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On restrictions of n-d systems to 1-d subspaces

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Abstract In this paper, we look into restrictions of the solution set of a system of PDEs to 1-d subspaces. We bring out its relation with certain intersection modules. We show that the restriction, which may not always be a solution set of differential equations, is always contained in a solution set of ODEs coming from the intersection module. Next, we focus our attention to restrictions of strongly autonomous systems. We first show that such a system always admits an equivalent first order representation given by an n-tuple of real square matrices called companion matrices. We then exploit this first order representation to show that the system corresponding to the intersection module has a state representation given by the restriction of a linear combination of the companion matrices to a certain invariant subspace. Using this result we bring out that the restriction of a strongly autonomous system is equal to the system corresponding to the intersection module. Then we look into restrictions of a general autonomous system, not necessarily strongly autonomous. We first define the notion of a free subspace of the domain—a 1-d subspace where every possible 1-d trajectory can be obtained by restricting the trajectories of the autonomous system. Then we give an algebraic characterization of free-ness of a 1-d subspace for a scalar autonomous system. Using this algebraic criterion we then give a full geometric characterization of free (and non-free) subspaces. As a consequence of this we show that the set of non-free 1-d subspaces is a closed linear set in the projective $(n-1)$-space. Finally, we show that restriction to a non-free subspace always equals the solution set of the ODEs coming from the intersection ideal. As a corollary to this we give a necessary and sufficient condition for a system to be stable in a given direction.

Keywords n-D systems · Autonomous systems · Intersection modules

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1 Introduction and preliminaries

In n-d systems, restriction of trajectories to smaller subsets of the domain $\mathbb{R}^n$ is of fundamental importance in various issues. For example, the theory of characteristic subsets (Valcher 2001), dissipativity/path-independence of quadratic functionals (Pillai and Willems 2002), stability theory (Valcher 2001; Kojima et al. 2010)—all of these issues are inextricably connected with the idea of restriction of n-d systems to certain smaller subsets of $\mathbb{R}^n$. Interestingly, it was shown in Zerz and Oberst (1993) that for discrete systems, *i.e.*, systems defined over $\mathbb{Z}^n$ instead of $\mathbb{R}^n$, restriction of trajectories to subsets of $\mathbb{Z}^n$ plays a crucial role in the Cauchy problem. In this paper, we look into restriction of n-d systems to 1-d subspaces. One of the most important results of this paper is that such restrictions can be analyzed by looking into an algebraic entity called *intersection submodule*. See Pal and Pillai (2011) where the idea of intersection submodules was introduced in this context. See also Avelli and Rocha (2010), where restriction of discrete n-d systems to sublattices of $\mathbb{Z}^n$ has been investigated in the context of autonomy degree of n-d autonomous systems. A preliminary version of some of the results in this paper, namely those contained in Sects. 2 and 3, was presented in Pal and Pillai (2011). In this paper, a substantial improvement on these sections has been carried out; the proofs have been rewritten to make the exposition clearer. Further, completely new results have been presented in Sect. 4. The remaining part of this section is devoted to some preliminary definitions and results which are essential for the rest of the paper.

The kind of systems we are concerned with in this paper are the ones described by linear partial differential equations (PDEs) with constant real coefficients. We use ∂ to denote the n-tuple partial differential operators $\{\partial_1, \partial_2, \ldots, \partial_n\}$. Following Polderman and Willems (1998), we call the solution set of such systems of PDEs *behaviors* and denote them by $\mathfrak{B}$. Thus

$$\mathfrak{B} := \left\{ w \in \mathcal{W}^{\mathtt{w}} \mid R(\partial)\, w = 0 \right\}, \tag{1}$$

where $R(\partial)$ is a matrix having $\mathtt{w}$ number of columns with entries from the n-variable polynomial ring $\mathbb{R}[\partial]$. Here, the matrix $R(\partial)$ 'acts' on w by differentiation. That is, let $r(\partial) = \left[r_1(\partial)\ r_2(\partial) \cdots r_{\mathtt{w}}(\partial) \right]$ with $r_1(\partial), r_2(\partial), \ldots, r_{\mathtt{w}}(\partial) \in \mathbb{R}[\partial]$, be a row of $R(\partial)$. Then this row acts on $w = \mathrm{col}\,[w_1, w_2, \ldots, w_{\mathtt{w}}]$ as

$$r(\partial)w = r_1(\partial)w_1 + r_2(\partial)w_2 + \cdots + r_{\mathtt{w}}(\partial)w_{\mathtt{w}}. \tag{2}$$

