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# Nyquist plots, finite gain/phase margins & dissipativity[☆]

Debasattam Pal [a], Madhu N. Belur [b,*]

[a] Department of Electrical Engineering, Indian Institute of Technology Guwahati, Guwahati 781039, Assam, India

[b] Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai 400076, India

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ABSTRACT

The relation between the small gain theorem and 'infinite phase margin' is classical; in this paper we formulate a novel supply rate, called the 'not-out-of-phase' supply rate, to first prove that 'infinite *gain margin*' (i.e. non-intersection of the Nyquist plot of a transfer function and the negative half of the real axis) is equivalent to dissipativity with respect to this supply rate. Capturing non-intersection of *half-line* makes the supply rate system-dependent: a novel feature unobserved in the supply rates considered in the literature so far.

We then show that the traditional finite and positive gain/phase margin conditions for stability are *equivalent* to dissipativity with respect to a frequency weighted convex combination of the not-out-of-phase supply rate and the small-gain supply rate; both frequency weightings and combining two supply-rates/performance-indices have been investigated in the literature in different contexts, but only as *sufficient* conditions.



## 1. Introduction and notation

In [1] the classical Luré problem of the stability of the interconnection of an LTI system with a nonlinearity was addressed using energy-like quantities called integral quadratic constraints (IQCs). Classic results about stability in *nonlinear* systems – like circle criteria, Popov criteria, and passivity theorem – were shown to be special cases of dissipation property of the interconnected subsystems with respect to various IQCs. Later in [2] more general energy-like functionals involving higher order derivatives of the system-variables were considered using the notion of quadratic differential forms (QDFs) of [3] to give a further unification of the results addressing the Luré stability problem. One interesting fact about the dissipativity approach is that the classic results (circle/Popov criteria, passivity), when applied to LTI systems, turn out to be only sufficient conditions for stability—special cases of the classical Nyquist stability criteria. One of these criteria is the classical positive gain/phase margin conditions. In this paper we ask the question whether closed loop stability due to finite and positive gain and phase margins is *equivalent* to a combination of 'small-gain-like' and 'passivity-like' dissipativities? This paper makes this question precise and resolves it (Theorem 3.5). We first propose a novel 'supply rate' that captures non-intersection of the negative real axis as a dissipativity property. We call this supply rate the 'Not-Out-of-Phase' (NOP) supply rate because a system with transfer function $G(s)$ which is dissipative with respect to the NOP supply rate is such that the input and output of this system are never 180° out of phase for sinusoidal input of any frequency. For a reasonably large class of systems, the gain/phase margin criterion is necessary for closed loop stability (see Proposition 3.1). For this class of systems, our main result Theorem 3.5 provides a necessary and sufficient condition for stability in terms of dissipativity with respect to a polynomially convex combination of small-gain and NOP supply rates.

The significance of relating Nyquist plot properties to dissipativity is manifold. For example, a systematic method to prove the stability of the interconnection of an LTI system and a nonlinearity is by using dissipativity properties of the two systems. As demonstrated in [1,2] and elsewhere, the formulation of the Nyquist plot property of an LTI system as a dissipativity property makes this property extremely useful[1] when dealing with the stability of interconnection with a class of nonlinearities also constructed from

[1] This point cannot be overemphasized: for example, restating the containment inside the unit circle of the Nyquist plot of a stable LTI SISO system $G(s)$ as dissipativity (w.r.t. the small-gain supply rate) results in the stability conclusion of the closed loop w.r.t., not just the unity feedback, but also *any contractive nonlinearity*. Same holds true for the circle/Popov criteria and the passivity results. Mere restating as dissipativity makes such an LTI system property applicable to nonlinear stability analysis too.

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* Corresponding author. Tel.: +91 22 2576 7404; fax: +91 22 2572 3707.
*E-mail addresses:* debasattam@iitg.ernet.in (D. Pal), belur@ee.iitb.ac.in, madhu.n.belur@gmail.com (M.N. Belur).

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the dissipativity supply rate. Another benefit of dissipativity property is that one can utilize the computational advantages of the theory of Linear Matrix Inequalities (LMIs) for checking dissipativity (see [4]). Furthermore, our main result can, in fact, be used to define phase-crossover frequencies for MIMO systems by finding the frequency $\omega_p$ at which $\Sigma_\epsilon^{\mathrm{NOP}}(\zeta,\eta)$-dissipativity is lost: we do not pursue this in this paper.

