# On arbitrary assignability of initial conditions for a discrete autonomous $n$-D system

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***Abstract*— The issue of initial/boundary conditions for a general system of partial difference equations – called a discrete n-D system – is resolved through the notion of characteristic sets: the restriction of a solution trajectory to a minimal characteristic set can be considered to be the initial/boundary condition to the corresponding trajectory. In the recent paper [8], it has been shown that every autonomous n-D system admits a minimal characteristic set that is the union of a sublattice (of rank equal to the Krull dimension of the system) and its finitely many parallel translates. Treating the restrictions of every solution trajectories to such a characteristic set as initial/boundary conditions, in this paper, we provide a full parametrization of the set of all initial/boundary conditions of a discrete autonomous n-D system. The key to this parametrization is that the set of allowable initial/boundary conditions itself can be viewed as a discrete d-D system, where d is the Krull dimension of the system: we prove this result here. An upshot of this parametrization is the answer to free assignability of these initial/boundary conditions.**

***Index Terms*— n-D systems, characteristic sets, partial difference equations, initial conditions, algebraic analysis.**

## I. Introduction

### A. Motivation and Objectives

By a discrete $n$-D system, we mean a system of partial difference equations (pdes) with real constant coefficients having $n$ independent variables. 'Initial/boundary conditions' play an important role in analyzing and explicitly solving such a system of pdes. In the conventional pde literature, a physical process modeled using pdes is often supplemented with initial and/or boundary conditions. On the contrary, in the $n$-D systems approach to analyzing pdes, it is often customary to deduce 'initial/boundary data' using the system of equations. For example, in [12], the canonical Cauchy problem [25], the Oberst-Riquier algorithm [11], and for computing trajectories in 2-D systems [16], initial data is specified by assigning trajectories on a subset of the domain, where the subset depends on the given system of equations. Following the standard practice, we use *characteristic sets* [19] to formalize the notion of initial/boundary[1] data for discrete $n$-D systems.

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[1]For ease of referencing we use 'initial condition(s)/data' to mean 'initial/boundary condition(s)/data' in the sequel.

A characteristic set is a proper subset of the domain – here $\mathbb{Z}^n$ – with the defining property that for every trajectory, the knowledge of the trajectory restricted to this set uniquely determines the trajectory over the whole domain [19]. It is important to note that a system of pdes having *free variables* cannot have a proper subset of the domain as a characteristic set. Therefore, in the context of initial data, we consider systems without free variables; such systems are called *autonomous* $n$-D systems.

It is known that for a discrete autonomous 1-D system described by a set of ordinary difference equations, a characteristic set is always given by finitely many points on the domain [20]. Initial data, in this case, is obtained by freely choosing the value of the solution trajectory at these finitely many points. However, for an $n$-D system a characteristic set is often infinite in size and can be of various shapes: characteristic sets having algebraic structures, such as half-spaces, cones, sublattices and finite unions of sublattices, have been considered in the literature [8], [9], [13], [19]. Subsets having the algebraic structure of sublattices and finite unions of sublattices play an important role in the theory of characteristic sets as they address the issue of *minimal* initial data for discrete autonomous $n$-D systems. Furthermore, it has been recently shown in [8] that *every* discrete autonomous $n$-D system admits a characteristic set given by a union of a sublattice and finitely many parallel translates of it. This is a key result that we base our paper on: we consider such characteristic sets.

Naturally, once a characteristic set is obtained, initial conditions can be obtained by restricting trajectories to the characteristic set. However, this requires an explicit knowledge of the solution trajectories. This requirement is unrealistic and apparently impossible. It would be more practical if one could arbitrarily assign the values of the trajectories on a characteristic set so that they serve as initial conditions. In other words, it is more reasonable to be able to choose initial conditions freely. Therefore, the most pertinent question in this regard is: *can initial conditions (i.e., restrictions of trajectories on a pre-identified characteristic set) be obtained without the explicit knowledge of the trajectories?* In this paper, we provide a complete answer to this question for the case when the characteristic set is given by a union of a sublattice and finitely many parallel translates of it.

### B. Contributions

The main contributions of this note are the following.

1) For a characteristic set given by a union of a sublattice and finitely many parallel translates of it, the collection of restrictions of trajectories to this characteristic set is called the *set of allowable initial conditions*. We show that this set of allowable initial conditions admits a characterization as a behavior (i.e., a set of solutions of a system of partial difference equations) over a domain of rank strictly less than $n$. Using this characterization, it is shown that, in general, initial conditions cannot be arbitrarily assigned for a discrete autonomous $n$-D system.
2) A parametrization of the allowable initial conditions using a free variable is provided.
3) A necessary and sufficient condition for arbitrary assignability of initial conditions is provided.

The results of [8] establish that every discrete autonomous $n$-D system admits a characteristic set given by a finite union of sublattices. The results of this note provide conditions as to how to specify trajectories on the characteristic set. Therefore, [8] in conjunction with the results developed here provide a complete answer to the open problem of obtaining minimal initial data required to solve a discrete autonomous $n$-D system. Preliminary results, relating to arbitrary assignability of initial conditions for the *scalar case*, has been published in [7].

## II. Notation and Preliminaries

### A. Notation

We use the symbols $\mathbb{Z}$, and $\mathbb{R}$, to denote the sets of integers, and real numbers, respectively. The symbols $\mathbb{Z}^n$, and $\mathbb{R}^n$, are used to denote the sets of $n$-tuples of integers, and real numbers, respectively. For a set $\Gamma$, $|\Gamma|$ denotes the cardinality of $\Gamma$. We use the symbol $\bullet$ to denote a quantity which is unspecified. For example, $R \in \mathbb{R}^{g\times\bullet}$ denotes a matrix with real entries having $g$ rows and an unspecified number of columns.

### B. System Description

We consider discrete $n$-D systems described by linear systems of pdes with real constant coefficients. A solution of a discrete $n$-D system is called a *trajectory*. For an $n$-D system having $q$ dependent variables, a trajectory, $w = (w_1, \dots, w_q)$, is a $q$-tuple of multi-indexed real valued sequences; i.e. $w : \mathbb{Z}^n \to \mathbb{R}^q$. We use the symbol $\left(\mathbb{R}^{\mathbb{Z}^n}\right)^q$ to denote the set of all possible trajectories. The collection of all trajectories satisfying a given system of pdes is called the *behavior* of the system and is denoted by $\mathfrak{B}$. Therefore, $\mathfrak{B} \subseteq \left(\mathbb{R}^{\mathbb{Z}^n}\right)^q$.

A system of pdes is described using the $n$-D shift operators $\sigma_1, \dots, \sigma_n$, where the $i$-th shift operator, $\sigma_i$, acts on a trajectory $w \in \left(\mathbb{R}^{\mathbb{Z}^n}\right)^q$ in the following way: for $\kappa := (\kappa_1, \dots, \kappa_n) \in \mathbb{Z}^n$,

$$(\sigma_i w)(\kappa_1, \dots, \kappa_n) := w(\kappa_1, \dots, \kappa_i + 1, \dots, \kappa_n). \quad (1)$$

Define $\sigma := (\sigma_1, \dots, \sigma_n)$, and $\sigma^{-1} := (\sigma_1^{-1}, \dots, \sigma_n^{-1})$, as the $n$-tuples of shift, and inverse shift operators, respectively. The Laurent polynomial ring in $n$ indeterminates with real coefficients is denoted by $\mathbb{R}[\xi_1, \xi_1^{-1}, \dots, \xi_n, \xi_n^{-1}]$. For brevity, we define $\mathcal{A} := \mathbb{R}[\xi, \xi^{-1}]$, where $\xi = (\xi_1, \dots, \xi_n)$ and $\xi^{-1} = (\xi_1^{-1}, \dots, \xi_n^{-1})$. Note that, $\mathcal{A}$ is a commutative ring.

