nu_\ell\} \subseteq \mathbb{Z}^n$. Define

$$\mathcal{C} := \bigcup_{\nu \in \Gamma} \mathcal{S}_\nu, \tag{5.1}$$

where $\mathcal{S}_\nu := \mathcal{S} + \nu$. Under the assumption that $0 \in \Gamma$, we have $\mathcal{S} \subseteq \mathcal{C}$. Recall that for a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, the sublattice algebra $\mathbb{R}[\mathcal{S}]$ is as defined in (3.3). Also recall that $\mathcal{A}^{1\times q}$ is an $\mathbb{R}[\mathcal{S}]$-module via the injection $\mathbb{R}[\mathcal{S}] \hookrightarrow \mathcal{A}$. For $j \in \{1, 2, \dots, q\}$, let e_j^T be the jth standard basis vector of $\mathcal{A}^{1\times q}$. That is, $e_j^T = \begin{bmatrix} 0 & 0 & \dots & 1 & \dots & 0 \end{bmatrix}$ with 1 appearing at the jth position. Define the set

$$\mathcal{G} := \left\{ \overline{\xi^{\nu_i} e_j^T} \mid \nu_i \in \Gamma, 1 \leqslant j \leqslant q \right\}, \tag{5.2}$$

where $\overline{\xi^{\nu_i} e_j^T}$ is the image of $\xi^{\nu_i} e_j^T$ under the canonical surjection $\mathcal{A}^{1\times q} \twoheadrightarrow \mathcal{M}$. Let $\mathcal{N}$ be the finitely generated $\mathbb{R}[\mathcal{S}]$-module generated by elements of $\mathcal{G}$. Note that $\mathcal{N} \subseteq \mathcal{M}$ as $\mathbb{R}[\mathcal{S}]$-modules. Also, $\mathcal{N}$ is a vector space over $\mathbb{R}$.

LEMMA 5.3. *Let $\mathcal{S}$ be a sublattice of rank r. Let $\mathcal{N}$ be the $\mathbb{R}[\mathcal{S}]$-module generated by elements of $\mathcal{G}$. Then the Krull dimension of $\mathcal{N}$ as an $\mathcal{A}$-module is less than or equal to r.*

Proof. Since $\mathcal{N}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$, there exists a surjective $\mathbb{R}[\mathcal{S}]$-module homomorphism $\Pi : \mathbb{R}[\mathcal{S}]^p \twoheadrightarrow \mathcal{N}$ for some $p \in \mathbb{Z}_{>0}$. Therefore, the Krull dimension of $\mathbb{R}[\mathcal{S}]^p$ is greater than or equal to the Krull dimension of $\mathcal{N}$ [5]. Using the fact that the Krull dimension of $\mathbb{R}[\mathcal{S}]^p$ as an $\mathbb{R}[\mathcal{S}]$-module is equal to the Krull dimension of $\mathbb{R}[\mathcal{S}]$, which is equal to the rank of $\mathcal{S}$, we have Krull dim $\mathcal{N} \leqslant r$, where r is equal to the rank of $\mathcal{S}$. ∎

We now prove Lemma 5.4, which plays a key role in proving Theorem 5.1. Recall the variant of Malgrange's theorem as stated in Proposition 3.5.

LEMMA 5.4. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete n-D autonomous system. Let $\mathcal{C} \subseteq \mathbb{Z}^n$ be as defined in* (5.1)*. Let $\mathcal{N}$ be the $\mathbb{R}[\mathcal{S}]$-module generated by elements of $\mathcal{G}$, where $\mathcal{G}$ is as defined in* (5.2)*. Suppose $\mu \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ and $\mathcal{N} \subseteq \ker \mu$. Recall the $\mathcal{A}$-module homomorphism $\Phi : \mathfrak{B} \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ as defined in Proposition* 3.5*. Define $w := \Phi^{-1}(\mu) \in \mathfrak{B}$. Then $w|_{\mathcal{C}} \equiv 0$.*

