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Every discrete 2D autonomous system admits a finite union of parallel lines as a characteristic set

Debasattam Pal[1]

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Abstract In this paper, we show that every discrete 2D autonomous system, that is described by a set of linear partial difference equations with constant real coefficients, admits a finite union of parallel lines as a characteristic set. In order to prove our claim, we first look at a special class of scalar discrete 2D systems and provide such characteristic sets for systems in this class. This special class has the property that systems in this class have their quotient rings to be finitely generated modules over a one-variable Laurent polynomial subring of the original two-variable Laurent polynomial ring in the shift operators. We show that such systems always admit a finite collection of horizontal lines for a characteristic set. We then extend this result to non-scalar discrete 2D autonomous systems. We achieve this in two steps: first, we show that *every* scalar discrete 2D system can be converted into a system in the above-mentioned class by a coordinate transformation on the independent variables set, $\mathbb{Z}^2$. Using this we then show that characteristic sets for the original system can be found by applying the inverse coordinate transformation on characteristic sets of the transformed system. Since the transformed system, by virtue of being in the special class, admits a finite union of horizontal lines as a characteristic set, the original system is guaranteed to admit a characteristic set that is a coordinate transformation applied to a finite union of horizontal lines. The coordinate transformation maps this union of horizontal lines to a union of parallel, but possibly tilted, lines. In the next step, we generalize the scalar case to the general vector case: that is, systems with more than one dependent variables. The main motivation for studying characteristic sets that are unions of finitely many parallel lines is that, arguably, such sets can be called "thin" in $\mathbb{Z}^2$ in comparison to the prevalent notions of convex cones and half-spaces as characteristic sets (see "Appendix 1").

Keywords 2D systems · Characteristic sets · Algebraic methods

✉ Debasattam Pal
debasattam@ee.iitb.ac.in

1 Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai, India

1 Introduction

Multidimensional systems are fast becoming ubiquitous in systems and control theory. Indeed, various state-of-the-art engineering applications—for example, earth-quake engineering (Valcher 2000), image/video processing, cognitive radio systems, and many others (Madanayake et al. 2013)—utilize space-time filters and other multidimensional filters. One of the fundamental issues in multidimensional systems theory is concerning their *characteristic sets*. Roughly speaking, characteristic sets are special subsets of the domain (the set over which the system trajectories evolve—usually $\mathbb{Z}^n$ or $\mathbb{R}^n$) with the defining property that, for every trajectory in the system, the knowledge of its values on the characteristic set *uniquely* identifies the trajectory over the whole domain. The question of finding out characteristic sets in a given system becomes crucial in many situations: for example, stability (Shankar 2000; Valcher 2000; Napp et al. 2011; Oberst 2006; Wood et al. 2005), Markovian-ness (Rocha and Willems 2006), the canonical Cauchy problem (Zerz and Oberst 1993), etc. In this paper, we deal with characteristic sets of discrete 2D systems—that is, systems that are described by linear partial difference equations in two independent variables with constant real coefficients. Note that the question of characteristic sets is irrelevant for systems having *inputs/free variables*; for the free variables can take arbitrary values over the entire domain $\mathbb{Z}^2$, therefore, no proper subset of $\mathbb{Z}^2$ can be a characteristic set. Systems having no free variables are called *autonomous*. As is expected, such systems always admit proper subsets of the domain as characteristic sets (Rocha and Willems 1989). In this paper, we show that *every* discrete 2D autonomous system admits characteristic sets that are much more than just proper subsets of $\mathbb{Z}^2$: these characteristic sets are *finite unions of parallel lines*. Arguably, such subsets of the domain can be called "thin" with respect to the whole domain (see "Appendix 1" for a discussion on thin-ness of these sets).

Characteristic sets for discrete 2D systems were analyzed in meticulous detail in Valcher's seminal work (Valcher 2000), where the typical characteristic sets considered are convex cones (a convex cone in $\mathbb{R}^2$ intersected with $\mathbb{Z}^2$) or half-spaces. Interestingly, the existing notion of characteristic sets of discrete 2D systems appears to have taken shape following (Valcher 2000): the characteristic sets of discrete 2D systems dealt with in the literature are predominantly either convex cones or half-spaces (see Napp et al. 2011; Rapisarda and Rocha 2012; Napp et al. 2011 for some recent examples that incorporate similar notions of characteristic sets). Perhaps the important significance of such characteristic sets lies in their applications in conic stability analysis. However, when it comes to *quantifying* the a priori knowledge that is required to uniquely specify trajectories in the system, characteristic sets such as convex cones or half-spaces appear to have a critical shortcoming. Both of these sets are—in a not-so-precise sense—2D subsets of the domain $\mathbb{Z}^2$, while a line (or a finite union of lines) is a 1D subset of $\mathbb{Z}^2$. Thus, intuitively, a line (or a finite union of lines) should be thin compared to convex cones or half-spaces. Consequently, characteristic sets that are finite unions of lines would mean a much less knowledge about a trajectory is required in order to deduce it completely. Such are the types of sets shown to be characteristic sets in this paper for every autonomous discrete 2D system. It should be noted at this point that this idea of a finite union of lines being thin compared to convex cones or half-spaces is indeed only intuitive, and is true only under special circumstances; unfortunately, making this idea precise is beyond the scope of this paper. See "Appendix 1", where this issue has been discussed briefly. We do not aim, in this paper, to settle this debatable issue of whether a finite union of parallel lines is indeed thin or not, but, instead, we just show that every autonomous discrete 2D system admits a characteristic set that is a finite union of parallel lines. However, one must also note that in discrete 1D autonomous systems, given by linear difference equations, the initial conditions that uniquely specify the solutions, are given by the values of the solution trajectories on finitely many points in the independent variable axis (see Willems 1991). Thus every autonomous 1D linear system that is described by difference equations admits a finite set of points as a characteristic set. Such a finite set of points is certainly thin in the whole domain $\mathbb{Z}$. Note, however, that the 1D analogue of a cone (or a half-space) in $\mathbb{Z}^2$ would be the half-line $\mathbb{Z}_+$ or $\mathbb{Z}_-$. While such sets do qualify as characteristic sets for discrete 1D autonomous systems, they are a little too bigger than what is required because, as already mentioned, these systems admit finite sets as characteristic sets. Therefore, it is only natural to expect that for 2D systems, too, cones and half-spaces as characteristic sets would be bigger than what is required. Thus, arguably, a true analogy of the 1D situation would necessitate discrete 2D autonomous systems to have characteristic sets that are thinner than cones or half-spaces. The main results of this paper (Theorems 20 and 24) show that this is indeed true for discrete 2D autonomous systems that are described by linear partial difference equations.

It is interesting to note that, for discrete 2D autonomous systems, the expectation of a characteristic set that is a finite union of lines is also supported by at least two other alternative approaches documented in the literature. The first one is the constructive approach to solving 2D partial difference equations utilizing variants of Gröbner basis methods (see Oberst 1990; Zerz and Oberst 1993; Oberst 2006). From this approach it follows that for autonomous systems, the entire solution can be constructed once the *initial condition* is specified. This initial condition, as it turns out, has a one-to-one correspondence with the *standard monomials* set of the constructed Gröbner basis of the system of partial difference equations. Thus a standard monomials set can be viewed as a characteristic set. The standard monomials set turns out to be much smaller than the entire $\mathbb{Z}^2$ (see Oberst 1990; Zerz and Oberst 1993; see also Pauer and Unterkircher 1999). The reason for this smallness of the standard monomials set is that they are the complements of finitely many translations of the 2D positive integer grid ($\mathbb{N}^2$) in $\mathbb{Z}^2$ (see Zerz and Oberst 1993; Pauer and Unterkircher 1999). Existence of characteristic sets that are much smaller compared to entire $\mathbb{Z}^2$ can hence be argued. However, unlike the case we prove in this paper, these standard monomials sets need not always be finite unions of *parallel* lines; see various examples constructed in Pauer and Unterkircher (1999).

The second approach involves the notion of *autonomy degree* introduced in Wood et al. (1998) and later extended in Avelli and Rocha (2010). The autonomy degree of a general nD autonomous system is $(n - d)$, where d is the highest possible dimension of a sublattice of $\mathbb{Z}^n$ having the property that the restriction of the solution trajectories to the sublattice is not autonomous. It has been shown in Wood et al. (1998) for continuous systems and in Avelli and Rocha (2010) for discrete systems that the autonomy degree of an nD autonomous system is equal to n minus the Krull dimension of the quotient module (see Sect. 2.5 for the definition of the quotient module of a discrete 2D system). For the 2D case, it follows from this result that autonomy degree of non-trivial autonomous discrete 2D systems must be either 2 or 1. Autonomy degree being 2 means that the trajectories are completely determined by their values on a finite set because restriction to no line can be free. On the other hand, autonomy degree being 1 would imply that restriction to only lines (that is 1D sublattices) can be free but not to higher dimensional sublattices. This fact is corroborated in this paper. However, what we show further is that for every discrete 2D autonomous system, there exists a line such that the restriction of a trajectory to the line and a finitely many parallel translates of it is enough to determine that trajectory uniquely.

2 Notation and preliminaries

In this section, we provide the notation used throughout the paper, and some preliminary notions and definitions required for the main text of the paper.

2.1 Notation

We use $\mathbb{R}, \mathbb{Z}$ to denote the field of real numbers, and the ring of integers, respectively. Consequently the n-dimensional Euclidean space over $\mathbb{R}$ is denoted by $\mathbb{R}^n$, while $\mathbb{Z}^2$ is used to denote the 2D integer grid. An ordered pair of integers in $\mathbb{Z}^2$ is denoted by (ν_1, ν_2). We denote the set of doubly indexed sequences over the real numbers by $\mathbb{R}^{\mathbb{Z}^2}$, i.e., $\mathbb{R}^{\mathbb{Z}^2} := \{w : \mathbb{Z}^2 \to \mathbb{R}\}$. To denote the set of vector-valued (q-tuple) doubly indexed sequences we use the symbol $(\mathbb{R}^q)^{\mathbb{Z}^2}$. The Laurent polynomial ring in two variables σ_1 and σ_2 is denoted by $\mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]$, and the same in one variable σ_1 by $\mathbb{R}[\sigma_1^{\pm 1}]$. In this paper, for brevity, we use $\mathcal{A}$ and $\mathcal{A}_1$ to denote $\mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]$ and $\mathbb{R}[\sigma_1^{\pm 1}]$, respectively. By $\mathcal{A}^q$ we denote the free $\mathcal{A}$-module of rank q; since elements from $\mathcal{A}^q$ "act" on elements of $(\mathbb{R}^q)^{\mathbb{Z}^2}$, we consider, as a convention, that elements of $\mathcal{A}^q$ are written as row vectors and those of $(\mathbb{R}^q)^{\mathbb{Z}^2}$ are written as column vectors. This enables us to write the action of $r(\sigma) \in \mathcal{A}^q$ on $w \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ as the product $r(\sigma)w$. For an integer pair $\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{Z}^2$, the symbol $\sigma^{\boldsymbol{\nu}}$ denotes the monomial $\sigma_1^{\nu_1}\sigma_2^{\nu_2}$. In this paper, we use the "bar" notation for equivalence classes, i.e., given a submodule $\mathcal{R} \subseteq \mathcal{A}^q$, the equivalence class of a row-vector $r(\sigma) \in \mathcal{A}^q$ belonging to the quotient module $\mathcal{A}^q/\mathcal{R}$ is denoted by $\overline{r(\sigma)}$. We use $\bullet$ to denote a number, size or dimension, which is unspecified.