A system of linear pdes with real constant coefficients having $q$ dependent variables is written as

$$R(\sigma, \sigma^{-1})w = 0, \quad (2)$$

where $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$. The behavior, $\mathfrak{B}$, of the system is equal to the kernel of $R(\sigma, \sigma^{-1})$, i.e.,

$$\mathfrak{B} = \left\{ w \in \left(\mathbb{R}^{\mathbb{Z}^n}\right)^q \,\middle|\, R(\sigma, \sigma^{-1})w = 0 \right\} = \ker\, R(\sigma, \sigma^{-1}). \quad (3)$$

A system represented as in (3) is called a *kernel representation* and $R(\xi, \xi^{-1})$ is called a *kernel representation matrix*. The terms behavior, discrete $n$-D system, or simply system are used interchangeably to denote a linear system of pdes having $n$ independent variables. We use the symbol $\mathfrak{L}^q$ to denote the set of all discrete $n$-D systems having $q$ dependent variables.

For the purpose of this paper, we use a more algebraic description of the system. For a system of pdes as in (2), let $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ denote the rowspan of $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$ over $\mathcal{A}$, i.e., $\mathcal{R} := \mathrm{rowspan}_{\mathcal{A}} R(\xi, \xi^{-1})$. Note that, $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$ is a submodule of the free module $\mathcal{A}^{1\times q}$ and is called the *equation module*. The behavior $\mathfrak{B}$, as in (3), is equivalently given by

$$\mathfrak{B} = \left\{ w \in \left(\mathbb{R}^{\mathbb{Z}^n}\right)^q \,\middle|\, r(\sigma, \sigma^{-1})w = 0 \; \forall\, r(\xi, \xi^{-1}) \in \mathcal{R} \right\} =: \mathfrak{B}(\mathcal{R}).$$

Note that, $\mathfrak{B}$ is a vector space over $\mathbb{R}$. Also, $\mathfrak{B}$ has the structure of an $\mathcal{A}$-module [21, Section 2.1].

Given an equation module $\mathcal{R} \subseteq \mathcal{A}^{1\times q}$, define the *quotient module* $\mathcal{M} := \mathcal{A}^{1\times q}/\mathcal{R}$. The quotient module $\mathcal{M}$ is naturally an $\mathcal{A}$-module by the operations of addition and scalar multiplication defined on $\mathcal{A}^{1\times q}$. Further, $\mathcal{M}$ being a module over the $\mathbb{R}$-algebra $\mathcal{A}$, $\mathcal{M}$ is naturally a vector space over $\mathbb{R}$. The action of $\mathcal{M}$ on $\mathfrak{B}$ plays an important role in this paper. Let $f \in \mathcal{A}^{1\times q}$ be such that $\overline{f} = m \in \mathcal{M}$. Then the action of $m$ on a trajectory $w \in \mathfrak{B}$ is defined as $mw := \big(f(\sigma, \sigma^{-1})w\big)$. It can be verified that this action of $\mathcal{M}$ on $\mathfrak{B}$ is well-defined.

### C. Free variables and autonomous n-D systems

For a trajectory $w = (w_1, \dots, w_q) \in \mathfrak{B}$, define the projection map

$$\Pi_{w_i} : \mathfrak{B} \to \mathbb{R}^{\mathbb{Z}^n}, (w_1, \dots, w_q) \mapsto w_i. \quad (4)$$

Then, $w_i$ is said to be a *free variable* if $\Pi_{w_i}(\mathfrak{B}) = \mathbb{R}^{\mathbb{Z}^n}$.

By an autonomous $n$-D system, we mean a discrete $n$-D system without free variables [21]. Discrete autonomous $n$-D systems have been characterized using various equivalent conditions in the literature [13], [15], [19], [22]. Some of the characterizations are summarized in Proposition 2.1 below. Proofs can be found in the above-mentioned references.

*Proposition 2.1:* Let $\mathfrak{B} \in \mathfrak{L}^q$ be a discrete $n$-D system. Then the following are equivalent:

1) $\mathfrak{B}$ is autonomous, i.e., $\mathfrak{B}$ has no free variables.
2) $\mathfrak{B} = \ker R(\sigma, \sigma^{-1})$, where $R(\xi, \xi^{-1}) \in \mathcal{A}^{\bullet\times q}$ has full column rank over $\mathcal{A}$.

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3) The quotient module $\mathcal{M}$ is a *torsion module*[2].
4) The annihilator ideal, $\mathrm{ann}_{\mathcal{A}}\mathcal{M} := \{f \in \mathcal{A} \mid fm = 0 \text{ for all } m \in \mathcal{M}\}$, is non-zero.

### D. Krull dimension of a system

Let $\mathcal{A}$ be a commutative ring. An ideal $\mathfrak{p} \subseteq \mathcal{A}$ is said to be a *prime ideal* if $\mathfrak{p}$ is not equal to the full ring and for every $f_1 f_2 \in \mathfrak{p}$ either $f_1 \in \mathfrak{p}$ or $f_2 \in \mathfrak{p}$. A chain of prime ideals in $\mathcal{A}$ of the form $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_\ell$ is said to be of length $\ell$. The Krull dimension of a ring $\mathcal{A}$ is defined to be the supremum of the lengths of chains of prime ideals in $\mathcal{A}$. The Krull dimension of modules is defined using the annihilator ideal (defined in Proposition 2.1).

*Definition 2.2:* [4, Chapter 9] The Krull dimension of an $\mathcal{A}$-module $\mathcal{M}$ is defined to be the Krull dimension of the quotient ring $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$, i.e.,

$$\text{Krull dim}\,(\mathcal{M}) := \text{Krull dim}\left(\frac{\mathcal{A}}{\mathrm{ann}_{\mathcal{A}}\mathcal{M}}\right).$$

For a discrete $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$, Krull dim$(\mathfrak{B}) :=$ Krull dim$(\mathcal{M})$.

### E. Integral ring extension

*Integrality* and *integral ring extension* play a crucial role in this paper. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be commutative rings such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ as a subring. Then an element $\alpha \in \mathcal{A}_2$ is said to be *integral* over $\mathcal{A}_1$ if $\alpha$ satisfies a monic polynomial equation with coefficients from $\mathcal{A}_1$. If every element of $\mathcal{A}_2$ is integral over $\mathcal{A}_1$ then $\mathcal{A}_2$ is said to be an *integral extension* of $\mathcal{A}_1$. Proposition 2.3 summarizes the results on integral ring extension required for this paper. (For details and proofs please see [1, Chapter 5].)

*Proposition 2.3:* Let $\mathcal{A}_1 \subseteq \mathcal{A}_2$ be commutative rings. Further, let $\mathcal{A}_2$ be a finitely generated algebra over $\mathcal{A}_1$. Then the following are equivalent.

1) $\mathcal{A}_2$ is integral over $\mathcal{A}_1$.
2) $\mathcal{A}_2$ is a finitely generated module over $\mathcal{A}_1$.

Krull dimension under integral ring extension is of special importance to us. It is well known that, when $\mathcal{A}_1 \subseteq \mathcal{A}_2$ is an integral ring extension, then Krull dimension of $\mathcal{A}_1$ is equal to the Krull dimension of $\mathcal{A}_2$ (see, for example [4, Proposition 9.2]). Proposition 2.4 states the converse relation. A proof can be found in [8, Lemma 4.6], [4, Corollary 10.13b].