Proof. Since $\mathcal{N} \subseteq \ker \mu$, $\mu(n) = 0$ for all $n \in \mathcal{N}$. Thus $(\Phi(w))(n) = 0$ for all $n \in \mathcal{N}$. In other words, using the definition of Φ, for all $n \in \mathcal{N}$, $(n(w))(0) = 0$. Let $\kappa \in \mathcal{C}$ be arbitrary. Then $\kappa = s + \nu$ for some $s \in \mathcal{S}$ and $\nu \in \Gamma$. Now

$$(\sigma^\kappa w)(0) = w(\kappa) = w_1(\kappa)e_1 + w_2(\kappa)e_2 + \dots + w_q(\kappa)e_q, \tag{5.3}$$

where $w_i : \mathbb{Z}^n \to \mathbb{R}$ is the ith component of w and e_i is the ith standard basis of $\mathbb{R}^q$. Note that

$$w_i(\kappa) = \left(\left(\sigma^\kappa e_i^T\right) w\right)(0) = \left(\left(\overline{\sigma^\kappa e_i^T}\right) w\right)(0) = \left(\left(\overline{\sigma^s \sigma^\nu e_i^T}\right) w\right)(0). \tag{5.4}$$

Observe that $\overline{\sigma^s \sigma^\nu e_i^T}$ is an element in $\mathcal{N}$ for all $i \in \{1, 2, \dots, q\}$. Therefore, using the fact that $(n(w))(0) = 0$ for all $n \in \mathcal{N}$, $w_i(\kappa) = 0$ for all $i \in \{1, 2, \dots, q\}$, which in turn implies $w(\kappa) = 0$. Since $\kappa \in \mathcal{C}$ was assumed to be arbitrary, $w|_{\mathcal{C}} \equiv 0$. ∎

Now we prove Theorem 5.1.

Proof of Theorem 5.1. We prove this by contradiction. Suppose $\mathcal{C}$, as defined in (5.1), is a characteristic set for $\mathfrak{B}$. Let $\mathcal{N}$ be the $\mathbb{R}[\mathcal{S}]$-module generated by elements of $\mathcal{G}$ as defined in (5.2). Note that $\mathcal{N} \subseteq \mathcal{M}$ as $\mathbb{R}[\mathcal{S}]$-modules. Indeed, $\mathcal{N} \subsetneq \mathcal{M}$. Otherwise, it contradicts the rank assumption $r < d$, because it follows from Lemma 5.3 that the Krull dimension of $\mathcal{N}$ is less than or equal to r. Let $\mu \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ be such that $\mathcal{N} \subseteq \ker \mu$. This is possible by choosing a Hamel basis [9, section 2] for $\mathcal{N}$ as a vector space over $\mathbb{R}$ and extending it to a Hamel basis of $\mathcal{M}$ in such a way that in $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$, $\mu(\mathcal{N}) = 0$ and $\mu(\mathcal{M} \setminus \mathcal{N}) \neq 0$. Using Malgrange's theorem, a trajectory $w \in \mathfrak{B}$ can be constructed as $w = \Phi^{-1}(\mu)$. It follows from Lemma 5.4 that $w|_{\mathcal{C}} \equiv 0$. However, as $\mathcal{N} \subsetneq \mathcal{M}$ and $\mu(\mathcal{M} \setminus \mathcal{N}) \neq 0$, we have $w \not\equiv 0$. It follows from [25, Lemma 2.3] that $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$ if and only if $w|_{\mathcal{C}} \equiv 0$ implies $w \equiv 0$. Here, we have shown the existence of a trajectory $w \in \mathfrak{B}$ such that $w|_{\mathcal{C}} \equiv 0$ but $w \not\equiv 0$, which contradicts the assumption that $\mathcal{C}$ is a characteristic set. ∎

6. Rank of $\mathcal{S}$ is more than the Krull dimension of the system. Unlike the situation in section 4, if the rank of $\mathcal{S}$ is strictly greater than the Krull dimension of the system, then the quotient module $\mathcal{M}$ may or may not be a finitely generated module over $\mathbb{R}[\mathcal{S}]$. The following lemma explains the conditions under which $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$.

LEMMA 6.1. *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$, with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice. Then $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$ if and only if $\mathbb{R}[\mathcal{S}]/\mathrm{ann}\ \mathcal{M} \cap \mathbb{R}[\mathcal{S}]$ has a Krull dimension equal to the Krull dimension of $\mathcal{M}$.*

Proof. Since $\mathbb{R} \subseteq \mathbb{R}[\mathcal{S}]$ and $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$ is a finitely generated $\mathbb{R}$-algebra, $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$ is a finitely generated $\mathbb{R}[\mathcal{S}]$-algebra. Consider the $\mathbb{R}[\mathcal{S}]$-module homomorphism $\phi : \mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathrm{ann}\ \mathcal{M}$ defined in the following manner:

$$\begin{array}{cccccc} \phi: & \mathbb{R}[\mathcal{S}] & \hookrightarrow & \mathcal{A} & \twoheadrightarrow & \mathcal{A}/\mathrm{ann}\ \mathcal{M}, \\ & p & \mapsto & p & \mapsto & \overline{p}. \end{array} \tag{6.1}$$

