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Algorithms for the theory of restrictions of scalar n-D systems to proper subspaces of $\mathbb{R}^n$

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Abstract In this paper, we study the restrictions of solutions of a scalar system of PDEs to a proper subspace of the domain $\mathbb{R}^n$. The object of study is associated with certain intersection ideals. In the paper, we provide explicit algorithms to calculate these intersection ideals. We next deal with when a given subspace is "free" with respect to the solution set of a system of PDEs—this notion of freeness is related to restrictions and intersection ideals. We again provide algorithms and checkable algebraic criterion to answer the question of freeness of a subspace. Finally, we provide an upper bound to the dimension of free subspaces that can be associated with the solution set of a system of PDEs.

Keywords Systems of PDEs · Restriction ideals · Computational algorithms

1 Introduction

By n-D systems, we mean systems of linear partial differential equations (PDEs) over reals in n independent variables. The word 'scalar' indicates that the number of dependent variables is just 1. As is the usual practice, we denote the independent variables by $x_1, x_2, \ldots, x_n$, and the dependent variable by w. A time-tested general approach to dealing with such systems has been to look into restrictions of the solutions to smaller subsets of the domain (that is, the n-dimensional Euclidean space $\mathbb{R}^n$). The well-known method of characteristics [see for example Renardy and Rogers (2004)] is an example of this approach. Interestingly, many key ideas in the analysis of such n-D systems fall broadly under the purview of this

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general approach of restriction. These include the theory of positivity/path-independence of various functionals (Pillai and Willems 2002), conic stability (Valcher 2001; Shankar 2000), Lyapunov-type stability of 2-D/n-D systems (Napp Avelli et al. 2011; Kojima et al. 2010), the notion of *autonomy degree* of Napp (2010), the problem of minimal initial conditions for the Cauchy problem (Zerz and Oberst 1993), etc. In this article, we look at restrictions of solutions of an n-D system to a special type of subsets of the domain $\mathbb{R}^n$, namely, its non-trivial subspaces. We provide various algebraic characterizations of such restrictions, with a focus on algorithms for computation of the relevant algebraic objects.

This article can be thought simultaneously as a companion to and an extension of Pal and Pillai (2014). In Pal and Pillai (2014) restrictions to only 1-D subspaces of $\mathbb{R}^n$ were considered. There it was shown that the dual operation to such restrictions yields the algebraic entity called *intersection ideals*. Intersection ideals are obtained from the ideal of the equations by intersecting it with a suitable subring of the ring of partial differential operators. In Pal and Pillai (2014), this suitable subring was determined by the chosen generator of the 1-D subspace to which restriction was done. In this article, we show that this operation extends to higher dimensional subspaces. We call such general intersection ideals for restrictions to r-D subspaces of $\mathbb{R}^n$ *r-D intersection ideals*. We provide Gröbner basis algorithms for computation of such r-D intersection ideals from a given equation ideal.

We also address the issue of *free subspaces* of a scalar n-D system. Scalar systems, being *autonomous systems*, have the property that its solutions are completely determined by their restrictions on a *proper subset* of $\mathbb{R}^n$ (Rocha 1990; Fornasini et al. 1993). The issue of freeness comes in connection to whether these restrictions can be arbitrary, or whether they satisfy certain equations. This issue was addressed in Pal and Pillai (2014) for restrictions of scalar n-D systems to 1-D subspaces. In this article we extend that result to higher dimensional subspaces. This notion of freeness plays an important role in half-line stability of n-D autonomous systems.

Notation: We use standard notation. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ denote the sets of real numbers, complex numbers and integers. $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{Z}^n$ denote the vector-space or free module (for $\mathbb{Z}$) with n copies of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$, respectively. We use bold-face letters to denote vectors; the components of a vector are denoted by the same letter in normal font with subscripted indices. For example, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. To denote the the ith partial derivative with respect to $\mathbf{x}$, that is, $\frac{\partial}{\partial x_i}$, we use ∂_{x_i}. For the n-tuple $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ we use ∂_x. Similarly, we use ∂_{t_i} to denote the ith partial derivative with respect to $\mathbf{t}$, that is $\frac{\partial}{\partial t_i}$, and for the r-tuple $\{\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_r}\}$, we use ∂_t. To denote multi-variable polynomial rings over $\mathbb{R}$, we use $\mathbb{R}[\partial_x]$ or $\mathbb{R}[\partial_t]$, where the variables are $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ or $\{\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_r}\}$, respectively. We consider mostly two kinds of function-spaces to look for solutions of the systems of PDEs; they are: the set of smooth functions, denoted by $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$, and that of real analytic functions of exponential type, denoted by $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$. We sometimes use $\mathcal{W}$ to denote a general function-space for discussions not specific to any particular function-space.

2 Restriction of solutions to a subspace

The central object of study in this article is the solution set of a system of linear PDEs. Following Willems' (1991) notation for 1-D systems, we denote such a solution set by $\mathfrak{B}$, and call it the *behavior* of the given system. We write a system of linear scalar PDEs in the vector form as

$$F(\partial_x)w = \begin{bmatrix} f_1(\partial_x) \\ f_2(\partial_x) \\ \vdots \\ f_d(\partial_x) \end{bmatrix} w = 0,$$

where $w \in \mathcal{W}$ and $F(\partial_x) \in \mathbb{R}[\partial_x]^{d\times 1}$. This gives us a succinct way to write down the behavior $\mathfrak{B}$ as

$$\mathfrak{B} = \{w \in \mathcal{W} \mid F(\partial_x)w = 0, F(\partial_x) \in \mathbb{R}[\partial_x]^{d\times 1}\} = \ker(F(\partial_x)). \tag{1}$$

Since we do not put any restriction (for example degree constraints) on the operators, we require $\mathcal{W}$ to be a module over $\mathbb{R}[\partial_x]$, where scalar multiplication is defined by partial differentiation. Note that both the function-spaces $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ have this property.

Equation (1) is called a *kernel representation* of $\mathfrak{B}$. Many different systems of equations may result in the same $\mathfrak{B}$. In fact, if $\widetilde{F}(\partial_x) \in \mathbb{R}[\partial_x]^{\widetilde{d}\times 1}$ is such that the ideal generated by the entries of $\widetilde{F}(\partial_x)$ is exactly equal to that generated by the entries of $F(\partial_x)$ then $\ker(\widetilde{F}(\partial_x)) = \ker(F(\partial_x))$. It is indeed the ideal generated by the equations that determines the behavior. This ideal generated by the equations plays an important role throughout this article; it is called the *equation ideal* of $\mathfrak{B}$. We denote the equation ideal by $\mathcal{I}$.

Although the non-uniqueness of kernel representation can be avoided by considering equation ideals, instead of only equations, still there can be distinct equation ideals resulting in the same behavior. This non-uniqueness is a consequence of the function-space. It follows from Oberst's theorem (Oberst 1990) that for both the function-spaces we are considering—that is, $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$—such a situation does not arise. In other words, we get a one-to-one correspondence between ideals of $\mathbb{R}[\partial_x]$ and scalar n-D behaviors. This allows us to associate another algebraic object with a behavior: the *quotient ring*, $\mathcal{M} := \mathbb{R}[\partial_x]/\mathcal{I}$.

The principal object of study in this article is the method of restriction of trajectories in a given behavior $\mathfrak{B}$ to a nontrivial subspace $\mathscr{S} \subseteq \mathbb{R}^n$. We now formalize the notion of restriction. Suppose $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$, and $\mathscr{S} \subseteq \mathbb{R}^n$ a subspace of dimension r. We denote the restriction of w to $\mathscr{S}$ as $w|_{\mathscr{S}}$ and define it as a function from $\mathscr{S}$ to $\mathbb{R}$ in the following manner

$$\begin{aligned} w|_{\mathscr{S}} : \mathscr{S} &\to \mathbb{R} \\ v &\mapsto w(v). \end{aligned}$$

By fixing a basis for $\mathscr{S}$, $w|_{\mathscr{S}}$ can be viewed as an element of $\mathfrak{C}^\infty(\mathbb{R}^r, \mathbb{R})$. This is done as follows. Suppose $S \in \mathbb{R}^{n\times r}$ is such that its columns form a basis of $\mathscr{S}$. So every element of $\mathscr{S}$ can be written uniquely as $S\mathbf{t}$ where $\mathbf{t} \in \mathbb{R}^r$ is a real parameter. This way $w|_{\mathscr{S}}$ can be identified uniquely with the function $\widetilde{w} \in \mathfrak{C}^\infty(\mathbb{R}^r, \mathbb{R})$ as

$$\widetilde{w}(\mathbf{t}) = (w|_{\mathscr{S}})(S\mathbf{t}) = w(S\mathbf{t}).$$

The smoothness of $\widetilde{w}$ follows from that of w.

It is important to note that it is possible that with a different choice of basis for $\mathscr{S}$, the same $w|_{\mathscr{S}}$ gets identified with a different $\widetilde{w}$. However, these distinct $\widetilde{w}$'s are equivalent modulo a coordinate change on $\mathbb{R}^r$. In this article, we do not attempt at making these restrictions representation-free. We assume that $\mathscr{S}$ is always specified by a fixed matrix S. Hence, we take the following as a definition of restriction:

$$(w|_{\mathscr{S}})(\mathbf{t}) := w(S\mathbf{t}) \in \mathfrak{C}^\infty\left(\mathbb{R}^r, \mathbb{R}\right). \tag{2}$$

With Eq. (2) we now define an important object central to this article.