*Proposition 2.4:* Let $\mathcal{A}_1 \subseteq \mathcal{A}_2$ be commutative rings. Further, let $\mathcal{A}_2$ be a finitely generated algebra over $\mathcal{A}_1$. Suppose Krull dimension of $\mathcal{A}_2$ is equal to the Krull dimension of $\mathcal{A}_1$. Then $\mathcal{A}_2$ is integral over $\mathcal{A}_1$.

*Remark 2.5:* [1, Chapter 5, Remark after Corollary 5.3] Let $g : \mathcal{A}_1 \hookrightarrow \mathcal{A}_2$ be a ring homomorphism. Then $g : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be integral if $\mathcal{A}_2$ is integral over $g(\mathcal{A}_1)$.

## III. Characterization of Initial Data Using Characteristic Sets

[2]An $\mathcal{A}$-module $\mathcal{M}$ is said to be a *torsion module* if for every element $m \in \mathcal{M}$, there exists a non-zero element $f \in \mathcal{A}$ such that $fm = 0 \in \mathcal{M}$.

*Definition 3.1:* Given a trajectory $w \in \left(\mathbb{R}^{\mathbb{Z}^n}\right)^q$ and a subset $\mathcal{C} \subseteq \mathbb{Z}^n$, the restriction of $w$ to $\mathcal{C}$, denoted by $w|_{\mathcal{C}}$, is defined as

$$w|_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}^q, k \mapsto w(k). \tag{5}$$

Applying Definition 3.1 to every trajectory $w \in \mathfrak{B}$, the restriction of $\mathfrak{B}$ to $\mathcal{C}$, denoted by $\mathfrak{B}|_{\mathcal{C}}$, is defined as

$$\mathfrak{B}|_{\mathcal{C}} := \{w|_{\mathcal{C}} \text{ such that } w \in \mathfrak{B}\}. \tag{6}$$

*Definition 3.2 ( [19]):* Consider a system $\mathfrak{B} \in \mathfrak{L}^q$, a subset $\mathcal{C}$ of $\mathbb{Z}^n$ is said to be a *characteristic set* for $\mathfrak{B}$ if for every trajectory $w \in \mathfrak{B}$, the restriction of $w$ to the set $\mathcal{C}$, allows to uniquely determine the remaining portion of $w$, i.e., $w|_{\mathbb{Z}^n \setminus \mathcal{C}}$ can be uniquely determined if $w|_{\mathcal{C}}$ is known.

At the heart of the theory presented in this paper lies a special kind of characteristic sets: sublattices of $\mathbb{Z}^n$ (see Definition 3.3 for the definition sublattices) and finite unions of their parallel translates. Such sets are particularly useful because the issue of minimality of characteristic sets gets naturally resolved via the *rank* of such sublattices (as a $\mathbb{Z}$-submodule of the free $\mathbb{Z}$-module $\mathbb{Z}^n$). For details on this property of such characteristic sets please see [8]. Importantly, it is also a fact that *every* discrete autonomous $n$-D system admits a characteristic set that is a finite union of parallel translates of a sublattice; in this regard, minimality is achieved when the sublattice is of rank equal to the Krull dimension of the autonomous system (see [8, Corollary 4.19], see also [7] for the special case of scalar systems).

By a sublattice of $\mathbb{Z}^n$, we mean a $\mathbb{Z}$-submodule of the free module $\mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a Noetherian module, every sublattice is finitely generated. Also, $\mathbb{Z}$ being a principal ideal domain, such a sublattice is freely generated [5, Chapter 3, section 7]. Therefore, we have the following definition.

*Definition 3.3:* A subset $\mathcal{S} \subseteq \mathbb{Z}^n$, is called a *sublattice of rank* $r \leqslant n$ if there exists a set $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{Z}^n$, of cardinality $r$, linearly independent over $\mathbb{Z}$, that generates $\mathcal{S}$ as a $\mathbb{Z}$-module:

$$\mathcal{S} = \{\lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_r s_r \mid \lambda_1, \ldots, \lambda_r \in \mathbb{Z}\}. \tag{7}$$

Given a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, a *parallel translate* of $\mathcal{S}$ is the set given by $\mathcal{S} + \nu := \{s + \nu \mid s \in \mathcal{S}\}$, where $\nu \in \mathbb{Z}^n$ is fixed.

*Remark 3.4:* Note that Definition 3.3 is more general than some existing notions of sublattices in the literature, for example, in [2].

Corresponding to a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, we define the *sublattice algebra*, $\mathbb{R}[\mathcal{S}]$, as

$$\mathbb{R}[\mathcal{S}] := \left\{ \sum_{\nu \in \mathcal{S}_1} \alpha_\nu \xi^\nu \;\middle|\; \mathcal{S}_1 \subseteq \mathcal{S},\ |\mathcal{S}_1| < \infty,\ \alpha_\nu \in \mathbb{R} \right\}. \tag{8}$$

In other words, $\mathbb{R}[\mathcal{S}]$ is the algebra formed by taking finite linear combinations of monomials corresponding to integer tuples in $\mathcal{S}$. Note that, $\mathbb{R}[\mathcal{S}]$ is a sub-algebra of $\mathcal{A}$.

Note that, for a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ and a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ of rank $d$, the restriction of $\mathfrak{B}$ to $\mathcal{S}$, $\mathfrak{B}|_{\mathcal{S}}$, is a $d$-D behavior [2]. For the above-mentioned special kind of characteristic sets (i.e., a finite union of a sublattice $\mathcal{S}$ and its parallel translates), it is crucial to have $\mathfrak{B}|_{\mathcal{S}}$ to be a *non-autonomous* $d$-D behavior. The importance of non-autonomy of $\mathfrak{B}|_{\mathcal{S}}$ lies in the minimality aspect of initial data.

For if $\mathfrak{B}|_{\mathcal{S}}$ is autonomous, a proper subset of $\mathcal{S}$ would be a characteristic set for $\mathfrak{B}|_{\mathcal{S}}$ and, by transitivity, the proper subset of $\mathcal{S}$ would be a characteristic set for $\mathfrak{B}$, too (see [8, Section 3.3] for more details). Note that, the equation module of $\mathfrak{B}|_{\mathcal{S}}$ as a $d$-D behavior is given by $\mathcal{R}_{\mathcal{S}} := \mathcal{R} \cap \mathbb{R}[\mathcal{S}]^{1\times q}$ (see [2]). Consequently, the corresponding quotient module is given by $\mathcal{M}_{\mathcal{S}} := \mathbb{R}[\mathcal{S}]^{1\times q} / \left(\mathcal{R} \cap \mathbb{R}[\mathcal{S}]^{1\times q}\right)$.

Before stating the result that gives a characteristic set for a discrete autonomous $n$-D system (i.e., Theorem 3.8) we go through some auxiliary results.

*Lemma 3.5:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice of rank $d$ such that $\mathfrak{B}|_{\mathcal{S}}$ is a $d$-D behavior with quotient module $\mathcal{M}_{\mathcal{S}}$. Then $\mathrm{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{M}_{\mathcal{S}} = (\mathrm{ann}_{\mathcal{A}}\mathcal{M}) \cap \mathbb{R}[\mathcal{S}]$.

**Proof:** See [8, Proposition 3.10]. □

*Proposition 3.6:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$. Then the following are equivalent.

1) $\mathfrak{B}|_{\mathcal{S}}$ is non-autonomous.
2) $(\mathrm{ann}_{\mathcal{A}}\mathcal{M}) \cap \mathbb{R}[\mathcal{S}] = \{0\}$.
3) $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a finitely generated *faithful*[3] module over $\mathbb{R}[\mathcal{S}]$.
4) $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$.

