International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

Lyapunov stability of n-D strongly autonomous systems

Debasattam Pal [a] & Harish K. Pillai [a]
[a] Department of Electrical Engineering, IIT Bombay, Mumbai, Maharashtra, India

Available online: 02 Nov 2011

To cite this article: Debasattam Pal & Harish K. Pillai (2011): Lyapunov stability of n-D strongly autonomous systems, International Journal of Control, 84:11, 1759-1768

To link to this article: http://dx.doi.org/10.1080/00207179.2011.624551

International Journal of Control
Vol. 84, No. 11, November 2011, 1759–1768

Lyapunov stability of *n*-D strongly autonomous systems

Debasattam Pal and Harish K. Pillai*

Department of Electrical Engineering, IIT Bombay, Mumbai, Maharashtra, India

(*Received 13 May 2011; final version received 14 September 2011*)

In this article we look into stability properties of strongly autonomous n-D systems, i.e. systems having finite-dimensional behaviour. These systems are known to have a first-order representation akin to 1-D state-space representation; we consider our systems to be already in this form throughout. We first define restriction of an n-D system to a 1-D subspace. Using this we define stability with respect to a given half-line, and then stability with respect to collections of such half-lines: proper cones. Then we show how stability with respect to a half-line, for the strongly autonomous case, reduces to a linear combination of the state representation matrices being Hurwitz. We first relate the eigenvalues of this linear combination with those of the individual matrices. With this we give an equivalent geometric criterion in terms of the real part of the characteristic variety of the system for half-line stability. Then we extend this geometric criterion to the case of stability with respect to a proper cone. Finally, we look into a Lyapunov theory of stability with respect to a proper cone for strongly autonomous systems. Each non-zero vector in the given proper cone gives rise to a linear combination of the system matrices. Each of these linear combinations gives a corresponding Lyapunov inequality. We show that the system is stable with respect to the proper cone if and only if there exists a common solution to all of these Lyapunov inequalities.

Keywords: strongly autonomous n-D systems; algebraic analysis; stability with respect to proper cones; Lyapunov theory

1. Introduction

Stability is one of the most important aspects in systems and control theory. In almost every design problem, stability remains one of the desirable criteria. While the history of stability of 1-D systems is very old, the n-D systems counterpart is quite recent. Stability is inextricably related to the notion of causality and, therefore, to a suitable partitioning of the domain space into two disjoint sets: 'past' and 'future'. Unlike the 1-D case, where there is a natural choice for this partitioning, there is still no unanimous such partitioning for the n-D case. In Curtain and Zwart (1995), Sasane, Thomas, and Willems (2002) and Wood, Sule, and Rogers (2005) one of the independent variables was treated as 'time', and thus, considering positive time as future, a corresponding stability theory has been built. On the other hand, in Valcher (Valcher 2001), the notion of future 'cone' for 2-D discrete systems was defined by introducing the idea of *characteristic cones*. Stability with respect to this idea was also presented there.

Lyapunov theory of stability has been a corner stone in 1-D systems theory for over a 100 years. It has provided a radically new point of view of looking at the notion of stability. Not only that, with the advent of strong LMI solving tools, it has also provided much more efficiently computable tests for stability. Lyapunov theory of stability for 2-D discrete systems has been done in Kojima, Rapisarda, and Takaba (2010), which was built around Valcher's theory of characteristic cones. In this article we shall look into a special kind of n-D systems and analyse its stability properties in a Lyapunov theory flavour. The kind of systems we shall look into are autonomous n-D systems, with $n \geqslant 2$, with the property that they have a finite 'characteristic variety'; such systems are called *strongly autonomous* (Pillai and Shankar 1998). See Avelli, Rapisarda, and Rocha (2011) for a recent paper that deals with a similar problem for the special case of 2-D discrete systems, although with an approach built on the theory of quadratic differential forms. Strongly autonomous systems are closer to 1-D systems for they are also finite-dimensional vector spaces like 1-D autonomous systems. Moreover, such systems allow a first-order representation, albeit with n-tuple of system matrices, like the state representation in 1-D case; following the 1-D terminology we shall call such a representation as a *state representation*. In this article we exploit this finite-dimensionality and state representation to bring out results about conic stability properties of strongly autonomous n-D systems.

*Corresponding author. Email: hp@ee.iitb.ac.in

ISSN 0020–7179 print/ISSN 1366–5820 online

http://dx.doi.org/10.1080/00207179.2011.624551
http://www.tandfonline.com

We start off by defining restriction of an n-D system to a 1-D subspace in Section 2. Then, with the idea of restriction we define stability with respect to a half-line given by the non-negative span of a non-zero vector in the domain. We show that the restriction of a strongly autonomous system to a 1-D subspace is governed by a certain linear combination of the n-tuple of state representation matrices; the coefficients in this linear combination are precisely the entries in the vector that spans the given 1-D subspace. Next we show (Theorem 2.4) how the eigenvalues of this linear combination are related with the eigenvalues of the individual matrices. This result, together with a consequence (Lemma 2.5) of observability assumption on the state representation, enables us to give a geometric criterion equivalent to stability with respect to a half-line (Theorem 2.6). In Section 3 we extend this result further to the case when stability is sought for a collection of half-lines, namely a *proper* cone. Finally, in Section 4, we state and prove a Lyapunov type equivalent criterion of stability with respect to a proper cone. It is a consequence of our results in the previous sections that corresponding to each half-line in the proper cone there is a different Lyapunov inequality, solvability of which is equivalent to stability with respect to that half-line. In this section, we show that stability with respect to the given proper cone is equivalent to existence of a *common* solution to all the Lyapunov inequalities given by the half-lines in the cone.

The notation we use is standard. We use $\mathbb{R}$ and $\mathbb{C}$ to denote the fields of real and complex numbers, respectively. The symbol $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ stands for the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^{\mathtt{w}}$. As a general strategy, if we use some letter to denote a tuple, then the same letter, but now indexed by natural numbers, is used to denote the individual entries in it, e.g. $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. We also use $\mathrm{col}(\xi_1, \xi_2, \dots, \xi_n)$ when we want to stack up the entries one above the other to make a column vector. The partial derivatives $\frac{\partial}{\partial x_i}$ are denoted simply by ∂_i, and the n-tuple $\{\partial_1, \partial_2, \dots, \partial_n\}$ is denoted by ∂. Thus $\mathbb{R}[\partial]$ denotes the polynomial ring $\mathbb{R}[\partial_1, \partial_2, \dots, \partial_n]$. Finally, the number of entries in a vector is denoted by the same letter as the variable itself but in typewriter font, e.g. the variable z takes values in $\mathbb{R}^{\mathtt{z}}$.