2.2 Discrete 2D systems

2D systems are those where the trajectories evolve over two independent variables; in discrete 2D systems the evolution is over $\mathbb{Z}^2$. In this paper, we are going to look at discrete 2D systems described by linear partial difference equations with constant real coefficients. These difference equations can be written succinctly using two shift operators σ_1 and σ_2. These shift operators act on discrete 2D trajectories in the following manner: $(\sigma_1^i w)(h,k) = w(h+i,k)$ and $(\sigma_2^j w)(h,k) = w(h,k+j)$ for all $h,i,j,k \in \mathbb{Z}$, where $w \in \mathbb{R}^{\mathbb{Z}^2}$ is a discrete 2D trajectory. This defines the action of a monomial $\sigma^{\boldsymbol{\nu}} := \sigma_1^{\nu_1}\sigma_2^{\nu_2}$ on a 2D trajectory $w \in \mathbb{R}^{\mathbb{Z}^2}$ as

$$\left(\sigma^{\boldsymbol{\nu}} w\right)(h,k) = w(h+\nu_1, k+\nu_2).$$

A Laurent polynomial in the shift operators σ_1 and σ_2 is a finite linear combination of various monomials in σ_1 and σ_2:

$$f(\sigma) = \sum_{\boldsymbol{\nu} \in \Gamma} \alpha_{\boldsymbol{\nu}} \sigma^{\boldsymbol{\nu}},$$

where $\Gamma \subseteq \mathbb{Z}^2$ is finite. Naturally, such an $f(\sigma) \in \mathcal{A}$ acts on $w \in \mathbb{R}^{\mathbb{Z}^2}$ as

$$f(\sigma)w = \sum_{\boldsymbol{\nu} \in \Gamma} \alpha_{\boldsymbol{\nu}} \sigma^{\boldsymbol{\nu}} w.$$

The action of a row vector of Laurent polynomial operators on a column vector of 2D discrete trajectories gets defined likewise: for $r(\sigma) = \left[r_1(\sigma), r_2(\sigma), \ldots, r_q(\sigma)\right] \in \mathcal{A}^q$ and $w = \mathrm{col}(w_1, w_2, \ldots, w_q) \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ we get

$$r(\sigma)w := \sum_{i=1}^{q} r_i(\sigma)w_i \in \mathbb{R}^{\mathbb{Z}^2}.$$

In this way, a system of linear 2D partial difference equations with real constant coefficients can be written as

$$R(\sigma)w = 0,$$

where $R(\sigma) \in \mathcal{A}^{\bullet \times q}$ is a matrix with entries that are Laurent polynomials in σ_1 and σ_2. Following Willems (1991), we call the set of all solutions of such a system of partial difference equations the *behavior* of the system, and denote it by the symbol $\mathfrak{B}$:

$$\mathfrak{B} := \left\{ w \in \left(\mathbb{R}^q\right)^{\mathbb{Z}^2} \,\middle|\, R(\sigma)w = 0 \right\},$$

where $R(\sigma) \in \mathcal{A}^{\bullet \times q}$. For example, the system given by the set of equations:

$$\begin{aligned} 2w(h+1,k) + w(h+2,k+1) + w(h,k) &= 0, \\ 3w(h+1,k+1) + w(h,k) &= 0, \end{aligned}$$

has its behavior $\mathfrak{B}$ given as

$$\mathfrak{B} := \left\{ w \in \mathbb{R}^{\mathbb{Z}^2} \,\middle|\, \begin{bmatrix} 2\sigma_1 + \sigma_1^2\sigma_2 + 1 \\ 3\sigma_1\sigma_2 + 1 \end{bmatrix} w = 0 \right\}.$$

The above description of $\mathfrak{B}$ is often written in short as $\mathfrak{B} = \ker R(\sigma)$. Such a representation is called a *kernel representation* of $\mathfrak{B}$, and $R(\sigma)$ is called a *kernel representation matrix*. In this paper, we denote the set of *all* discrete 2D systems having q number of dependent variables and are described by linear partial difference equations with constant real coefficients by the symbol $\mathfrak{L}^q_{2\mathrm{D}}$. We shall often abuse the notation slightly and write $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$ with the understanding that $\mathfrak{B}$ is the behavior of a system that is in $\mathfrak{L}^q_{2\mathrm{D}}$. Also, we shall often suppress the details and write just "discrete 2D system" to mean a system in $\mathfrak{L}^q_{2\mathrm{D}}$.

2.3 The Laurent polynomial ring in the shifts

The Laurent polynomial ring in the shift operators σ_1, σ_2, denoted by $\mathcal{A}$ in this paper, is the algebra generated by both *positive* and *negative* powers of the variables σ_1, σ_2 with coefficients from $\mathbb{R}$. It was shown in Willems (1991) that in discrete 1D systems, both positive and negative powers of the shift operator are legitimate operators on the trajectories, and hence, the operator algebra turns out to be $\mathcal{A}$, and not the polynomial ring. In this paper, we also need the one variable Laurent polynomial ring in the shift operator σ_1. We denote this ring by $\mathcal{A}_1$. This ring $\mathcal{A}_1$ is contained in $\mathcal{A}$ as a subring. The free modules of all row vectors of size q with entries in $\mathcal{A}$ ($\mathcal{A}_1$) is denoted by $\mathcal{A}^q$ ($\mathcal{A}_1^q$).

2.4 The equation module $\mathcal{R}$

The kernel representation of a behavior is not unique. The algebraic object that is uniquely associated with the behavior is the *module* generated by the equations. Suppose $R_1(\sigma), R_2(\sigma) \in \mathcal{A}^{\bullet \times q}$ are such that the modules generated by the rows of $R_1(\sigma)$ and those of $R_2(\sigma)$ (the row-spans of $R_1(\sigma)$ and $R_2(\sigma)$ over $\mathcal{A}$) are the same. Then, it easily follows that $\ker R_1(\sigma) = \ker R_2(\sigma)$. Remarkably, it follows from a strong result in Oberst (1990) that the converse is also true. That is, if $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^q_{2\mathrm{D}}$ are given by $\mathfrak{B}_1 = \ker R_1(\sigma)$,

$\mathfrak{B}_2 = \ker\ R_2(\sigma)$ and the modules generated by the rows of $R_1(\sigma)$, $R_2(\sigma)$ are $\mathcal{R}_1$, $\mathcal{R}_2$, respectively, then $\mathfrak{B}_1 = \mathfrak{B}_2$ *if and only if* $\mathcal{R}_1 = \mathcal{R}_2$. Note that $\mathcal{R}_1$ and $\mathcal{R}_2$ are both submodules of the free module $\mathcal{A}^q$. So, in other words, the submodules of $\mathcal{A}^q$ and discrete 2D behaviors in $\mathfrak{L}^q_{2\mathrm{D}}$ are in (an inclusion reversing) one-to-one correspondence with each other. Hence, given a behavior $\mathfrak{B} = \ker\ R(\sigma)$, the module generated by the rows of $R(\sigma)$ is uniquely associated with $\mathfrak{B}$, although the matrix $R(\sigma)$ need not be so. This module generated by the rows of a kernel representation matrix is called the *equation module* of $\mathfrak{B}$, and, in this paper, we denote this module by $\mathcal{R}$. Also, given a submodule $\mathcal{R} \subseteq \mathcal{A}^q$, we denote by $\mathfrak{B}(\mathcal{R})$ the behavior corresponding to the module $\mathcal{R}$:

$$\mathfrak{B}(\mathcal{R}) := \left\{ w \in \left(\mathbb{R}^q\right)^{\mathbb{Z}^2} \ \middle|\ f(\sigma)w = 0 \text{ for all } f(\sigma) \in \mathcal{R} \right\}. \tag{1}$$

In this paper, behaviors with a single dependent variable, that is, $q = 1$, play a significant role. Such behaviors are called *scalar behaviors*. It is worth mentioning here that for a scalar behavior the equation module turns out to be an ideal in $\mathcal{A}$. We denote this ideal by $\mathfrak{a}$ and call it the *equation ideal*.

2.5 The quotient module $\mathcal{M}$

Given a behavior $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$ let $\mathcal{R}$ be its equation module. The module obtained by quotienting $\mathcal{A}^q$ by $\mathcal{R}$ is of crucial importance in this paper. In fact, exploring and manipulating the structure of this quotient module are the keys behind the main results. We denote this module by $\mathcal{M}$, that is,

$$\mathcal{M} := \mathcal{A}^q/\mathcal{R},$$

and call it the *quotient module* of $\mathfrak{B}$. For an element $f(\sigma) \in \mathcal{A}^q$, we denote by $\overline{f(\sigma)}$ the image of $f(\sigma)$ under the canonical surjection $\mathcal{A}^q \twoheadrightarrow \mathcal{M}$. In the sequel, we shall often let elements of $\mathcal{M}$ act on the trajectories in the corresponding behavior. This is done as follows: let $m \in \mathcal{M}$, and $w \in \mathfrak{B}$. Then

$$m(w) := f(\sigma)w, \tag{2}$$

where $f(\sigma) \in \mathcal{A}^q$ is such that $\overline{f(\sigma)} = m$; such an $f(\sigma)$ is called a *lift* of m. Note that, if $f_1(\sigma), f_2(\sigma) \in \mathcal{A}^q$ are two distinct lifts of m then they must satisfy

$$0 \neq f_1(\sigma) - f_2(\sigma) \in \mathcal{R}.$$

Therefore, the actions of these distinct lifts on a w will result in the same:

$$\left(f_1(\sigma) - f_2(\sigma)\right) w = 0 \ \Rightarrow\ f_1(\sigma)w = f_2(\sigma)w,$$

because for $w \in \mathfrak{B}$, $f(\sigma)w = 0$ for all $f(\sigma) \in \mathcal{R}$ Eq. (1). Thus the action of $\mathcal{M}$ on $\mathfrak{B}$ as per Eq. (2) is well-defined.

2.6 Characteristic sets

Characteristic sets are the central objects of study in this paper. Given a $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$, a subset $\mathcal{S}$ of $\mathbb{Z}^2$ is said to be a *characteristic set* of $\mathfrak{B}$ if for every trajectory $w \in \mathfrak{B}$, the knowledge of w on $\mathcal{S}$ uniquely specifies w over all of $\mathbb{Z}^2$. In Definition 1 we give a precise definition of characteristic sets following the one given in Valcher (2000). For the purpose of this definition we need the notion of *restriction* of a trajectory to a subset $\mathcal{S}$ of $\mathbb{Z}^2$. Given a discrete 2D trajectory $w : \mathbb{Z}^2 \to \mathbb{R}^q$, and $\mathcal{S} \subseteq \mathbb{Z}^2$, the restriction of w to $\mathcal{S}$, denoted by $w|_{\mathcal{S}}$, is a map $w|_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^q$ defined as

$$w|_{\mathcal{S}}(h, k) = w(h, k) \text{ for all } (h, k) \in \mathcal{S}.$$

Definition 1 (Valcher 2000) Given $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$, a subset $\mathcal{S} \subseteq \mathbb{Z}^2$ is said to be a *characteristic set* for $\mathfrak{B}$ if for every trajectory $w \in \mathfrak{B}$, the restriction of w to the set $\mathcal{S}$, that is $w|_{\mathcal{S}}$, allows to uniquely determine the remaining portion of w, namely $w|_{\mathbb{Z}^2 \setminus \mathcal{S}}$.

In Fornasini et al. (1993), Rocha and Willems (1989), *autonomous discrete 2D systems* were defined as those which have a *proper* subset of $\mathbb{Z}^2$ for their characteristic sets. It was shown in Fornasini et al. (1993) that this property of autonomy is equivalent to the behavior admitting a kernel representation with a full column rank kernel representation matrix. Consequently, it was shown that autonomy is equivalent to $\mathcal{M}$ being a *torsion module*[1] (Pillai and Shankar 1998). This, in turn, is equivalent to the *annihilator ideal*, ann $\mathcal{M}$, defined as

$$\operatorname{ann} \mathcal{M} := \{f(\sigma) \in \mathcal{A} \mid f(\sigma)m = 0 \text{ for all } m \in \mathcal{M}\},$$

being non-zero. Thus, for $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$, there exits a proper subset $\mathcal{S} \subseteq \mathbb{Z}^2$ which is a characteristic set if and only if $\mathcal{M}$ is a torsion module, or, equivalently, ann $\mathcal{M} \neq \{0\}$. A special case of this is when the characteristic set turns out to be finite. Such systems are called *strongly autonomous* (Pillai and Shankar 1998). A behavior $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$ is strongly autonomous if and only if $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$. So, for strongly autonomous systems it is already known that thin characteristic sets (in fact, finite) exist. Therefore, in this paper we are concerned with characteristic sets of autonomous systems that are *not* strongly autonomous. It was shown in Valcher (2000) that such systems always admit convex cones as characteristic sets. But, convex cones, as argued earlier, are bigger compared to lines or finite unions of parallel lines. In this paper, we show that *all* discrete 2D systems, in particular, those which are not strongly autonomous, admit finite collections of parallel lines as characteristic sets. As pointed out earlier in this paper, such a finite collection of lines in $\mathbb{Z}^2$ can be viewed as thin compared to convex cones (see “Appendix 1”).