Definition 1 Let $\mathfrak{B}$ be the behavior of a given system of scalar PDEs, and let $\mathscr{S} \subseteq \mathbb{R}^n$ be a subspace with dimension equal to r. Further, let the columns of $S \in \mathbb{R}^{n\times r}$ be a basis of $\mathscr{S}$. Then by $\mathfrak{B}|_{\mathscr{S}}$ we denote the following set of trajectories in $\mathfrak{C}^\infty(\mathbb{R}^r, \mathbb{R})$ and call it the *$\mathscr{S}$-restricted behavior*:

$$\mathfrak{B}|_{\mathscr{S}} := \{w|_{\mathscr{S}} \mid w \in \mathfrak{B}\} \subseteq \mathfrak{C}^\infty\left(\mathbb{R}^r, \mathbb{R}\right) \tag{3}$$

Example 2 Consider the 3-D system of equations $(\partial_{x_1} - \alpha_1)w = (\partial_{x_2} - \alpha_2)w = (\partial_{x_3} - \alpha_3)w = 0$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Then every solution $w \in \mathfrak{B}$ is of the form

$$w(x_1, x_2, x_3) = ke^{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}$$

for some constant $k \in \mathbb{R}$. Suppose $\mathscr{S} = \text{span}\{e_1, e_1 + e_2\}$, where e_1, e_2 are the first two standard basis vectors of $\mathbb{R}^3$. Then trajectories in $\mathfrak{B}|_{\mathscr{S}}$ are given by

$$w|_{\mathscr{S}}(t_1, t_2) = ke^{\alpha_1 t_1 + (\alpha_1+\alpha_2)t_2}$$

for $k \in \mathbb{R}$.

3 Intersection ideal

It was shown in Pal and Pillai (2014) that for the case when dimension of $\mathscr{S}$ is equal to 1, the $\mathscr{S}$-restricted behavior is always contained in a certain 1-D behavior. The equation ideal of this 1-D behavior is obtained by intersecting the original equation ideal with a certain subring of $\mathbb{R}[\partial_x]$ determined by the spanning vector of $\mathscr{S}$. In this section we extend this result to the case when dim $\mathscr{S} > 1$. In order to do this, we require the following construction.

Let $\mathscr{S} \subseteq \mathbb{R}^n$ have a basis $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{R}^n$. These basis vectors can be used to define an $\mathbb{R}$-algebra homomorphism $\Phi_{\mathscr{S}} : \mathbb{R}[\partial_t] \to \mathbb{R}[\partial_x]$ as

$$\Phi_{\mathscr{S}} : f(\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_r}) \mapsto f(s_1^{\mathrm{T}}\partial_x, s_2^{\mathrm{T}}\partial_x, \ldots, s_r^{\mathrm{T}}\partial_x). \tag{4}$$

With the help of this map $\Phi_{\mathscr{S}}$ we now define the $\mathscr{S}$-intersection ideal of a given ideal $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$.

Definition 3 Let $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ be an ideal, and $\mathscr{S} \subseteq \mathbb{R}^n$ a subspace with basis $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{R}^n$. Then the $\mathscr{S}$-intersection ideal $\mathcal{I}_{\mathscr{S}} \subseteq \mathbb{R}[\partial_t]$ is defined as:

$$\mathcal{I}_{\mathscr{S}} := \{f(\partial_t) \mid \Phi_{\mathscr{S}}(f(\partial_t)) \in \mathcal{I}\} \subseteq \mathbb{R}[\partial_t]. \tag{5}$$

Remark 4 Note that Definition 3 does not explicitly state that $\mathcal{I}_{\mathscr{S}}$ is an *ideal* of $\mathbb{R}[\partial_t]$. However, it is a consequence of the definition of $\Phi_{\mathscr{S}}$ that $\mathcal{I}_{\mathscr{S}}$ as defined above is indeed an ideal of $\mathbb{R}[\partial_t]$.

Related with $\mathcal{I}_{\mathscr{S}}$ is the following r-D behavior.

$$\mathfrak{B}_{\mathscr{S}} := \{w \in \mathfrak{C}^\infty\left(\mathbb{R}^r, \mathbb{R}\right) \mid f(\partial_t)w = 0 \text{ for all } f(\partial_t) \in \mathcal{I}_{\mathscr{S}}\}. \tag{6}$$

Equivalently, $\mathfrak{B}_{\mathscr{S}}$ can be defined as

$$\mathfrak{B}_{\mathscr{S}} := \ker(F_{\mathscr{S}}(\partial_t)), \tag{7}$$

where $F_{\mathscr{S}}(\partial_t) \in \mathbb{R}[\partial_t]^{\bullet\times 1}$ is such that its rows generate $\mathcal{I}_{\mathscr{S}}$ as an ideal of $\mathbb{R}[\partial_t]$. Since $\mathbb{R}[\partial_t]$ is a noetherian ring, $\mathcal{I}_{\mathscr{S}}$ is finitely generated by elements from $\mathbb{R}[\partial_t]$. These elements constitute the column vector $F_{\mathscr{S}}(\partial_t)$. With this we now state our first main result.

Theorem 5 *Let $\mathfrak{B}$ be a scalar n-D behavior with equation ideal $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$. Further, let $\mathscr{S} \subseteq \mathbb{R}^n$ be an r-dimensional subspace with basis $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{R}^n$. Suppose $\mathfrak{B}|_{\mathscr{S}}$ is the $\mathscr{S}$-restricted behavior, and let $\mathcal{I}_{\mathscr{S}} \subseteq \mathbb{R}[\partial_t]$ and $\mathfrak{B}_{\mathscr{S}} \subseteq \mathfrak{C}^{\infty}(\mathbb{R}^r, \mathbb{R})$ be as defined by Eqs. (5) and (6), respectively. Then we have the following:*

$$\mathfrak{B}|_{\mathscr{S}} \subseteq \mathfrak{B}_{\mathscr{S}}. \tag{8}$$

Proof Let $v \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ be a typical element in $\mathfrak{B}|_{\mathscr{S}}$. This means there exists $w \in \mathfrak{B}$ such that $v(\mathbf{t}) = w(S\mathbf{t})$, where $S \in \mathbb{R}^{n\times r}$ is the matrix with $\{s_1, s_2, \ldots, s_r\}$ as its columns. In order to show that $v \in \mathfrak{B}_{\mathscr{S}}$ it is enough that we show $f(\partial_t)v = 0$ for all $f(\partial_t) \in \mathcal{I}_{\mathscr{S}}$. First observe that for any $w \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ and any $g(\partial_t) \in \mathbb{R}[\partial_t]$ we have the following identity by applying chain rule of differentiation:

$$g(\partial_t)(w(S\mathbf{t})) = (\Phi_{\mathscr{S}}(g(\partial_t)))w)(S\mathbf{t}). \tag{9}$$

It then follows from Eq. (9) that for $f(\partial_t) \in \mathcal{I}_{\mathscr{S}}$ and $v = w|_{\mathscr{S}} \in \mathfrak{B}|_{\mathscr{S}}$ we have

$$\begin{aligned} f(\partial_t)(w|_{\mathscr{S}}(\mathbf{t})) = f(\partial_t)w(S\mathbf{t}) &= (\Phi_{\mathscr{S}}(f(\partial_t)))w)(S\mathbf{t}) \\ &= 0, \end{aligned}$$

as $f(\partial_t) \in \mathcal{I}_{\mathscr{S}}$ implies $\Phi_{\mathscr{S}}(f(\partial_t)) \in \mathcal{I}$ and $g(\partial_x)w = 0$ for all $g(\partial_x) \in \mathcal{I}$. □

Example 6 Consider the situation of Example 2. Here the $\mathscr{S}$-intersection ideal is given by

$$\mathcal{I}_{\mathscr{S}} = \langle(\partial_{t_1} - \alpha_1), (\partial_{t_2} - \alpha_1 - \alpha_2)\rangle.$$

Clearly, the behavior of $\mathcal{I}_{\mathscr{S}}$ contains the $\mathscr{S}$-restricted behavior $\mathfrak{B}|_{\mathscr{S}}$. In fact, in this case, the two are equal. This is a consequence of the fact that $\mathfrak{B}$ is finite dimensional. See Pal and Pillai (2014) where this result was derived for the case when $\dim \mathscr{S} = 1$.

Remark 7 At this point, we would like to emphasize an important open issue in connection with the contrast between the result of Theorem 5 and its discrete systems analogue. It has been shown in Napp (2010) that for discrete systems (that is, nD systems with $\mathbb{Z}^n$ as the domain), the restriction of a behavior $\mathfrak{B}$ to any proper sublattice of $\mathbb{Z}^n$ is also a behavior. Further, it was also shown that the restricted behavior is equal to the behavior given by the corresponding intersection submodule. For continuous systems, however, it is not clear whether the inclusion, $\mathfrak{B}|_{\mathscr{S}} \subseteq \mathfrak{B}_{\mathscr{S}}$, shown in Theorem 5 is a proper inclusion or not. In fact, it is not even clear whether $\mathfrak{B}|_{\mathscr{S}}$ is a behavior. Extension of the methods used in Napp (2010) to show that $\mathfrak{B}|_{\mathscr{S}}$ is a behavior in the discrete setting (Theorem 6) do not work for the continuous case. This is due to the fact, that showing $\mathfrak{B}|_{\mathscr{S}}$ is a closed set in the $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$-topology (the topology of uniform convergence over compact sets), does not guarantee $\mathfrak{B}|_{\mathscr{S}}$ is a behavior. For continuous systems, it was shown in Willems (1991, Footnote 12) a closed shift-invariant set need not be a differential behavior. Thus, strengthening the result of Theorem 5 remains an important open issue.