**Proof:** ($1 \Leftrightarrow 2$): Recall that, when rank of $\mathcal{S}$ is equal to $d$, $\mathfrak{B}|_{\mathcal{S}}$ is a $d$-D behavior with quotient module $\mathcal{M}_{\mathcal{S}}$. It follows from Proposition 2.1 that, $\mathfrak{B}|_{\mathcal{S}}$ is non-autonomous if and only if $\mathrm{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{M}_{\mathcal{S}} = \{0\}$. Using Lemma 3.5, it then follows that $\mathfrak{B}|_{\mathcal{S}}$ is non-autonomous if and only if $(\mathrm{ann}_{\mathcal{A}}\mathcal{M}) \cap \mathbb{R}[\mathcal{S}] = \{0\}$.

($2 \Rightarrow 3$): Consider the chain of $\mathbb{R}[\mathcal{S}]$-linear maps $\mathbb{R}[\mathcal{S}] \hookrightarrow \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$. Assuming 2, i.e., $(\mathrm{ann}_{\mathcal{A}}\mathcal{M}) \cap \mathbb{R}[\mathcal{S}] = \{0\}$, the composite map $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is injective. Thus, $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a faithful $\mathbb{R}[\mathcal{S}]$-module. Now, $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ being a finitely generated $\mathbb{R}[\mathcal{S}]$-algebra, using the rank condition ($\mathrm{rank}(\mathcal{S}) = \text{Krull}\dim(\mathcal{M})$), it follows from Proposition 2.4 that, the $\mathbb{R}$-linear map $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is integral as well. Thus, using Proposition 2.3, $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a finitely generated $\mathbb{R}[\mathcal{S}]$-module.

($3 \Rightarrow 2$): $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a faithful module over $\mathbb{R}[\mathcal{S}]$ implies that $(\mathrm{ann}_{\mathcal{A}}\mathcal{M}) \cap \mathbb{R}[\mathcal{S}] = \{0\}$.

($3 \Rightarrow 4$): Note that, $\mathcal{M}$ is naturally a finitely generated faithful module over $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$. Assuming 3, $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$. Therefore, by transitivity, $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$.

($4 \Rightarrow 3$): Since $\mathcal{M}$ contains a copy of $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ as a submodule, the fact that $\mathcal{M}$ is a finitely generated $\mathbb{R}[\mathcal{S}]$-module implies that $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ must also be a finitely generated $\mathbb{R}[\mathcal{S}]$-module. Further, $\mathcal{M}$ being a faithful $\mathbb{R}[\mathcal{S}]$-module implies that the $\mathbb{R}[\mathcal{S}]$-linear map $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is injective. Combining these two implications we have $\mathbb{R}[\mathcal{S}] \to \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ to be injective and integral. Thus from [1, Corollary 5.2], it follows that $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a finitely generated and faithful $\mathbb{R}[\mathcal{S}]$-module. □

Proposition 3.6 asserts that, when the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$, $\mathcal{M}$ is a finitely generated faithful module over $\mathbb{R}[\mathcal{S}]$. We now explicitly construct a finite generating set for $\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module. The significance of this generating set is that it enables us to construct a characteristic set of the special form for the given system. A generating set for $\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module can be constructed using a generating set for $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module as we show in Proposition 3.7 below. An important assumption required in constructing a generating set for $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module is that the sublattice $\mathcal{S}$ needs to be a *direct summand.*[4] The property of $\mathcal{S}$ being a direct summand is used to establish the fact that $\mathcal{A} = \mathbb{R}[\mathcal{S}, \mathcal{S}']$ (see [8, Lemma 4.8]). This is crucial for constructing a generating set for $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module. We do not show the construction of this generating set here; please see [8, Lemma 4.10] for the same. Let

$$\mathcal{G} := \{\overline{\xi^\nu} \mid \nu \in \Gamma \subseteq \mathbb{Z}^n\} \subseteq \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M} \tag{9}$$

denote the finite generating set for $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module. Note that, $\Gamma \subseteq \mathbb{Z}^n$ is a finite set. Using $\mathcal{G}$, Proposition 3.7 provides a generating set for $\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module.

*Proposition 3.7:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that $\mathcal{S}$ is a direct summand of $\mathbb{Z}^n$ and the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$. Further, let $\mathfrak{B}|_{\mathcal{S}}$ be non-autonomous. Then a generating set for $\mathcal{M}$ as a module over $\mathbb{R}[\mathcal{S}]$ is given by

$$\mathcal{G}_{\mathcal{M}} := \left\{ \overline{\xi^\nu e_j^T} \;\middle|\; \nu \in \Gamma \subseteq \mathbb{Z}^n, |\Gamma| < \infty, 1 \leqslant j \leqslant q \right\} \subseteq \mathcal{M}, \tag{10}$$

where $\Gamma \subseteq \mathbb{Z}^n$ is as defined in (9) and $e_j$ is the $j$-th standard basis vector in $\mathcal{A}^q$.

**Proof:** Note that, $\mathcal{M}$ is naturally finitely generated by $\left\{ \overline{e_j^T} \;\middle|\; 1 \leqslant j \leqslant q \right\}$ as a module over $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$. When $\mathfrak{B}|_{\mathcal{S}}$ is non-autonomous and the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$ it follows from Proposition 3.6 that, $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ is a finitely generated $\mathbb{R}[\mathcal{S}]$-module. Let $\mathcal{G} := \left\{ \overline{\xi^\nu} \;\middle|\; \nu \in \Gamma \subseteq \mathbb{Z}^n, |\Gamma| < \infty \right\} \subseteq \mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ be a finite generating set for $\mathcal{A}/\mathrm{ann}_{\mathcal{A}}\mathcal{M}$ as module over $\mathbb{R}[\mathcal{S}]$. Therefore, $\left\{ \overline{\xi^\nu e_j^T} \;\middle|\; \nu \in \Gamma, |\Gamma| < \infty, 1 \leqslant j \leqslant q \right\}$ is a finite generating set for $\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module. □

We now prove Theorem 3.8 that gives a characteristic set for a discrete autonomous $n$-D system.

*Theorem 3.8:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be a sublattice such that $\mathcal{S}$ is a direct summand and the rank of $\mathcal{S}$ is equal to the Krull dimension of $\mathcal{M}$. Further, let $\mathfrak{B}|_{\mathcal{S}}$ be non-autonomous. Then a union of $\mathcal{S}$ and finitely many parallel translates of $\mathcal{S}$ is a characteristic set for $\mathfrak{B}$.

**Proof:** Note that, for a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, a parallel translate of $\mathcal{S}$ is defined by $\mathcal{S} + \nu := \{s + \nu \mid s \in \mathcal{S}\}$, where $\nu \in \mathbb{Z}^n$ is fixed. Then for a finite set $\Gamma \subseteq \mathbb{Z}^n$,

$$\mathcal{C} := \bigcup_{\nu \in \Gamma} \mathcal{S} + \nu \tag{11}$$

denotes a union of $\mathcal{S}$ and finitely many parallel translates of it. To show $\mathcal{C} \subseteq \mathbb{Z}^n$ is a characteristic set for $\mathfrak{B}$, it is sufficient

[3] An $\mathcal{A}$-module $\mathcal{M}$ is said to be a *faithful module* over $\mathcal{A}$ if $\mathrm{ann}_{\mathcal{A}}\mathcal{M} = \{0\}$.