Note that $\ker \phi = \mathrm{ann}\ \mathcal{M} \cap \mathbb{R}[\mathcal{S}]$. Consider the $\mathbb{R}[\mathcal{S}]$-module homomorphism

$$\widetilde{\phi} : \frac{\mathbb{R}[\mathcal{S}]}{\mathrm{ann}\ \mathcal{M} \cap \mathbb{R}[\mathcal{S}]} \to \frac{\mathcal{A}}{\mathrm{ann}\ \mathcal{M}}. \tag{6.2}$$

It is easy to check that $\widetilde{\phi}$ is well-defined. It can also be verified that $\widetilde{\phi}$ is injective.

(If) To show that $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$, note that $\mathcal{M}$ is naturally a finitely generated module over $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$. Therefore, in order to show that $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$, it is enough to show that $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$. This is because if $\mathcal{M}$ is generated by $\{h_1, h_2, \dots, h_y\}$ as a module over $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$ and if $\{f_1, f_2, \dots, f_z\}$ is a generating set for $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$ as module over $\mathbb{R}[\mathcal{S}]$, then $\{h_i f_j \mid 1 \leqslant i \leqslant y, 1 \leqslant j \leqslant z\}$ is a generating set for $\mathcal{M}$ as a module over $\mathbb{R}[\mathcal{S}]$. That $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$ is finitely generated over $\mathbb{R}[\mathcal{S}]$ can be proved in the following manner. Recall that the Krull dimension of $\mathcal{M}$ is by definition equal to the Krull dimension of $\mathcal{A}/\mathrm{ann}\ \mathcal{M}$. From the assumption that the Krull dimension of $\mathbb{R}[\mathcal{S}]/\mathrm{ann}\ \mathcal{M} \cap \mathbb{R}[\mathcal{S}]$ is equal to the Krull dimension of $\mathcal{M}$ it follows that $\widetilde{\phi}$ is integral. Therefore, using Proposition 4.5, $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$.

(Only if) Since $\mathcal{M}$ is finitely generated as an $\mathbb{R}[\mathcal{S}]$-module, it follows that $\mathbb{R}[\mathcal{S}]/\text{ann}\,\mathcal{M}$, too, must be a finitely generated $\mathbb{R}[\mathcal{S}]$-module. Thus by (6.2) and Proposition 4.5 we must have $\widetilde{\phi} : \mathbb{R}[\mathcal{S}]/\text{ann}\,\mathcal{M} \cap \mathbb{R}[\mathcal{S}] \to \mathcal{A}/\text{ann}\,\mathcal{M}$ to be injective and integral. Hence, the Krull dimensions of $\mathbb{R}[\mathcal{S}]/\text{ann}\,\mathcal{M} \cap \mathbb{R}[\mathcal{S}]$ and $\mathcal{A}/\text{ann}\,\mathcal{M}$ must be the same. □

PROPOSITION 6.2. *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that $\mathcal{S}$ is a direct summand of $\mathbb{Z}^n$, and the Krull dimension of $\mathcal{M}$ is strictly less than the rank of $\mathcal{S}$. Further, let the Krull dimension of $\mathcal{M}$ be equal to the Krull dimension of $\mathbb{R}[\mathcal{S}]/\text{ann}\,\mathcal{M}\cap\mathbb{R}[\mathcal{S}]$. Then there exists $g \in \mathbb{N}$ such that a union of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ up to the gth one is a characteristic set for $\mathfrak{B}$.*

Proof. It follows from Lemma 6.1 that $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$. When $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$ under the additional assumption that $\mathcal{S}$ is also a direct summand of $\mathbb{Z}^n$, Lemma 4.8, Corollary 4.9, and Lemma 4.10 apply verbatim, except the faithfulness of $\mathcal{M}$ over $\mathbb{R}[\mathcal{S}]$. Therefore, a union of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ is a characteristic set for $\mathfrak{B}$. □

Proposition 6.2 seems to suggest that the situation with rank of $\mathcal{S}$ being strictly greater than the Krull dimension is identical to that with rank of $\mathcal{S}$ being equal to the Krull dimension. However, we show in Lemma 6.3 below that, with rank of $\mathcal{S}$ strictly bigger than the Krull dimension, the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$ will always be autonomous. This is in contrast to the case when rank of $\mathcal{S}$ is equal to the Krull dimension (see Theorem 4.1).