Remark 8 However, it can be shown that $\mathfrak{B}_{\mathscr{S}}$ happens to be the 'smallest' differential behavior containing $\mathfrak{B}|_{\mathscr{S}}$. The proof follows essentially by the techniques used in proving Theorem 7 of Napp (2010). We elaborate on this now. By the 'smallest' behavior containing $\mathfrak{B}|_{\mathscr{S}}$ we mean that $\mathfrak{B}_{\mathscr{S}}$ is such that if $\widetilde{\mathfrak{B}}$ is any r-D behavior that contains $\mathfrak{B}|_{\mathscr{S}}$ as a subset (r is the dimension of $\mathscr{S}$) then $\widetilde{\mathfrak{B}}$ must also contain $\mathfrak{B}_{\mathscr{S}}$ as a *subbehavior*. In order to show this, let us first define

$$\mathcal{I}(\mathfrak{B}|_{\mathscr{S}}) := \{f(\partial_t) \in \mathbb{R}[\partial_t] \mid f(\partial_t)w|_{\mathscr{S}} = 0 \text{ for all } w \in \mathfrak{B}\}.$$

By the inclusion reversing one-to-one correspondence between scalar behaviors and ideals of $\mathbb{R}[\partial_x]$, the claim of $\mathfrak{B}_{\mathscr{S}}$ being the smallest behavior containing $\mathfrak{B}|_{\mathscr{S}}$ follows if we show that $\mathcal{I}(\mathfrak{B}|_{\mathscr{S}}) = \mathcal{I}_{\mathscr{S}}$. Note that Theorem 5 essentially means that $\mathcal{I}(\mathfrak{B}|_{\mathscr{S}}) \supseteq \mathcal{I}_{\mathscr{S}}$. Therefore, it is enough that we show $\mathcal{I}(\mathfrak{B}|_{\mathscr{S}}) \subseteq \mathcal{I}_{\mathscr{S}}$. Suppose $f(\partial_t) \in \mathcal{I}(\mathfrak{B}|_{\mathscr{S}})$. So, for all $w \in \mathfrak{B}$, we must have by the chain rule of differentiation

$$f(\partial_t)(w|_{\mathscr{S}}) = (\Phi_{\mathscr{S}}(f(\partial_t))w)\,|_{\mathscr{S}} = 0. \tag{10}$$

Suppose that $f(\partial_t) \notin \mathcal{I}_{\mathscr{S}}$. This means $\Phi_{\mathscr{S}}(f(\partial_t)) = f(s_1^{\mathrm{T}}\partial_x, s_2^{\mathrm{T}}\partial_x, \ldots, s_r^{\mathrm{T}}\partial_x) \notin \mathcal{I}$. Therefore, there exist a $w \in \mathfrak{B}$ and a $v \in \mathbb{R}^n$ such that $(\Phi_{\mathscr{S}}(f(\partial_t))w)\,(v) \neq 0$. Now, since $\mathfrak{B}$ is given by PDEs with constant coefficients, it is shift-invariant. Hence, the trajectory $\widetilde{w}$—a shifted version of w by v, that is $\widetilde{w}(\mathbf{x}) := w(\mathbf{x}+v)$ for all $\mathbf{x} \in \mathbb{R}^n$—too is an element of $\mathfrak{B}$. Therefore, $(\Phi_{\mathscr{S}}(f(\partial_t))\widetilde{w})\,(\mathbf{0}) = (\Phi_{\mathscr{S}}(f(\partial_t))w)\,(v) \neq 0$. Since $\mathbf{0} \in \mathscr{S}$, this means $\widetilde{w} \in \mathfrak{B}$ is such that $(\Phi_{\mathscr{S}}(f(\partial_t))\widetilde{w})\,|_{\mathscr{S}}$ is not the zero trajectory. This is a contradiction to Eq. (10).

We would like to point out that this issue of whether $\mathfrak{B}_{\mathscr{S}}$ is the smallest behavior containing $\mathfrak{B}|_{\mathscr{S}}$ for a 1-D subspace $\mathscr{S}$ was also raised in Pal and Pillai (2014, Remark 7). The discussion above settles this issue.

In the rest of this section we provide algorithms for computing the $\mathscr{S}$-intersection ideal from a given ideal $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ and a subspace $\mathscr{S} \subseteq \mathbb{R}^n$. The method of obtaining $\mathcal{I}_{\mathscr{S}}$ relies crucially on the following observation.

Proposition 9 *Suppose $\mathcal{I}$ is given by the row-span of the matrix $F(\partial_x) \in \mathbb{R}[\partial_x]^{g\times 1}$. Also suppose that the subspace $\mathscr{S}$ has basis $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{R}^n$. Then consider the following column-vector:*

$$F_{\mathrm{col}}(\partial_x, \partial_t) := \begin{bmatrix} \partial_{t_1} - s_1^{\mathrm{T}}\partial_x \\ \partial_{t_2} - s_2^{\mathrm{T}}\partial_x \\ \vdots \\ \partial_{t_r} - s_r^{\mathrm{T}}\partial_x \end{bmatrix} \in \mathbb{R}[\partial_x, \partial_t]^{r\times 1}.$$

And now, consider the following augmented matrix

$$F_{\mathrm{aug}}(\partial_x, \partial_t) := \begin{bmatrix} F(\partial_x) \\ F_{\mathrm{col}}(\partial_x, \partial_t) \end{bmatrix} \in \mathbb{R}[\partial_x, \partial_t]^{(g+r)\times 1}. \tag{11}$$

Let the ideal generated by the entries of $F_{\mathrm{aug}}(\partial_x, \partial_t)$ over the bigger ring $\mathbb{R}[\partial_x, \partial_t]$ be called $\mathcal{I}_{\mathrm{aug}}$. Then

$$\mathcal{I}_{\mathscr{S}} = \mathcal{I}_{\mathrm{aug}} \cap \mathbb{R}[\partial_t].$$

Proof Suppose $f(\partial_t) \in \mathbb{R}[\partial_t]$. Consider the polynomial

$$q(\partial_x, \partial_t) := f(\partial_t) - \Phi_{\mathscr{S}}(f(\partial_t)).$$

Note that $q(\partial_x, \partial_t)$ vanishes at $\{\partial_{t_i} = s_i^{\mathrm{T}}\partial_x\}_{i\in\{1,2,\ldots,r\}}$. Hence it follows that there exist polynomials $\alpha_i(\partial_x, \partial_t) \in \mathbb{R}[\partial_x, \partial_t]$ for $i \in \{1, 2, \ldots, r\}$ such that

$$q(\partial_x, \partial_t) = f(\partial_t) - \Phi_{\mathscr{S}}(f(\partial_t)) = \sum_{i=1}^{r} \alpha_i(\partial_x, \partial_t)(\partial_{t_i} - s_i^{\mathrm{T}}\partial_x). \tag{12}$$

Since the right most side of Eq. (12) is in $\mathcal{I}_{\mathrm{aug}}$, it follows that $f(\partial_t) \in \mathcal{I}_{\mathrm{aug}}$ if and only if $\Phi_{\mathscr{S}}(f(\partial_t)) \in \mathcal{I}_{\mathrm{aug}}$. Note that $f(\partial_t) \in \mathbb{R}[\partial_t]$ and $\Phi_{\mathscr{S}}(f(\partial_t)) \in \mathbb{R}[\partial_x]$. Therefore, (12) in fact implies that $f(\partial_t) \in \mathcal{I}_{\mathrm{aug}} \cap \mathbb{R}[\partial_t]$ if and only if $\Phi_{\mathscr{S}}(f(\partial_t)) \in \mathcal{I}_{\mathrm{aug}} \cap \mathbb{R}[\partial_x]$. The result then follows by noting $\mathcal{I}_{\mathrm{aug}} \cap \mathbb{R}[\partial_x] = \mathcal{I}$. □

Proposition 10 below is a well-known result in Gröbner basis theory. It is usually called the Elimination Theorem; see Cox et al. (2007) for a proof. We use this result to give an algorithm for computing intersection ideals.

Proposition 10 (Elimination Theorem) *Let* $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ *be an ideal, and* $\prec$ *a term ordering in* $\mathbb{R}[\partial_x]$. *Further, let* $\mathcal{G} \subseteq \mathbb{R}[\partial_x]$ *be a Gröbner basis of* $\mathcal{I}$ *under the term ordering* $\prec$. *Suppose* $\prec$ *is such that* $\partial_i \prec \partial_j$ *for all* $1 \leqslant i < j \leqslant n$. *Then* $\mathcal{G}_r := \mathcal{G} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}]$ *is a Gröbner basis for* $\mathcal{I}_r = \mathcal{I} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}]$.