The idea of combining two supply rates has been investigated before: see [1, Remark 4], and, for recent examples, [5–8]. In [5], the small gain and passivity-type supply rates have been combined with frequency dependent weights to obtain a sufficient condition for loop stability. The central result in our paper is the *equivalence*[2] of

- the Nyquist plot not intersecting the real axis to the left of the critical point '−1', and
- dissipativity with respect to the frequency weighted combination of two key supply rates.

The paper is organized as follows. The next section contains some preliminaries about Quadratic Differential Forms (QDFs) and dissipativity. Section 3 contains the main results of this paper: Theorems 3.2 and 3.5. These results are proved in Section 4, where additional auxiliary results are formulated and proved for this purpose. Example 4.2 contains an example to illustrate the main results in this paper. We end the paper with concluding remarks in Section 5. The rest of this section is about notation that we follow in this paper.

*Notation:* The set $\mathbb{R}$ stands for the real numbers, while $\mathbb{C}$ stands for the complex numbers. The point '−1' on the complex plane is important for stability (w.r.t. negative unity feedback configuration): we call it the critical point. $\mathbb{R}[s]$ and $\mathbb{R}^{2\times2}[\zeta,\eta]$ denote respectively the sets of polynomials in $s$ and $2\times2$ polynomial matrices in variables $\zeta$ and $\eta$, with real coefficients. The space of infinitely often differentiable functions is denoted by $\mathfrak{C}^\infty$, and its subspace containing compactly supported functions is denoted by $\mathfrak{D}$. For a complex function $f$, we use $f^*$ to denote its complex conjugate.

## 2. Preliminaries

In this paper, we deal with only SISO systems, and hence various notions and results from [3] about dissipativity and Quadratic Differential Forms (QDFs) are specialized to the SISO case below. A two-variable polynomial matrix $\Phi(\zeta,\eta)\in\mathbb{R}^{2\times2}[\zeta,\eta]$ with $\Phi(\zeta,\eta):=\sum_{i,k}\Phi_{ik}\zeta^i\eta^k$, and $\Phi_{ik}=\Phi_{ik}^T\in\mathbb{R}^{2\times2}$, defines a *supply rate*[3] $Q_\Phi:(\mathfrak{C}^\infty)^2\to\mathfrak{C}^\infty$ as follows:

$$Q_\Phi(u,y):=\sum_{i,k}\begin{bmatrix}\frac{d^iu}{d^it}\\ \frac{d^iy}{d^it}\end{bmatrix}^T\Phi_{ik}\begin{bmatrix}\frac{d^ku}{d^kt}\\ \frac{d^ky}{d^kt}\end{bmatrix}.$$

We require the one-variable polynomial matrix $\Phi(-s,s)$ obtained from $\Phi(\zeta,\eta)$: define $\partial\Phi(s):=\Phi(-s,s)$.

Consider a system $G$ with input $u$ and output $y$: we write $y=Gu$. Suppose $n(s)$ and $d(s)$ are the numerator and denominator polynomials of the transfer function $G$, respectively. System $G$ is called *dissipative* on $\mathbb{R}$ with respect to a supply rate defined by the symmetric two-variable polynomial matrix $\Phi(\zeta,\eta)$ if for every $u,y\in\mathfrak{D}$ such that $y=Gu$ we have[4]

$$\int_{\mathbb{R}}Q_\Phi(u,y)\mathrm{d}t\geqslant0. \tag{1}$$

For brevity, we say the system $G$ is $\Phi$-dissipative. For the purpose of this paper[5] the system $G$ is said to be *strictly* dissipative if the integral in inequality (1) holds with equality only when $u=0$.

We will make use of the following result from [3] that relates the time-domain dissipativity of a system to the non-negativity of a certain polynomial on the imaginary axis.

**Proposition 2.1** ([3]). *Consider the system $G=\frac{n(s)}{d(s)}$ and $\Phi\in\mathbb{R}^{2\times2}[\zeta,\eta]$. Then, system $G$ is dissipative with respect to $\Phi(\zeta,\eta)$ on $\mathbb{R}$ if and only if*

$$\begin{bmatrix}d^*(i\omega)\\ n^*(i\omega)\end{bmatrix}^T\partial\Phi(i\omega)\begin{bmatrix}d(i\omega)\\ n(i\omega)\end{bmatrix}\geqslant0\quad\textit{for all }\omega\in\mathbb{R}. \tag{2}$$

*Furthermore, the system is strictly dissipative if and only if the above inequality is strict for almost all $\omega\in\mathbb{R}$.*