Since a key factor in establishing the main result of this paper is showing existence of the claimed type of characteristic sets for scalar systems, we point out the following fact about scalar systems. Kernel representation matrices of scalar systems are column vectors, and any column vector with at least one non-zero entry is clearly full column rank. Therefore, every scalar discrete 2D system (with a nonzero kernel representation matrix) is autonomous.

3 The special case: strongly σ_2-relevant scalar systems

We first consider a special class of scalar discrete 2D systems called *strongly σ_2-relevant systems* (Pal and Pillai 2013). As mentioned in Pal and Pillai (2013), the concept of strongly σ_2-relevant systems is inspired by that of “time/space-relevant systems” introduced in Napp et al. (2011). In Definition 2 below, we adapt the definition of time/space-relevant systems of Napp et al. (2011) to suit the purpose of this paper. To avoid confusion with strong σ_2-relevance, we call the adapted version of time/space relevance of Napp et al. (2011), the *(weak) σ_2-relevance*.

[1] For every element $m \in \mathcal{M}$, there is a $0 \neq f(\sigma) \in \mathcal{A}$ such that $f(\sigma)m = 0 \in \mathcal{M}$.

Definition 2 ((*Weak*) σ_2-*relevance*) (Napp et al. 2011) A 2D system with behavior $\mathfrak{B}$ is called *(weakly)* σ_2-*relevant* if for every $k \in \mathbb{Z}$ the subset of $\mathbb{Z}^2$ of the form

$$\mathcal{S}_k := \{(\nu_1, \nu_2) \in \mathbb{Z}^2 | \nu_2 \leqslant k\} \tag{3}$$

is a characteristic set of $\mathfrak{B}$.

Note that these characteristic sets $\mathcal{S}_k$, *half-spaces* as they are often called, consist of infinitely many horizontal lines.

In contrast to this trajectory level definition of (weak) σ_2-relevance, the notion of strongly σ_2-relevant systems, as introduced in Pal and Pillai (2013), is defined using a purely algebraic property.

Definition 3 (*Strong* σ_2-*relevance*) Consider $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ with equation ideal $\mathfrak{a}$. Then $\mathfrak{B}$ is said to be *strongly* σ_2-*relevant* if the quotient ring $\mathcal{M} = \mathcal{A}/\mathfrak{a}$ is a finitely generated module over $\mathcal{A}_1$.

Note that the quotient ring is trivially a module over $\mathcal{A}_1$; what makes the $\mathcal{M}$ of a strongly σ_2-relevant system special is that $\mathcal{M}$ is *finitely generated* as a module over $\mathcal{A}_1$.

Although, Definition 3 is purely algebraic, Theorem 13 brings out the trajectory level meaning of strong σ_2-relevance: Theorem 13 shows that a scalar strongly σ_2-relevant system admits a *finite* collection of *horizontal* lines for its characteristic set. Proposition 4 below gives another algebraic property equivalent to $\mathfrak{B}$ being strongly σ_2-relevant. This result, proved in Pal and Pillai (2013), will be useful for us in the sequel.

Proposition 4 *Suppose* $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ *has equation ideal* $\mathfrak{a}$. *Then* $\mathfrak{B}$ *is strongly* σ_2-*relevant if and only if* $\mathfrak{a}$ *contains an element, say* $f(\sigma) \in \mathcal{A}$, *of the form*

$$f(\sigma) = \sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1), \tag{4}$$

where $L \in \mathbb{Z}$ *and* $L > 0$, $a_0(\sigma_1), a_1(\sigma_1), \ldots, a_{L-1}(\sigma_1) \in \mathcal{A}_1$, *with* $a_0(\sigma_1)$ *being a unit in* $\mathcal{A}_1$.

Remark 5 ((weak) σ_2 -relevance versus strong σ_2 -relevance) At this point, it is important to note the difference between (weakly) σ_2-relevant systems and *strongly* σ_2-relevant ones. According to Napp et al. (2011), Proposition 6, a 'square'[2] behavior $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ is (weakly) σ_2-relevant if and only if it admits a kernel representation $\mathfrak{B} = \ker f(\sigma)$ where $f(\sigma)$ is of the form of Eq. (4) with no condition on the term $a_0(\sigma_1)$ being a unit. It then clearly follows from Proposition 4 that every square strongly σ_2-relevant $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ is also (weakly) σ_2-relevant, but the converse is not true. Here is an example of a square $\mathfrak{B}$ that is (weakly) σ_2-relevant, but *not* strongly σ_2-relevant: $\mathfrak{B} = \ker\,(\sigma_2 + \sigma_1 + 1)$.

Remark 6 (Evolution: unidirectional versus bidirectional) Note that (weak) σ_2-relevance is equivalent to saying that the behavior is composed entirely of trajectories that 'evolve' in the σ_2-direction from an 'initial condition' specified on the half-spaces, $\mathcal{S}_k$, of the type described in Eq. (3). Although, the definition requires the whole half-space $\mathcal{S}_k$ to be a characteristic set, it follows from Napp et al. (2011), Proposition 6, that the restriction of any trajectory $w \in \mathfrak{B}$ to only a finite collection of horizontal lines contained in $\mathcal{S}_k$' uniquely determines $w(\nu_1, \nu_2)$ for all $\nu_2 \geqslant k + 1$. Thus, it appears that every (weakly) σ_2-relevant system would admit a finite collection of horizontal lines as a characteristic set. However, such an inference

[2] 'Square' means the equation ideal is principal.

would in general be erroneous. This is because w, by this restriction on a finitely many horizontal lines, gets specified uniquely only in the 'future' ($\nu_2 \geqslant k+1$) and not in the 'past' ($\nu_2 \leqslant k-L-1$, where L is the so called *time lag*, Napp et al. 2011). For example, consider again the (weakly) σ_2-relevant behavior $\mathfrak{B} = \ker\,(\sigma_2+\sigma_1+1)$. Here, every trajectory w satisfies the equation $w(\nu_1,\nu_2+1) = -w(\nu_1+1,\nu_2) - w(\nu_1,\nu_2)$. Hence, restriction of w on the line $\{(\nu_1,0) \mid \nu_1 \in \mathbb{Z}\}$ uniquely determines $w(\nu_1,\nu_2)$ for all $\nu_2 \geqslant 1$. However, observe that the information of w restricted to this line cannot uniquely determine $w(\nu_1,\nu_2)$ for $\nu_2 \leqslant -1$. Theorem 13 shows that for strongly σ_2-relevant systems, restriction of every trajectory w to finitely many horizontal lines uniquely determines both the 'future' and the 'past' of w.

Remark 7 (Is it enough to look at only the square part of $\mathfrak{B}$?) The following observation has played a crucial role in the literature on characteristic sets: a set $\mathcal{S} \subseteq \mathbb{Z}^2$, which is a *proper cone*[3], is a characteristic set for $\mathfrak{B}$ *if and only if* it is a characteristic set for $\mathfrak{B}^{\mathrm{sq}}$, the 'square part' of $\mathfrak{B}$ (Valcher 2000, Proposition 2.6). However, when it comes to dealing with the type of characteristic sets considered in this paper, it is observed that the above-mentioned situation changes drastically. There exists $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ such that a set $\mathcal{S}$, which is not a proper cone, is a characteristic set for $\mathfrak{B}^{\mathrm{sq}}$, but $\mathcal{S}$ is not a characteristic set for $\mathfrak{B}$. Here is an example: $\mathfrak{B} = \ker \begin{bmatrix} \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1 \\ \sigma_2^2 - 2\sigma_2 + 1 \end{bmatrix}$. Here, $\mathfrak{B}^{\mathrm{sq}} = \ker\,(\sigma_2 - 1)$. Clearly, $\mathcal{S} = \{(\nu_1,0) \mid \nu_1 \in \mathbb{Z}\}$ is a characteristic set for $\mathfrak{B}^{\mathrm{sq}}$. However, $\mathcal{S}$ is *not* a characteristic set for $\mathfrak{B}$. To see this, consider the trajectory

$$w(\nu_1,\nu_2) = -\nu_2.$$

It can be checked easily that $w \in \mathfrak{B}$, but $w|_{\mathcal{S}} \equiv 0$. Therefore, $\mathcal{S}$ *cannot* be a characteristic set for $\mathfrak{B}$ because if it were so then $w|_{\mathcal{S}} \equiv 0$ would have implied w is the zero trajectory (Valcher 2000, Lemma 2.3), which it clearly is not. This example shows that reducing the question of finding characteristic sets to only square autonomous systems by applying (Valcher 2000, Proposition 2.6) does not work out for all types of subsets of $\mathbb{Z}^2$. The method works well, provided the purported characteristic set in question is a proper cone. Thus, while dealing with characteristic sets such as lines or finite unions of lines, it is imperative that the corresponding algebraic criteria must have provisions for non-square systems. Note that this quality is already present in the definition of strong σ_2-relevance (Definition 3).

Remark 8 (Strongly autonomous systems are strongly σ_2 -relevant) Systems that are strongly autonomous are also strongly σ_2-relevant. For in case of strongly autonomous systems, the quotient ring $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$, and thus, trivially, a finitely generated module over $\mathcal{A}_1$. However, the two notions are clearly not equivalent.

Our main tool for proving Theorem 13 is a certain representation formula for trajectories in strongly σ_2-relevant discrete 2D autonomous behaviors provided in Pal and Pillai (2013). We paraphrase this result of Pal and Pillai (2013) as Proposition 10 here. However, before we get to Proposition 10 we need to familiarize with certain objects associated with it. The existence of these objects follow from the assumption that $\mathcal{M}$ is finitely generated as a module over $\mathcal{A}_1$.

3.1 Consequences of $\mathcal{M}$ being a finitely generated module over $\mathcal{A}_1$

The quotient ring $\mathcal{M}$ being finitely generated as a module over $\mathcal{A}_1$ means that one can fix a finite generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. Suppose $\{g_1, g_2, \ldots, g_n\} \subseteq \mathcal{M}$ is such a

[3] A proper cone is a closed, pointed and solid convex cone in $\mathbb{R}^2$ intersected with $\mathbb{Z}^2$. See Valcher (2000) for more details.

generating set. Once a generating set is fixed it can be used to set up a map ψ from the free module $\mathcal{A}_1^n$ to $\mathcal{M}$ as shown below:

$$\begin{aligned} \psi : \mathcal{A}_1^n &\to \mathcal{M} \\ e_i \quad &\mapsto g_i \text{ for all } 1 \leqslant i \leqslant n, \end{aligned} \tag{5}$$

where e_i is the standard ith basis row-vector[4] of the free module $\mathcal{A}_1^n$. Note that the map ψ is an $\mathcal{A}_1$-module homomorphism, and is surjective. Using this map ψ, we construct the following two matrices: $A(\sigma_1)$, $C(\sigma_1)$.

3.1.1 *The flow matrix* $A(\sigma_1)$

Consider the map $\mu : \mathcal{M} \to \mathcal{M}$ given by $\mu(m) = \overline{\sigma_2} m$ for $m \in \mathcal{M}$. Clearly μ is an $\mathcal{A}_1$-module homomorphism. Now, since $\mathcal{M}$ is finitely generated as an $\mathcal{A}_1$-module, this map μ can be represented by a matrix. This is done as follows: suppose $\{g_1, g_2, \ldots, g_n\} \subseteq \mathcal{M}$ is a generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. For $1 \leqslant i \leqslant n$ consider the action of μ on the generator g_i. Since $\{g_1, g_2, \ldots, g_n\}$ is a generating set, the image of g_i under μ can be expressed as an $\mathcal{A}_1$-linear combination of these $g_1, g_2, \ldots, g_n$. That is, for $1 \leqslant i \leqslant n$, there exist $a_{i,1}(\sigma_1), a_{i,2}(\sigma_1), \ldots, a_{i,n}(\sigma_1) \in \mathcal{A}_1$ such that

$$\mu(g_i) = a_{i,1}(\sigma_1) g_1 + a_{i,2}(\sigma_1) g_2 + \cdots + a_{i,n}(\sigma_1) g_n.$$

Using this observation, we define the *flow matrix* $A(\sigma_1)$ as

$$A(\sigma_1) := \left[a_{i,j}(\sigma_1)\right]_{1 \leqslant i,j \leqslant n} \in \mathcal{A}_1^{n \times n}. \tag{6}$$

Remark 9 It has been shown in Pal and Pillai (2013) that there always exists a generating set for which the corresponding $A(\sigma_1)$ turns out to be invertible in $\mathcal{A}_1$ (that is, det $A(\sigma_1)$ is a unit in $\mathcal{A}_1$).