Utilizing the result of Proposition 9 and the Elimination Theorem (Proposition 10) the following algorithm can be given for computing the $\mathscr{S}$-intersection ideal $\mathcal{I}_{\mathscr{S}}$.

Algorithm 11 (computation of $\mathcal{I}_{\mathscr{S}}$)
Input:

- $F(\partial_x) \in \mathbb{R}[\partial_x]^{g\times 1}$.
- $\mathscr{S} \subseteq \mathbb{R}^n$ with a basis $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{R}^n$ ($r = \dim \mathscr{S}$).

Computation:

- Form the matrix $F_{\text{aug}}(\partial_x, \partial_t) \in \mathbb{R}[\partial_x, \partial_t]^{(g+r)\times 1}$.
- Fix a term ordering $\prec$ in $\mathbb{R}[\partial_x, \partial_t]$ such that $\partial_{t_j} \prec \partial_{x_i}$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$.
- With this ordering of $\mathbb{R}[\partial_x, \partial_t]$ compute a Gröbner basis $\mathcal{G} \subseteq \mathbb{R}[\partial_x, \partial_t]$ of the ideal $\mathcal{I}_{\text{aug}} = \text{rowspan}(F_{\text{aug}}(\partial_x, \partial_t))$ over $\mathbb{R}[\partial_x, \partial_t]$.
- Define $\mathcal{G}_{\mathscr{S}} := \mathcal{G} \cap \mathbb{R}[\partial_t]$.

Output: The column-vector $F_{\mathscr{S}} \in \mathbb{R}[\partial_t]^{\bullet\times 1}$ whose entries are the elements of $\mathcal{G}_{\mathscr{S}}$.

4 Free subspaces

In this section we explore the notion of free subspaces of scalar n-D systems. Scalar n-D systems are a special class of what are called *autonomous systems*. Autonomous systems are characterized by the property that their behaviors (that is, the sets of solutions) are completely determined by their restrictions to a *proper subset* of $\mathbb{R}^n$ (Rocha 1990; Valcher 2001). We say that a given subspace $\mathscr{S} \subseteq \mathbb{R}^n$ is *free* in a given autonomous system if the restriction of its behavior $\mathfrak{B}$ to the subspace $\mathscr{S}$, that is, $\mathfrak{B}|_{\mathscr{S}}$, is the whole function-space $\mathcal{W}$ with domain $\mathbb{R}^r$, r being the dimension of $\mathscr{S}$. The notion of free subspaces of an autonomous system is important from the point of view of explicitly solving for the given system of equations. It gives us an idea about the 'size' of the initial data required for a complete description of the behavior. For example, if the restriction of a behavior to an r dimensional subspace is free then the initial data set is at least as big as the function space with $\mathbb{R}^r$ as the domain. See Wood et al. (1998) where these questions have been dealt with in the context of discrete systems. We make the notion of free subspaces precise below, but before that a few technical details are in order for further development.

In this section we consider a subspace of smooth functions for the solutions, namely real analytic functions of exponential type, in short, *exponential functions*. We denote this set by $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$. Elements in $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ are characterized by possessing convergent power series expansions. That is, $w(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ if $w(\mathbf{x})$ can be written in the following form

$$w(\mathbf{x}) = \sum_{\nu\in\mathbb{N}^n} a_\nu \frac{\mathbf{x}^\nu}{\nu!}, \tag{13}$$

where $a_\nu \in \mathbb{R}$ for all ν, and the symbols $\mathbf{x}^\nu$, $\nu!$ are shorthand notation for $x_1^{\nu_1} x_2^{\nu_2} \ldots x_n^{\nu_n}$, $\nu_1! \nu_2! \ldots \nu_n!$, respectively. Moreover, the power series expression on the right-hand-side of Eq. (13) is such that $w(\mathbf{x}) \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Definition 12 Let $\mathscr{S} \subseteq \mathbb{R}^n$ be a subspace of dimension $r \neq 0$. Then a given autonomous system with behavior $\mathfrak{B}$ is said to be *$\mathscr{S}$-free* if

$$\mathfrak{B}|_{\mathscr{S}} = \mathfrak{Exp}\left(\mathbb{R}^r, \mathbb{R}\right).$$

In this case $\mathscr{S}$ is said to be *free in* $\mathfrak{B}$.

4.1 Characterization of free subspaces of scalar systems

We now characterize free subspaces for scalar systems. This characterization is based on an algorithm for finding formal power series solutions to a scalar n-D system using a Gröbner basis of the equations. This algorithm has been elaborated in Oberst (1990, 2006). Algorithm 14 is a short description of this method for formal integration of PDEs. In Pal and Pillai (2014), free 1-D subspaces for scalar systems were characterized using the same strategy. In this article, we extend the result to subspaces of higher dimensions.

In the above-mentioned method, first a Gröbner basis $\mathcal{G}$, of the equation ideal $\mathcal{I}$ is computed for some fixed term ordering, say $\prec$. We denote by $\text{in}_\prec(\mathcal{I})$ the initial ideal of $\mathcal{I}$ with respect to the term ordering $\prec$. The monomials *not* belonging to $\text{in}_\prec(\mathcal{I})$ are called the *standard monomials*. We denote the set of standard monomials by $\Gamma_\prec(\mathcal{I})$. (Note that there is a bijection between monomials in $\mathbb{R}[\partial_x]$ and the lattice of non-negative integers $\mathbb{N}^n$. We often consider $\Gamma_\prec(\mathcal{I}) \subseteq \mathbb{N}^n$ without explicitly mentioning it since there is no risk of ambiguity). In Pommaret (1994), Oberst (2006), an idea similar to Algorithm 14 was traced back to the works of Riquier (1910). Interestingly, in the 1920's, French mathematician Maurice Janet obtained algorithmic techniques for solving nonlinear PDEs, which, when applied to linear PDEs, turn out to be equivalent to the above mentioned method based on Gröbner basis (Plesken and Robertz 2005). This fact, too, was brought out in Pommaret (1994). We provide below a sketch of the working principle of this algorithm; details can be found in Oberst (1990), Oberst (2006), Pommaret (1994), Plesken and Robertz (2005).

The idea behind Algorithm 14 stems from the algebraic fact that each element in $\mathbb{R}[\partial_x]$, modulo the ideal $\mathcal{I}$, can be written as a unique $\mathbb{R}$-linear combination of the standard monomials. Recall that every exponential solution can be written as a convergent power series

$$w(\mathbf{x}) = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu,$$

where $w_\nu \in \mathbb{R}$. This means that, for any $\nu \in \mathbb{N}^n$, the action of the monomial ∂_x^ν on w must follow the equation

$$\left(\partial_x^\nu w\right)(\mathbf{0}) = w_\nu.$$

Now suppose the monomial ∂_x^ν, upon division by the Gröbner basis $\mathcal{G}$, reduces to $\sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} \partial_x^{\nu'}$. In other words,

$$\partial_x^\nu = \sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} \partial_x^{\nu'} + q(\partial_x),$$

where $q(\partial_x) \in \mathcal{I}$. It then follows that if w is a solution to the given set of PDEs it must satisfy the following: for all $\nu \in \mathbb{N}^n$,

$$\begin{aligned} w_\nu = \left(\partial_x^\nu w\right)(\mathbf{0}) &= \left(\sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} \partial_x^{\nu'} + q(\partial_x) \right) w(\mathbf{0}) \\ &= \left(\sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} \partial_x^{\nu'} \right) w(\mathbf{0}) \ \ (\text{since } q(\partial_x) \in \mathcal{I}, q(\partial_x) w = 0) \\ &= \sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} w_{\nu'}. \end{aligned} \tag{14}$$

On the other hand, a power series $w(\mathbf{x}) = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu$ that satisfies Eq. (14) will be a solution to the given set of PDEs. This essentially follows from the fact that the standard monomials form a *basis* for the quotient ring $\mathbb{R}[\partial_x]/\mathcal{I}$ as a vector space over $\mathbb{R}$. This is the key idea behind the following Algorithm 14. In the sequel, for notational convenience, we use just Γ to denote $\Gamma_\prec(\mathcal{I})$ when the ideal and the term ordering are clear from the context.

Algorithm 13 (Gröbner basis of equations)
Input: A set of PDEs $f_1(\partial_x)w = 0, f_2(\partial_x)w = 0, \ldots, f_d(\partial_x)w = 0$.
Computation:

- Fix a term ordering $\prec$ in $\mathbb{R}[\partial_x]$.
- Compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} := \langle f_1, f_2, \ldots, f_d \rangle$.
- Construct the set of standard monomials $\Gamma := \{\nu \in \mathbb{N}^n \mid \partial_x^\nu \notin \text{in}_\prec(\mathcal{I})\}$.

Output: *Standard monomial set* Γ.

Algorithm 14 (Formal integration of the PDEs)
Input:

- A Gröbner basis $\mathcal{G}$ of the equation ideal $\mathcal{I}$.
- The set of standard monomials $\Gamma := \{\nu \in \mathbb{N}^n \mid \partial_x^\nu \notin \text{in}_\prec(\mathcal{I})\}$.
- Initial data: $\{w_\nu \in \mathbb{R}\}_{\nu \in \Gamma}$.