## 3. Main results

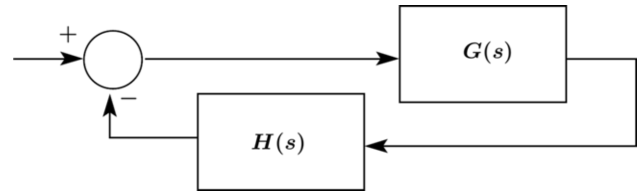



**Fig. 1.** Standard unity feedback configuration.

Consider the negative unity feedback configuration shown in Fig. 1. The Nyquist plot of $GH$ does not encircle the point '−1' on the complex plane if the transfer function $GH$ satisfies any one of the following two conditions:

1. the magnitude $|GH(i\omega)|<1$ for all real $\omega$: 'infinite phase margin' condition;
2. the angle[6] $|\angle GH(i\omega)|<180^\circ$ for all real $\omega$: 'infinite gain margin' condition.

We state below as a proposition, a slight variant of the classical Nyquist stability criterion under the assumption of open loop asymptotic stability: a transfer function $G(s)$ is called asymptotically stable if all poles of $G$ are in the open left half complex plane.

**Proposition 3.1.** *Let $G$ and $H$ be two asymptotically stable proper real rational transfer functions. Further, suppose that the loop gain at the zero frequency is positive, and the Nyquist plot of $GH$ has at most one intersection with the negative real axis. Then the closed loop is asymptotically stable if and only if*

$$\angle GH(i\omega)=180^\circ\Rightarrow|GH(i\omega)|<1. \tag{3}$$

[2] See the results in [9,6] for other equivalent relations between polynomially convex combination and Nyquist plot conditions.

[3] The coefficients $\Phi_{ik}$ are assumed symmetric matrices: this is without loss of generality for *quadratic* supply rates.

[4] Note that $y=Gu$, and $u,y\in\mathfrak{D}$ do not rule out *unstable* systems: see [3, Theorem 6.4] for additional conditions under which dissipativity does imply stability.

[5] This definition of strict dissipativity meets the purpose of this paper. There are other more stringent definitions of strict dissipativity; see [3].

[6] We assume throughout this paper that the transfer functions $G$ and $H$ have no poles/zeros on the imaginary axis.

Note that Eq. (3) is satisfied when the small-gain condition holds: i.e. $GH$ is dissipative with respect to $Q_{sg}(u, y) := u^2 - y^2$. Similarly, dissipativities of systems $G$ and $H$ with respect to $Q_{pa}(u, y) := uy$ also cause Eq. (3) to hold; this is the passivity theorem. However, both these dissipativity conditions are only sufficient for Eq. (3) to hold. Note that while small-gain theorem is applied to the loop gain, the passivity theorem is applied to $G$ and $H$ separately. Passivity applied to the loop gain means $|\angle GH| < 180°$. In Theorem 3.5 below we show that dissipativity with respect to a combination of these two cases is equivalent to Eq. (3). We first formulate a supply rate called $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ (the 'Not-Out-of-Phase' supply rate) and show that dissipativity w.r.t. $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ is equivalent to non-intersection of the negative-real axis by the Nyquist plot of $GH$; this is our first main result, Theorem 3.2. Then, as mentioned above, in Theorem 3.5 we show that Eq. (3) is equivalent to the existence of polynomial weighting functions such that $GH$ is dissipative w.r.t. the weighted combination of $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ and $\Sigma_{\text{sg}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. By Proposition 3.1 it follows that dissipativity w.r.t. this combination of two supply rates is equivalent to closed loop stability for the class of systems considered in Proposition 3.1.

**Theorem 3.2.** *Consider the feedback interconnection shown in Fig. 1. Suppose*[7] *$H(s) = 1$ and $G(s) = \frac{n(s)}{d(s)}$ proper, with no poles/zeros on the imaginary axis. Define $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta) \in \mathbb{R}^{2\times 2}[\zeta, \eta]$ as*

$$\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta) := \begin{bmatrix} n(\zeta)n(\eta) & -d(\zeta)n(\eta) + \epsilon \\ -n(\zeta)d(\eta) + \epsilon & d(\zeta)d(\eta) \end{bmatrix}. \tag{4}$$

*Then, the following are equivalent.*

1. *The Nyquist plot of $G$ does not intersect the negative real axis.*
2. *$|\angle G(i\omega)| < 180°$ for all $\omega \in \mathbb{R}$.*
3. *There exists an $\epsilon > 0$ such that system $G$ is strictly $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ dissipative.*

*In particular, if $G$ has no poles in the closed RHP and $G$ satisfies any one of the conditions* 1–3, *then the closed loop is also stable.*