Further, $\mathcal{W}$ (the solution space) is an $\mathbb{R}$-vector space of trajectories which contains the solutions of the differential equations. We denote by $\mathfrak{L}^{\mathtt{w}}$ the set of all behaviors, as described above, where the codomain of the trajectories is $\mathbb{R}^{\mathtt{w}}$. In this paper, we think of elements from $\mathbb{R}[\partial]^{\mathtt{w}}$ as row-vectors and elements from $\mathcal{W}^{\mathtt{w}}$ as column-vectors, so that the action of $r(\partial) \in \mathbb{R}[\partial]^{\mathtt{w}}$ on $w \in \mathcal{W}^{\mathtt{w}}$ follows Eq. (2). We often require to write down elements from $\mathbb{R}[\partial]^{\mathtt{w}}$ in an expanded form. For this purpose, given a positive integer vector $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$, we denote by ∂^ν the monomial $\partial_1^{\nu_1}\partial_2^{\nu_2}\cdots\partial_n^{\nu_n}$. So for a typical $r(\partial) \in \mathbb{R}[\partial]^{\mathtt{w}}$ we can write it as

$$r(\partial) = \sum_{\nu \in \mathbb{N}^n} \partial^\nu \alpha_\nu, \tag{3}$$

where $\{\alpha_\nu \in \mathbb{R}^{\mathtt{w}}_{\mathrm{row}}\}_{\nu\in\mathbb{N}^n}$ are row-vectors of $\mathtt{w}$-tuple of real numbers. Moreover, only a finite number of these α_νs are non-zero.

A crucial observation is that there is an alternative description of $\mathfrak{B}$: if we denote by $\mathcal{R}$ the row-span of the matrix R over $\mathbb{R}[\partial]$, then $\mathfrak{B}$ can also be written as

$$\mathfrak{B}(\mathcal{R}) := \left\{ w \in \mathcal{W}^{\mathtt{w}} \mid r(\partial)w = 0, \text{ for all } r(\partial) \in \mathcal{R} \right\}. \tag{4}$$

Thus, given a submodule $\mathcal{R}$ of the free module $\mathbb{R}[\partial]^{\mathtt{w}}$, we can associate with $\mathcal{R}$ the behavior $\mathfrak{B}(\mathcal{R})$ given by Eq. (4). Similarly, given a set of trajectories in $\mathcal{W}^{\mathtt{w}}$, one can define all $r(\partial) \in \mathbb{R}[\partial]^{\mathtt{w}}$, such that the action of $r(\partial)$ on the set of trajectories is zero. In particular, given a behavior $\mathfrak{B}$, we define

$$\mathcal{R}(\mathfrak{B}) := \{ r \in \mathbb{R}[\partial]^{\mathtt{w}} \mid r(\partial)w = 0 \text{ for all } w \in \mathfrak{B} \}.$$

In Oberst (1990), Oberst shows that $\mathfrak{B}(\bullet)$ and $\mathcal{R}(\bullet)$ are inverses of each other whenever the signal space is a *large injective cogenerator*. This shows that the correspondence between submodules of $\mathbb{R}[\partial]^{\mathtt{w}}$ and behaviors $\mathfrak{B}$ is one-to-one. By this one-to-one correspondence, we call the submodule $\mathcal{R}(\mathfrak{B})$ *the equation module* of $\mathfrak{B}$. In this paper, we shall restrict ourselves to the following signal spaces : space of infinitely differentiable functions, denoted by $\mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$, and space of real entire analytic functions of exponential type, denoted by $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$. It is shown in Oberst (1990) that both these signal spaces are large injective cogenerators. The exponential type functions $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ happen to be a subspace of $\mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$. We restrict ourselves to this subspace only in Sect. 4.

We often talk about elements in the quotient module $\mathcal{M}(\mathfrak{B}) := \mathbb{R}[\partial]^{\mathtt{w}}/\mathcal{R}(\mathfrak{B})$ acting on the trajectories in the behavior $\mathfrak{B}$. By this we mean the action of a lift of that element in $\mathbb{R}[\partial]^{\mathtt{w}}$ on the trajectories in $\mathfrak{B}$. Although these lifts are not unique, their actions on $\mathfrak{B}$ are: two distinct lifts always differ by an element in $\mathcal{R}(\mathfrak{B})$, and the action of $\mathcal{R}(\mathfrak{B})$ on the trajectories in $\mathfrak{B}$ produces the zero trajectory.