Computation:
for $\nu \notin \Gamma$

- Compute by division algorithm using $\mathcal{G}$ to obtain

$$\partial_x^\nu \equiv \sum_{i=1, \nu_i \in \Gamma} \alpha_i \partial_x^{\nu_i} \text{ modulo } \mathcal{I}.$$

- Set $w_\nu = \sum_{i=1}^k \alpha_i w_{\nu_i}$.

end
Output: The sequence[1] $w := \{w_\nu\}_{\nu \in \mathbb{N}^n}$.

In Oberst (1990, 2006) it was shown that the output of the above algorithm, written as a formal power series $w(\mathbf{x}) = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu$, is indeed a solution to the given set of PDEs. And, conversely, every formal power series solution is obtained from this algorithm by giving

[1] We came to know from an anonymous reviewer that Janet had obtained an explicit way of returning this (usually infinite) sequence.

different initial conditions $\{w_\nu\}_{\nu\in\Gamma}$. However, Algorithm 14 says nothing about *convergence* of the solution. In Oberst (2006), Oberst and Pauer (2001), it was proved that if the initial data itself is an exponential trajectory then the solution obtained following Algorithm 14 is guaranteed to be an exponential one. We paraphrase this result in the following proposition.[2]

Proposition 15 (Oberst and Pauer 2001, Theorems 24 and 26) *Given a set of PDEs* $f_1(\partial_x)w = 0,\ f_2(\partial_x)w = 0, \ldots, f_d(\partial_x)w = 0$*, and a term ordering* $\prec$ *of* $\mathbb{R}[\partial_x]$*, let* Γ *be the set of standard monomials, that is, monomials that do not belong to* $\text{in}_\prec(\langle f_1, f_2, \ldots, f_d\rangle)$*. Further, let* $w_{\text{in}} := \{w_\nu\}_{\nu\in\Gamma}$ *be an arbitrary sequence of real numbers indexed by* Γ*. With this* w_{in} *as the initial data, let* $\{w_\nu\}_{\nu\in\mathbb{N}^n}$ *be the output of Algorithm 14. Suppose the following formal power series*

$$\hat{w}(\mathbf{x}) := \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$$

obtained from w_{in} *converges for all* $\mathbf{x} \in \mathbb{R}^n$*. Then so does the power series*

$$w(\mathbf{x}) := \sum_{\nu\in\mathbb{N}^n} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$$

obtained from the solution of Algorithm 14. That is, $\hat{w}(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ *implies* $w(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$.

Keeping the above result in mind, we call an initial condition w_{in} (or $\hat{w}(\mathbf{x}) = \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$) *valid* if $\hat{w}(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$. Now, define the following sub-monoid of the monoid $\mathbb{N}^n$:

$$\widetilde{\Gamma}_r := \{\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n \mid \nu_i = 0 \text{ for } r+1 \leqslant i \leqslant n\}, \tag{15}$$

where $1 \leqslant r < n$ is a positive integer. Further, define the following power series expression in the variables $\mathbf{t} = (t_1, t_2, \ldots, t_r)$

$$\widetilde{w}(\mathbf{t}) = \sum_{\widetilde{\nu}\in\mathbb{N}^r} w_{\widetilde{\nu}} \frac{\mathbf{t}^{\widetilde{\nu}}}{\widetilde{\nu}!}. \tag{16}$$

Note that $\mathfrak{Exp}(\mathbb{R}^r, \mathbb{R})$ *injects* into $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ by mapping t_i to x_i for $1 \leqslant i \leqslant r$. That is the map

$$\begin{aligned} \iota : \mathfrak{Exp}(\mathbb{R}^r, \mathbb{R}) &\to \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R}) \\ t_i &\mapsto x_i \end{aligned} \tag{17}$$

is an injection. In that case $\widetilde{w}(\mathbf{t})$ of Eq. (16) gets mapped to

$$\begin{aligned} \iota(\widetilde{w}(\mathbf{t})) = \hat{w}(\mathbf{x}) &= \sum_{\widetilde{\nu}\in\mathbb{N}^r} w_{\widetilde{\nu}} \frac{x_1^{\widetilde{\nu}_1} x_2^{\widetilde{\nu}_2} \ldots x_r^{\widetilde{\nu}_r}}{\widetilde{\nu}!} \\ &= \sum_{\nu\in\widetilde{\Gamma}_r} w_\nu \frac{\mathbf{x}^\nu}{\nu!}, \end{aligned}$$

where $w_\nu = w_{\widetilde{\nu}}$ with $\nu = (\widetilde{\nu}, \mathbf{0}) \in \widetilde{\Gamma}_r$.

[2] It was brought to our notice by an anonymous reviewer that this result, too, was contained in Janet's work on more general classes of systems of PDEs.

Lemma 16 *Given a set of PDEs* $f_1(\partial_x)w = 0,\ f_2(\partial_x)w = 0, \ldots, f_d(\partial_x)w = 0$*, and a term ordering* $\prec$ *of* $\mathbb{R}[\partial_x]$*, let* Γ *be the set of standard monomials, that is, monomials that do not belong to* $\mathrm{in}_\prec(\langle f_1, f_2, \ldots, f_d\rangle)$*. Let* $\mathfrak{B}$ *be the behavior of the system, and* $\widetilde{\Gamma}_r$ *be as defined by Eq.* (15)*. Consider the subspace* $\mathscr{S} = \mathrm{span}\{e_1, e_2, \ldots, e_r\}$*. If* $\widetilde{\Gamma}_r \subseteq \Gamma$ *then* $\mathscr{S}$ *is free in* $\mathfrak{B}$*.*

Proof Suppose $\widetilde{w}(\mathbf{t})$ is an arbitrary trajectory in $\mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$, which is given by Eq. (16). Let $\hat{w}(\mathbf{x})$ be the image of this $\widetilde{w}(\mathbf{t}) \in \mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$ under the map $\iota : \mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R}) \to \mathfrak{Exp}\,(\mathbb{R}^n, \mathbb{R})$ defined in Eq. (17). Therefore, we can write $\hat{w}(\mathbf{x})$ as

$$\hat{w}(\mathbf{x}) = \sum_{\nu \in \widetilde{\Gamma}_r} w_\nu \frac{\mathbf{x}^\nu}{\nu!},$$

where $w_\nu = w_{\widetilde{\nu}}$ with $\nu = (\widetilde{\nu}, \mathbf{0}) \in \widetilde{\Gamma}_r$. Since $\widetilde{\Gamma}_r \subseteq \Gamma$, $\hat{w}(\mathbf{x})$ can also be written as

$$\hat{w}(\mathbf{x}) = \sum_{\nu \in \Gamma} w_\nu \frac{\mathbf{x}^\nu}{\nu!},$$

where

$$w_\nu = \begin{cases} w_{\widetilde{\nu}} \text{ when } \nu \in \widetilde{\Gamma}_r \text{ and } \nu = (\widetilde{\nu}, \mathbf{0}), \\ 0 \quad \text{when } \nu \notin \widetilde{\Gamma}_r. \end{cases} \tag{18}$$

Therefore, the data given by Eq. (18) is an initial condition specified on the standard monomial set Γ. Further, since $\hat{w}(\mathbf{x}) \in \mathfrak{Exp}\,(\mathbb{R}^n, \mathbb{R})$, it is a *valid* initial condition. Hence, by Proposition 15 the solution $w(\mathbf{x})$ obtained by Algorithm 14 with $\hat{w}(\mathbf{x})$ as the initial condition is in $\mathfrak{Exp}\,(\mathbb{R}^n, \mathbb{R})$. Moreover, this solution $w(\mathbf{x})$ is such that

$$w|_{\mathscr{S}}(\mathbf{t}) = \widetilde{w}(\mathbf{t}).$$

Since $\widetilde{w}(\mathbf{t})$ can be chosen arbitrarily in $\mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$, it follows that $w|_{\mathscr{S}}(\mathbf{t})$ is arbitrary in $\mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$. Thus, for every trajectory $\widetilde{w}(\mathbf{t}) \in \mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$ there exists $w(\mathbf{x}) \in \mathfrak{B} \cap \mathfrak{Exp}\,(\mathbb{R}^n, \mathbb{R})$ such that the restriction of w to $\mathscr{S}$, $w|_{\mathscr{S}}(\mathbf{t}) = \widetilde{w}(\mathbf{t})$. In other words, $\mathfrak{B}|_{\mathscr{S}} = \mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$, that is $\mathscr{S}$ is free in $\mathfrak{B}$. □

The next result is a technical lemma required in the sequel. The lemma deals with the effect on the equation ideal and the behavior due to a change of basis in the domain. This is closely related to the differential geometric notions of *push-forward* and *pull-back* of a map between two smooth manifolds.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. We call the coordinate functions of the domain and the co-domain spaces $\mathbf{x}$ and $\mathbf{y}$, respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are related by $\mathbf{y} = T\mathbf{x}$. This induces a map between the tangent spaces, $T_* : \mathcal{T}_{\mathbf{x}}\mathbb{R}^n \to \mathcal{T}_{\mathbf{y}}\mathbb{R}^n$, as follows. Let $\mathbf{y} \mapsto w(\mathbf{y})$ be in $\mathfrak{C}^\infty\,(\mathbb{R}^n, \mathbb{R})$. Define for all $1 \leqslant i \leqslant n$

$$\left(T_* \frac{\partial}{\partial x_i}\right)(w(\mathbf{y})) := \frac{\partial}{\partial x_i} w(T\mathbf{x}).$$