**Remark 3.3.** The role that $\epsilon$ plays in $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ dissipativity definition can be understood as follows. Loosely speaking, the intersection of $\mathbb{R}_-$ by the Nyquist plot occurs at only finitely many points. Since dissipativity is an integral inequality condition, it fails to capture this. Instead of ruling out just $\mathbb{R}_-$ intersection, we now also rule out the intersection of a thin band about $\mathbb{R}_-$ of thickness that is controlled by $\epsilon$, with the thickness decreasing rapidly as we go closer to the origin. We show later that ruling out this intersection is achieved by requiring existence of $\epsilon > 0$ such that $\Pi_\epsilon(\omega) > 0$ for all $\omega \in \mathbb{R}$ with $\Pi_\epsilon(\omega)$ defined as

$$2\epsilon \text{Real } (n^*(i\omega)d(i\omega)) + 4(\text{Imag } (n^*(i\omega)d(i\omega)))^2. \tag{5}$$

For the case when the Nyquist plot of $G$ intersects $\mathbb{R}_-$, it is easy to see that there does not exist a positive $\epsilon$ that ensures $\Pi_\epsilon(\omega) > 0$ for all $\omega \in \mathbb{R}$. See Example 4.2 for a system $G$ where $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$-dissipativity is true for very small $\epsilon$ only.

**Remark 3.4.** A noteworthy point about the supply rate $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ corresponding to non-intersection of the negative real axis by the Nyquist plot of a transfer function $G$ is that $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ depends on $G$, unlike other supply rates such as $\Sigma_{\text{sg}}$ and $\Sigma_{\text{pa}} := \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ that induces $Q_{pa}(u, y) = uy$. The latter two supply rates respectively correspond to non-intersection of the unit circle and the imaginary axis. Note that in the complex plane, lines and circles are essentially the same; indeed, in the extended complex plane, linear fractional transformations take lines and circles to lines and circles. Half-lines, on the other hand, are intrinsically different. Ruling out intersection of a *half-line* is inevitable since gain margin applies to only positive gain $k$: perhaps this forces the supply rate to become system dependent. In spite of this 'disadvantage', expressing a system property in terms of dissipativity has its advantages, some of which have been summarized in Section 1.

We now state the second main result of this paper: dissipativity of $G$ w.r.t. a polynomial combination of the small gain and the $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ supply rates is equivalent to $G$ having gain and phase margins finite and positive.

**Theorem 3.5.** *Consider a SISO LTI system given by the real rational proper transfer function $G(s) = \frac{n(s)}{d(s)}$, with no poles/zeros on the imaginary axis. Define $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta) \in \mathbb{R}^{2\times 2}[\zeta, \eta]$ as in Eq. (4) above and let $\Sigma_{\text{sg}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then the following two statements are equivalent:*

1. *there exist $p, q \in \mathbb{R}[s]$ and $\epsilon > 0$ such that system $G$ is strictly dissipative with respect to*

   $$\Phi_\epsilon(\zeta, \eta) := p(\zeta)\Sigma_{\text{sg}}p(\eta) + q(\zeta)\Sigma_\epsilon^{\text{NOP}}q(\eta),$$

2. *for each $\omega \in \mathbb{R}$, either $|G(i\omega)| < 1$, or $|\angle(G(i\omega))| < 180°$, or both.*

*In particular, if $G$ has no poles in the closed RHP, and $G$ satisfies condition* 1, *then the closed loop is also stable.*

Notice that the second condition in the above theorem rules out encirclements of the critical point $-1$ by the Nyquist plot of $G$.

**Remark 3.6.** The polynomials $p$ and $q$ of Theorem 3.5 play the role of multipliers in the context of Popov/circle criteria. This role is more familiar and easily seen in the context polynomial convex combination of $\Sigma_{\text{sg}}$ and $\Sigma_{\text{pa}}$: hence we describe only for this situation. Suppose a system $G$ is dissipative w.r.t. $p(\zeta)\Sigma_{\text{sg}}p(\eta) + q(\zeta)\Sigma_{\text{pa}}q(\eta)$. Define $F_1(s) := \frac{p(s)}{q(s)+p(s)}$ and $F_2(s) := \frac{q(s)-p(s)}{q(s)+p(s)}$. Then it can be shown that the system with transfer function $\frac{F_1(s)+F_2(s)G(s)}{1+F_1(s)G(s)}$ is $\Sigma_{\text{pa}}$-dissipative.