3.1.2 *The output matrix* $C(\sigma_1)$

Let $C(\sigma_1) \in \mathcal{A}_1^n$ be such that $\psi(C(\sigma_1)) = \overline{1}$, the image of $1 \in \mathcal{A}$ under the canonical map $\mathcal{A} \twoheadrightarrow \mathcal{M}$. We call $C(\sigma_1)$ the *output matrix*.

With the matrices $A(\sigma_1)$, $C(\sigma_1)$ defined as above we now state a slightly modified version of Theorem 3.7 of Pal and Pillai (2013) as Proposition 10 below. It is important at this point to recall that a row-vector $r(\sigma_1)$ of operators from $\mathcal{A}_1^n$ acts on a column vector of 1D trajectories $x \in \left(\mathbb{R}^{\mathbb{Z}}\right)^n$ to produce a trajectory in $\mathbb{R}^{\mathbb{Z}}$: for $r(\sigma_1) = [r_1(\sigma_1), r_2(\sigma_1), \ldots, r_n(\sigma_1)] \in \mathcal{A}_1^n$ and $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \left(\mathbb{R}^{\mathbb{Z}}\right)^n$, the action $r(\sigma_1)x$ is given by

$$r(\sigma_1)x = \sum_{i=1}^{n} r_i(\sigma_1) x_i.$$

Proposition 10 *Suppose* $\mathfrak{B} \in \mathfrak{L}^1_{\mathrm{2D}}$, *let* $\mathfrak{B}$ *be strongly* σ_2*-relevant, that is the quotient ring* $\mathcal{M}$ *is finitely generated as a module over* $\mathcal{A}_1$. *Fix a generating set* $\{g_1, g_2, \ldots, g_n\}$ *for* $\mathcal{M}$ *as an* $\mathcal{A}_1$*-module. Let the matrices* $A(\sigma_1)$, $C(\sigma_1)$ *be as defined above in Sects.* 3.1.1, 3.1.2,

[4] As a convention, we consider the elements in $\mathcal{A}_1^n$ to be written as row-vectors.

respectively. Assume that $\{g_1, g_2, \ldots, g_n\}$ *is chosen so as to ensure* $A(\sigma_1)$ *is invertible in* $\mathcal{A}_1^{n\times n}$. *Let* $w \in \mathfrak{B}$ *be arbitrary. Define the vector 1D trajectory* $x \in \left(\mathbb{R}^{\mathbb{Z}}\right)^n$ *as*

$$x(\nu_1) := \left(\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} w \right) (\nu_1, 0), \textit{ for all } \nu_1 \in \mathbb{Z}.$$

Then $w \in \mathfrak{B}$ *can be written as*

$$w(\nu_1, \nu_2) = \left(C(\sigma_1)A(\sigma_1)^{\nu_2} x\right)(\nu_1) \tag{7}$$

at all points $(\nu_1, \nu_2) \in \mathbb{Z}^2$.

Proposition 10 is the key to proving Theorem 13. The representation formula (7) asserts that if the vector-valued 1D trajectory $x(\bullet)$ is known then w can be uniquely derived from it. Note that $x(\bullet)$ is derived from $w(\bullet, 0)$ by making a vector of (Laurent) polynomials of difference operators act on it. Thus, it appears that the construction of x requires only a portion of the knowledge of w, which, in turn, uniquely specifies w. A natural question that arises now is: *can this portion of knowledge of* w *be identified with* $w|_{\mathcal{S}}$ *for* $\mathcal{S} \subseteq \mathbb{Z}^2$, *where* $\mathcal{S}$ *has desirable properties like being a line or a finite union of lines, etc?* We show in Theorem 13 that this is indeed the case: a suitable set of generators $\{g_1, g_2, \ldots, g_n\}$ results in x being nothing but the restriction of w on a finite union of horizontal lines. For this purpose, we need the crucial algebraic result Lemma 11. In Lemma 11, we establish that whenever the quotient ring $\mathcal{M}$ is finitely generated as a module over $\mathcal{A}_1$ (that is, the corresponding behavior is strongly σ_2-relevant), the desired suitable generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module exists. Due to the technical nature of the proof, it has been presented in the "Appendix 2".

Lemma 11 *Let* $\mathfrak{a} \subseteq \mathcal{A}$ *be an ideal such that the quotient ring* $\mathcal{M} := \mathcal{A}/\mathfrak{a}$ *is finitely generated as a module over* $\mathcal{A}_1$. *Then there exists a positive integer* L *such that*

$$\{\overline{1}, \overline{\sigma_2}, \overline{\sigma_2}^2, \ldots, \overline{\sigma_2}^{L-1}\}$$

is a generating set for $\mathcal{M}$ *as a module over* $\mathcal{A}_1$.

Remark 12 To proceed any further, it is crucially important to verify whether the chosen generating set as per Lemma 11 leads to a flow matrix $A(\sigma_1)$ that is invertible. We show here that this is indeed the case. By Proposition 4 we know that the equation ideal $\mathfrak{a}$ contains a polynomial $f(\sigma)$ of the form given by Eq. (4). It then follows from the construction of $A(\sigma_1)$, as delineated in Sect. 3.1.1, that under the generating set $\{\overline{\sigma_2}^i\}_{0\leqslant i\leqslant L-1}$, the matrix $A(\sigma_1)$ gets the following form:

$$A(\sigma_1) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_0(\sigma_1) & -a_1(\sigma_1) & \cdots & -a_{L-1}(\sigma_1) \end{bmatrix},$$

where $a_i(\sigma_1)$ come from the expression of $f(\sigma)$ of Eq. (4). As a consequence of this, we get that $\det A(\sigma_1) = (-1)^L a_0(\sigma_1)$, which is a unit because $a_0(\sigma_1)$ is a unit as per Proposition 4. Therefore, this $A(\sigma_1)$ is invertible in $\mathcal{A}_1^{L\times L}$.

With the above results we now show that every strongly σ_2-relevant $\mathfrak{B} \in \mathfrak{L}^1_{\mathrm{2D}}$ admits a finite number of horizontal lines as a characteristic set.

Theorem 13 *Let $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ be strongly σ_2-relevant. Then $\mathfrak{B}$ admits a finite number of horizontal lines as a characteristic set.*

Proof Since $\mathfrak{B}$ is strongly σ_2-relevant, the quotient ring $\mathcal{M}$ is a finitely generated module over $\mathcal{A}_1$. By Lemma 11 the set $\{\overline{1}, \overline{\sigma_2}, \ldots, \overline{\sigma_2}^{L-1}\}$ is a generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. Let the matrices $A(\sigma_1)$, $C(\sigma_1)$ be constructed as in Sects. 3.1.1, 3.1.2, respectively. By Remark 12, $A(\sigma_1)$ can be constructed to be invertible in $\mathcal{A}_1^{L\times L}$. Let $w \in \mathfrak{B}$ be arbitrary. Following Proposition 10 we define the vector 1D trajectories $x \in \left(\mathbb{R}^{\mathbb{Z}}\right)^L$ as

$$x(\nu_1) := \left(\begin{bmatrix} \overline{1} \\ \overline{\sigma_2} \\ \vdots \\ \overline{\sigma_2}^{L-1} \end{bmatrix} w \right)(\nu_1, 0), \text{ for all } \nu_1 \in \mathbb{Z}. \tag{8}$$

Then it follows from Proposition 10 that this w admits the following representation

$$w(\nu_1, \nu_2) = \left(C(\sigma_1) A(\sigma_1)^{\nu_2} x \right)(\nu_1)$$

at all points $(\nu_1, \nu_2) \in \mathbb{Z}^2$. Therefore, the knowledge of x (derived from w) uniquely specifies w. Now recall Eq. (2) that defines the action of elements in $\mathcal{M}$ on $w \in \mathfrak{B}$. Using Eq. (2), Eq. (8) can be simplified as

$$x(\nu_1) = \left(\begin{bmatrix} 1 \\ \sigma_2 \\ \vdots \\ \sigma_2^{L-1} \end{bmatrix} w \right)(\nu_1, 0) = \begin{bmatrix} w(\nu_1, 0) \\ w(\nu_1, 1) \\ \vdots \\ w(\nu_1, L-1) \end{bmatrix}.$$

Since $w \in \mathfrak{B}$ was chosen as arbitrary, it follows that the set

$$\mathcal{S} := \{(\nu_1, \nu_2) \in \mathbb{Z}^2 \,|\, 0 \leqslant \nu_2 \leqslant L-1\},$$

is a characteristic set for $\mathfrak{B}$, which consists of finitely many horizontal lines. $\square$

We illustrate the result of Theorem 13 by an example below.

Example 14 Consider the behavior

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2^2 + 5\sigma_2 + 6 \\ \sigma_1\sigma_2 + 2\sigma_1 - \sigma_2 - 2 \end{bmatrix}.$$

The equation ideal is given by

$$\mathfrak{a} = \langle \sigma_2^2 + 5\sigma_2 + 6, \sigma_1\sigma_2 + 2\sigma_1 - \sigma_2 - 2 \rangle.$$

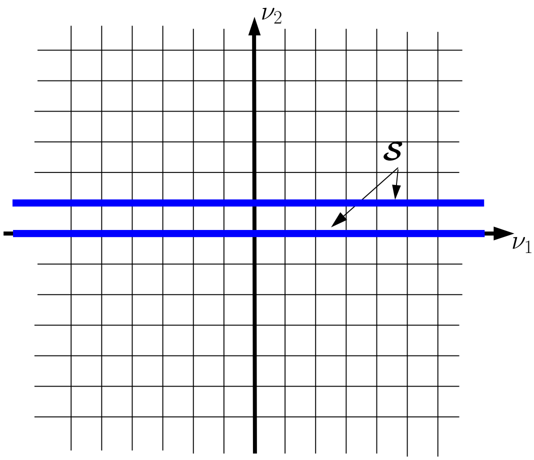
Presence of the polynomial $\sigma_2^2 + 5\sigma_2 + 6$ in $\mathfrak{a}$ makes the quotient ring $\mathcal{M}$ a finitely generated module over $\mathcal{A}_1$ (Proposition 4). It follows from Lemma 11 that $\{\overline{1}, \overline{\sigma_2}\}$ is a generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. With this generating set we get the following matrices:

$$A(\sigma_1) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, C(\sigma_1) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Therefore, any $w \in \mathfrak{B}$ is given by

$$w(\nu_1, \nu_2) = \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^{\nu_2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)(\nu_1), \tag{9}$$

Fig. 1 Characteristic set for $\mathfrak{B}$ in Example 14



where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\nu_1) = \begin{bmatrix} w(\nu_1, 0) \\ w(\nu_1, 1) \end{bmatrix}$. This means the entire trajectory w can be found once we find its values at the points on the lines $\{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid \nu_2 = 0\}$ and $\{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid \nu_2 = 1\}$. This means the set $\mathcal{S}$ defined below is a characteristic set for $\mathfrak{B}$:

$$\mathcal{S} = \{(\nu_1, 0) \mid \nu_1 \in \mathbb{Z}\} \cup \{(\nu_1, 1) \mid \nu_1 \in \mathbb{Z}\}.$$

This is shown pictorially in Fig. 1.