T_* is called the *push-forward* of the map T. For the case when T is linear, T_* naturally turns out to be linear too. In fact, by making $T_* \frac{\partial}{\partial x_i}$ act on the coordinate functions y_j's, we can get an expression for $T_* \frac{\partial}{\partial x_i}$'s in terms of derivatives in $\mathbf{y}$ coordinates, *i.e.*, $\frac{\partial}{\partial y_j}$'s. Let T be given by the matrix (more precisely, the Jacobian of the linear map T)

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}.$$

Then it follows from the definition of T_* that

$$\left(T_* \frac{\partial}{\partial x_i}\right) y_j = \frac{\partial}{\partial x_i} \sum_{k=1}^{n} t_{jk} x_k = t_{ji}.$$

It then follows by varying j that

$$\left(T_* \frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{n} t_{ji} \frac{\partial}{\partial y_j}.$$

This can be written in the matrix-vector form as

$$T_* \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} = T^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix}. \tag{19}$$

For ease of explanation, and to avoid cumbersome notation, we use ∂_x and ∂_y to denote the n-tuples of partial derivatives $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ and $\left\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right\}$, respectively. These partial derivatives correspond to the two coordinate functions $\mathbf{x}$ and $\mathbf{y}$, which are related by the linear coordinate transformation T. Note that the push-forward can be naturally extended to an automorphism of polynomial rings as

$$\begin{aligned} T_* : \mathbb{R}[\partial_x] &\to \mathbb{R}[\partial_y] \\ \partial_x &\mapsto T^{\mathrm{T}} \partial_y. \end{aligned} \tag{20}$$

The linear map T also induces a map from $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ to itself in the following manner:

$$\begin{aligned} T^* : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}) &\to \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}) \\ w(\mathbf{y}) &\mapsto w(T\mathbf{x}). \end{aligned} \tag{21}$$

This map is called the *pull-back* of T.

Lemma 17 *Let $T \in \mathbb{R}^{n\times n}$ define an invertible linear change of coordinates of $\mathbb{R}^n$ by $\mathbf{x} \mapsto T\mathbf{x} =: \mathbf{y}$. Let $T_* : \mathbb{R}[\partial_x] \to \mathbb{R}[\partial_y]$ and $T^* : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be the push-forward and pull-back of T as defined by Eqs.* (20) *and* (21)*, respectively. Suppose $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ is an ideal, and let $\mathfrak{B}$ be its behavior. Then we have the following:*

$$T^*(\mathfrak{B}(T_*(\mathcal{I}))) = \mathfrak{B}. \tag{22}$$

Further, let $\mathscr{S} \subseteq \mathbb{R}^n$ be a subspace. Then

$$\mathfrak{B}(T_*(\mathcal{I}))|_{T\mathscr{S}} = \mathfrak{B}|_{\mathscr{S}}. \tag{23}$$

Finally, the intersection ideals satisfy the following:

$$T_*(\mathcal{I})|_{T\mathscr{S}} = \mathcal{I}_{\mathscr{S}}. \tag{24}$$

Proof First observe that it follows from Eq. (19) and the definition of push-forward that for $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(T\mathbf{x}) = T_* \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(\mathbf{y}) = T^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} w(\mathbf{y}). \tag{25}$$

Therefore, we have

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(T\mathbf{x}) = T^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} w(\mathbf{y}).$$

More generally, for $m(\partial_x) \in \mathbb{R}[\partial_x]$

$$m(\partial_x) w(T\mathbf{x}) = T_*(m)(\partial_y) w(\mathbf{y}) = m(T^{\mathrm{T}} \partial_y) w(\mathbf{y}).$$

Hence it follows that $w(\mathbf{y}) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is in the kernel of $T_*(m)(\partial_y)$ if and only if $T^*(w(\mathbf{y})) = w(T\mathbf{x})$ is in the kernel of $m(\partial_x)$. Therefore, for the ideal $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$, $T^*(w(\mathbf{y})) \in \mathfrak{B}(\mathcal{I})$ if and only if $w(\mathbf{y}) \in \mathfrak{B}(T_*(\mathcal{I}))$. That is, $T^*(\mathfrak{B}(T_*(\mathcal{I}))) \subseteq \mathfrak{B}(\mathcal{I})$. Since T is invertible, we get the converse inclusion by substituting T by T^{-1} and $\mathcal{I}$ by $T_*(\mathcal{I})$.

For the restriction, we first show that $\mathfrak{B}(T_*(\mathcal{I}))|_{T\mathscr{S}} \subseteq \mathfrak{B}|_{\mathscr{S}}$. Let the columns of the matrix $S \in \mathbb{R}^{n\times r}$ form a basis of $\mathscr{S}$. A typical element from $\mathfrak{B}(T_*(\mathcal{I}))|_{T\mathscr{S}}$ is given by $w(TS\mathbf{t})$, where $w(\mathbf{y}) \in \mathfrak{B}(T_*(\mathcal{I}))$. Note that $w(TS\mathbf{t}) = (T^*w)(S\mathbf{t})$. But, by Eq. (22) $T^*w \in \mathfrak{B}(\mathcal{I})$. Therefore, $w(TS\mathbf{t}) = (T^*w)(S\mathbf{t}) \in \mathfrak{B}(\mathcal{I})|_{\mathscr{S}}$.

In order to show the inclusion $\mathfrak{B}(T_*(\mathcal{I}))|_{T\mathscr{S}} \supseteq \mathfrak{B}|_{\mathscr{S}}$ we first note that it follows from Eq. (22) that every element in $\mathfrak{B}(\mathcal{I})$ can be written as $v(\mathbf{x}) = w(T\mathbf{x})$ with $w(\mathbf{y}) \in \mathfrak{B}(T_*(\mathcal{I}))$. Hence every element in $\mathfrak{B}(\mathcal{I})|_{\mathscr{S}}$ is of the form $v(S\mathbf{t}) = w(TS\mathbf{t}) \in \mathfrak{B}(T_*(\mathcal{I}))|T\mathscr{S}$.

For the third part, that is, Eq. (24), it follows from the definition of intersection ideals that

$$\begin{aligned} \mathcal{I}_{\mathscr{S}} &= \{ f(\partial_t) \in \mathbb{R}[\partial_t] \mid f(s_1^{\mathrm{T}} \partial_x, \ldots, s_r^{\mathrm{T}} \partial_x) \in \mathcal{I} \} \\ &= \{ f(\partial_t) \in \mathbb{R}[\partial_t] \mid f(s_1^{\mathrm{T}} T^{\mathrm{T}} \partial_y, \ldots, s_r^{\mathrm{T}} T^{\mathrm{T}} \partial_y) \in T_*(\mathcal{I}) \} \\ &= T_*(\mathcal{I})_{T\mathscr{S}}. \end{aligned}$$

□

Theorem 18 *Let $\mathfrak{B}$ be the behavior of a scalar autonomous system defined by the equation ideal $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ and let $\mathscr{S} \subseteq \mathbb{R}^n$ be a subspace of dimension $r \neq 0$. Then the following conditions are equivalent:*

1. *$\mathscr{S}$ is free in $\mathfrak{B}$.*
2. *The intersection ideal $\mathcal{I}_{\mathscr{S}} \subseteq \mathbb{R}[\partial_t]$ is the zero ideal.*
3. *The $\mathbb{R}$-algebra homomorphism φ in the following commutative diagram is an injection.*

$$\begin{array}{ccc} \mathbb{R}[\partial_x] & \twoheadrightarrow & \mathbb{R}[\partial_x]/\mathcal{I}\ . \\ {\scriptstyle \Phi_{\mathscr{S}}} \big\uparrow & \nearrow_{\varphi} & \\ \mathbb{R}[\partial_t] & & \end{array}$$

Proof (**1** ⇒ **2**) Consider the behavior $\mathfrak{B}_{\mathscr{S}}$, the behavior corresponding to the intersection ideal $\mathcal{I}_{\mathscr{S}}$. By Theorem 5, $\mathfrak{B}|_{\mathscr{S}} \subseteq \mathfrak{B}_{\mathscr{S}}$. Since $\mathfrak{B}$ is $\mathscr{S}$-free, we have $\mathfrak{B}|_{\mathscr{S}} = \mathfrak{Exp}(\mathbb{R}^r, \mathbb{R})$. Therefore, $\mathfrak{Exp}(\mathbb{R}^r, \mathbb{R}) \subseteq \mathfrak{B}_{\mathscr{S}}$. Now consider the set

$$\mathcal{I}(\mathfrak{B}_{\mathscr{S}}) := \{f(\partial_t) \mid f(\partial_t)w = 0 \text{ for all } w \in \mathfrak{B}_{\mathscr{S}}\}.$$

Since $\mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R}) \subseteq \mathfrak{B}_{\mathscr{S}}$, it follows that $\mathcal{I}(\mathfrak{B}_{\mathscr{S}}) = \{0\}$. But, $\mathcal{I}(\mathfrak{B}_{\mathscr{S}}) = \mathcal{I}_{\mathscr{S}}$ because $\mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$ is a large injective cogenerator. Thus we get $\mathcal{I}_{\mathscr{S}} = \{0\}$.

(**2** $\Leftrightarrow$ **3**) This follows from the fact that $\ker \varphi = \mathcal{I}_{\mathscr{S}}$.