LEMMA 6.3. *Consider a discrete autonomous n-D system $\mathfrak{B} \in \mathfrak{L}^q$ with equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ and corresponding quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that the Krull dimension of $\mathcal{M}$ is strictly less than the rank of $\mathcal{S}$, say, r. Then, the r-D behavior $\mathfrak{B}|_{\mathcal{S}}$ is autonomous.*

Proof. We prove this by contradiction. Suppose $\mathfrak{B}|_{\mathcal{S}}$ is not autonomous. Using statement 2 of Proposition 3.10 we have $\text{ann}\,\mathcal{M} \cap \mathbb{R}[\mathcal{S}] = \{0\}$. This implies that the $\mathbb{R}[\mathcal{S}]$-module homomorphism ϕ as defined in (6.1) is injective. Therefore the Krull dimension of $\mathcal{M}$ is greater than or equal to the rank of $\mathcal{S}$. This contradicts the assumption that the Krull dimension of $\mathcal{M}$ is strictly less than the rank of $\mathcal{S}$. □

A striking consequence of Lemma 6.3 is the following: in the situation of Proposition 6.2, let

$$\mathcal{C} := \bigcup_{\nu\in\Gamma} \mathcal{S} + \nu$$

with $\Gamma \subseteq \mathbb{Z}^n$, $|\Gamma| < \infty$, be a characteristic set for $\mathfrak{B}$, which is guaranteed to exist according to Proposition 6.2. It then follows that there exists a sublattice $\mathcal{S}' \subsetneq \mathcal{S}$ and a finite set $\Gamma' \subseteq \mathbb{Z}^n$ such that

$$\mathcal{C}' := \bigcup_{\nu\in\Gamma'} \mathcal{S}' + \nu \subseteq \mathcal{C}$$

is a characteristic set for $\mathfrak{B}$. This is the content of Theorem 6.6 below, the main result of this section. We prove this in three steps, starting with Lemma 6.4 below.

LEMMA 6.4. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete autonomous n-D system and $\mathcal{S} \subseteq \mathbb{Z}^n$ a sublattice of rank r. Consider the r-D behavior $\mathfrak{B}|_{\mathcal{S}}$, and let $\mathcal{C}' \subseteq \mathcal{S}$ be a characteristic set for $\mathfrak{B}|_{\mathcal{S}}$. For any $\nu \in \mathbb{Z}^n$ and $w \in \mathfrak{B}$, we must have $w|_{\mathcal{S}+\nu} \equiv 0$ if and only if $w|_{\mathcal{C}'+\nu} \equiv 0$.*

Proof. The "only if" part is trivial. We prove the "if" part here. First note that for every $w \in \mathfrak{B}$, there exists a unique $w' \in \mathfrak{B}$ defined as $w' := \sigma^\nu w$ such that $w'(s) = w(s+\nu)$ for all $s \in \mathcal{S}$. Now suppose $w \in \mathfrak{B}$ is such that $w|_{\mathcal{C}'+\nu} \equiv 0$. It then follows that the corresponding w' satisfies $w'|_{\mathcal{C}'} \equiv 0$. But, since $\mathcal{C}'$ is a characteristic set for $\mathfrak{B}|_{\mathcal{S}}$, this must imply that $w'|_{\mathcal{S}} \equiv 0$. Therefore, the corresponding w must satisfy that $w|_{\mathcal{S}+\nu} \equiv 0$. ◻

LEMMA 6.5. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete autonomous n-D system and $\mathcal{S} \subseteq \mathbb{Z}^n$ a sublattice such that*

$$\mathcal{C} := \bigcup_{\nu \in \Gamma} \mathcal{S} + \nu$$

with $\Gamma \subseteq \mathbb{Z}^n$, $|\Gamma| < \infty$, a characteristic set for $\mathfrak{B}$. Suppose $\mathcal{C}' \subseteq \mathcal{S}$ is a characteristic set for the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$. Then

$$\widetilde{\mathcal{C}} := \bigcup_{\nu \in \Gamma} \mathcal{C}' + \nu$$

is a characteristic set for $\mathfrak{B}$.