4 The first general case: arbitrary scalar discrete 2D systems

As is expected, there is a significantly large number of scalar discrete 2D systems that are *not* strongly σ_2-relevant. Indeed, for example, the behavior $\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)$ is not strongly σ_2-relevant. Naturally, for these systems, the theory presented so far in this paper does not hold. Consequently, for these systems, existence of characteristic sets of the type shown in Sect. 3 cannot be inferred directly. On the contrary, however, using the existing results in the literature it can be shown that the exemplary system, that is, $\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)$, *cannot* admit finitely many *horizontal* lines as a characteristic set. By Valcher (2000), Lemma 2.7, it follows that the entire lower half-space of any horizontal line, that is, $\mathcal{S}_k = \{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid \nu_2 \leqslant k\}$, cannot be a characteristic set of $\mathfrak{B}$ for any $k \in \mathbb{Z}$. Consequently, since any finite union of horizontal lines must be a subset of $\mathcal{S}_k$ for some k, it turns out that such a finite union cannot be a characteristic set for $\mathfrak{B}$.

It then seems reasonable to ask: *does a system that is not strongly σ_2-relevant, too, admit a characteristic set that is a union of finitely many parallel lines?*

An immediate partial answer to this question can be obtained by noting that in our analysis so far, the roles of σ_1 and σ_2 can be interchanged, and the analysis will still hold perfectly well. Thus, any system that perhaps is not strongly σ_2-relevant, but happens to be *strongly σ_1-relevant* can be shown to admit a finite union of *vertical* lines for a characteristic set. This, however, is in no way a complete answer to the above-mentioned question. For there are still a large number of systems that are neither strongly σ_2-relevant nor strongly σ_1-relevant. For example, the behavior $\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)$ considered above is such a behavior. In the remaining part of this paper, we resolve this issue in complete generality. In this section, we show that *every scalar* discrete 2D system admits a finite union of parallel lines (need not always be vertical or horizontal, could be tilted) as a characteristic set. And later in Sect. 5, we show the same holds for *every autonomous* discrete 2D system with more than one dependent variables. The trick in obtaining the result for general scalar systems lies in doing a suitable invertible coordinate change in the domain $\mathbb{Z}^2$.

4.1 Coordinate transformations on $\mathbb{Z}^2$ and their effects

By a coordinate transformation on $\mathbb{Z}^2$ we mean a mapping $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ that is bijective and satisfies the property of $\mathbb{Z}$-linearity: for all $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$, T must satisfy

$$T(\boldsymbol{\nu} + \boldsymbol{\nu}') = T(\boldsymbol{\nu}) + T(\boldsymbol{\nu}'), \text{ and } T(k\boldsymbol{\nu}) = kT(\boldsymbol{\nu}).$$

When the elements of $\mathbb{Z}^2$ are written as column vectors, then coordinate transforms are represented by (2×2) integer matrices whose determinants are ± 1. Such matrices are known as unimodular integer matrices.

Given a coordinate transformation T on $\mathbb{Z}^2$ (or, equivalently, a unimodular matrix $T \in \mathbb{Z}^{2\times 2}$), it induces a map $T^* : \mathbb{R}^{\mathbb{Z}^2} \to \mathbb{R}^{\mathbb{Z}^2}$ in the following manner: for an arbitrary $w \in \mathbb{R}^{\mathbb{Z}^2}$ and $\boldsymbol{\nu} \in \mathbb{Z}^2$

$$\left(T^*(w)\right)(\boldsymbol{\nu}) := w(T(\boldsymbol{\nu})). \tag{10}$$

The map T^* is often called the *pull-back* of the coordinate transformation T. Note that the bijectivity of T forces T^* to be bijective, too. Indeed, $(T^*)^{-1} = (T^{-1})^*$. Another noteworthy property of T^* is that it is $\mathbb{R}$-linear. It then follows that, for a $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$, since $\mathfrak{B}$ is a subspace of $\mathbb{R}^{\mathbb{Z}^2}$, the image of $\mathfrak{B}$ under T^*, that is, $T^*(\mathfrak{B})$, must also be a subspace of $\mathbb{R}^{\mathbb{Z}^2}$. Interestingly, it turns out that $T^*(\mathfrak{B})$ is more than just a subspace of $\mathbb{R}^{\mathbb{Z}^2}$, it is, in fact, a scalar discrete 2D behavior. We elaborate on this fact now.

The fact that $T^*(\mathfrak{B}) \in \mathfrak{L}^1_{2\mathrm{D}}$ can be inferred by showing that $T^*(\mathfrak{B})$ is the solution set of a system of 2D linear partial difference equations. It turns out that the ideal of these equations is related to the original equation ideal through yet another map induced by the coordinate transformation T. This map is known as the *push-forward* of T, denoted as T_*, and is defined thus:

$$T_* : \mathcal{A} \to \mathcal{A}, \quad T_*(\sigma^{\boldsymbol{\nu}}) = \sigma^{T(\boldsymbol{\nu})}. \tag{11}$$

Since every polynomial in $\mathcal{A}$ is a finite $\mathbb{R}$-linear combination of the monomials $\sigma^{\boldsymbol{\nu}}$, the action of T_* on every element of $\mathcal{A}$ gets uniquely specified once it is defined on the monomials $\sigma^{\boldsymbol{\nu}}$ as done in Eq. (11) above, and it is imposed that T_* must be $\mathbb{R}$-linear. Thus, T_* is a map of $\mathbb{R}$-algebras that keeps the base field $\mathbb{R}$ fixed. The coordinate transformation being unimodular forces T_* to be bijective, too. Hence, the image of an ideal $\mathfrak{a} \subseteq \mathcal{A}$ under this map, $T_*(\mathfrak{a})$, too, turns out to be an ideal.

Now, to see the relation between the two maps T^* and T_*, first observe that for any $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathbb{Z}^2$, and $w \in \mathbb{R}^{\mathbb{Z}^2}$, we get the following:

$$\left(\sigma^{\boldsymbol{\nu}}(T^*(w))\right)(\boldsymbol{\nu}') = T^*\left(\sigma^{T(\boldsymbol{\nu})}(w)\right)(\boldsymbol{\nu}'). \tag{12}$$

Indeed, $\left(\sigma^{\boldsymbol{\nu}}(T^*(w))\right)(\boldsymbol{\nu}') = (T^*(w))(\boldsymbol{\nu} + \boldsymbol{\nu}') = w(T(\boldsymbol{\nu} + \boldsymbol{\nu}')) = \left(\sigma^{T(\boldsymbol{\nu})}w\right)(T(\boldsymbol{\nu}')) = T^*\left(\sigma^{T(\boldsymbol{\nu})}(w)\right)(\boldsymbol{\nu}')$. Equation (12) can be rewritten using the push-forward as

$$\sigma^{\boldsymbol{\nu}}(T^*(w)) = T^*\left(T_*\left(\sigma^{\boldsymbol{\nu}}\right)(w)\right). \tag{13}$$

The above equation shows equality of trajectories: $\boldsymbol{\nu}'$ is omitted because Eq. (12) holds for arbitrary $\boldsymbol{\nu}' \in \mathbb{Z}^2$. Following this we get that, for a general Laurent polynomial $f(\sigma) =$

$\sum_{\boldsymbol{\nu}\in\Gamma} a_{\boldsymbol{\nu}}\sigma^{\boldsymbol{\nu}}$, where $\Gamma \subseteq \mathbb{Z}^2$ is finite,

$$\begin{aligned}
f(\sigma)\left(T^*(w)\right) &= \sum_{\boldsymbol{\nu}\in\Gamma} a_{\boldsymbol{\nu}}\sigma^{\boldsymbol{\nu}}\left(T^*(w)\right) \\
&= \sum_{\boldsymbol{\nu}\in\Gamma} a_{\boldsymbol{\nu}} T^*\left(T_*\left(\sigma^{\boldsymbol{\nu}}\right)(w)\right) \\
&= T^*\left(\sum_{\boldsymbol{\nu}\in\Gamma} a_{\boldsymbol{\nu}} T_*\left(\sigma^{\boldsymbol{\nu}}\right)(w)\right) \\
\Rightarrow f(\sigma)\left(T^*(w)\right) &= T^*\left(T_*\left(f(\sigma)(w)\right)\right).
\end{aligned} \tag{14}$$

Proposition 15 *Consider $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ with equation ideal $\mathfrak{a}$. Further, let $T \in \mathbb{Z}^{2\times 2}$ be a unimodular matrix defining a coordinate transformation on $\mathbb{Z}^2$. Define the two maps $T^* : \mathbb{R}^{\mathbb{Z}^2} \to \mathbb{R}^{\mathbb{Z}^2}$ and $T_* : \mathcal{A} \to \mathcal{A}$ as in Eqs. (10) and (11), respectively. Then we have*

$$\mathfrak{B} = T^*\left(\mathfrak{B}\left(T_*(\mathfrak{a})\right)\right). \tag{15}$$

Proof Follows easily from Eq. (14): see the proof of Pal and Pillai (2013), Theorem 2.6. □

Proposition 16 below shows the effect of coordinate transformations on characteristic sets. Given a coordinate transformation T, and two behaviors $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^1_{2\mathrm{D}}$ such that $\mathfrak{B}_2 = T^*(\mathfrak{B}_1)$, it turns out that characteristic sets of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are directly related by the coordinate transformation T.

Proposition 16 *Let $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ be a coordinate transformation. Further, let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^1_{2\mathrm{D}}$ be related to each other by $\mathfrak{B}_2 = T^*(\mathfrak{B}_1)$. Then a subset $\mathcal{S} \subseteq \mathbb{Z}^2$ is a characteristic set for $\mathfrak{B}_1$ if and only if $T^{-1}(\mathcal{S})$ is a characteristic set for $\mathfrak{B}_2$.*

Proof (Only if part) Suppose $\mathcal{S} \subseteq \mathbb{Z}^2$ is a characteristic set for $\mathfrak{B}_1$, we have to show that $T^{-1}(\mathcal{S})$ is a characteristic set for $\mathfrak{B}_2$. In order for that, let us assume $w \in \mathfrak{B}_2$ is arbitrary, then we must show that $w|_{T^{-1}(\mathcal{S})}$ uniquely determines w over the whole of $\mathbb{Z}^2$. Now, since $w \in \mathfrak{B}_2 = T^*(\mathfrak{B}_1)$, there exists a unique $\widetilde{w} \in \mathfrak{B}_1$ such that $w = T^*(\widetilde{w})$. That is, for all $\boldsymbol{\nu} \in \mathbb{Z}^2$, $w(\boldsymbol{\nu}) = \widetilde{w}(T(\boldsymbol{\nu}))$. The uniqueness follows from the fact that unimodularity of T forces its pull-back T^* to be bijective. The bijectivity of T^* also means that $\widetilde{w}$, too, uniquely determines w.

Suppose now that $\boldsymbol{\nu} \in T^{-1}(\mathcal{S})$, that is, $\boldsymbol{\nu} = T^{-1}(\boldsymbol{\nu}')$, where $\boldsymbol{\nu}' \in \mathcal{S}$. Then

$$w(\boldsymbol{\nu}) = \widetilde{w}(T(\boldsymbol{\nu})) = \widetilde{w}(T(T^{-1}(\boldsymbol{\nu}'))) = \widetilde{w}(\boldsymbol{\nu}').$$

Hence it follows that $w|_{T^{-1}(\mathcal{S})}$ uniquely determines $\widetilde{w}|_{\mathcal{S}}$. Since, $\mathcal{S}$ has been assumed to be a characteristic set for $\mathfrak{B}_1$, $\widetilde{w}|_{\mathcal{S}}$ uniquely determines $\widetilde{w}$, which in turn, determines w uniquely because T^* is bijective. Therefore, effectively, $w|_{T^{-1}(\mathcal{S})}$ determines w uniquely. Since w was taken to be an arbitrary element of $\mathfrak{B}_2$, it follows that for all $w \in \mathfrak{B}_2$, the restriction $w|_{T^{-1}(\mathcal{S})}$ uniquely determines w; hence, $T^{-1}(\mathcal{S})$ is a characteristic set for $\mathfrak{B}_2$.