[4] A sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$ is called a *direct summand* of $\mathbb{Z}^n$ if there exists another sublattice of $\mathbb{Z}^n$, say $\mathcal{S}'$, such that $\mathbb{Z}^n = \mathcal{S} \oplus \mathcal{S}'$.

to show that for any $w \in \mathfrak{B}$, $w|_{\mathcal{C}} = 0$ implies $w \equiv 0$ ( [19, Lemma 2.3]). Let $\kappa \in \mathbb{Z}^n$ be arbitrary, then

$$w(\kappa) = \sum_{i=1}^{q} w_i(\kappa) e_i, \tag{12}$$

where $w_i(\kappa) = \left(\left(\overline{\sigma^{\kappa} e_i^T}\right) w\right)(0)$ is the $i$-th component of $w$. Using Proposition 3.7, for all $i \in \{1, 2, \dots, q\}$, $\overline{\xi^{\kappa} e_i^T}$ can be written as a finite $\mathbb{R}[\mathcal{S}]$-linear combination of elements of $\mathcal{G}_{\mathcal{M}}$, where $\mathcal{G}_{\mathcal{M}}$ is as defined in (10). Suppose

$$\overline{\xi^{\kappa} e_i^T} = \sum_{\nu \in \Gamma} \alpha_{\nu_i} m_{\nu_i}, \tag{13}$$

where $\alpha_{\nu_i} \in \mathbb{R}[\mathcal{S}]$ and $m_{\nu_i} \in \mathcal{G}_{\mathcal{M}}$.

Recall that, elements in $\mathcal{G}_{\mathcal{M}}$ are of the form $\overline{\xi^{\nu} e_j^T}$, where $\nu \in \Gamma \subseteq \mathbb{Z}^n$ and since $\alpha_{\nu_i} \in \mathbb{R}[\mathcal{S}]$, $\alpha_{\nu_i} = \sum_{\widetilde{\nu} \in \mathcal{S}} \beta_{\widetilde{\nu}} \xi^{\widetilde{\nu}}$, where $\beta_{\widetilde{\nu}} \in \mathbb{R}$. Therefore, the $i$-th component of $w$ can be written as

$$w_i(\kappa) = \left(\left(\overline{\sigma^{\kappa} e_i^T}\right) w\right)(0) = \left(\left(\begin{bmatrix} f_{i_1}(\sigma) & \dots & f_{i_q}(\sigma)\end{bmatrix}\right) w\right)(0), \tag{14}$$

where each $f_{i_j}(\xi)$ is of the form

$$f_{i_j} = \sum_{\nu \in \Gamma} \sum_{\widetilde{\nu} \in \mathcal{S}} \beta_{\widetilde{\nu}} \overline{\xi^{\widetilde{\nu}+\nu}} = \sum_{\nu \in \mathcal{C}} \widetilde{\beta}_{\nu} \overline{\xi^{\nu}},$$

where $\widetilde{\beta}_{\nu} \in \mathbb{R}$. Therefore, (14) can be written as

$$\begin{aligned} w_i(\kappa) = \left(\sum_{j=1}^{q} f_{i_j}(\sigma) w_j\right)(0) \quad &= \quad \left(\sum_{j=1}^{q} \sum_{\nu_j \in \mathcal{C}} \widetilde{\beta}_{\nu_j} \overline{\sigma^{\nu_j}} w_j\right)(0) \\ &= \quad \sum_{j=1}^{q} \sum_{\nu_j \in \mathcal{C}} \widetilde{\beta}_{\nu_j} w_j(\nu_j). \end{aligned}$$

Since $w|_{\mathcal{C}} = 0$, $w_j(\nu_j) = 0$ for all $j \in \{1, \dots, q\}$ and $\nu_j \in \mathcal{C}$. Therefore, $w_i(\kappa) = 0$ and from (12), $w(\kappa) = 0$. Thus, $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$. □

*Remark 3.9:* The existence of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, satisfying the specifications of Theorem 3.8, can always be guaranteed for any discrete autonomous $n$-D system [8, Theorem 4.18].

Stated in plain terms, if $\mathcal{C}$ is a characteristic set for $\mathfrak{B}$ then the elements in $\mathfrak{B}|_{\mathcal{C}}$ are in one-to-one correspondence with the trajectories in $\mathfrak{B}$. Thus, each element in $\mathfrak{B}|_{\mathcal{C}}$ can be viewed as an initial condition (say $w|_{\mathcal{C}}$) whence the trajectory (i.e., $w$) can be uniquely determined (as done in the proof of Theorem 3.8 above). Such an initial condition is impractical; for it is necessary to know $w$ in order to know $w|_{\mathcal{C}}$. However, upon deeper analysis, it is found that the set $\mathfrak{B}|_{\mathcal{C}}$ itself has $n$-D systems-like structure, which can be exploited to parametrize it, and thus the initial conditions $w|_{\mathcal{C}}$ can be obtained from the system of pdes without knowing $w$. We elaborate on this in the sequel.

## IV. SET OF ALLOWABLE INITIAL CONDITIONS

Given a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$, let $\mathcal{C} \subseteq \mathbb{Z}^n$, as defined in (11), denote a characteristic set for $\mathfrak{B}$ given by a union of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, of rank $d$, and finitely many parallel translates of it. Let $\gamma := |\Gamma|$. Then

$$\mathcal{C} = \bigcup_{i=1}^{\gamma} (\mathcal{S} + \nu_i), \quad \nu_i \in \Gamma. \tag{15}$$

To analyze the behavior of the system restricted to $\mathcal{C}$, we first define the restriction of $\mathfrak{B} \in \mathfrak{L}^q$ to the sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, of rank $d$. Let $\mathcal{S} \subseteq \mathbb{Z}^n$ be freely generated by $\{s_1, \dots, s_d\} \subseteq \mathbb{Z}^n$. Note that, $\mathcal{S}$ is isomorphic to $\mathbb{Z}^d$ as $\mathbb{Z}$-modules. For an element $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{Z}^d$, let $s_1\kappa_1 + \dots + s_d\kappa_d =: s$. Then the restriction of a trajectory $w \in \mathfrak{B}$ to the sublattice $\mathcal{S}$ is defined in the following manner: for $\kappa \in \mathbb{Z}^d$,

$$x_1(\kappa) := w(s_1\kappa_1 + \dots + s_d\kappa_d) = w(s) = (\sigma^s w)(0). \tag{16}$$

Note that, $x_1(\kappa)$ is a $q$-tuple of real valued sequences evolving over $\mathbb{Z}^d$, i.e., $x_1 \in (\mathbb{R}^q)^{\mathbb{Z}^d}$.

The restriction of a trajectory $w \in \mathfrak{B}$ to the parallel translate $\mathcal{S} + \nu$ is defined as: for $\kappa \in \mathbb{Z}^d$,

$$x_\nu(\kappa) := w(s_1\kappa_1 + \dots + s_d\kappa_d + \nu) = \left(\sigma^{s+\nu} w\right)(0), \tag{17}$$

where $s = \sum_{i=1}^{d} s_i\kappa_i$. Note that, $x_\nu(\kappa)$ is a $q$-tuple of real valued sequences evolving over $\mathbb{Z}^d$, i.e., $x_\nu \in (\mathbb{R}^q)^{\mathbb{Z}^d}$ and $x_\nu = w|_{\mathcal{S}+\nu}$.