2. Strongly autonomous n-D systems and their restrictions to 1-D subspaces

As mentioned earlier, we consider a special kind of autonomous n-D systems, called *strongly autonomous*: behaviours for which the characteristic variety is a discrete set of finitely many points. It has been shown in Pillai and Shankar (1998) and Rocha and Willems (2006) that strongly autonomous systems are finite-dimensional vector spaces over $\mathbb{R}$. It can also be shown that they admit a first-order representation as follows:

$$\begin{aligned} \partial_1 z &= A_1 z,\\ \partial_2 z &= A_2 z,\\ &\vdots\\ \partial_n z &= A_n z,\\ w &= Cz, \end{aligned} \tag{1}$$

where $\{A_1, A_2, \dots, A_n\} \subseteq \mathbb{R}^{\mathtt{z}\times\mathtt{z}}$ commute pairwise, and $C \in \mathbb{R}^{\mathtt{w}\times\mathtt{z}}$. The set of all trajectories $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ that satisfy the above equation is called the *behaviour* of the system, and we denote this set by $\mathfrak{B}$. In mathematical terms

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid \exists z \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \text{ such that } (z, w) \text{ satisfy Equation (1)}\}.$$

We often identify a system with its behaviour and call $\mathfrak{B}$ a strongly autonomous system. Further, when we write $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ we mean $\mathfrak{B}$ is a behaviour with $\mathtt{w}$ number of manifest variables. In the sequel, we are going to consider systems that are already given in the first-order form as in Equation (1). Moreover, we may also assume that the 'state' variables z are observable from the manifest variable w. In terms of the above $\{A_1, A_2, \dots, A_n\}$ and C matrices this is equivalent to

$$\operatorname{rank}\begin{bmatrix} \xi_1 I - A_1\\ \xi_2 I - A_2\\ \vdots\\ \xi_n I - A_n\\ C \end{bmatrix} = \mathtt{z} \quad \text{for all } (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{C}^n. \tag{2}$$

The above state space description of strongly autonomous n-D systems reveals that all the trajectories in such systems are exponential and they look like

$$w(x) = C\exp(A_1x_1 + A_2x_2 + \cdots + A_nx_n)z(0), \tag{3}$$

where $z(0) \in \mathbb{R}^{\mathtt{z}}$ is an initial condition.

At the heart of the question of stability of n-D autonomous systems lies the idea of restricting a trajectory in the system to a given half-line in the domain space $\mathbb{R}^n$. Presently, we look into this idea of stability along a given half-line, then in Sections 3 and 4 we shall consider stability with respect to a special collection of such half-lines, namely closed convex cones. Suppose $v \in \mathbb{R}^n$ is non-zero, then by restriction of a trajectory $w(x) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ to the 1-D space spanned by v we mean

$$w|_v = w(vt) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}),$$

where $t \in \mathbb{R}$ is a real parameter. We now define stability with respect to the half-line spanned by a given non-zero vector v.

Definition 2.1: Given $0 \neq v \in \mathbb{R}^n$, an autonomous system $\mathfrak{B}$ is said to be *v-stable* if for all $w \in \mathfrak{B}$, $w|_v$ approaches zero as t becomes large; in other words,

$$\lim_{t\to\infty} w(vt) = 0.$$

Now, Equation (3) can be used to deal with restrictions of a strongly autonomous system. Putting $x = vt$ in Equation (3), we get

$$w|_v = C\exp((v_1A_1 + v_2A_2 + \cdots + v_nA_n)t)z(0). \qquad (4)$$

This is clearly a 1-D exponential trajectory determined by the action of the matrix exponential $\exp((v_1A_1 + v_2A_2 + \cdots + v_nA_n)t)$ on the initial condition $z(0)$. So, in order to infer about v-stability of a strongly autonomous system, it becomes crucial to know where the eigenvalues of the linear combination of the A_i matrices, that is $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$, are located. Our main result of this section, Theorem 2.4, answers this question. But, before we state and prove Theorem 2.4 we need to review some background theory from commutative algebra and algebraic geometry.

Recall the defining first-order equations of a strongly autonomous system (Equation (1)), and consider the equations involving only the state variable z:

$$R_z(\partial) := \begin{bmatrix} \partial_1 I - A_1 \\ \partial_2 I - A_2 \\ \vdots \\ \partial_n I - A_n \end{bmatrix}. \qquad (5)$$

We define the following complex affine variety:

$$\mathbb{V}_z := \{\xi \in \mathbb{C}^n \mid \operatorname{rank}(R_z(\xi)) < \mathtt{z}\}. \qquad (6)$$

We also define $\mathcal{I}_z \subseteq \mathbb{R}[\partial]$ to be the ideal generated by all the $\mathtt{z} \times \mathtt{z}$ minors of $R_z(\partial)$. For the rest of this section we are going to change our base field to complex numbers. Although it is possible to carry on without this change, the proofs become simpler and shorter with the base field being $\mathbb{C}$. In the subsequent sections we switch back to the real field. The main result of this section, Theorem 2.4, however, remains applicable in the sequel irrespective of this change of the base field. Note that, if $\mathcal{I}_z^{\mathbb{C}}$ denotes the extension of $\mathcal{I}_z$ to $\mathbb{C}[\partial]$, that is,

$$\mathcal{I}_z^{\mathbb{C}} := \mathbb{C}[\partial]\mathcal{I}_z,$$

then it follows from the definitions of $\mathcal{I}_z$ and $\mathbb{V}_z$ that

$$\mathbb{V}_z = \mathbb{V}(\mathcal{I}_z^{\mathbb{C}}) := \{\xi \in \mathbb{C}^n \mid f(\xi) = 0 \text{ for all } f(\partial) \in \mathcal{I}_z^{\mathbb{C}}\}.$$

We now define another ideal of $\mathbb{C}[\partial]$, which will play an important role in the proof of Theorem 2.4. Note that the matrices $\{A_1, A_2, \ldots, A_n\}$ commute pairwise. Therefore, it makes sense to talk about polynomials in these matrices. It is straightforward to check that the following subset of $\mathbb{C}[\partial]$ is in fact an ideal:

$$\begin{aligned}\mathfrak{a} := \{&f(\partial_1, \partial_2, \ldots, \partial_n) \in \mathbb{C}[\partial] \mid f(A_1, A_2, \ldots, A_n) \\ &= 0 \in \mathbb{C}^{\mathtt{z}\times\mathtt{z}}\}.\end{aligned} \qquad (7)$$

We show in the following result how the two ideals $\mathcal{I}_z^{\mathbb{C}}$ and $\mathfrak{a}$, and the variety $\mathbb{V}_z$ are related. The result is a special case of a result regarding characteristic ideals and annihilator ideals (Pommaret and Quadrat 1999; Shankar 1999). However, we state the result in a form more suited for our purpose, and we prove it here to make our exposition self-contained.

Theorem 2.2: *Let $\mathcal{I}_z$, $\mathfrak{a}$ and $\mathbb{V}_z$ be defined as above, then the following hold*:

(1) $\mathcal{I}_z^{\mathbb{C}} \subseteq \mathfrak{a}$,
(2) $\sqrt{\mathcal{I}_z^{\mathbb{C}}} = \sqrt{\mathfrak{a}}$,
(3) $\mathbb{V}_z = \mathbb{V}(\mathfrak{a}) := \{\xi \in \mathbb{C}^n \mid f(\xi) = 0$ *for all* $f(\partial) \in \mathfrak{a}\}$.