## 4. Proof of main results

In this section we prove the two main results. Recall the definition of the supply rate $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ given by Eq. (4), and that Theorem 3.2 states equivalence of $\Sigma_\epsilon^{\text{NOP}}(\zeta, \eta)$ dissipativity of a system $G$ with non-intersection of the negative real axis by the Nyquist plot of $G$. For the proof of Theorem 3.2 the relative rate of approaching zero of two rational functions in $\omega$ as $\omega$ approaches $\infty$ plays an important role. The notion of valuation at the point $\infty$ makes proving this easier. Valuation of a rational function $f$ at the point $\omega = \infty$ (denoted by $v_\infty$) is defined as the multiplicity of zero at $\omega = \infty$ and $v_\infty(f) := \infty$ for $f \equiv 0$. See [10, page 454] for an elaborate treatment on valuations at infinity. Using the notion of valuations, the proposition below follows from a routine count of degrees of numerator and denominator of the rational functions involved.

**Proposition 4.1.** *Consider functions $f_i$ and $f_r$ that are real rational in $\omega$ and assume*

- *$f_i(\omega) \to 0$ as $\omega \to \infty$.*
- *$f_i(\omega) > 0$ for all $\omega$ sufficiently large.*

*Consider $v_\infty(f_i)$ and $v_\infty(f_r)$, the valuations at $\omega = \infty$ of $f_i$ and $f_r$ respectively. Then the following are equivalent.*

[7] Since it is only the loop gain $GH$ that affects closed loop stability conditions, we assume without loss of generality that $H = 1$. Further, $G(s)$ is assumed to have no poles/zeros on the imaginary axis just to ensure that $\angle G(i\omega)$ is defined for all $\omega$.

- *There exist $\epsilon > 0$ and $\omega_0 > 0$ such that $f_i(\omega) + \epsilon f_r(\omega) > 0$ for all $\omega > \omega_0$.*
- $v_\infty(f_r) \geqslant v_\infty(f_i)$.

We now prove Theorem 3.2.

**Proof of Theorem 3.2.** ($1 \Leftrightarrow 2$): The equivalence of statements (1) and (2) in Theorem 3.2 is clear: non-intersection of the Nyquist plot of $G(i\omega)$ with the negative real axis is equivalent to $|\angle G(i\omega)| < 180^\circ$.

($3 \Rightarrow 2$): Assuming $\Sigma_\epsilon^{\mathrm{NOP}}(\zeta, \eta)$-dissipativity of $G$, we want to show that $|G(i\omega)| < 180^\circ$ for all $\omega \in \mathbb{R}$. Recall the definition of $\Pi_\epsilon(\omega)$ in Eq. (5) in Remark 3.3. With some simple algebraic manipulations it follows that $G$ is $\Sigma_\epsilon^{\mathrm{NOP}}(\zeta, \eta)$-dissipative if and only if $\Pi_\epsilon(\omega) > 0$ for all $\omega \in \mathbb{R}$. Assume there exists $\epsilon > 0$ such that $\Pi_\epsilon(\omega) > 0$ for all $\omega \in \mathbb{R}$. We show that this implies that for any $\omega$ such that Imag $G(i\omega) = 0$, we have Real $G(i\omega) > 0$, thus proving that there are no negative real axis intersections. Observe that Imag $(G(i\omega)) = 0$ implies Imag $(n^*(i\omega)d(i\omega)) = 0$. Now use $\Pi_\epsilon(\omega) > 0$, $\epsilon > 0$ and Imag $(n^*(i\omega)d(i\omega)) = 0$ to infer Real $(n^*(i\omega)d(i\omega)) > 0$, i.e. Real $(G(i\omega)) > 0$. This proves that there are no negative real axis intersections, and hence proves ($3 \Rightarrow 2$).

($2 \Rightarrow 3$): We now prove the converse: assuming the Nyquist plot of $G$ does not intersect the negative real axis, we prove the existence of an $\epsilon > 0$ such that $\Pi_\epsilon(\omega) > 0$ for all $\omega \in \mathbb{R}$. This proof has two parts: we fix a suitable $\omega_0$ and prove the existence of $\epsilon > 0$ for each of the following cases.

*Case* 1: For all $\omega \in [-\omega_0, \omega_0]$ (the finite $\omega$ case).
*Case* 2: For all $|\omega| > \omega_0$ (the asymptotic case).