(**2** $\Rightarrow$ **1**) In order to prove this implication we will first prove a simpler case, and then we will make use of an earlier result, Lemma 17, that will render the general case into the simpler one.

Case 1 ($\mathscr{S} = \text{span}\{e_1, e_2, \ldots, e_r\}$): First notice that in this case the map $\Phi_{\mathscr{S}}$ maps each of the ∂_{t_i} to the corresponding ∂_{x_i}. Thus, the intersection ideal can be identified with $\mathcal{I} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}]$. Therefore, the problem here reduces to proving $\mathcal{I} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}] = \{0\}$ implies that $\mathfrak{B}$ is $\mathscr{S}$-free. We claim that $\mathcal{I} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}] = \{0\}$ implies there exists a term ordering such that the standard monomials set Γ contains $\widetilde{\Gamma}_r$, where $\widetilde{\Gamma}_r$ is defined by Eq. (15). By Lemma 16, it will then follow that $\mathfrak{B}$ is $\mathscr{S}$-free. Indeed, if we take a term ordering with $\partial_{x_i} \succ \partial_{x_j}$ for all $r+1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$, then a Gröbner basis for $\mathcal{I}$, say $\mathcal{G}$, with this term ordering will have no element whose leading term will be purely in the variables $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}\}$. For if $\mathcal{G}$ had a polynomial, say $f(\partial_x) \in \mathbb{R}[\partial_x]$, with leading term purely in $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}\}$, then since $\partial_{x_i} \succ \partial_{x_j}$ for all $r+1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant r$, the rest of the monomials in $f(\partial_x)$ will also be in $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}\}$ only. But this would mean $f(\partial_x) \in \mathcal{I} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}]$. This contradicts the assumption that $\mathcal{I} \cap \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}] = \{0\}$. Now since $\mathcal{G}$ has no element with leading term purely in $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}\}$, the initial ideal $\text{in}_{\prec}(\mathcal{I})$, too, does not contain any monomial purely in $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_r}\}$. In other words, the standard monomial set $\Gamma \supseteq \widetilde{\Gamma}_r$.

Case 2 (general $\mathscr{S}$**)**: For the general case we make use of Lemma 17. Let $\mathscr{S}$ be given by

$$\mathscr{S} = \text{im} S,$$

where $S \in \mathbb{R}^{n \times r}$ is of full column-rank. Since S has full column-rank it can be completed into a square non-singular matrix by adding $(n-r)$ columns. That is, there exist $S' \in \mathbb{R}^{n \times (n-r)}$ full column-rank, such that $\begin{bmatrix} S & S' \end{bmatrix}$ is invertible. Define

$$T := \begin{bmatrix} S & S' \end{bmatrix}^{-1}.$$

We use this T to do a change of coordinates. Recall from Eqs. (20) and (21) that this T induces two maps, namely the push-forward T_* and the pull-back T^*. Note that under this change of coordinates

$$TS = \begin{bmatrix} e_1 & e_2 & \ldots & e_r \end{bmatrix}.$$

So $T\mathscr{S} = \text{span}\{e_1, e_2, \ldots, e_r\}$. Thus, with $T_*(\mathcal{I})$ as the equation ideal and $T\mathscr{S}$ as the subspace of restriction, we are exactly in the situation of Case 1. By Lemma 17,

$$T_*(\mathcal{I})_{T\mathscr{S}} = \mathcal{I}_{\mathscr{S}} = \{0\}.$$

Therefore, by Case 1, $T\mathscr{S}$ is free in $\mathfrak{B}(T_*(\mathcal{I}))$. That is, $\mathfrak{B}(T_*(\mathcal{I}))|_{T\mathscr{S}} = \mathfrak{Exp}\,(\mathbb{R}^r, \mathbb{R})$ But, by Lemma 17,

$$\mathfrak{B}|_{\mathscr{S}} = \mathfrak{B}(T_*(\mathcal{I}))|_{T\mathscr{S}} = \mathfrak{Exp}\left(\mathbb{R}^r, \mathbb{R}\right).$$

That is, $\mathscr{S}$ is free in $\mathfrak{B}$. □

4.2 Maximally free subspace

A relevant fundamental question concerning free subspaces is: *what should be the highest dimension possible for a free subspace?* Obviously, scalar systems, being autonomous ones, cannot have the full space $\mathbb{R}^n$ as free. So the dimension of a free subspace is always upper-bounded by $n-1$. This, however, is a very slack upper-bound; we would like to have a tighter one. In this subsection, we show that the tightest upper-bound is the Krull dimension of the quotient ring $\mathcal{M}$.

Krull dimension of a commutative ring is defined as the length of the longest chain of prime ideals. For the quotient ring $\mathcal{M} = \mathbb{R}[\partial_x]/\mathcal{I}$, this number turns out to be equal to the length of the longest chain of prime ideals in $\mathbb{R}[\partial_x]$ containing the equation ideal $\mathcal{I}$. We illustrate the relation between the Krull dimension of $\mathcal{M}$ and dimensions of free subspaces in the following example.

Example 19 Consider the following system of equations in 3-D:

$$\partial_{x_1} w = \partial_{x_2} w = 0.$$

So the equation ideal is $\mathcal{I} = \langle \partial_{x_1}, \partial_{x_2} \rangle$. The Krull dimension of the quotient ring $\mathcal{M} = \mathbb{R}[\partial_{x_1}, \partial_{x_2}, \partial_{x_3}]/\mathcal{I}$ can be shown to be equal to 1. Using the result of Theorem 18 we can readily see that x_3-axis is a free subspace. We claim that no 2-dimensional subspace can be free in this case.

In order to see this consider a generic 2-dimensional subspace $\mathscr{S}$ spanned by vectors

$$\left\{ \begin{bmatrix} s_{11} \\ s_{12} \\ s_{13} \end{bmatrix}, \begin{bmatrix} s_{21} \\ s_{22} \\ s_{23} \end{bmatrix} \right\}.$$

If either of s_{13} or s_{23} is zero then either $(s_{11}\partial_{x_1} + s_{12}\partial_{x_2} + s_{13}\partial_{x_3}) \in \mathcal{I}$, or, respectively, $(s_{21}\partial_{x_1} + s_{22}\partial_{x_2} + s_{23}\partial_{x_3}) \in \mathcal{I}$. In either case the intersection ideal $\mathcal{I}_{\mathscr{S}} \neq \{0\}$. Hence, by Theorem 18, $\mathscr{S}$ cannot be free. On the other hand, if both s_{13} and s_{23} are nonzero, then the non-zero polynomial

$$-\frac{s_{23}}{s_{13}}(s_{11}\partial_{x_1} + s_{12}\partial_{x_2} + s_{13}\partial_{x_3}) + (s_{21}\partial_{x_1} + s_{22}\partial_{x_2} + s_{23}\partial_{x_3}) \in \mathcal{I}.$$

This, again, means the intersection ideal $\mathcal{I}_{\mathscr{S}} \neq \{0\}$, and hence $\mathscr{S}$ cannot be free.

Remark 20 It is interesting to note that the results we derive in this subsection are analogous to their respective discrete versions derived in Wood et al. (1998): Lemma 7.3 and Theorem 7.4. However, the results are markedly different in their approach. For one, in Wood et al. (1998) restrictions of an autonomous behavior have been done to gradually lower dimensional sublattices of $\mathbb{Z}^n$ and it has been checked whether the autonomy is retained. Whereas, in this article, we look at restrictions to gradually higher dimensional subspaces and check whether the restriction is free. However, finally the notion of autonomy degree of Wood et al. (1998), that is the lowest possible dimension of a sublattice on which the restriction is autonomous, and that of the dimension of a maximally free subspace coincides. Furthermore, as per our knowledge, no straightforward extension of the techniques used in Wood et al. (1998) to the continuous case is possible in order to obtain the analogous results that we derive here.

Our first observation of this subsection is that the Krull dimension of $\mathcal{M}$ is an upper-bound on the dimension of a free subspace. In order to prove this result we need the following alternative formulation of Krull dimension of finitely generated algebras over a field. The proof of this result can be found in textbooks like (Cox et al. 2007; Eisenbud 1995).