Our proof is via another claim: equality of the ideal $\mathfrak{a}$ to the annihilator ideal of a certain $\mathbb{C}[\partial]$-module. First, consider the free module $\mathbb{C}[\partial]^{\mathtt{z}}$ by taking $\mathtt{z}$ copies of the polynomial ring $\mathbb{C}[\partial]$; the elements in $\mathbb{C}[\partial]^{\mathtt{z}}$ are written as rows of $\mathtt{z}$-tuples of elements in $\mathbb{C}[\partial]$. Let $\mathcal{R}_z$ denote the submodule of $\mathbb{C}[\partial]^{\mathtt{z}}$ generated by the rows of the polynomial matrix $R_z(\partial)$ defined in Equation (8), i.e.

$$\mathcal{R}_z := \operatorname{rowspan}\begin{bmatrix} \partial_1 I - A_1 \\ \partial_2 I - A_2 \\ \vdots \\ \partial_n I - A_n \end{bmatrix}. \qquad (8)$$

We define the following quotient module:

$$\mathcal{M}_z := \mathbb{C}[\partial]^{\mathtt{z}}/\mathcal{R}_z, \qquad (9)$$

and its annihilator ideal

$$\begin{aligned}\operatorname{ann}(\mathcal{M}_z) := \{&f(\partial) \in \mathbb{C}[\partial] \mid \overline{f(\partial)m(\partial)} = 0 \\ &\text{for all } m(\partial) \in \mathbb{C}[\partial]^{\mathtt{z}}\},\end{aligned} \qquad (10)$$

where $\overline{f(\partial)m(\partial)}$ denotes the class of $f(\partial)m(\partial)$ in $\mathcal{M}_z$.

Lemma 2.3: *Let $\mathcal{M}_z$ and $\operatorname{ann}(\mathcal{M}_z)$ be as defined above, and let $\mathfrak{a}$ be as defined in equation (7). Then*

$$\mathfrak{a} = \operatorname{ann}(\mathcal{M}_z). \qquad (11)$$

Proof: First, note that the $\mathbb{C}$-vector space $\mathbb{C}^{\mathtt{z}}_{\text{row}}$, with elements written as rows of $\mathtt{z}$-tuples, injects into $\mathcal{M}_z$.

Suppose now that the matrix A_i for $i \in \{1, 2, \ldots, n\}$ is given by

$$A_i = \begin{bmatrix} a_{11,i} & a_{12,i} & \cdots & a_{1\mathtt{z},i} \\ a_{21,i} & a_{22,i} & \cdots & a_{2\mathtt{z},i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mathtt{z}1,i} & a_{\mathtt{z}2,i} & \cdots & a_{\mathtt{zz},i} \end{bmatrix},$$

and let $e_j \in \mathbb{C}^{\mathtt{z}}$ be a column with all entries zero except the jth position where it is 1. Now, since the equation $\partial_i I - A_i = 0$ is satisfied in $\mathcal{M}_z$, it follows that,

$$\partial_i e_j^{\mathrm{T}} = a_{j1,i} e_1^{\mathrm{T}} + a_{j2,i} e_2^{\mathrm{T}} + \cdots + a_{j\mathtt{z},i} e_{\mathtt{z}}^{\mathrm{T}}.$$

This clearly implies that for $v^{\mathrm{T}} \in \mathbb{C}^{\mathtt{z}}_{\mathrm{row}}$

$$\partial_i v^{\mathrm{T}} = v^{\mathrm{T}} A_i. \tag{12}$$

There is, however, another more striking consequence of the above observation: given any $m(\partial) \in \mathbb{C}[\partial]^{\mathtt{z}}$, it can be written as

$$m(\partial) = f_1(\partial) e_1^{\mathrm{T}} + f_2(\partial) e_2^{\mathrm{T}} + \cdots + f_{\mathtt{z}}(\partial) e_{\mathtt{z}}^{\mathrm{T}},$$

where $\{f_1(\partial), f_2(\partial), \ldots, f_z(\partial)\} \subseteq \mathbb{C}[\partial]$. By Equation (12) above, it follows that in $\mathcal{M}_z$, $\overline{m(\partial)}$ must be equal to

$$e_1^{\mathrm{T}} f_1(A_1, A_2, \ldots, A_n) + e_2^{\mathrm{T}} f_2(A_1, A_2, \ldots, A_n) + \cdots + e_{\mathtt{z}}^{\mathrm{T}} f_{\mathtt{z}}(A_1, A_2, \ldots, A_n) \in \mathbb{C}^{\mathtt{z}}_{\mathrm{row}}.$$

This identification gives an onto map from $\mathcal{M}_z$ to $\mathbb{C}^{\mathtt{z}}_{\mathrm{row}}$. Thus the injection map $\mathbb{C}^{\mathtt{z}}_{\mathrm{row}} \hookrightarrow \mathcal{M}_z$ is not only one-to-one, but also onto and $\mathbb{C}$-linear. This makes $\mathcal{M}_z$ isomorphic to $\mathbb{C}^{\mathtt{z}}_{\mathrm{row}}$ as $\mathbb{C}$-vector spaces, where multiplication by ∂_i in $\mathcal{M}_z$ is represented by right multiplication by A_i in $\mathbb{C}^{\mathtt{z}}_{\mathrm{row}}$.

With the above observation, it is now straightforward to prove our claim. We first show $\mathfrak{a} \subseteq \mathrm{ann}(\mathcal{M}_z)$. Let $f(\partial) \in \mathfrak{a}$. This means $f(A_1, A_2, \ldots, A_n)$ is the zero matrix. Therefore, for $j \in \{1, 2, \ldots, \mathtt{z}\}$,

$$\begin{aligned} \overline{f(\partial) e_j^{\mathrm{T}}} &= e_j^{\mathrm{T}} f(A_1, A_2, \ldots, A_n) \\ &= 0 \text{ since } f(A_1, A_2, \ldots, A_n) \text{ is the zero matrix.} \end{aligned}$$

Hence, $f(\partial) \in \mathrm{ann}(\mathcal{M}_z)$.

Conversely, if $f(\partial) \in \mathrm{ann}(\mathcal{M}_z)$, then for all $j \in \{1, 2, \ldots, \mathtt{z}\}$,

$$\begin{aligned} e_j^{\mathrm{T}} f(A_1, A_2, \ldots, A_n) &= \overline{f(\partial) e_j^{\mathrm{T}}} \\ &= 0 \text{ because } f(\partial) \in \mathrm{ann}(\mathcal{M}_z) \\ \Rightarrow e_j^{\mathrm{T}} f(A_1, A_2, \ldots, A_n) &= 0. \end{aligned}$$

Therefore, $f(A_1, A_2, \ldots, A_n)$ is the zero matrix, and hence $f(\partial) \in \mathfrak{a}$. Thus the claim $\mathfrak{a} = \mathrm{ann}(\mathcal{M}_z)$ is proved. □

Equipped with Lemma 2.3, we now prove Theorem 2.2. Recall the definitions of $\mathcal{R}_z$, $\mathcal{M}_z$ and $\mathrm{ann}(\mathcal{M}_z)$ defined in Equations (8), (9) and (10), respectively.