For the characteristic set $\mathcal{C} \subseteq \mathbb{Z}^n$, given by (15), following (17), the restriction of a trajectory $w \in \mathfrak{B}$ to $\mathcal{C}$ is defined in the following manner: for $\kappa \in \mathbb{Z}^d$

$$x(\kappa) = \begin{bmatrix} x_1(\kappa) \\ \vdots \\ x_\gamma(\kappa) \end{bmatrix} := \left(\begin{bmatrix} \sigma^{s+\nu_1} \\ \vdots \\ \sigma^{s+\nu_\gamma} \end{bmatrix} w\right)(0) = \left(\begin{bmatrix} \overline{\sigma^{s+\nu_1}} \\ \vdots \\ \overline{\sigma^{s+\nu_\gamma}} \end{bmatrix} w\right)(0), \tag{18}$$

where $s = \sum_{i=1}^{d} s_i\kappa_i$. Here, each component of $x(\kappa)$ is a $q$-tuple of real valued sequences evolving over $\mathbb{Z}^d$. Thus, $x \in \left((\mathbb{R}^q)^{\mathbb{Z}^d}\right)^{\gamma}$ and as a whole $x$ can be identified with the restriction of a trajectory $w$ to $\mathcal{C}$, i.e., $x = w|_{\mathcal{C}}$.

Recall, from (10), the generating set, $\mathcal{G}_{\mathcal{M}}$, of $\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module. Then, (18) is rewritten by acting the elements of $\mathcal{G}_{\mathcal{M}}$ on a trajectory $w \in \mathfrak{B}$, i.e., for $\mathcal{G} := \{g_i \mid 1 \leqslant i \leqslant \gamma\}$,

$$x(\kappa) = \begin{bmatrix} x_1(\kappa) \\ \vdots \\ x_\gamma(\kappa) \end{bmatrix} = \left(\begin{bmatrix} g_1 \overline{I_q} \\ \vdots \\ g_\gamma \overline{I_q} \end{bmatrix} w\right)(0). \tag{19}$$

The collection of all trajectories in $\mathfrak{B}$ restricted to $\mathcal{C}$ forms the *set of allowable initial conditions*, i.e.,

$$\mathfrak{X} := \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_\gamma \end{bmatrix} \in \left((\mathbb{R}^q)^{\mathbb{Z}^d}\right)^{\gamma} \,\middle|\, x = \begin{bmatrix} w|_{\mathcal{S}+\nu_1} \\ \vdots \\ w|_{\mathcal{S}+\nu_\gamma} \end{bmatrix}, w \in \mathfrak{B} \right\}. \tag{20}$$

Recall, from Proposition 3.6, that $\mathcal{M}$ is a finitely generated module over $\mathbb{R}[\mathcal{S}]$. Let the finite generating set for $\mathcal{M}$ as an $\mathbb{R}[\mathcal{S}]$-module be given by $\{m_1, \dots, m_{q\gamma}\} \subseteq \mathcal{M}$. Then there exists a surjective $\mathbb{R}[\mathcal{S}]$-module homomorphism $\phi$ defined in the following manner: for $i \in \{1, \dots, q\gamma\}$, and $m_i \in \mathcal{M}$,

$$\begin{array}{rccc} \phi: & \mathbb{R}[\mathcal{S}]^{1\times q\gamma} & \twoheadrightarrow & \mathcal{M} \\ & e_i^T & \mapsto & m_i, \end{array} \tag{21}$$

where $e_i^T$ is the $i$-th standard basis vector of $\mathbb{R}[\mathcal{S}]^{1\times q\gamma}$. It follows from the first isomorphism theorem [1] that,

$$\mathcal{M} \cong \frac{\mathbb{R}[\mathcal{S}]^{1\times q\gamma}}{\ker\,\phi} \text{ as } \mathbb{R}[\mathcal{S}]\text{-modules.} \tag{22}$$

Define the $\mathbb{R}[\mathcal{S}]$-module $\mathcal{M}_\phi := \mathbb{R}[\mathcal{S}]^{1\times q\gamma}/\ker\,\phi$. Let $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$ denote the algebraic dual of $\mathcal{M}_\phi$, i.e., the set of all $\mathbb{R}$-linear maps from $\mathcal{M}_\phi$ to $\mathbb{R}$. Note that, $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$ is an $\mathbb{R}[\mathcal{S}]$-module. Theorem 4.1 shows that the set of allowable initial conditions, $\mathfrak{X}$, is isomorphic as an $\mathbb{R}[\mathcal{S}]$-module to $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$. This gives an algebraic characterization of the set of allowable initial conditions.

*Theorem 4.1:* Let $\mathfrak{X}$ be the set of allowable initial conditions as defined in (20). Let $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$ be the algebraic dual of $\mathcal{M}_\phi$, where $\mathcal{M}_\phi := \mathbb{R}[\mathcal{S}]^{1\times q\gamma}/\ker\,\phi$. Define the $\mathbb{R}[\mathcal{S}]$-module homomorphism $\Psi^\star : \mathfrak{X} \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$ in the following manner: for $x \in \mathfrak{X}$ and $f \in \mathbb{R}[\mathcal{S}]^{1\times q\gamma}$,

$$\begin{array}{rccc} \Psi^\star: & \mathfrak{X} & \to & \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R}) \\ & (\Psi^\star(x))\left(\overline{f}\right) & := & \left(f(\sigma,\sigma^{-1})(x)\right)(0). \end{array} \tag{23}$$

Then $\Psi^\star$ is an isomorphism of $\mathbb{R}[\mathcal{S}]$-modules.

**Proof:** It is easy to verify that $\Psi^\star$ is well-defined and $\mathbb{R}[\mathcal{S}]$-linear. To show $\Psi^\star$ is an isomorphism we show that $\Psi^\star$ is injective and surjective.

*($\Psi^\star$ is injective)* Let $\Psi^\star(x) = 0 \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$. This implies $(\Psi^\star(x))\left(\overline{f}\right) = 0$ for all $\overline{f} \in \mathcal{M}_\phi$. Let $\rho : \mathcal{M} \to \mathcal{M}_\phi$ denote the isomorphism in (22). Then for every element $\{m_1, \ldots, m_{q\gamma}\} \subseteq \mathcal{M}$ there exist unique elements $\rho(m_1), \ldots, \rho(m_{q\gamma})$ in $\mathcal{M}_\phi$. As $(\Psi^\star(x)) = 0 \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$, in particular $(\Psi^\star(x))(\rho(m_i)) = 0$ for all $i \in \{1, \ldots, q\gamma\}$. Therefore, for $w \in \mathfrak{B}$ and $i \in \{1, \ldots, q\gamma\}$, $m_i(w) = 0$. This implies $x \equiv 0$ .

*($\Psi^\star$ is surjective)* Note that, $\mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$ as $\mathbb{R}[\mathcal{S}]$-modules as $\mathcal{M} \cong \mathcal{M}_\phi$ ((22)). It follows from a variant of the well-known Malgrange's theorem that $\mathfrak{B} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ as $\mathcal{A}$-modules [6, Proposition 6]. Since $\mathbb{R}[\mathcal{S}] \subseteq \mathcal{A}$, $\mathfrak{B} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R})$ as $\mathbb{R}[\mathcal{S}]$-modules, as well. Therefore, we obtain the following chain of isomorphisms of $\mathbb{R}[\mathcal{S}]$-modules

$$\mathfrak{B} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R}). \tag{24}$$

To show $\Psi^\star$ is surjective, let $\mu \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R})$. Using (24), there exists a unique $w^\mu \in \mathfrak{B}$. Defining the action of the elements $\{m_1, \ldots, m_{q\gamma}\} \subseteq \mathcal{M}$ on $w^\mu$, we obtain $x \in \mathfrak{X}$. Thus $\Psi^\star(x) = \mu$. □

Now, using the variant of Malgrange's Theorem [6, Proposition 6], we define the behavior $\mathfrak{B}_\phi$ as

$$\mathfrak{B}_\phi := \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}_\phi, \mathbb{R}). \tag{25}$$

Thus, $\mathfrak{B}_\phi$ is a $d$-D behavior, where $d = \mathrm{rank}(\mathcal{S})$, with equation module $\ker\,\phi \subseteq \mathbb{R}[\mathcal{S}]^{1\times q\gamma}$ and corresponding quotient module $\mathcal{M}_\phi$. Therefore, using (25), the isomorphism in (23) can be rewritten as

$$\mathfrak{X} \cong \mathfrak{B}_\phi \text{ as } \mathbb{R}[\mathcal{S}]\text{-modules.} \tag{26}$$

Because of the above identification of the set of allowable initial conditions $\mathfrak{X}$ with a $d$-D behavior $\mathfrak{B}_\phi$, the question of parametrization, *vis-à-vis* its freeness, boils down to the question as it applies to behaviors. The parametrization of $n$-D behaviors is one of the central problems in systems theory and has been long resolved via controllability (see [15], [20], [13], [14], [21] among others). This observation plays a key role in the next section.