(If part) Suppose $T^{-1}(\mathcal{S})$ is a characteristic set for $\mathfrak{B}_2$, we have to show that $\mathcal{S}$ is a characteristic set for $\mathfrak{B}_1$. Note that $\mathfrak{B}_2 = T^*(\mathfrak{B}_1)$ implies that $\mathfrak{B}_1 = (T^*)^{-1}(\mathfrak{B}_2) = \left(T^{-1}\right)^*(\mathfrak{B}_2)$. It then follows from the proof of the only if part that $T(T^{-1}(\mathcal{S})) = \mathcal{S}$ is a characteristic set of $\mathfrak{B}_1$. □

4.2 Discrete Noether's normalization lemma

Recall that a system is not strongly σ_2-relevant if and only if its quotient ring, $\mathcal{M} = \mathcal{A}/\mathfrak{a}$, is not a finitely generated module over $\mathcal{A}_1$. In this subsection, we shall see that if $\mathcal{M}$ is not a finitely generated module over $\mathcal{A}_1$, it can be made so by the push-forward, T_*, of a suitable coordinate transformation T. We state this result in Theorem 18. Theorem 18 can be viewed as an analogue of Noether's normalization lemma (see Eisenbud 1995 to get a comprehensive exposition on the conventional Noether's normalization lemma). We refer to Theorem 18 by discrete Noether's normalization lemma (DNNL). DNNL follows from Lemma 17 below, which shows that given a 2D Laurent polynomial, there exists a unimodular $T \in \mathbb{Z}^{2\times 2}$ such that under T_* the given Laurent polynomial is mapped to a Laurent polynomial with a special structure: when written as a Laurent polynomial in σ_2 with coefficients from $\mathcal{A}_1$, these coefficients are all *units* in $\mathcal{A}_1$. A similar result can be found in Park (2004), where the result was used in the context of designing 2D filters. See Pal and Pillai (2013), Lemma 4.1, for a proof of Lemma 17.

Lemma 17 *Let* $0 \neq f(\sigma) \in \mathcal{A}$ *be given by*

$$f(\sigma) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^2} \alpha_{\boldsymbol{\nu}}\sigma^{\boldsymbol{\nu}}, \quad \alpha_{\boldsymbol{\nu}} \in \mathbb{R},$$

with only finitely many $\alpha_{\boldsymbol{\nu}} \neq 0$. *Then there exists a unimodular* $T \in \mathbb{Z}^{2\times 2}$ *such that under the push-forward* T_* *given by Eq.* (11)*, we have*

$$T_*(f(\sigma)) = \left(\sum_{k=0}^{\delta} u_k(\sigma_1)\sigma_2^k\right)u(\sigma_2), \tag{16}$$

where $u_0(\sigma_1), u_1(\sigma_1), \ldots, u_\delta(\sigma_1) \in \mathcal{A}_1$ *and* $u(\sigma_2) \in \mathbb{R}[\sigma_2^{\pm 1}]$ *are all units in* $\mathcal{A}$ *and* δ *is some finite positive integer.*

We now state and prove the DNNL adapted from Pal and Pillai (2013) to suit the purpose of this paper.

Theorem 18 **(DNNL)** *Let* $\mathfrak{a} \subseteq \mathcal{A}$ *be a nonzero ideal such that* $\mathcal{A}/\mathfrak{a}$ *is not finitely generated as a module over* $\mathcal{A}_1$. *Then there exists* $T \in \mathbb{Z}^{2\times 2}$ *unimodular, such that under its push-forward map* $T_* : \mathcal{A} \to \mathcal{A}$ *we have* $\mathcal{A}/T_*(\mathfrak{a})$ *to be a finitely generated module over* $\mathcal{A}_1$.

Proof Let $0 \neq f(\sigma) \in \mathfrak{a}$. By Lemma 17 above, there exists a unimodular matrix T such that $T_*(f(\sigma))$ has the form of Eq. (16). Let us define $\mathfrak{b} := T_*(\mathfrak{a})$. As we have mentioned earlier, $\mathfrak{b}$ is an ideal. Note that $T_*(f(\sigma)) \in \mathfrak{b}$. Since $T_*(f(\sigma)) \in \mathfrak{b}$, and $u(\sigma_2)$, $u_\delta(\sigma_1)$ in Eq. (16) are units in $\mathcal{A}$, we also have $g(\sigma) := u(\sigma_2)^{-1}u_\delta(\sigma_1)^{-1}T_*(f(\sigma)) \in \mathfrak{b}$. Now note that $g(\sigma)$ is of the following form:

$$g(\sigma) = \sigma_2^{\delta} + u_\delta(\sigma_1)^{-1}u_{\delta-1}(\sigma_1)\sigma_2^{\delta-1} + \cdots + u_\delta(\sigma_1)^{-1}u_0(\sigma_1).$$

Thus, $g(\sigma_2)$ is a monic polynomial in σ_2 with coefficients from $\mathcal{A}_1$ such that the constant term is $u_\delta(\sigma_1)^{-1}u_0(\sigma_1)$, which is a unit in $\mathcal{A}_1$. It then follows from Proposition 4 that $\mathcal{A}/\mathfrak{b}$ is a finitely generated module over $\mathcal{A}_1$. □

4.3 Finite union of parallel lines as a characteristic set for arbitrary $\mathfrak{B} \in \mathfrak{L}_{2D}^{1}$

With the help of DNNL we are now in a position to prove the first main result of this paper: every scalar discrete 2D system admits a characteristic set that is a finite union of parallel lines. The key idea behind the result is the following three observations: suppose $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$, with equation ideal $\mathfrak{a}$, is *not* strongly σ_2-relevant, then

(i) by Theorem 18 (DNNL) there exists a coordinate transformation T of $\mathbb{Z}^2$, such that $\mathcal{A}/T_*(\mathfrak{a})$ is finitely generated as a module over $\mathcal{A}_1$.
(ii) The behavior $\widetilde{\mathfrak{B}}$ corresponding to the ideal $T_*(\mathfrak{a})$ is then strongly σ_2-relevant. Hence it admits a finite union of horizontal lines, say $\mathcal{S}$, as a characteristic set.
(iii) By Proposition 15 we have $\mathfrak{B} = T^*(\widetilde{\mathfrak{B}})$, and by Proposition 16 we have the inverse-image of $\mathcal{S}$ under T must be a characteristic set for $\mathfrak{B}$. Since $\mathcal{S}$ is a finite union of horizontal lines, its inverse image under T must be a finite union of parallel lines.

In Theorems 19 and 20 we state and prove these observations formally.

Theorem 19 *Consider $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$ with equation ideal $\mathfrak{a}$. Then there exists a unimodular matrix $T \in \mathbb{Z}^{2\times 2}$ and a strongly σ_2-relevant $\widetilde{\mathfrak{B}} \in \mathfrak{L}^1_{2\mathrm{D}}$ such that*

$$\mathfrak{B} = T^*\left(\widetilde{\mathfrak{B}}\right). \tag{17}$$

Proof Suppose $\mathfrak{B}$ is already strongly σ_2-relevant. Then the result trivially holds by taking $T := \mathrm{diag}(1, 1)$, and $\widetilde{\mathfrak{B}} := \mathfrak{B}$. The non-trivial case is when $\mathfrak{B}$ is *not* strongly σ_2-relevant. In that case, the quotient ring $\mathcal{M}$ is not a finitely generated module over $\mathcal{A}_1$. By Theorem 18, then, there exists a unimodular matrix $T \in \mathbb{Z}^{2\times 2}$ such that under the push-forward, T_*, we get $\mathcal{A}/T_*(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1$. Define

$$\widetilde{\mathfrak{B}} := \mathfrak{B}(T_*(\mathfrak{a})).$$

Since $\mathcal{A}/T_*(\mathfrak{a})$ is finitely generated as a module over $\mathcal{A}_1$, it follows that $\widetilde{\mathfrak{B}}$ is strongly σ_2-relevant. Further, from Proposition 15 we get that

$$\mathfrak{B} = T^*(\widetilde{\mathfrak{B}}).$$

Hence, for every scalar discrete 2D behavior $\mathfrak{B}$ there exist $T \in \mathbb{Z}^{2\times 2}$ unimodular and a strongly σ_2-relevant scalar discrete 2D behavior $\widetilde{\mathfrak{B}}$ such that $\mathfrak{B} = T^*(\widetilde{\mathfrak{B}})$. □

Theorem 20 *Every scalar discrete 2D system admits a finite union of parallel lines as a characteristic set.*

Proof Given a $\mathfrak{B} \in \mathfrak{L}^1_{2\mathrm{D}}$, by Theorem 19, there exist $T \in \mathbb{Z}^{2\times 2}$ unimodular, and a strongly σ_2-relevant $\widetilde{\mathfrak{B}} \in \mathfrak{L}^1_{2\mathrm{D}}$, such that

$$\mathfrak{B} = T^*(\widetilde{\mathfrak{B}}).$$

Now, since $\widetilde{\mathfrak{B}}$ is strongly σ_2-relevant, it follows from Theorem 13 that $\widetilde{\mathfrak{B}}$ admits a characteristic set, say $\mathcal{S}$, that consists of finitely many horizontal lines. But, by Proposition 16, since $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ are related by the pull-back of the coordinate transformation T, it follows that if $\mathcal{S}$ is a characteristic set for $\widetilde{\mathfrak{B}}$ then $T^{-1}(\mathcal{S})$ must be a characteristic set for $\mathfrak{B}$. Since $\mathcal{S}$ is a union of finitely many horizontal lines, $T^{-1}(\mathcal{S})$ is a union of finitely many parallel (but, possibly, tilted) lines. Thus $\mathfrak{B}$ admits a finite union of parallel lines for a characteristic set. □

We illustrate the result of Theorem 20 in the following example.

Example 21 Suppose $\mathfrak{B} = \ker\,(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)$. We have seen at the beginning of this section that this $\mathfrak{B}$ is neither strongly σ_2-relevant nor strongly σ_1-relevant. Here the equation ideal, $\mathfrak{a} = \langle \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1\rangle$. Let us choose coordinate transformation matrix $T = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ to apply the DNNL. Under the corresponding push-forward, T_*, the generator of $\mathfrak{a}$ goes to

$\sigma_1\sigma_2^3 - \sigma_1\sigma_2^2 - \sigma_2 + 1$. Since σ_1 is a unit, we get that the ideal $T_*(\mathfrak{a}) = \langle \sigma_2^3 - \sigma_2^2 - \sigma_1^{-1}\sigma_2 + \sigma_1^{-1} \rangle$. It then follows from Proposition 4 that $\mathcal{A}/T_*(\mathfrak{a})$ is finitely generated as an $\mathcal{A}_1$-module.

Define $\widetilde{\mathfrak{B}} := \ker\,(\sigma_2^3 - \sigma_2^2 - \sigma_1^{-1}\sigma_2 + \sigma_1^{-1})$. Since $\mathcal{A}/T_*(\mathfrak{a})$ is a finitely generated $\mathcal{A}_1$-module, $\widetilde{\mathfrak{B}}$ is strongly σ_2-relevant. And, by Proposition 15, $\mathfrak{B} = T^*(\widetilde{\mathfrak{B}})$.

Now, by Lemma 11, $\{\overline{1}, \overline{\sigma_2}, \overline{\sigma_2}^2\}$ is a generating set for $\mathcal{A}/T_*(\mathfrak{a})$ as an $\mathcal{A}_1$-module. Thus we get

$$A(\sigma_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma_1^{-1} & \sigma_1^{-1} & 1 \end{bmatrix}, \; C(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Applying Proposition 10 now we get that every $\widetilde{w} \in \widetilde{\mathfrak{B}}$ is given by

$$\widetilde{w}(\nu_1, \nu_2) = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma_1^{-1} & \sigma_1^{-1} & 1 \end{bmatrix}^{\nu_2} x \right)(\nu_1),$$

where $x(\nu_1) := \left(\begin{bmatrix} \overline{1} \\ \overline{\sigma_2} \\ \overline{\sigma_2}^2 \end{bmatrix} \widetilde{w} \right)(\nu_1, 0) = \begin{bmatrix} \widetilde{w}(\nu_1,0) \\ \widetilde{w}(\nu_1,1) \\ \widetilde{w}(\nu_1,2) \end{bmatrix}$. Therefore, by Theorem13 we have the following subset $\mathcal{S}$ of $\mathbb{Z}^2$

$$\mathcal{S} := \{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid 0 \leqslant \nu_2 \leqslant 2\}$$

as a characteristic set of $\widetilde{\mathfrak{B}}$. By Proposition 16 we get that $T^{-1}(\mathcal{S})$ must be a characteristic set for $\mathfrak{B}$. An explicit description of $T^{-1}(\mathcal{S})$ can be given as

$$\begin{aligned} T^{-1}(\mathcal{S}) &= \{(\nu_1, -2\nu_1 + \nu_2) \in \mathbb{Z}^2 \mid \nu_1 \in \mathbb{Z}, 0 \leqslant \nu_2 \leqslant 2\} \\ &= \{(\nu_1, -2\nu_1)\} \cup \{(\nu_1, -2\nu_1 + 1)\} \cup \{(\nu_1, -2\nu_1 + 2)\}. \end{aligned}$$

These characteristic sets of $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ are shown in Fig. 2.