Proof of Theorem 2.2: (1) Note that by Equation (11) in Lemma 2.3, it is enough to prove that $\mathcal{I}_z^{\mathbb{C}} \subseteq \mathrm{ann}(\mathcal{M}_z)$. Suppose $f(\partial) \in \mathcal{I}_z^{\mathbb{C}}$. Since $\mathcal{I}_z^{\mathbb{C}}$ is generated by all the $\mathtt{z} \times \mathtt{z}$ determinants of $R_z(\partial)$, it easily follows that there exists a matrix $E(\partial) \in \mathbb{C}[\partial]^{\mathtt{z} \times n\mathtt{z}}$ such that

$$E(\partial) R_z(\partial) = f(\partial) I.$$

Going modulo $\mathcal{R}_z$ we get, for all $j \in \{1, 2, \ldots, \mathtt{z}\}$,

$$\overline{f(\partial) e_j^{\mathrm{T}}} = 0,$$

which means $f(\partial) \in \mathrm{ann}(\mathcal{M}_z)$.

(2) From statement 1 it follows by taking radicals that $\sqrt{\mathcal{I}_z^{\mathbb{C}}} \subseteq \sqrt{\mathrm{ann}(\mathcal{M}_z)}$. We want to show that $\sqrt{\mathcal{I}_z^{\mathbb{C}}} \supseteq \sqrt{\mathrm{ann}(\mathcal{M}_z)}$, and this will prove statement 2 by Equation (11). It is enough to show that $\sqrt{\mathcal{I}_z^{\mathbb{C}}} \supseteq \mathrm{ann}(\mathcal{M}_z)$ (because $\sqrt{\sqrt{\mathcal{I}_z^{\mathbb{C}}}} = \sqrt{\mathcal{I}_z^{\mathbb{C}}}$). Suppose $f(\partial) \in \mathrm{ann}(\mathcal{M}_z)$. This means, for all $j \in \{1, 2, \ldots, \mathtt{z}\}$,

$$\overline{f(\partial) e_j^{\mathrm{T}}} = 0.$$

Lifting this to $\mathbb{C}[\partial]^{\mathtt{z}}$, we get that there exists $E(\partial) \in \mathbb{C}[\partial]^{\mathtt{z} \times n\mathtt{z}}$ such that

$$E(\partial) R_z(\partial) = f(\partial) I.$$

By taking determinants on both sides and using Cauchy–Binnet formula, we get

$$f(\partial)^{\mathtt{z}} \in \mathcal{I}_z^{\mathbb{C}} \Rightarrow f(\partial) \in \sqrt{\mathcal{I}_z^{\mathbb{C}}}.$$

(3) This is equivalent to statement 2 by Hilbert's Nullstellensatz (Cox, Little, and O'Shea 2007). □

Remark 1: The upshot of Theorem 2.2 is statement 3. It relates the affine variety $\mathbb{V}_z$, which is defined to be the collection of n-tuple of complex numbers where the matrix $R_z(\xi)$ loses rank, with the affine variety of the ideal $\mathfrak{a}$ of $\mathbb{C}[\partial]$, which is the collection of all complex polynomials wherein if ∂_is are replaced by the real square matrices A_is then the zero matrix results. This result to n-D strongly autonomous system is what the famous Cayley–Hamilton theorem is to 1-D autonomous systems.

We are now in a position to state and prove our main result of this section: it tells us that the eigenvalues of the linear combination of A_is, that is $(v_1 A_1 + v_2 A_2 + \cdots + v_n A_n)$, where $v = \mathrm{col}(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is non-zero, are given by projecting the affine variety $\mathbb{V}_z$ onto the complex 1-D subspace spanned by v.

Theorem 2.4: *Let $\{A_1, A_2, \dots, A_n\} \subseteq \mathbb{R}^{\mathtt{z}\times\mathtt{z}}$ be a collection of pairwise commuting matrices and let $\mathbb{V}_z$ and $\mathfrak{a}$ be as defined by Equations* (6) *and* (7), *respectively. Suppose $0 \neq v = \mathrm{col}(v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is given. Define the following two sets of complex numbers*:

$$\Pi_v(\mathbb{V}_z) := \{\lambda \in \mathbb{C} \mid \lambda = v^{\mathrm{T}}\xi,\ \xi \in \mathbb{V}_z\},$$
$$\mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n) := \{\lambda \in \mathbb{C} \mid \mathrm{rank}(\lambda I - (v_1A_1 + v_2A_2 + \cdots + v_nA_n)) < \mathtt{z}\}.$$

Then

$$\Pi_v(\mathbb{V}_z) = \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n).$$

Proof: We first show the inclusion $\Pi_v(\mathbb{V}_z) \subseteq \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$. Suppose $\lambda \in \Pi_v(\mathbb{V}_z)$. This means there exists $\xi \in \mathbb{V}_z$ such that $\lambda = v^{\mathrm{T}}\xi$. It follows from the definition of $\mathbb{V}_z$ that

$$\mathrm{rank}\begin{bmatrix} \xi_1 I - A_1 \\ \xi_2 I - A_2 \\ \vdots \\ \xi_n I - A_n \end{bmatrix} < \mathtt{z}.$$

This implies that there exists $0 \neq \alpha \in \mathbb{C}^{\mathtt{z}}$ such that

$$\begin{bmatrix} \xi_1 I - A_1 \\ \xi_2 I - A_2 \\ \vdots \\ \xi_n I - A_n \end{bmatrix}\alpha = 0 \qquad (13)$$
$$\Rightarrow (\xi_i I - A_i)\alpha = 0 \text{ for all } i \in \{1, 2, \dots, n\}.$$

It now easily follows from Equation (13) that

$$\begin{aligned} & v_1(\xi_1 I - A_1)\alpha + v_2(\xi_2 I - A_2)\alpha + \cdots + v_n(\xi_n I - A_n)\alpha = 0 \\ & \Rightarrow [(v_1\xi_1 + v_2\xi_2 + \cdots + v_n\xi_n)I \\ & \quad - (v_1A_1 + v_2A_2 + \cdots + v_nA_n)]\alpha = 0 \\ & \Rightarrow [\lambda I - (v_1A_1 + v_2A_2 + \cdots + v_nA_n)]\alpha = 0, \end{aligned} \qquad (14)$$

which means $\lambda \in \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$.

We prove the converse, that is $\Pi_v(\mathbb{V}_z) \supseteq \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$, by contradiction. Suppose, $\lambda \in \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$, but $\lambda \notin \Pi_v(\mathbb{V}_z)$. Since $\lambda \notin \Pi_v(\mathbb{V}_z)$, the linear polynomial $f(\partial) = \lambda - (v_1\partial_1 + v_2\partial_2 + \cdots + v_n\partial_n)$ is non-zero on $\mathbb{V}_z$, that is

$$f(\xi) \neq 0 \quad \text{for all } \xi \in \mathbb{V}_z.$$

This implies that $\mathbb{V}(f(\partial)) \cap \mathbb{V}_z = \emptyset$. Since $\mathbb{V}_z = \mathbb{V}(\mathfrak{a})$, as shown in Theorem 2.2, by the weak form of Hilbert's Nullstellensatz (Cox et al. 2007), we get that there exist $g(\partial) \in \mathbb{C}[\partial]$ and $h(\partial) \in \mathfrak{a}$ such that

$$1 = g(\partial)f(\partial) + h(\partial) \Rightarrow g(\partial)f(\partial) - 1 \in \mathfrak{a}.$$

But, from the definition of the ideal $\mathfrak{a}$ the above equation says that

$$g(A_1, A_2, \dots, A_n)(\lambda I - (v_1A_1 + v_2A_2 + \cdots + v_nA_n)) - I = 0 \in \mathbb{C}^{\mathtt{z}\times\mathtt{z}},$$

which means $(\lambda\ I - (v_1A_1 + v_2A_2 + \cdots + v_nA_n))$ is invertible – this is a contradiction to our assumption that $\lambda \in \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$. Thus $\Pi_v(\mathbb{V}_z) \supset \mathrm{eig}(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$. □

From Equation (4) it follows that for a strongly autonomous system $\mathfrak{B}$ to be v-stable, it is sufficient that the eigenvalues of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ have negative real parts. Our next main result, Theorem 2.6, shows that this is even necessary. In order to prove this result we require the following lemma.