Proposition 21 *Let $\mathbb{K}$ be a field and consider the n-variable polynomial ring $\mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]$. Let $\mathcal{I} \subseteq \mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]$ be an ideal. Then the Krull dimension of $\mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]/\mathcal{I}$ is equal to the cardinality of a maximal set of elements in $\mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]/\mathcal{I}$ that is algebraically independent over $\mathbb{K}$.*

Theorem 22 *Let $\mathfrak{B}$ be the behavior of a scalar n-D system with equation ideal $\mathcal{I}$. Further, let $\mathscr{S} \subseteq \mathbb{R}^n$ be a free subspace in $\mathfrak{B}$. Suppose the dimension of $\mathscr{S}$ is r and the Krull dimension of the quotient ring $\mathcal{M} = \mathbb{R}[\partial_x]/\mathcal{I}$ is d. Then we have*

$$r \leqslant d.$$

Proof Suppose $\{s_1, s_2, \ldots, s_r\} \subseteq \mathbb{R}^n$ be a basis of $\mathscr{S}$. From Theorem 18 it follows that $\mathscr{S}$ is free if and only if the $\mathbb{R}$-algebra homomorphism φ in the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}[\partial_x] & \twoheadrightarrow & \mathbb{R}[\partial_x]/\mathcal{I} \; . \\ {\scriptstyle \Phi_{\mathscr{S}}} \uparrow & \nearrow {\scriptstyle \varphi} & \\ \mathbb{R}[\partial_t] & & \end{array}$$

is an injection. This, in turn, is equivalent to saying that the canonical $\mathbb{R}$-algebra homomorphism $\mathbb{R}[s_1^{\mathrm{T}}\partial_x, s_2^{\mathrm{T}}\partial_x, \ldots, s_r^{\mathrm{T}}\partial_x] \rightarrow \mathbb{R}[\partial_x]/\mathcal{I}$ is injective. Therefore, the images of $s_1^{\mathrm{T}}\partial_x, s_2^{\mathrm{T}}\partial_x, \ldots, s_r^{\mathrm{T}}\partial_x$ in $\mathbb{R}[\partial_x]/\mathcal{I}$, that is, $\{\overline{s_1^{\mathrm{T}}\partial_x}, \overline{s_2^{\mathrm{T}}\partial_x}, \ldots, \overline{s_r^{\mathrm{T}}\partial_x}\} \subseteq \mathbb{R}[\partial_x]/\mathcal{I}$, form a set of elements algebraically independent over $\mathbb{R}$. By Proposition 21, this means the Krull dimension of $\mathbb{R}[\partial_x]/\mathcal{I}$ is *at least* r. This proves the desired inequality. □

Our next result shows that if the quotient ring of a scalar system has Krull dimension equal to d, then one can always find a free subspace of dimension equal to d. This result follows from a much stronger result of commutative algebra, known as Noether's Normalization Lemma. We quote only a portion of the general result here. Full details can be found in textbooks, for example (Kreuzer and Robbiano 2000, Tutorial 78).

Proposition 23 *Let $\mathbb{K}$ be an infinite field. Consider $\mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]$, the n-variable polynomial ring over $\mathbb{K}$, and let $\mathcal{I} \subseteq \mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]$ be an ideal such that Krull dimension of $\mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]/\mathcal{I}$ is d. Then there exists an upper triangular matrix $A \in \mathbb{K}^{n \times n}$ such that if we define*

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} = A \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix},$$

then the canonical $\mathbb{K}$-algebra homomorphism $\mathbb{K}[\eta_1, \eta_2, \ldots, \eta_d] \rightarrow \mathbb{K}[\xi_1, \xi_2, \ldots, \xi_n]/\mathcal{I}$ is an injection.

With this we now state and prove the second main result of this subsection.

Theorem 24 *Let $\mathfrak{B}$ be the behavior of a scalar n-D system with equation ideal $\mathcal{I}$. Suppose the Krull dimension of the quotient ring $\mathcal{M} = \mathbb{R}[\partial_x]/\mathcal{I}$ is equal to d. Then there exists a d-dimensional subspace $\mathscr{S} \subseteq \mathbb{R}^n$ that is free in $\mathfrak{B}$.*

Proof By Proposition 23, there exists an upper triangular matrix $A \in \mathbb{R}^{n\times n}$ such that if we define

$$\begin{bmatrix} \partial_{t_1} \\ \partial_{t_2} \\ \vdots \\ \partial_{t_n} \end{bmatrix} = A \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

then the canonical $\mathbb{R}$-algebra homomorphism $\mathbb{R}[\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_d}] \to \mathbb{R}[\partial_x]/\mathcal{I}$ is an injection. Suppose A is written row-wise as

$$A = \begin{bmatrix} a_1^{\mathrm{T}} \\ a_2^{\mathrm{T}} \\ \vdots \\ a_n^{\mathrm{T}} \end{bmatrix},$$

where $a_i \in \mathbb{R}^n$. Then, note that for $1 \leqslant i \leqslant n$ we have $\partial_{t_i} = a_i^{\mathrm{T}}\partial_x$. Therefore, the above-mentioned canonical $\mathbb{R}$-algebra homomorphism being injective is equivalent to the following $\mathbb{R}$-algebra homomorphism

$$\mathbb{R}[a_1^{\mathrm{T}}\partial_x, a_2^{\mathrm{T}}\partial_x, \ldots, a_d^{\mathrm{T}}\partial_x] \to \mathbb{R}[\partial_x]/\mathcal{I} \tag{26}$$

being *injective*. Define the d-dimensional subspace

$$\mathscr{S} := \operatorname{span}\{a_1, a_2, \ldots, a_d\} \subseteq \mathbb{R}^n.$$

By Theorem 18 then it follows from Eq. (26) that $\mathscr{S}$ is free in $\mathfrak{B}$. □

5 Concluding remarks

In this article we looked at certain issues related to the method of restrictions of solutions of a scalar system of PDEs to a proper subspace of the domain $\mathbb{R}^n$. One crucial finding is the close relation between the method of restrictions and an algebraic object called intersection ideal. We showed that the set of restricted trajectories is always contained in the solution set corresponding the intersection ideal. We then gave an algorithm for computing the intersection ideal from a given ideal and a subspace of $\mathbb{R}^n$. We then concentrated on an important question concerning restrictions, namely, the question of whether a given subspace is free. We gave an algebraic criterion for answering this question. For this we made use of an existing algorithm for formal solutions of PDEs; the algorithm requires computing a Gröbner basis of the ideal of the given PDEs. This algebraic criterion also happens to involve the intersection ideal. Finally, we considered the question of the highest possible dimension of free subspaces. Using the algebraic characterization of free subspaces and some existing results in commutative algebra we showed that this highest dimension is equal to the Krull dimension of the quotient ring corresponding to the equation ideal.

An immediate next step in the exploration of the kind of algebraic analysis of restrictions done in the article would be extending the results general non-scalar systems.

References

Cox, D., Little, J., & O'Shea, D. (2007). *Ideals, varieties, and algorithms: Undergraduate texts in mathematics* (3rd ed.). Berlin: Springer.

Eisenbud, D. (1995). *Commutative algebra with a view toward algebraic geometry*. Berlin: Springer.

Fornasini, E., Rocha, P., & Zampieri, S. (1993). State space realization of 2-D finite-dimensional behaviours. *SIAM Journal on Control and Optimization*, *31*(6), 1502–1517.

Kojima, C., Rapisarda, P., & Takaba, K. (2010). Lyapunov stability analysis of higher order 2-D systems. *Multidimensional Systems and Signal Processing*, *22*, 287–302.

Kreuzer, M., & Robbiano, L. (2000). *Computational commutative algebra 2*. Berlin: Springer.

Napp Avelli, D., & Rocha, P. (2010). Autonomous multidimensional systems and their implementation by behavioral control. *Systems and Control Letters*, *59*(34), 203–208.

Napp Avelli, D., Rapisarda, P., & Rocha, P. (2011). Lyapunov stability of 2D finite-dimensional behaviours. *International Journal of Control*, *84*(4), 737–745.

Oberst, U. (1990). Multidimensional constant linear systems. *Acta Applicandae Mathematicae*, *20*, 1–175.

Oberst, U., & Pauer, F. (2001). The constructive solution of linear systems of partial difference and differential equations with constant coefficients. *Multidimensional Systems and Signal Processing*, *12*, 253–308.

Oberst, U. (2006). The constructive solution of linear systems of partial difference and differential equations with constant coefficients. In B. Hanzon & M. Hazewinkel (Eds.), *Constructive algebra and systems theory* (pp. 205–233). Amsterdam: Royal Netherlands Academy of Arts and Sciences.

Pal, D., & Pillai, H. K. (2014). On restrictions of n-D systems to 1-D subspaces. *Multidimensional Systems and Signal Processing*, *25*, 115–144.

Pillai, H. K., & Willems, J. C. (2002). Lossless and dissipative distributed systems. *SIAM Journal on Control and Optimization*, *40*(5), 1406–1430.

Plesken, W., & Robertz, D. (2005). Janets approach to presentations and resolutions for polynomials and linear pdes. *Archiv der Mathematik*, *84*(1), 22–37.

Pommaret, J.-F. (1994). *Partial differential equations and group theory: New perspectives for applications*. Dordrecht: Kluwer.

Renardy, M., & Rogers, R. C. (2004). *An introduction to partial differential equations*. New York: Springer.

Riquier, C. (1910). *Les Systèmes d'Équations aux Dérivées Partielles*. Paris: Gauthiers-Villars.

Rocha, P. (1990) Structure and representation of 2-D systems. PhD thesis, University of Groningen, The Netherlands

Shankar, S. (2000). Can one control the vibrations of a drum? *Multidimensional Systems and Signal Processing*, *11*, 67–81.

Valcher, M. E. (2001). Characteristic cones and stability properties of two-dimensional autonomous behaviors. *IEEE Transactions on Circuits and Systems-Part I: Fundamental Theory and Applications*, *47*(3), 290–302.

Willems, J. C. (1991). Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, *36*(3), 259–294.

Wood, J., Rogers, E., & Owens, D. H. (1998). A formal theory of matrix primeness. *Mathematics of Control, Signals, and Systems*, *11*, 40–78.

Zerz, E., & Oberst, U. (1993). The canonical cauchy problem for linear systems of partial difference equations with constant coefficients over the complete r-dimensional integral lattice $\mathbb{Z}^r$. *Acta Applicandae Mathematicae.*, *31*, 249–273.

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