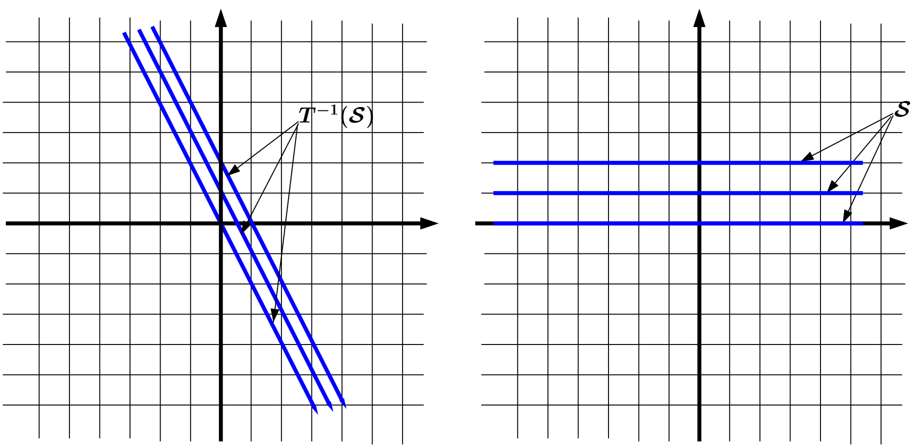


Fig. 2 Characteristic sets for $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$

5 The second general case: autonomous (non-scalar) discrete 2D systems

In this section, we prove the second main result of this paper: the existence of a union of finitely many parallel lines as a characteristic set for a general (non-scalar) discrete 2D autonomous system. This is achieved by reducing a given non-scalar autonomous discrete 2D behavior $\mathfrak{B}$ to a corresponding auxiliary scalar behavior, called $\mathfrak{B}_{\mathrm{sc}}$ in this paper, and then applying Theorem 20 to this $\mathfrak{B}_{\mathrm{sc}}$. The key observation is that a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$ is also a characteristic set for the corresponding $\mathfrak{B}$. By Theorem 20, the auxiliary behavior $\mathfrak{B}_{\mathrm{sc}}$, being scalar, admits a finite union of parallel lines as a characteristic set, and hence so does the original non-scalar autonomous behavior $\mathfrak{B}$. This key observation was suggested by an anonymous reviewer, we thank the reviewer for this crucial suggestion.

Given an autonomous behavior $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$, we start by showing the constructions of $\mathfrak{B}_{\mathrm{sc}}$—the above-mentioned auxiliary scalar behavior, and of $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}} \in \mathfrak{L}^q_{2\mathrm{D}}$ an auxiliary non-scalar autonomous behavior that is constructed out of $\mathfrak{B}_{\mathrm{sc}}$. Since the given $\mathfrak{B}$ is autonomous, as mentioned in Sect. 2.6, it follows that the quotient module $\mathcal{M}$ is a torsion module, and consequently, the annihilator ideal, ann $\mathcal{M}$ is a non-zero proper ideal. Define the auxiliary scalar behavior $\mathfrak{B}_{\mathrm{sc}}$ now as

$$\mathfrak{B}_{\mathrm{sc}} := \mathfrak{B}(\mathrm{ann}\ \mathcal{M}).$$

With $\mathfrak{B}_{\mathrm{sc}}$, now define $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}} \in \mathfrak{L}^q_{2\mathrm{D}}$ as

$$\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}} := \underbrace{\mathfrak{B}_{\mathrm{sc}} \times \mathfrak{B}_{\mathrm{sc}} \times \cdots \times \mathfrak{B}_{\mathrm{sc}}}_{q \text{ times}}. \tag{18}$$

Note that $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ is indeed a behavior in $\mathfrak{L}^q_{2\mathrm{D}}$ and its equation module $\mathcal{R}^{\mathrm{vec}}_{\mathrm{sc}}$ is given by

$$\mathcal{R}^{\mathrm{vec}}_{\mathrm{sc}} = \left\{ \left[f_1(\sigma)\ f_2(\sigma)\ \cdots\ f_q(\sigma) \right] \in \mathcal{A}^q \mid f_i(\sigma) \in \mathrm{ann}\ \mathcal{M} \text{ for all } i = 1, 2, \ldots, q \right\}.$$

The following observation will be important for proving the main result, Theorem 24.

Proposition 22 *Suppose $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$ is autonomous with equation module $\mathcal{R}$ and quotient module $\mathcal{M}$. Define $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ as in Eq.* (18). *Then*

$$\mathfrak{B} \subseteq \mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}.$$

Proof Let $w = \mathrm{col}(w_1, w_2, \ldots, w_q) \in \mathfrak{B}$ be arbitrary. In order to prove $w \in \mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ it is enough to show that

$$w_i \in \mathfrak{B}_{\mathrm{sc}} \text{ for all } i = 1, 2, \ldots, q. \tag{19}$$

Equation (19) will be proven true if we show that

$$f(\sigma)w_i = 0 \text{ for all } f(\sigma) \in \mathrm{ann}\ \mathcal{M}, \text{ and for all } i = 1, 2, \ldots, q.$$

Let $f(\sigma) \in \mathrm{ann}\ \mathcal{M}$ be arbitrary. Note that, since $f(\sigma) \in \mathrm{ann}\ \mathcal{M}$, we must have $f(\sigma)e_i \in \mathcal{R}$ for all $i = 1, 2, \ldots, q$, where $e_i \in \mathcal{A}^q$ is the ith standard basis row-vector. It then follows that

$$f(\sigma)w_i = f(\sigma)e_i w = 0,$$

because, as $w \in \mathfrak{B}$, with $\mathcal{R}$ being the equation module of $\mathfrak{B}$, we must have $r(\sigma)w = 0$ for all $r(\sigma) \in \mathcal{R}$. □

It has been shown in Valcher (2000), Lemma 2.5 that if $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^q_{2\mathrm{D}}$ are autonomous with $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ then $\mathcal{S} \subseteq \mathbb{Z}^2$ is a characteristic set for $\mathfrak{B}_1$ if $\mathcal{S}$ is a characteristic set for $\mathfrak{B}_2$. Therefore, in order to prove $\mathfrak{B}$ has a finite union of parallel lines as a characteristic set, it is sufficient that we show that the corresponding $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ has such a characteristic set because $\mathfrak{B} \subseteq \mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ by Proposition 22. Lemma 23 below shows that this is indeed the case: it is a consequence of the construction of $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ that it must have a finite union of parallel lines as a characteristic set.

Lemma 23 *Let $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$ be autonomous with quotient module $\mathcal{M}$. Define $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ as done in Eq.* (18)*. Then $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ has a finite union of parallel lines as a characteristic set.*

Proof Recall Eq. (18), where $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ has been defined as

$$\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}} := \underbrace{\mathfrak{B}_{\mathrm{sc}} \times \mathfrak{B}_{\mathrm{sc}} \times \cdots \times \mathfrak{B}_{\mathrm{sc}}}_{q \text{ times}},$$

with $\mathfrak{B}_{\mathrm{sc}} := \mathfrak{B}(\mathrm{ann}\,\mathcal{M})$. From this definition, it follows that if $\mathcal{S} \subseteq \mathbb{Z}^2$ is a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$ then $\mathcal{S}$ must be a characteristic set for $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$, too. Indeed, suppose $w = \mathrm{col}(w_1, w_2, \ldots, w_q) \in \mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ is such that

$$w|_{\mathcal{S}} = 0,$$

where $\mathcal{S} \subseteq \mathbb{Z}^2$. Let $\mathcal{S}$ be a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$. Now, $w|_{\mathcal{S}} = 0$ means $w_i|_{\mathcal{S}} = 0$ for all $i = 1, 2, \ldots, q$. From the structure of $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ it follows that for each $i = 1, 2, \ldots, q$, the scalar trajectory $w_i \in \mathfrak{B}_{\mathrm{sc}}$. But, by assumption, $\mathcal{S}$ is a characteristic set for $\mathfrak{B}_{\mathrm{sc}}$, therefore, $w_i|_{\mathcal{S}} = 0$ implies $w_i \equiv 0$ for all $i = 1, 2, \ldots, q$. Thus it follows that $w \equiv 0$. Hence we get that for all $w \in \mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$, $w|_{\mathcal{S}} = 0$ implies $w \equiv 0$, which is equivalent to saying $\mathcal{S}$ is a characteristic set for $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ (see Valcher 2000, Lemma 2.3).

Now, by the above argument, in order to show $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ has a characteristic set that is a finite union of parallel lines, it is enough that we show $\mathfrak{B}_{\mathrm{sc}}$ has a characteristic set that is a finite union of parallel lines. But this is true by Theorem 13 because $\mathfrak{B}_{\mathrm{sc}} \in \mathfrak{L}^1_{2\mathrm{D}}$. Hence $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ admits a characteristic set that is a finite union of parallel lines. □

With these results, the proof of the main result Theorem 24 now follows immediately.

Theorem 24 *Let $\mathfrak{B} \in \mathfrak{L}^q_{2\mathrm{D}}$ be autonomous. Then $\mathfrak{B}$ admits a characteristic set that is a finite union of parallel lines.*

Proof Suppose $\mathfrak{B}$ has quotient module $\mathcal{M}$. Since $\mathfrak{B}$ is autonomous we have $\mathrm{ann}\,\mathcal{M} \neq \{0\}$. This allows us to define $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ as in Eq. (18). By Lemma 23, $\mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ admits a characteristic set, say $\mathcal{S}$, that is a finite union of parallel lines. Since $\mathfrak{B} \subseteq \mathfrak{B}^{\mathrm{vec}}_{\mathrm{sc}}$ (Proposition 22), it follows from Valcher (2000), Lemma 2.5, that $\mathcal{S}$ must be a characteristic set for $\mathfrak{B}$, too. This concludes the proof. □

6 Conclusion

In this paper we have shown that every discrete 2D autonomous system admits a characteristic set that is composed of only finitely many parallel lines. We have argued that such characteristic sets are ‘thin’ in $\mathbb{Z}^2$ compared to the prevalent notion of characteristic sets which are either convex cones or half-spaces. We arrived at this result broadly in two steps. First we showed that a certain special class of scalar discrete 2D systems, called strongly σ_2-relevant, always admit a finite union of horizontal lines as their characteristic sets. This was done, in this paper, by utilizing a representation formula of trajectories in discrete 2D autonomous systems derived in Pal and Pillai (2013). After that, we showed that every scalar discrete 2D system can be converted into a strongly σ_2-relevant scalar discrete 2D system by a suitable coordinate transformation on the domain $\mathbb{Z}^2$. Such coordinate transformations, as we showed in this paper, map characteristic sets of one system to those of the transformed system. Thus, we get a characteristic set for the original system by applying the inverse coordinate transformation on a characteristic set for the transformed system. Since the transformed system is strongly σ_2-relevant, it admits finitely many horizontal lines as a characteristic set. Thus we get a characteristic set for the original system by applying the inverse coordinate transformation on these finitely many horizontal lines. This set turns out to be a finite union of parallel (possibly tilted) lines. This constituted our first main result of this paper, Theorem 20. Theorem 20 was then extended to non-scalar systems that are autonomous. Using the well-known relation between an autonomous behavior, $\mathfrak{B}$, and the corresponding scalar behavior given by the annihilator ideal ann $\mathcal{M}$, and the result of Theorem 20, we showed in our second main result, Theorem 24, that $\mathfrak{B}$, too, admits a characteristic set that is a finite union of parallel lines.