Lemma 2.5: *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a strongly autonomous system, described by Equation* (1). *Suppose $(C, \{A_1, A_2, \dots, A_n\})$ is observable in the sense of Equation* (2), *that is*

$$\mathrm{rank}\begin{bmatrix} \xi_1 I - A_1 \\ \xi_2 I - A_2 \\ \vdots \\ \xi_n I - A_n \\ C \end{bmatrix} = \mathtt{z} \quad \textit{for all } (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{C}^n.$$

Further, let $v = \mathrm{col}(v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ be non-zero. Then for every $\lambda \in \mathrm{eig}\ (v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ there exists a non-zero $\alpha \in \mathbb{C}^{\mathtt{z}}$, which is an eigenvector of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ corresponding to eigenvalue λ, such that $C\alpha \neq 0$.

Proof: It follows from Theorem 2.4 that every eigenvalue of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ is obtained by projecting the points in $\mathbb{V}_z$ onto the complex 1-D subspace spanned by v. Therefore, if $\lambda \in \mathrm{eig}\ (v_1A_1 + v_2A_2 + \cdots + v_nA_n)$, then there exists $\xi := \mathrm{col}(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{V}_z$ such that $\lambda = v_1\xi_1 + v_2\xi_2 + \cdots + v_n\xi_n$. Now, observe that since $\xi \in \mathbb{V}_z$,

$$\mathrm{rank}\begin{bmatrix} \xi_1 I - A_1 \\ \xi_2 I - A_2 \\ \vdots \\ \xi_n I - A_n \end{bmatrix} < \mathtt{z}.$$

This means there exists a non-zero $\alpha \in \mathbb{C}^{\mathtt{z}}$ such that

$$\begin{bmatrix} \xi_1 I - A_1 \\ \xi_2 I - A_2 \\ \vdots \\ \xi_n I - A_n \end{bmatrix}\alpha = 0$$
$$\Rightarrow (\xi_i I - A_i)\alpha = 0 \quad \text{for all } i \in \{1, 2, \dots, n\}.$$

As we have already seen in Equation (14), if we now take a linear combination of the equations $(\xi_i I - A_i)\alpha = 0$, we get

$$[\lambda I - (v_1 A_1 + v_2 A_2 + \cdots + v_n A_n)]\alpha = 0.$$

Therefore, α is an eigenvector of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ corresponding to eigenvalue λ. However, because of observability assumption, $C\alpha \neq 0$. This proves the existence of the desired α. □

We are now in a position to state and prove Theorem 2.6. The theorem gives a geometric criterion equivalent to stability with respect to a given half-line. In what follows, by $\Pi_{\mathbb{R}}(\mathbb{V}_z)$ we mean the real part of the complex affine variety $\mathbb{V}_z$.

Theorem 2.6: *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a strongly autonomous system described by an observable state equation* (1). *Further, let $v = \mathrm{col}(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ be non-zero. Define the following open half-space*:

$$H_v := \{y \in \mathbb{R}^n \mid v^{\mathrm{T}} y < 0\}.$$

Then $\mathfrak{B}$ is v-stable if and only if $\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq H_v$.

Proof: (If): First, note that we have the following equivalence as a consequence of Theorem 2.4:

$$\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq H_v \Leftrightarrow (v_1A_1 + v_2A_2 + \cdots + v_nA_n) \text{ is Hurwitz.} \tag{15}$$

Indeed, suppose $\xi = \xi_{\mathrm{re}} + i\xi_{\mathrm{imag}} \in \mathbb{V}_z$, where $\xi_{\mathrm{re}}, \xi_{\mathrm{imag}} \in \mathbb{R}^n$ are real and imaginary parts of ξ, respectively. Then $\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq H_v$ means $v^{\mathrm{T}}\xi_{\mathrm{re}} < 0$. This in turn means $v^{\mathrm{T}}\xi \in \mathbb{C}$ has negative real part. It now easily follows by Theorem 2.4 that if $\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq H_v$ then every eigenvalue of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ has negative real part, and conversely. Now from Equation (4), if $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ is Hurwitz, then $\lim_{t\to\infty} w(vt) = 0$ for all $w \in \mathfrak{B}$. Therefore, by Equation (15), $\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq H_v$ implies $\mathfrak{B}$ is v-stable.

(Only if): We prove this part by contradiction. Suppose $\Pi_{\mathbb{R}}(\mathbb{V}_z) \nsubseteq H_v$, we will show that this implies the existence of a $w \in \mathfrak{B}$ such that $\lim_{t\to\infty} w(vt) \neq 0$. Since $\Pi_{\mathbb{R}}(\mathbb{V}_z) \nsubseteq H_v$, we must have $\xi \in \mathbb{V}_z$ such that $v^{\mathrm{T}}\xi \notin \mathbb{C}^-$. By Theorem 2.4 this $v^{\mathrm{T}}\xi$ is an eigenvalue of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$, and by Lemma 2.5 there exists $0 \neq \alpha \in \mathbb{C}^{\mathtt{z}}$ which is an eigenvector of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$ corresponding to this eigenvalue, such that $C\alpha \neq 0$. Let $\alpha = \alpha_{\mathrm{re}} + i\alpha_{\mathrm{imag}}$, where $\alpha_{\mathrm{re}}, \alpha_{\mathrm{imag}} \in \mathbb{R}^{\mathtt{z}}$ are the real and imaginary parts of α, respectively. Then substituting $z(0) = \alpha_{\mathrm{re}}$ in Equation (3), we get $w = C\exp(A_1x_1 + A_2x_2 + \cdots + A_nx_n)\alpha_{\mathrm{re}} \in \mathfrak{B}$. Because $\alpha_{\mathrm{re}} + i\alpha_{\mathrm{imag}}$ is an eigenvector of $(v_1A_1 + v_2A_2 + \cdots + v_nA_n)$, the restriction of this w is such that

$$\begin{aligned} w(vt) &= C\exp((v_1A_1 + v_2A_2 + \cdots + v_nA_n)t)\alpha_{\mathrm{re}} \\ &= Ce^{\lambda_{\mathrm{re}}t}((\cos(\lambda_{\mathrm{imag}}t))\alpha_{\mathrm{re}} - (\sin(\lambda_{\mathrm{imag}}t))\alpha_{\mathrm{imag}}), \end{aligned}$$

where $v^{\mathrm{T}}\xi = \lambda_{\mathrm{re}} + i\lambda_{\mathrm{imag}}$. Now since $C\alpha \neq 0$, the right-hand side of the above equation is non-zero. Moreover, since $v^{\mathrm{T}}\xi \notin \mathbb{C}^-$, which means $\lambda_{\mathrm{re}} \geqslant 0$, we have $\lim_{t\to\infty} w(vt) \neq 0$. Thus, $\mathfrak{B}$ is not v-stable. □

3. Conic stability of strongly autonomous systems

One prime difficulty in defining the stability of an n-D autonomous system, with $n \geqslant 2$, stems from the fact that there is no well-defined direction of evolution. As discussed in the last section, one can look for stability in a given direction. An obvious next step would be to look for stability with respect to a collection of half-lines. In this section, we look for a special class of such collections, namely *closed convex cones*.