*Example 4.2:* We illustrate the results of this section by a simple example of a discrete 2-D autonomous system. In order to get such a system of practical importance, we consider the following system from iterative learning control (ILC) [3] described by a family of differential algebraic equations (DAEs).

$$\begin{array}{rcl} \begin{bmatrix} Y_1^{(i)}(z) \\ Y_2^{(i)}(z) \end{bmatrix} & = & \begin{bmatrix} \frac{1}{z^2+0.5z+0.5} & 0 \\ 0 & \frac{1}{z+1} \end{bmatrix} \begin{bmatrix} U_1^{(i)}(z) \\ U_2^{(i)}(z) \end{bmatrix} \\ Y_1^{(i)}(z) & = & Y_2^{(i)}(z). \end{array} \tag{27}$$

Here $i$ is the index of iteration, $Y_j(z)$ and $U_j(z)$ are $z$-transforms[5] of $y_j(k)$ and $u_j(k)$, respectively, for $j = 1, 2$. For the purpose of tracking the zero trajectory, suppose the following iterative learning control is provided $u_1^{(i)}(k) = 0$ for all $k$ and all $i$, and $u_2^{(i)}(k) = -y_2^{(i-1)}(k)$ for all $k$. With this control, the closed-loop ILC system can be rewritten as a 2-D system. Indeed, using the standard 2-D system notation $y_j(i,k)$ to denote $y_j^{(i)}(k)$, the closed-loop ILC system is equal to a 2-D system that satisfies the following equations.

$$\left.\begin{array}{rcl} y_1(i,k+2) + 0.5y_1(i,k+1) + 0.5y_1(i,k) & = & 0 \\ y_2(i,k+1) + y_2(i,k) + y_2(i-1,k) & = & 0 \\ y_1(i,k) - y_2(i,k) & = & 0. \end{array}\right\} \tag{28}$$

Defining $y(i,k) := y_1(i,k) = y_2(i,k)$ we get that the 2-D system of (28) is equivalent to the scalar 2-D behavior $\mathfrak{B} :=$

$$\left\{ y \in \mathbb{R}^{\mathbb{Z}^2} \mid \begin{bmatrix} y(i,k+2) + 0.5y(i,k+1) + 0.5y(i,k) \\ y(i,k+1) + y(i,k) + y(i-1,k) \end{bmatrix} = 0 \right\}.$$

It easily follows from these equations that $y(i,0)$ and $y(i,1)$ together forms the initial condition for the 2-D system. Equivalently, $\mathcal{C} := \{\mathrm{span}_{\mathbb{Z}}(1,0)\} \cup \{(0,1) + \mathrm{span}_{\mathbb{Z}}(1,0)\}$ is a characteristic set for $\mathfrak{B}$. However, it must be noted that $y(i,0)$ and $y(i,1)$ both cannot be chosen to be free simultaneously. Indeed, the second equation necessitates that $y(i,0)$ and $y(i,1)$ must satisfy $y(i,1) + y(i,0) + y(i-1,0) = 0$, i.e.,

$$\begin{bmatrix} (1+\sigma_1^{-1}) & 1 \end{bmatrix} \begin{bmatrix} y(i,0) \\ y(i,1) \end{bmatrix} = 0. \tag{29}$$

In the language of Theorem 4.1, equation (29) translates to

$$\mathfrak{X} \cong \ker \begin{bmatrix} (1+\sigma^{-1}) & 1 \end{bmatrix},$$

where $\mathfrak{X} := \left\{ \begin{bmatrix} y(i,0) \\ y(i,1) \end{bmatrix} \mid y \in \mathfrak{B} \right\} = \mathfrak{B}|_{\mathcal{C}}$ is the set of allowable initial conditions.

Note that, in order for the initial conditions to be free the 2-D system must admit an equation ideal that is principal (see [10, Theorem 6.10]). It can be easily verified that the 2-D behavior $\mathfrak{B}$ does not admit such an equation ideal.

[5]Note that, the $z$-transform requires signals to be defined over an infinite time horizon. However, for ILC, the signals are over finite duration. Thus, the $z$-domain representation is an approximation of the ILC system [3] which is often used in practice to derive various useful properties of the original ILC system.

## V. FREE-NESS OF THE SET OF ALLOWABLE INITIAL CONDITIONS

Recall that, for a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ having a characteristic set $\mathcal{C} \subseteq \mathbb{Z}^n$ given by a union of a sublattice $\mathcal{S} \subseteq \mathbb{Z}^n$, of rank $d$, and finitely many parallel translates of it (see (15)), the set of allowable initial conditions $\mathfrak{X}$, as defined in (20), is isomorphic as a $d$-D behavior to $\mathfrak{B}_\phi$ (defined in (25)).

*Proposition 5.1:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{C} \subseteq \mathbb{Z}^n$, given by (15), be a characteristic set for $\mathfrak{B}$ and $\mathfrak{X}$ be the set of allowable initial conditions for $\mathfrak{B}$. Then $\mathfrak{X}$ has free variable(s) if and only if $\mathcal{M}$ is a faithful module over $\mathbb{R}[\mathcal{S}]$.

**Proof:** Note that, it follows from Statement 1 of Proposition 2.1 that, a behavior is non-autonomous if and only if it has free variables. Using the isomorphism $\mathfrak{X} \cong \mathfrak{B}_\phi$, as defined in (23), $\mathfrak{X}$ having free variables is equivalent to saying that $\mathfrak{B}_\phi$ is non-autonomous. Now, $\mathrm{ann}_{\mathbb{R}[\mathcal{S}]}\mathcal{M}_\phi = \{0\}$ is equivalent to saying $\mathfrak{B}_\phi$ is non-autonomous (Statement 4 of Proposition 2.1). Using the isomorphism $\mathcal{M} \cong \mathcal{M}_\phi$ as $\mathbb{R}[\mathcal{S}]$-modules ((22)) $(\mathrm{ann}_{\mathcal{A}}\mathcal{M}) \cap \mathbb{R}[\mathcal{S}] = \{0\}$, i.e., $\mathcal{M}$ is a faithful module over $\mathbb{R}[\mathcal{S}]$. □

*Remark 5.2:* It follows from Propositions 3.6 and 5.1 that the set of allowable initial conditions always has free variables. In other words, for an autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$, the set of allowable initial conditions $\mathfrak{X}$, or equivalently $\mathfrak{B}_\phi$, is non-autonomous, i.e., there exists some freedom in choosing initial conditions.

In Theorem 5.3, using the notion of controllability, we provide a parametrization of the allowable initial conditions using free variables. We simply utilize the well-known fact that every controllable $n$-D system admits an equivalent image representation (see [21] for the $n$-D discrete case).