We believe that the idea of thin characteristic sets—like finite unions of lower dimensional sublattices—should also prevail in nD discrete autonomous systems with general n. This could be a possible direction of future research. Also, in this paper, we have not concentrated on algorithms for implementation of the main results. This will be done in future.

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Appendix 1: On "thin"ness of finite unions of parallel lines in $\mathbb{Z}^2$

We have mentioned that a finite union of parallel lines is a thin set in $\mathbb{Z}^2$. While this statement intuitively appears to be true, it is not beyond all reasonable doubts and hence requires a proof. In order for this, a suitable notion of thin sets in $\mathbb{Z}^2$ must first be formulated because the above-mentioned sets are thin only under special circumstances. However, giving a precise definition of thin sets in $\mathbb{Z}^2$ turns out to be a rather tricky issue, and is, unfortunately, beyond the scope of this article. The main difficulty seems to arise from the fact that various sets like—a half-space in $\mathbb{Z}^2$, a quadrant in $\mathbb{Z}^2$, a line in $\mathbb{Z}^2$ (e.g., an axis of $\mathbb{Z}^2$), or the entire $\mathbb{Z}^2$—all of them have the same cardinality: the countable infinity, $\aleph_0$. Hence it follows that these sets can be put into a one-to-one correspondence with each other. Therefore, from this point of view, a trajectory defined over the whole of $\mathbb{Z}^2$, or over a half-space, or over a quadrant, or over an axis, would all require countably infinite amount of data. Thus, if we go by this notion of quantifying the information required to specify a trajectory, a characteristic set that is finitely many lines would not be any better than a characteristic set that is a half-space or a quadrant.

Having said this, one must also note that specifying a 2D trajectory over an axis amounts to specifying a 1D trajectory, and that way, having a characteristic set that is a line is indeed better for it cuts down the 'dimension' of the set, over which a trajectory is being specified. In

Wood et al. (1998) this fact has been called the “order of magnitude” of the initial condition set. Subsequently, in Wood et al. (1998) for continuous nD autonomous systems, and in Avelli and Rocha (2010) for discrete nD autonomous systems, it was shown that this order of magnitude is strictly smaller than the dimension of the indexing set, that is n. Our main results of this paper, too, echo this fact.

To promote this view-point of finite unions of parallel lines as characteristic sets as a possible improvement over the current state-of-the-art, that is, convex cones, quadrants and half-spaces as characteristic sets, we provide below two constructions that indicate thin-ness of finitely many parallel lines in $\mathbb{Z}^2$ in comparison to quadrants, cones and half-spaces.

First, suppose $A \in \mathbb{Z}^{2\times 2}$ is such that $\det(A) = \pm 1$, that is, A is a *unimodular* integer matrix (see Sect. 4.1). Such an A acts on an integer vector $\nu = \text{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$ as $A\nu$ to produce another vector in $\mathbb{Z}^2$. This transformation is invertible, because A is invertible and its inverse, too, is an integer matrix. Thus, A defines a linear change of coordinates on $\mathbb{Z}^2$. Now, for $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{Z}^2$, let us define $\mathcal{S}_1$ to be *equivalent* to $\mathcal{S}_2$ if there exists $A \in \mathbb{Z}^{2\times 2}$ unimodular and $\hat{\nu} \in \mathbb{Z}^2$ such that

$$\mathcal{S}_2 = A(\mathcal{S}_1) + \hat{\nu}.$$

It can be checked that it is indeed an equivalence relation. With this equivalence relation it can then be shown that if $\mathcal{S}$ is a line or a finite union of parallel lines then $\mathbb{Z}^2$ can never be written as a *finite* union of sets that are equivalent to $\mathcal{S}$. On the other hand, if $\mathcal{S}$ is a convex cone, or a quadrant or a half-space, then a finite union of $\mathcal{S}$ and its equivalent sets covers the entire $\mathbb{Z}^2$.

The second construction that indicates thinness of a finite union of parallel lines compared to quadrants or half-spaces or convex cones is as follows. Define the following family of finite subsets of $\mathbb{Z}^2$. For $i = 0, 1, 2, \ldots$,

$$\mathcal{V}_i := \left\{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid -i \leqslant \nu_1 \leqslant i, \ -i \leqslant \nu_2 \leqslant i\right\}.$$

Note that $\mathcal{V}_i$’s form an ascending chain that ultimately covers the entire $\mathbb{Z}^2$:

$$\{(0,0)\} = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \mathcal{V}_2 \subsetneq \cdots$$

and

$$\bigcup_{i=1}^{\infty} \mathcal{V}_i = \mathbb{Z}^2.$$

Now, let $\mathcal{S}$ be any subset of $\mathbb{Z}^2$. Then for $i = 1, 2, \ldots$, define

$$\rho_i(\mathcal{S}) := \frac{|\mathcal{S} \cap \mathcal{V}_i|}{|\mathcal{V}_i|},$$

where $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$. It can then be easily checked that if $\mathcal{S}$ is a finite union of parallel lines then

$$\lim_{i\to\infty} \rho_i(\mathcal{S}) = 0,$$

but, if $\mathcal{S}$ is a convex cone, or a quadrant, or a half-space then

$$\lim_{i\to\infty} \rho_i(\mathcal{S}) > 0.$$

Appendix 2: Proof of Lemma 11

We need the following standard result—Euclidean division algorithm over polynomial overrings—for the proof.

Proposition 25 *Let $\mathcal{A}$ be an arbitrary commutative ring with $1 \in \mathcal{A}$, and let ξ be transcendental over $\mathcal{A}$. Suppose $p(\xi) \in \mathcal{A}[\xi]$ is a monic polynomial, that is,*

$$p(\xi) = \xi^L + a_{L-1}\xi^{L-1} + \cdots + a_1\xi + a_0,$$

where L is a finite positive integer and $a_{L-1}, \ldots, a_1, a_0 \in \mathcal{A}$. Then for all $f(\xi) \in \mathcal{A}[\xi]$ there exist $q(\xi) \in \mathcal{A}[\xi]$ and $r_0, r_1, \ldots, r_{L-1} \in \mathcal{A}$ such that

$$f(\xi) = q(\xi)p(\xi) + \sum_{i=0}^{L-1} r_i\xi^i.$$

Proof of Lemma 11 Since $\mathcal{M}$ is finitely generated as a module over $\mathcal{A}_1$, by Proposition 4, $\mathfrak{a}$ must contain a polynomial $p(\sigma)$ of the form $p(\sigma) = \sigma_2^L + a_{L-1}(\sigma_1)\sigma_2^{L-1} + \cdots + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1)$, where L is a finite positive integer, $a_{L-1}(\sigma_1), \ldots, a_1(\sigma_1), a_0(\sigma_1) \in \mathcal{A}_1$ and $a_0(\sigma_1)$ is a unit in $\mathcal{A}_1$. Then, by Proposition 25, for every $f(\sigma) \in \mathcal{A}_1[\sigma_2]$ (that is $f(\sigma) \in \mathcal{A}$ whose terms *do not* contain negative powers of σ_2), there exist $q(\sigma) \in \mathcal{A}_1[\sigma_2]$ and $r_0(\sigma_1), r_1(\sigma_1), \ldots, r_{L-1}(\sigma_1) \in \mathcal{A}_1$ such that

$$f(\sigma) = q(\sigma)p(\sigma) + \sum_{i=0}^{L-1} r_i(\sigma_1)\sigma_2^i. \tag{20}$$

Because $a_0(\sigma_1)$ is a unit we can multiply $p(\sigma)$ by $a_0(\sigma_1)^{-1}$ and σ_2^{-1} to get that $\sigma_2^{-1}a_0(\sigma_1)^{-1}p(\sigma) = a_0(\sigma_1)^{-1}\sigma_2^{L-1} + b_{L-1}(\sigma_1)\sigma_2^{L-2} + \cdots + b_1(\sigma_1) + \sigma_2^{-1} \in \mathfrak{a}$, where $b_i(\sigma_1) = a_0(\sigma_1)^{-1}a_i(\sigma_1)$ for all $1 \leqslant i \leqslant L-1$. Note that in the above expression every term except the last one has non-negative powers in σ_2. Therefore, defining $g(\sigma) := \sigma_2^{-1}a_0(\sigma_1)^{-1}p(\sigma)$ and $h(\sigma) := -\left(a_0(\sigma_1)^{-1}\sigma_2^{L-1} + b_{L-1}(\sigma_1)\sigma_2^{L-2} + \cdots + b_1(\sigma_1)\right)$ we can write

$$\sigma_2^{-1} = g(\sigma) + h(\sigma). \tag{21}$$

Note here that in Eq. (21), we have $g(\sigma) \in \mathfrak{a}$ and $h(\sigma) \in \mathcal{A}_1[\sigma_2]$ (that is, $h(\sigma)$ contains only non-negative powers of σ_2).

By taking positive powers on both sides of Eq. (21) and utilizing the binomial theorem, it follows that for every positive integer i there exists $g_i(\sigma) \in \mathfrak{a}$ and $h_i(\sigma) \in \mathcal{A}_1[\sigma_2]$ such that

$$\sigma_2^{-i} = g_i(\sigma) + h_i(\sigma). \tag{22}$$

Since any Laurent polynomial $f(\sigma) \in \mathcal{A}$ can be viewed as a finite linear combination of negative and positive powers of σ_2 with coefficients coming from $\mathcal{A}_1$, we can write from Eq. (22) above that for every $f(\sigma) \in \mathcal{A}$ there exist $g(\sigma) \in \mathfrak{a}$ and $h(\sigma) \in \mathcal{A}_1[\sigma_2]$ such that

$$f(\sigma) = g(\sigma) + h(\sigma). \tag{23}$$

The right hand side of Eq. (23) can be further broken up by applying Eq. (20) to $h(\sigma) \in \mathcal{A}_1[\sigma_2]$. That is, there exist $q(\sigma) \in \mathcal{A}$ and $r_i(\sigma_1) \in \mathcal{A}_1$ such that $h(\sigma) = q(\sigma)p(\sigma) + \sum_{i=0}^{L-1} r_i(\sigma_1)\sigma_2^i$. This leads us to conclude that for every $f(\sigma) \in \mathcal{A}$ there exist $g(\sigma) \in \mathfrak{a}$ and

$r_0(\sigma_1), \ldots, r_{L-1}(\sigma_1) \in \mathcal{A}_1$ such that

$$f(\sigma) = g(\sigma) + \sum_{i=0}^{L-1} r_i(\sigma_1)\sigma_2^i. \tag{24}$$

Note that, since $p(\sigma) \in \mathfrak{a}$ we have $q(\sigma)p(\sigma) \in \mathfrak{a}$, too. Therefore $q(\sigma)p(\sigma) + g(\sigma) \in \mathfrak{a}$. We have utilized this fact in the right-hand-side of Eq. (24) to merge $q(\sigma)p(\sigma)$ and $g(\sigma)$ together and call this sum as $g(\sigma)$.

Now under the canonical surjection $\mathcal{A} \twoheadrightarrow \mathcal{M}$ Eq. (24) translates to

$$\overline{f(\sigma)} = \sum_{i=0}^{L-1} \overline{r_i(\sigma_1)}\,\overline{\sigma_2}^i. \tag{25}$$

Thus every element in $\mathcal{M}$ can be written as a linear combination of $\{\overline{\sigma_2}^i\}_{0\leqslant i\leqslant L-1}$ with coefficients from $\mathcal{A}_1/\mathfrak{a} \cap \mathcal{A}_1$. In other words, $\{\overline{\sigma_2}^i\}_{0\leqslant i\leqslant L-1}$ generates $\mathcal{M}$ as a module over $\mathcal{A}_1$. $\square$

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Debasattam Pal received his Bachelor of Engineering (B.E.) degree from the Department of Electrical Engineering of Jadavpur University, Kolkata, in 2005. He received his M.Tech. and Ph.D. degrees from the Department of Electrical Engineering, IIT Bombay, in the years 2007 and 2012, respectively. He then worked as an Assistant Professor in IIT Guwahati from July, 2012 to May, 2014. After that, he joined IIT Bombay in June, 2014, as an Assistant Professor in the EE Department. His main area of research is systems and control theory. More specifically, his areas of interest are: multidimensional systems theory, algebraic analysis of systems, dissipative systems, optimal control and computational algebra.