Consider the set $\Omega \subseteq [-\omega_0, \omega_0]$ defined as

$$\Omega := \{\omega \in [-\omega_0, \omega_0] \mid \text{Real } (n^*(i\omega)d(i\omega)) \leqslant 0\},$$

which turns out to be a compact[8] (possibly empty) set. Define

$$\epsilon_1 := \frac{\min_{\omega\in\Omega}(\text{Imag } (n^*(i\omega)d(i\omega)))^2}{\max_{\omega\in\Omega} |\text{Real } (n^*(i\omega)d(i\omega))|}.$$

(If the set $\Omega$ is empty or only a finite[9] number of points, then define $\epsilon_1 := \infty$.) Notice that $\epsilon_1 > 0$ since the absence of negative real axis intersections ensures that the numerator above is positive. For any positive $\epsilon \leqslant \epsilon_1$ inequality (5) holds for all $\omega \in [-\omega_0, \omega_0]$: this follows from the definition of $\epsilon_1$, thus proving Case 1.

For the asymptotic case, the existence of an $\epsilon_2$ and $\omega_0$ is inferred using Proposition 4.1. Notice that $\Pi_\epsilon(\omega) = (d^*d)^2(2\frac{\epsilon}{d^*d}\text{Real } (G(i\omega)) + 4(\text{Imag } (G(i\omega)))^2)$. Using positivity $d^*(i\omega)d(i\omega)$ for each $\omega$, we use properness of $G$ to show that there exist $\epsilon_2 > 0$ and $\omega_0 > 0$ such that

$$\frac{2\epsilon}{d^*(i\omega)d(i\omega)}\text{Real } (G(i\omega)) + 4(\text{Imag } (G(i\omega)))^2 > 0$$

for all $\omega > \omega_0$. Define $f_i(\omega) := 4(\text{Imag } (G(i\omega)))^2$ and $f_r(\omega) := 2\frac{\text{Real } (G(i\omega))}{d^*(i\omega)d(i\omega)}$. Clearly, $f_i(\omega) > 0$ and since Imag $(G(i\omega)) \to 0$ as $\omega \to \infty$ so does $f_i(\omega)$ as $\omega \to \infty$. So the assumptions in Proposition 4.1 are satisfied. In order to use Proposition 4.1 to conclude the existence of $\epsilon_2$ and $\omega_0$ as required, it remains to show that the valuations at $\infty$ of $f_i$ and $f_r$ are related by $v_\infty(f_r) \geqslant v_\infty(f_i)$. Notice that $v_\infty(f_r) = v_\infty(2\text{Real } (G(i\omega))) + v_\infty(\frac{1}{d^*d})$. Since $G(s)$ is proper, $v_\infty(\text{Real } (G(i\omega))) \geqslant 0$. Thus $v_\infty(f_r) \geqslant v_\infty(\frac{1}{d^*d})$. On the other hand, again using properness of $G(s)$, $v_\infty((\text{Imag } (G(i\omega)))^2) \leqslant v_\infty(\frac{1}{d^*d})$. Combining these two inequalities we have the required relation $v_\infty(f_r) \geqslant v_\infty(f_i)$. Thus there exists $\epsilon_2 > 0$ and $\omega_0$ such that $\Pi_\epsilon(\omega) > 0$ for all $\omega > \omega_0$. Since $\Pi_\epsilon(\omega)$ is an even function of $\omega$, clearly, $\Pi_\epsilon(\omega)$ with $\epsilon = \epsilon_2$ is positive for all $\omega \notin [-\omega_0, \omega_0]$. Like in the finite $\omega$ case, once positivity of $\Pi_\epsilon(\omega)$ is established for an $\epsilon_2 > 0$, it is satisfied for all lower and positive values of $\epsilon$ also. It then follows that if we take $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ then $\Pi_\epsilon(\omega)$ is positive for all $\omega \in \mathbb{R}$. This completes the proof of ($2 \Rightarrow 3$). □

We now prove Theorem 3.5.

**Proof of Theorem 3.5.** ($2$) $\Rightarrow$ ($1$): Using Proposition 2.1, Statement 1 is equivalent to the inequality

$$\begin{bmatrix} d^*(i\omega) \\ n^*(i\omega) \end{bmatrix}^T \partial\Phi_\epsilon(i\omega) \begin{bmatrix} d(i\omega) \\ n(i\omega) \end{bmatrix} > 0 \quad \text{for all } \omega \in \mathbb{R}.$$

The LHS simplifies to

$$|p(i\omega)|^2 \left(|d(i\omega)|^2 - |n(i\omega)|^2\right) + |q(i\omega)|^2 \left(2\epsilon \text{Real } (n^*(i\omega)d(i\omega)) + 4(\text{Imag } (n^*(i\omega)d(i\omega)))^2\right).$$