*Theorem 5.3:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{C} \subseteq \mathbb{Z}^n$, given by (15), be a characteristic set for $\mathfrak{B}$ and $\mathfrak{X}$ be the set of allowable initial conditions for $\mathfrak{B}$. Then the following are equivalent.

1) $\mathcal{M}$ is torsion-free as an $\mathbb{R}[\mathcal{S}]$-module.
2) $\mathfrak{X}$ is controllable.
3) $\mathfrak{X}$ has an image representation.
4) There exists a Laurent polynomial matrix $M \in \mathbb{R}[\mathcal{S}]^{q\gamma\times\bullet}$ such that $\mathfrak{X} = \left\{x = M(\sigma,\sigma^{-1})\ell \mid \ell \in \left(\mathbb{R}^{\mathbb{Z}^d}\right)^{\bullet}\right\}$.

**Proof:** Follows from the characterization of controllability; see, for example, [13], [22, Theorem 5], [21, Theorem 5], [14, Section 3, Corollary 2] among others. □

Statement 4 of Theorem 5.3 provides a parametrization of the initial conditions $w|_{\mathcal{C}}$ by a free variable $\ell$. The free variable $\ell$ is much akin to a potential function. Note, however, that this does not guarantee freeness of the initial conditions. It turns out that a stronger notion of controllability guarantees that the initial condition vector $x \in \mathfrak{X}$ can be partitioned into a free part and a non-free part such that the set $\mathfrak{X}$ can be parametrized by the free part of $x$. Once again, this notion of controllability is well-known in $n$-D systems theory; it is often called *strong controllability* (see [17], [18], [24], [23]).

*Theorem 5.4:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let $\mathcal{C} \subseteq \mathbb{Z}^n$, given by (15), be a characteristic set for $\mathfrak{B}$ and $\mathfrak{X}$ be the set of allowable initial conditions for $\mathfrak{B}$. Then the following are equivalent.

1) $\mathcal{M}$ is free as an $\mathbb{R}[\mathcal{S}]$-module.
2) $\mathfrak{X}$ is strongly controllable.
3) $\mathfrak{X}$ has an observable image representation.
4) For every allowable initial condition $x \in \mathfrak{X}$, there exists an input-output[6] partitioning of $x$ as $x = (x_u, x_y)$ such that $x_y = M_2 M_1^{-1} x_u$, where $x_u$ is free.

**Proof:** Follows from the characterization of strong controllability. □

The significance of an observable image representation is the following: $\mathfrak{X}$ has an observable image representation implies that for $x = M(\sigma,\sigma^{-1})\ell$, $M \in \mathbb{R}[\mathcal{S}]^{q\gamma\times\bullet}$ is zero-right-prime[7]. Let $\ell \in \left(\mathbb{R}^{\mathbb{Z}^d}\right)^{\lambda}$. Then the ideal formed by the $\lambda \times \lambda$ minors of $M$ is equal to $\mathbb{R}[\mathcal{S}]$. Thus, there exists a submatrix $M_1 \in \mathbb{R}[\mathcal{S}]^{\lambda\times\lambda}$ which is co-maximal with respect to all other $\lambda \times \lambda$ minors of $M$. This leads to an input-output partitioning of $x$. Let $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$. Partitioning $x$ as $x = (x_u, x_y)$, $x_y$ can be written as $x_y = M_2 M_1^{-1} x_u$, where $M_2 M_1^{-1}$ is a transfer function matrix (see [22, Section 6] for more details). The fact that the operator $M_2 M_1^{-1}$ is well-defined and is a legitimate operator follows from the fact that $M$ is zero-right-prime. Thus, $x_y$ can be inferred from $x_u$.

Note that, Theorem 5.4 provides a stronger condition as compared to Theorem 5.3. However, for the condition when the Krull dimension of the system is equal to one, the conditions given by Theorems 5.3 and 5.4 are equivalent. We state this in Corollary 5.5 below.

*Corollary 5.5:* Consider a discrete autonomous $n$-D system $\mathfrak{B} \in \mathfrak{L}^q$ with quotient module $\mathcal{M}$. Let the Krull dimension of $\mathcal{M}$ be equal to one. Let $\mathcal{C} \subseteq \mathbb{Z}^n$, given by (15), be a characteristic set for $\mathfrak{B}$ and $\mathfrak{X}$ be the set of allowable initial conditions for $\mathfrak{B}$. Then $\mathfrak{X}$ is controllable if and only if $\mathfrak{X}$ has an observable image representation.

**Proof:** The Krull dimension of $\mathcal{M}$ being equal to one implies that $\mathrm{rank}(\mathcal{S}) = 1$. Therefore, $\mathbb{R}[\mathcal{S}]$ is a principal ideal domain (PID) and thus $\mathcal{M}$ is torsion-free if and only if $\mathcal{M}$ is free as an $\mathbb{R}[\mathcal{S}]$-modules. □

*Example 5.6:* Continuing with the simplistic 2-D system of Example 4.2, recall that the set of allowable initial conditions $\mathfrak{X}$, as a 1-D behavior, is given by $\ker \begin{bmatrix}(1+\sigma^{-1}) & 1\end{bmatrix}$. This is a controllable 1-D system, therefore, it admits an observable image representation, as well as representation by a transfer function. Indeed, $\mathfrak{X} =$

[6] An input is defined to be a maximal set of free components. The notion of causality is not considered.

[7] A matrix $M \in \mathcal{A}^{g\times q}$, with $g > q$ is said to be *zero-right-prime* if all the $q \times q$ minors of $M$ do not have a common zero in $(\mathbb{C}^*)^n$.

$\left\{ \begin{bmatrix} y(\cdot,0) \\ y(\cdot,1) \end{bmatrix} = \begin{bmatrix} 1 \\ -(1+\sigma^{-1}) \end{bmatrix} \ell \mid \ell \in \mathbb{R}^{\mathbb{Z}} \right\}$ is an observable image representation. For the representation of $\mathfrak{X}$ by a transfer function, note that if $y(\cdot,1)$ is taken as input (i.e., free) and $y(\cdot,0)$ as output (i.e., non-free) then the input-output transfer function is given by $-\frac{1}{1+z^{-1}}$.

## VI. Concluding remarks

An essentially complete answer to the open problem of characterizing minimal initial data for discrete autonomous $n$-D systems was recently proposed in [8] using the notion of characteristic sets. However, the procedure of obtaining initial conditions, by assigning trajectories on a characteristic set, was not addressed. In this paper, we provided a complete answer to the question of assignability of initial conditions for a given discrete autonomous $n$-D system having a characteristic set given by a union of a sublattice and finitely many parallel translates of it. In this regard, an algebraic characterization of the set of allowable initial conditions was given. Using this characterization, it was shown that initial conditions, in general, cannot be arbitrarily chosen. This bears an analogy to choosing initial conditions for a descriptor 1-D system where initial conditions need to satisfy some algebraic constraints. For an $n$-D system a parametrization of the set of allowable initial conditions was provided which showed that every allowable initial condition can be obtained using an arbitrary function. Finally, a necessary and sufficient condition as to when trajectories can be arbitrarily assigned was provided. With these results in place, the algorithms for computing solutions of discrete $n$-D systems can be implemented in a better way.

It is evident that assigning initial conditions on a characteristic set, based on the results developed in this paper, is not always possible by inspection. Developing implementable algorithms requires the computation of syzygies, which is a standard problem in computational commutative algebra and can be done using tools, such as, Gröbner basis. This is a matter of future research. Other possible future directions include stability analysis of discrete $n$-D systems in this context of initial conditions.

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