Definition 3.1: A subset S of $\mathbb{R}^n$ is said to be a cone if for all $v \in S$, we have $\lambda\ v \in S$ for all $\lambda \geq 0$.

All the cones that we shall consider here are *closed* in the Euclidean topology of $\mathbb{R}^n$, and *convex*, meaning for all v_1, $v_2 \in S$,

$$\lambda v_1 + (1-\lambda)v_2 \in S \quad \text{for all } \lambda \in [0, 1].$$

For our purpose in this article, we need the cone S to satisfy one more condition: the *lineality space* $S \cap (-S)$ consists of only the origin. Such a cone, which is closed, convex and with only the origin as its lineality space, is said to be a *proper* cone. We shall consider only proper cones. With this, now we define the conic stability of an n-D autonomous system as follows.

Definition 3.2: Given a proper cone $S \subseteq \mathbb{R}^n$, an autonomous system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *stable with respect to S* (or simply *S-stable*) if for all non-zero $v \in S$ and $w \in \mathfrak{B}$, we have

$$\lim_{t\to\infty} w(vt) = 0.$$

In other words, $\mathfrak{B}$ is S-stable if for all $0 \neq v \in S$, $\mathfrak{B}$ is v-stable.

Theorem 3.3 is a straightforward extension of Theorem 2.6 to the case of conic stability. The crucial part here is the fact that if $S \in \mathbb{R}^n$ is a proper cone, then the following set is non-empty:

$$(S)_< := \{y \in \mathbb{R}^n \mid v^{\mathrm{T}} y \leqslant 0 \text{ for all } v \in S\} \neq \emptyset.$$

The set $(S)_<$ is called the *polar cone* of S. We now state and prove the main result of this section. We denote by int $(S)_<$ the interior of the set $(S)_<$ in Euclidean topology.

Theorem 3.3: *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a strongly autonomous system given by an observable state equation* (1) *with $\mathbb{V}_z$ the corresponding complex affine variety as defined by equation* (6). *Further, let $S \subseteq \mathbb{R}$ be a proper cone and let $\Pi_{\mathbb{R}}(\mathbb{V}_z)$ denote the real part of $\mathbb{V}_z$. Then $\mathfrak{B}$ is S-stable if and only if*

$$\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \text{int } (S)_< .$$

Proof: Following exactly the same chain of arguments as in the proof of Theorem 2.6, it can be shown that

$$\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \text{int } (S)_< \Leftrightarrow (v_1 A_1 + v_2 A_2 + \cdots + v_n A_n) \text{ is Hurwitz for all non-zero } v = \text{col}(v_1, v_2, \ldots, v_n) \in S. \tag{16}$$

Once again, by making use of Lemma 2.5, it follows that $\mathfrak{B}$ is v-stable for all non-zero $v \in S$, if and only if $(v_1 A_1 + v_2 A_2 + \cdots + v_n A_n)$ is Hurwitz for all non-zero $v \in S$, which in turn is equivalent to $\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \text{int } (S)_<$ by Equation (16). □

4. Lyapunov theory of stability of strongly autonomous systems

In Lyapunov theory of stability of 1-D linear systems, we look for a positive definite storage function, then its rate of change being negative along all trajectories is equivalent to the system being asymptotically stable. We mimic this idea in the n-D case. We look for a positive definite storage function, and then (in)stability with respect to a half-line can be inferred by looking into the sign of the directional derivative of this storage function along that half-line. Our main result shows that stability with respect to a proper cone is equivalent to the existence of a common storage function whose directional derivative along every half-line in the cone is negative. Sufficiency of this storage function condition is not difficult (as we shall see in the proof), what is more striking is that it is necessary too. We now state this result below. We postpone the proof till we state and prove an auxiliary lemma.

Theorem 4.1: *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a strongly autonomous system given by an observable state equation* (1). *Further, let $S \subseteq \mathbb{R}^n$ be a proper cone. Then $\mathfrak{B}$ is S-stable if and only if there exists $P = P^{\mathrm{T}} \in \mathbb{R}^{\mathtt{z} \times \mathtt{z}}$, and $P > 0$, such that for all non-zero* $\text{col}(v_1, v_2, \ldots, v_n) \in S$ *we have*

$$\left(\sum_{i=1}^{n} v_i A_i^{\mathrm{T}}\right) P + P \left(\sum_{i=1}^{n} v_i A_i\right) < 0. \tag{17}$$

As mentioned earlier, the if part of the above theorem is easier than the only if part. One of the difficulties for the only if part arises out of the requirement to obtain a common solution to a possibly (uncountably) infinite number of simultaneous Lyapunov inequalities. However, it so happens that the finiteness of the variety $\mathbb{V}_z$ renders the situation down to a question of solving only a finite number of Lyapunov inequalities. A crucial observation that plays a key role in this reduction is the fact that for a strongly autonomous system the proper cone S can be assumed to be *polyhedral* without loss of generality. A proper cone is said to be *polyhedral* if it can be written as a finite intersection of half-spaces of the form $H = \{y \in \mathbb{R}^n \mid \beta^{\mathrm{T}} y \geqslant 0\}$ for some given $\beta \in \mathbb{R}^n$. Equivalently, a polyhedral cone is one which is generated as a non-negative hull of finitely many vectors in $\mathbb{R}^n$.

Lemma 4.2: *Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, a strongly autonomous behaviour given by an observable state equation* (1), *is S-stable for some proper cone $S \subseteq \mathbb{R}^n$. Then there exists a polyhedral cone $S_1 \supseteq S$ such that $\mathfrak{B}$ is stable with respect to S_1.*

Proof: In order to show the existence of the desired S_1, it is enough to show that there exists a proper polyhedral cone, say $\widehat{S}$, contained in the polar cone $(S)_<$, such that the real part of $\mathbb{V}_z$ is contained in the interior of $\widehat{S}$. In that case the polar cone $(\widehat{S})_<$ is going to be nonempty because $\widehat{S}$ is proper. Moreover, $(\widehat{S})_<$ will be polyhedral because $\widehat{S}$ is. We can take this polar cone $(\widehat{S})_<$ as a candidate for S_1. Here, since

$$(S_1)_< = ((\widehat{S})_<)_< = \widehat{S},$$

we have $\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \text{int } (S_1)_<$ by construction. Thus stability with respect to S_1 will follow from Theorem 3.3.