Thus Statement 1 is equivalent to

$$\begin{bmatrix} p^*(i\omega) \\ q^*(i\omega) \end{bmatrix}^T \begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi_\epsilon(\omega) \end{bmatrix} \begin{bmatrix} p(i\omega) \\ q(i\omega) \end{bmatrix} > 0, \quad \text{for all } \omega \in \mathbb{R} \tag{6}$$

where $\Gamma(\omega) := |d(i\omega)|^2 - |n(i\omega)|^2$ and $\Pi_\epsilon(\omega)$ is as defined in Eq. (5). We now assume Statement 2 and prove the existence of polynomials $p$ and $q$ such that inequality (6) is satisfied. Statement 2 implies that $\begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi_\epsilon(\omega) \end{bmatrix}$ has inertia[10] (1, 1) or (0, 2). The latter case requires no proof since any pair of polynomials $(p, q)$ satisfying coprimeness on the imaginary axis also satisfies inequality (6), thus proving Statement (1). For the former case, we use[11] the results from [11] to show that there exist matrices $L(s)$ and $K(s) \in \mathbb{R}^{\bullet\times 2}[s]$, with $K$ square and nonsingular such that

$$\begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi_\epsilon(\omega) \end{bmatrix} = K^T(-i\omega)\Sigma_{\mathrm{sg}}K(i\omega) + L^T(-i\omega)L(i\omega). \tag{7}$$

As done in [11], Eq. (7) is used to construct $p, q$ that meet the requirements of Statement 1 as follows.

Choose any pair[12] of polynomials $(p', q')$ such that $p'(-i\omega)p'(i\omega) - q'(-i\omega)q'(i\omega) > 0$ for all $\omega$. Construct the adjugate adj $K(s)$ of $K(s)$. The required $p, q$ are given by

$$\begin{bmatrix} p(s) \\ q(s) \end{bmatrix} := (\text{adj } K(s)) \begin{bmatrix} p'(s) \\ q'(s) \end{bmatrix}.$$

Using this $p$ and $q$ (after cancelling common factors, if any), one can reverse the chain of arguments before inequality (6) in order to conclude strict dissipativity. This concludes the proof of ($2 \Rightarrow 1$) of Theorem 3.5.

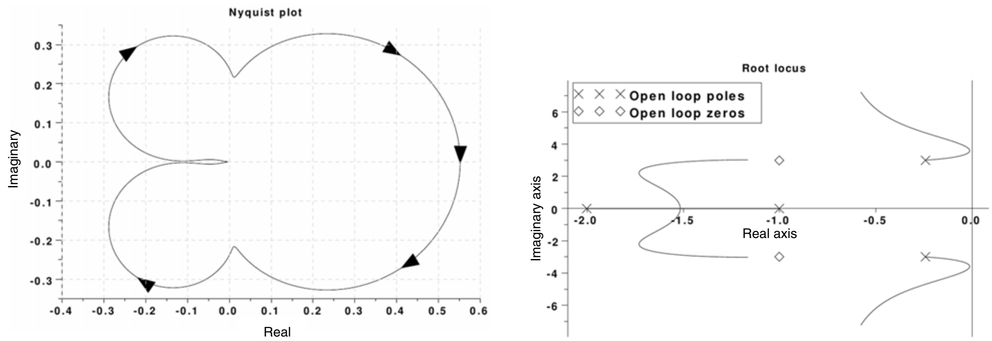
8 Compactness is used for the existence of the maximum and minimum in the definition of $\bar{\epsilon}$.

9 The case when both real and imaginary parts of $n^*(i\omega)d(i\omega)$ are zero, i.e. when the imaginary axis contains a finite zero of $G(s)$, has been ruled out: see footnote 6 above.

10 The inertia of a nonsingular Hermitian matrix $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is the pair $(\sigma_-(S), \sigma_+(S))$, the number of negative and positive eigenvalues of $S$ respectively.

11 The details of this existence is straightforward when we use the notion of 'worst inertia' as introduced before [11, Theorem 3.6]: see [9] for these details.

12 Any transfer function $\frac{q'(s)}{p'(s)} \in \mathbb{R}(s)$ with $\mathcal{L}_\infty$ norm strictly less than one will give such $p'$ and $q'$. (The $\mathcal{L}_\infty$-norm $\|G\|_{\mathcal{L}_\infty}$ of a proper transfer function $G$ with no poles on $i\mathbb{R}$ is defined as $\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in\mathbb{R}} |G(i\omega)|$.)




(a) Nyquist plot.

(b) Root locus.