Proof. It is enough to show that for any $w \in \mathfrak{B}$ if $w|_{\widetilde{\mathcal{C}}} \equiv 0$, then $w \equiv 0$. First note that by Lemma 6.4, for any $w \in \mathfrak{B}$ and any $\nu \in \mathbb{Z}^n$, we must have $w|_{\mathcal{S}+\nu} \equiv 0$ if and only if $w|_{\mathcal{C}'+\nu} \equiv 0$. Clearly, $w|_{\widetilde{\mathcal{C}}} \equiv 0$ implies that $w|_{\mathcal{C}'+\nu} \equiv 0$ for all $\nu \in \Gamma$. Therefore, Lemma 6.4 implies that $w|_{\mathcal{S}+\nu} \equiv 0$ for all $\nu \in \Gamma$. Thus, $w|_{\widetilde{\mathcal{C}}} \equiv 0$ implies that $w|_{\mathcal{C}} \equiv 0$. But, this means $w \equiv 0$ because $\mathcal{C}$ has been assumed to be a characteristic set for $\mathfrak{B}$. ◻

We are now in a position to prove the main result of this section.

THEOREM 6.6. *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete autonomous n-D system, and $\mathcal{S} \subseteq \mathbb{Z}^n$ a sublattice whose rank is bigger than the Krull dimension of $\mathfrak{B}$. Suppose further that*

$$\mathcal{C} := \bigcup_{\nu \in \Gamma} \mathcal{S} + \nu$$

with $\Gamma \subseteq \mathbb{Z}^n$, $|\Gamma| < \infty$ is a characteristic set for $\mathfrak{B}$. Then there exists a proper sublattice $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$ and a finite set $\widetilde{\Gamma} \subseteq \mathbb{Z}^n$ such that

$$\widetilde{\mathcal{C}} := \bigcup_{\nu \in \widetilde{\Gamma}} \widetilde{\mathcal{S}} + \nu$$

is a characteristic set for $\mathfrak{B}$.

Proof. Consider the restricted behavior $\mathfrak{B}|_{\mathcal{S}}$. Since $\operatorname{rank}(\mathcal{S})$ is strictly greater than the Krull dimension of $\mathfrak{B}$, we must have that $\mathfrak{B}|_{\mathcal{S}}$ is autonomous (Lemma 6.3). By Theorems 4.1 and 4.18, there exists a proper sublattice (having rank strictly smaller than that of $\mathcal{S}$) $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$ and a finite set $\Gamma' \subseteq \mathcal{S}$ such that

$$\mathcal{C}' := \bigcup_{\nu \in \Gamma'} \widetilde{\mathcal{S}} + \nu$$

is a characteristic set for $\mathfrak{B}|_{\mathcal{S}}$. (Note that $\mathcal{C}' \subseteq \mathcal{S}$.) It then follows from Lemma 6.5 that

$$\widetilde{\mathcal{C}} := \bigcup_{\nu \in \Gamma} \mathcal{C}' + \nu$$

is a characteristic set for $\mathfrak{B}$. Defining $\widetilde{\Gamma} := \{\nu + \nu' \mid \nu \in \Gamma, \nu' \in \Gamma'\}$ we get the result. ∎

7. Concluding remarks. We have provided an essentially complete answer to the question of minimal initial data required to solve an overdetermined system of linear PDEs with real constant coefficients using the notion of characteristic sets. First, we emphasized the fact that sublattices are the most suitable subsets to answer the question of minimality, primarily because of the rank associated to a sublattice. We proved a variant of the well-known Malgrange's theorem for the behavior restricted to a sublattice. Using this we gave a necessary and sufficient condition for a sublattice to be a characteristic sublattice. From this characterization it follows that a necessary condition for a sublattice to be a characteristic sublattice is to have a sublattice with rank equal to the Krull dimension of the system. We proved that, for this case, a union of a sublattice and finitely many parallel translates of it is a characteristic set for a given system of PDEs. We further showed that such a characteristic set can always be constructed for a given system.

We then addressed the cases when the rank of the sublattice is not equal to the Krull dimension of the system. We showed that when the rank of the sublattice is strictly less than the Krull dimension of the system, neither the sublattice nor a union of finitely many parallel translates of it can be a characteristic set. For the case when the rank of the sublattice is strictly greater than the Krull dimension of the system, we showed that a union of finitely many parallel translates of the sublattice may turn out to be a characteristic set for the system. But, unlike the case when the rank of the sublattice is equal to the Krull dimension of the system, here a finite union of parallel translates of a sublattice (of the given sublattice) having strictly smaller rank qualifies as a characteristic set for the system.

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