We now show existence of the above-mentioned $\widehat{S}$. By assumption $\mathfrak{B}$ is S-stable. Therefore, by Theorem 3.3, the real part of $\mathbb{V}_z$ is contained in the interior of $(S)_<$:

$$\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \text{int } (S)_< .$$

This implies that for each point in $\Pi_{\mathbb{R}}(\mathbb{V}_z)$ there exists a closed hypercube, containing the point in its interior, small enough to be fully contained in $(S)_<$. We take the convex hull of the vertices of these hypercubes. Since $\mathbb{V}_z$ (and therefore, $\Pi_{\mathbb{R}}(\mathbb{V}_z)$) contains only finitely many points, this convex hull, say $\mathcal{P}$, is a convex polytope. Note that it follows from the construction of $\mathcal{P}$ that

$$\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \text{int}(\mathcal{P}).$$

We consider the cone obtained by taking the non-negative hull of the vertices of this convex polytope, and call it $\widehat{S}$. This cone is clearly closed and convex, and polyhedral by construction. Moreover, since $\mathcal{P}$ is contained in $(S)_<$, the cone $\widehat{S}$ too is contained in $(S)_<$. This is because all the vertices of $\mathcal{P}$ are contained in $(S)_<$, which is a convex cone. Therefore, the non-negative hull of these points is also contained in $(S)_<$. But this means $\widehat{S}$ must also have $\widehat{S} \cap (-\widehat{S}) = \{0\}$. Thus $\widehat{S}$ is proper. Further, note that the way $\widehat{S}$ is constructed, it contains the convex polytope $\mathcal{P}$. But this polytope in turn contains $\Pi_{\mathbb{R}}(\mathbb{V}_z)$ in its interior. Thus

$$\Pi_{\mathbb{R}}(\mathbb{V}_z) \subseteq \operatorname{int}(\mathcal{P}) \subseteq \operatorname{int}(\widehat{S}).$$

Now, by the discussion at the beginning of this proof it follows that $S_1 := (\widehat{S})_<$ meets the requirement of the lemma. $\square$

We have mentioned that the above lemma enables us to look for a common solution to a finitely many Lyapunov inequalities. In the following proof we utilise Lemma 4.2 to first bring down the situation to a finitely many Lyapunov inequalities and then we give a constructive solution to this simultaneous inequalities.

Proof of Theorem 4.1: (If): Suppose there is a $P > 0$ such that $\left(\sum_{i=1}^n v_i A_i^{\mathrm{T}}\right)P + P\left(\sum_{i=1}^n v_i A_i\right) < 0$ for all $0 \neq v := \operatorname{col}(v_1, v_2, \ldots, v_n) \in S$. We want to show that $\mathfrak{B}$ is S-stable, that is, for all $0 \neq v \in S$, $\lim_{t\to\infty} w(vt) = 0$ for all $w \in \mathfrak{B}$. Recall that $\mathfrak{B}$ admits an observable state representation given by Equation (1).

We first define a quadratic form on the state variable z by $V(z) := z^{\mathrm{T}}Pz$. Note that when evaluated along a trajectory $z \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$, the quadratic form $V(z)$ becomes a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. In other words, a smooth 0-form. Then it makes sense to talk about the directional derivative of this 0-form along a non-zero $v \in \mathbb{R}^n$. Moreover, derivative of $V(z(vt))$ (i.e. the restriction of $V(z(x))$ to the 1-D subspace spanned by v) with respect to t is equal to the directional derivative of $V(z)$ along v:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(z(vt)) = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} (\partial_1 V(z))(vt) \\ (\partial_2 V(z))(vt) \\ \vdots \\ (\partial_n V(z))(vt) \end{bmatrix}$$
$$= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} [(\partial_1 z)^{\mathrm{T}} Pz + z^{\mathrm{T}} P \partial_1 z](vt) \\ [(\partial_2 z)^{\mathrm{T}} Pz + z^{\mathrm{T}} P \partial_2 z](vt) \\ \vdots \\ [(\partial_n z)^{\mathrm{T}} Pz + z^{\mathrm{T}} P \partial_n z](vt) \end{bmatrix}$$
$$= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} [z^{\mathrm{T}} A_1^{\mathrm{T}} Pz + z^{\mathrm{T}} P A_1 z](vt) \\ [z^{\mathrm{T}} A_2^{\mathrm{T}} Pz + z^{\mathrm{T}} P A_2 z](vt) \\ \vdots \\ [z^{\mathrm{T}} A_n^{\mathrm{T}} Pz + z^{\mathrm{T}} P A_n z](vt) \end{bmatrix}$$
$$= z(vt)^{\mathrm{T}} \left[\left(\sum_{i=1}^n v_i A_i^{\mathrm{T}} \right) P + P \left(\sum_{i=1}^n v_i A_i \right) \right] z(vt).$$

Now, for all non-zero $v \in S$, by assumption we have $\left[\left(\sum_{i=1}^n v_i A_i^{\mathrm{T}}\right)P + P\left(\sum_{i=1}^n v_i A_i\right)\right] < 0$. So, for all $t \in \mathbb{R}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V(z(vt)) = z(vt)^{\mathrm{T}} \left[\left(\sum_{i=1}^n v_i A_i^{\mathrm{T}} \right) P + P \left(\sum_{i=1}^n v_i A_i \right) \right] z(vt) < 0.$$

A straightforward application of the fundamental theorem of integral calculus to the above inequality shows that $V(z(vt))$ is a monotonically decreasing positive function of t. Therefore, $V(z(vt))$ goes to zero as t tends to infinity. Since $V(z)$ is positive definite, the last assertion implies $z(vt)$ itself tends to zero as t goes to infinity. Thus $\lim_{t\to\infty} w(vt) = 0$, in other words, $\mathfrak{B}$ is v-stable for all non-zero $v \in S$. Therefore $\mathfrak{B}$ is S-stable.

(Only if): We assume that $\mathfrak{B}$ is S-stable and show that there exists $P = P^{\mathrm{T}} \in \mathbb{R}^{\mathtt{z}\times\mathtt{z}}$, $P > 0$, satisfying inequality (17) for all $0 \neq v = \operatorname{col}(v_1, v_2, \ldots, v_n) \in S$. First, note that by Lemma 4.2, there exists a polyhedral cone $S_1 \supseteq S$ such that $\mathfrak{B}$ is S_1-stable. We shall show that there exits a P that satisfies inequality (17) for all $0 \neq v \in S_1$. Since $S \subseteq S_1$, this P will satisfy our requirement.