**Fig. 2.** $G(s) = \frac{s^2+3s+11.25}{(s^2+2ps+9+p^2)(s^2+3s+2)}$ with $p = 0.243$ (plotted in Scilab).

(1) ⇒ (2): Here, we first assume polynomials $p$ and $q$ exist such that system $G$ is strictly dissipative with respect to supply rate $\Phi_\epsilon := p(\zeta)\Sigma_{sg}p(\eta) + q(\zeta)\Sigma_\epsilon^{\mathrm{NOP}}q(\eta)$.

Assume that there exist polynomials $p, q \in \mathbb{R}[s]$ and $\epsilon > 0$ such that system $G$ is strictly $\Phi_\epsilon$-dissipative. Like in the first part of this proof, this implies

$$\begin{bmatrix} p(i\omega) \\ q(i\omega) \end{bmatrix}^* \begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi_\epsilon(\omega) \end{bmatrix} \begin{bmatrix} p(i\omega) \\ q(i\omega) \end{bmatrix} > 0 \tag{8}$$

where $\Gamma(\omega)$ and $\Pi_\epsilon(\omega)$ are as defined above. Since both $p(-i\omega)p(i\omega)$ and $q(-i\omega)q(i\omega)$ are non-negative for every $\omega \in \mathbb{R}$, the above inequality rules out the existence of any $\omega_0$ such that $\Gamma(\omega_0) < 0$ and $\Pi_\epsilon(\omega_0) < 0$. Therefore, for almost all $\omega \in \mathbb{R}$, either $|d(i\omega)|^2 - |n(i\omega)|^2 > 0$ or $2\epsilon\mathrm{Real}\ (n(-i\omega)d(i\omega)) + 4(\mathrm{Imag}\ n(-i\omega)d(i\omega))^2 > 0$ or both. In order to infer statement 2, it remains to show

(A) $|d(i\omega)|^2 - |n(i\omega)|^2 > 0 \Leftrightarrow |G(i\omega)| < 1$
(B) $2\epsilon\mathrm{Real}\ (n(-i\omega)d(i\omega)) + 4(\mathrm{Imag}\ n(-i\omega)d(i\omega))^2 > 0 \Leftrightarrow |\angle G(i\omega)| < 180°$.

Statement (A) is straightforward/well-known: special case of the small gain theorem; and Statement (B) is precisely Theorem 3.2 and was proved above. This completes the proof of the (1 ⇒ 2) part of Theorem 3.5. □

**Example 4.2.** We bring out the significance of the parameter $\epsilon$ in the supply rate $\Sigma_\epsilon^{\mathrm{NOP}}(\zeta, \eta)$ by choosing an example where for a particular $\omega$ value, the imaginary part of $G(i\omega)$ is small in magnitude but nonzero, and the real part is negative; the Nyquist plot does not intersect the negative real axis thus suggesting the existence of an $\epsilon > 0$ due to Theorem 3.2. Consider the transfer function $G(s) = \frac{s^2+3s+11.25}{(s^2+2ps+9+p^2)(s+1)(s+2)}$ with $p$ a parameter. The poles and zeros of this system are at $-1, -2, -p \pm 3i$ and at $-1.5 \pm 3i$ respectively. Fig. 2 shows the Nyquist plot and the root locus for the case of $p = 0.243$. The figure indicates that the closed loop is stable for all $k > 0$ in the feedback configuration of Fig. 1 above. The value of $p$ has been chosen such that there is no negative real axis intersection though the magnitude of the imaginary part is very small ($=\frac{2.48}{1000}$) at $\omega_0 = 3.65$ rad/s, and the real part there is 0.111. Using these values, we infer that the system is $\Sigma_\epsilon^{\mathrm{NOP}}$-dissipative if and only if $\epsilon \in (0, \frac{2(2.48)^2}{111000})$; see Eq. (5).

## 5. Concluding remarks

We proposed a new supply rate (called the Not-Out-of-Phase (NOP) supply rate) such that dissipativity with respect to this is equivalent to Nyquist plot's non-intersection of the negative real axis, and hence infinite gain margin: Theorem 3.2. A polynomially convex combination of the two supply rates yields the traditional result that, assuming open loop stability, finite and positive gain and phase margin conditions on the open loop results in closed loop stability (Theorem 3.5).

An interesting direction of future investigation is whether the supply rate's dependence on the system is inevitable due to the requirement of non-intersection demanded on a *half-line*. Another question that arises in the context of derivatives of system variables playing a role in the $\Sigma_\epsilon^{\mathrm{NOP}}(\zeta, \eta)$ supply rate is whether expressing the supply rate in terms of the *states* of the system helps by not having to differentiate any variable.

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