Now, the advantage of working with S_1, rather than S itself, is that S_1 is polyhedral, and therefore, there exist vectors $\{v^1, v^2, \ldots, v^r\} \subseteq \mathbb{R}^n$ whose non-negative hull is S_1. For each $j \in \{1, 2, \ldots, r\}$, we define

$$\widetilde{A}_j := \sum_{i=1}^n v_i^j A_i \in \mathbb{R}^{\mathtt{z}\times\mathtt{z}},$$

where $v^j = \operatorname{col}(v_1^j, v_2^j, \ldots, v_n^j)$. Thus we get a collection of r pairwise commutative square $\mathtt{z}\times\mathtt{z}$ real matrices $\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_r\}$. If we look for a simultaneous solution P for the inequalities

$$\widetilde{A}_j^{\mathrm{T}} P + P\widetilde{A}_j < 0 \quad \text{for all } j \in \{1, 2, \ldots, r\}, \tag{18}$$

then it follows that a solution P, if it exists, works for all non-zero $v \in S_1$. To see this let $0 \neq v = \operatorname{col}(v_1, v_2, \ldots, v_n) \in S_1$. Then, because S_1 is the non-negative hull of the vectors $\{v^1, v^2, \ldots, v^r\}$, there exist non-negative real numbers $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, not all zeros, such that $v = \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_r v^r$. Let P be a solution to the simultaneous Lyapunov inequalities (18). It then follows that

$$\left(\sum_{i=1}^{r} v_i A_i^{\mathrm{T}}\right)P + P\left(\sum_{i=1}^{r} v_i A_i\right) = \alpha_1\left(\widetilde{A}_1^{\mathrm{T}}P + P\widetilde{A}_1\right) + \alpha_2\left(\widetilde{A}_2^{\mathrm{T}}P + P\widetilde{A}_2\right) + \cdots + \alpha_r\left(\widetilde{A}_r^{\mathrm{T}}P + P\widetilde{A}_r\right).$$

Since P is a common solution to the simultaneous inequalities (18), the right-hand side of the above equation is a non-negative combination of negative definite matrices with at least one of the α_is non-zero. This forces the right-hand side to be negative definite because the set of negative definite matrices is a convex cone.

Thus, for the purpose of this proof it suffices to show that we have a common solution to simultaneous inequalities (18). Since $\mathfrak{B}$ is S_1-stable, in particular, it is stable with respect to v^{j} for all $j \in \{1, 2, \ldots, r\}$. We have already seen in the proof of Theorem 2.6 that this is true if and only if the matrices $\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_r,\}$ are Hurwitz. We now define the following real symmetric $\mathtt{z} \times \mathtt{z}$ matrix P that satisfies the simultaneous inequalities (18):

$$P_j := \int_0^{\infty} \exp(\widetilde{A}_j^{\mathrm{T}}\tau_j)\exp(\widetilde{A}_j\tau_j)\mathrm{d}\tau_j,$$

$$P := \prod_{j=1}^{r} P_j,$$

where $\{\tau_1, \tau_2, \ldots, \tau_r\}$ are auxiliary variables of integration. Note that since each $\widetilde{A}_j$ is Hurwitz, for all $j \in \{1, 2, \ldots, r\}$, P_j is real symmetric positive-definite matrix. Moreover, since $\widetilde{A}_j$s commute with each other, so do the P_js. Therefore, the product P is also real symmetric and positive-definite. We now show that P satisfies the simultaneous inequalities (18). The following is a consequence of the fact that P_js commute with each other and with $\widetilde{A}_j$s.

$$\begin{aligned}
\widetilde{A}_j^{\mathrm{T}}P + P\widetilde{A}_j &= (\widetilde{A}_j^{\mathrm{T}}P_j + P_j\widetilde{A}_j)\prod_{i\neq j}^{r} P_i \\
&= \left(\int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}\tau_j}\left(\exp(\widetilde{A}_j^{\mathrm{T}}\tau_j)\exp(\widetilde{A}_j\tau_j)\right)\mathrm{d}\tau_j\right)\prod_{i\neq j}^{r} P_i \\
&= \Big(\exp(\widetilde{A}_j^{\mathrm{T}}\tau_j)\exp(\widetilde{A}_j\tau_j)|_{\tau_j=\infty} \\
&\quad - \exp(\widetilde{A}_j^{\mathrm{T}}\tau_j)\exp(\widetilde{A}_j\tau_j)|_{\tau_j=0}\Big)\prod_{i\neq j}^{r} P_i \\
&= -\prod_{i\neq j}^{r} P_i < 0.
\end{aligned}$$

□

5. Concluding remarks

In this article we looked into stability properties of a special kind of autonomous n-D systems, called strongly autonomous systems. We started with the fact that our systems are given by an observable first-order representation with n-tuple of real constant square matrices which commute pairwise. It is already known that strongly autonomous systems admit such a representation. A straightforward consequence of this representation is that now restrictions of trajectories to 1-D subspaces are given by matrix exponentials of certain linear combinations of these n-tuple of system matrices. In order to relate stability with respect to a half-line with the eigenvalues of these matrices, we showed in our main result, Theorem 2.4, how the eigenvalues of the linear combination are related with the eigenvalues of the individual matrices. As a natural consequence of this, we showed in Theorem 2.6, that, stability with respect to a half-line is equivalent to the real part of the characteristic variety being contained in the half-space polar to the half-line of stability. We then extended this result to the case for stability with respect to a closed convex pointed cone (which we have called a proper cone). Our next result provides a Lyapunov theory for strongly autonomous systems. We showed that stability of such a system with respect to a proper cone is equivalent to existence of a real symmetric positive definite matrix which satisfies simultaneous Lyapunov inequalities obtained from non-zero vectors in the given proper cone.

References

Avelli, D.N., Rapisarda, P., and Rocha, P. (2011), 'Lyapunov Stability of 2D Finite-dimensional Behaviours', *International Journal of Control*, 84, 737–745.

Cox, D., Little, J., and O'Shea, D. (2007), *Ideals, Varieties, and Algorithms* (3rd ed.), New York, Springer. Undergraduate Texts in Mathematics.

Curtain, R., and Zwart, H. (1995), *An Introduction to Infinite Dimensional Linear Systems Theory*, New York: Springer-Verlag.

Kojima, C., Rapisarda, P., and Takaba, K. (2010), 'Lyapunov Stability Analysis of Higher Order 2-D Systems', *Multidimensional Systems and Signal Processing*, 20, 287–302.

Pillai, H.K., and Shankar, S. (1998), 'A Behavioural Approach to Control of Distributed Systems', *SIAM Journal on Control and Optimization*, 37, 388–408.

Pommaret, J., and Quadrat, A. (1999), 'Algebraic Analysis of Linear Multidimensional Control Systems', *IMA Journal of Mathematical Control and Information*, 16, 275–297.

Rocha, P., and Willems, J. (2006), 'Markov Properties for Systems Described by PDEs and First-order Representations', *Systems and Control Letters*, 55, 538–542.

Sasane, A., Thomas, E., and Willems, J. (2002), 'Time Autonomy Versus Time Controllability', *Systems and Control Letters*, 45, 145–153.
Shankar, S. (1999), 'The Nullstellensatz for Systems of PDE', *Advances in Applied Mathematics*, 23, 360–374.
Valcher, M. (2001), 'Characteristic Cones and Stability Properties of Two-dimensional Autonomous Behaviours', *IEEE Transactions on Circuits and Systems-Part I: Fundamental Theory and Applications*, 47, 290–302.
Wood, J., Sule, V., and Rogers, E. (2005), 'Causal and Stable Input/Output Structures on Multi-dimensional Behaviours', *SIAM Journal on Control and Optimization*, 43, 1493–1520.