It was proved in Pillai and Shankar (1998), Pommaret and Quadrat (1999), Pommaret (2005) that a behavior is controllable if and only if the quotient module $\mathcal{M}(\mathfrak{B})$ is *torsion-free*[1]. With this idea of controllability, one defines an *autonomous* behavior to be one which does not contain any non-trivial controllable behavior within itself. It then follows, as was shown in Pommaret and Quadrat (1999), Pillai and Shankar (1998), that an autonomous behavior is characterized by a quotient module that is a *torsion* module. This algebraic property of the quotient module gives rise to two fundamental invariants of an autonomous behavior $\mathfrak{B}$, namely the *annihilator ideal* of $\mathcal{M}(\mathfrak{B})$, which we denote by $\operatorname{ann}(\mathcal{M}(\mathfrak{B}))$, and the *characteristic ideal* of $\mathfrak{B}$, which we denote by $\mathcal{I}(\mathfrak{B})$. The characteristic ideal is defined as follows: given a behavior $\mathfrak{B}$ and its corresponding equation module $\mathcal{R}(\mathfrak{B})$, let $R \in \mathbb{R}[\partial]^{\mathtt{g} \times \mathtt{w}}$ be a matrix whose rows generate $\mathcal{R}(\mathfrak{B})$. Then define the ideal generated by the $(\mathtt{w} \times \mathtt{w})$ minors of R to be $\mathcal{I}(\mathfrak{B})$[2].

For an autonomous behavior, there is another invariant, a geometric one, called the *characteristic variety* and denoted by $\mathbb{V}(\mathfrak{B})$. By the characteristic variety of an autonomous behavior we mean the following set of complex n-tuples.

$$\begin{aligned}\mathbb{V}(\mathfrak{B}) &:= \{\xi \in \mathbb{C}^n \mid f(\xi) = 0 \text{ for all } f \in \mathcal{I}(\mathfrak{B})\} = \mathbb{V}(\mathcal{I}(\mathfrak{B})) \\ &= \{\xi \in \mathbb{C}^n \mid f(\xi) = 0 \text{ for all } f \in \operatorname{ann}(\mathcal{M}(\mathfrak{B}))\} = \mathbb{V}(\operatorname{ann}(\mathcal{M}(\mathfrak{B}))).\end{aligned}$$

The second equality follows by applying Hilbert's Nullstellensatz to the fact that the radicals of $\mathcal{I}(\mathfrak{B})$ and $\operatorname{ann}(\mathcal{M}(\mathfrak{B}))$ are the same. We sum up all these important results in

[1] Controllability roughly means the ability to 'patch-up' two trajectories; see Polderman and Willems (1998), Pillai and Shankar (1998) for details.

[2] Note that although the definition relies on a matrix representation of $\mathcal{R}(\mathfrak{B})$, the ideal $\mathcal{I}(\mathfrak{B})$ is independent of this representation and depends only on the submodule $\mathcal{R}(\mathfrak{B})$, and hence, because of the one-to-one correspondence between $\mathfrak{B}$ and $\mathcal{R}(\mathfrak{B})$, on the behavior $\mathfrak{B}$.

the form of a proposition below (the result can be found in the literature, see for example Shankar 1999).

Proposition 1 *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. *Then*

1. $\mathfrak{B}$ *is autonomous if and only if* $\mathcal{M}(\mathfrak{B})$ *is a torsion module.*
2. *If* $\mathfrak{B}$ *is autonomous then* $\sqrt{\mathcal{I}(\mathfrak{B})} = \sqrt{\mathrm{ann}(\mathcal{M}(\mathfrak{B}))}$.
3. *If* $\mathfrak{B}$ *is autonomous then* $\mathbb{V}(\mathfrak{B})$ *is a proper subset of* $\mathbb{C}^n$.

Remark 2 For the special case when $n = 1$, $\mathbb{R}[\partial]$ turns out to be a principal ideal domain (PID). So both the characteristic ideal and the annihilator ideal are principal, and hence each is generated by a polynomial. The unique monic generators are, in fact, the characteristic and minimal polynomials of the system, respectively. The above result reasserts the well-known fact for a system of ODEs: the characteristic and minimal polynomials have the same roots with possibly different multiplicities.

From the next section onwards we shall drop the argument $\mathfrak{B}$ from $\mathcal{R}(\mathfrak{B})$, $\mathcal{M}(\mathfrak{B})$, $\mathcal{I}(\mathfrak{B})$, and use just $\mathcal{R}$, $\mathcal{M}$ and $\mathcal{I}$, respectively, whenever the behavior $\mathfrak{B}$ is clear from the context.

2 Restriction of a behavior to a 1-dimensional subspace

Our prime concern in this paper is to analyze an autonomous behavior when restricted to a given 1-dimensional subspace in its domain space. In this section we make this idea of restriction precise. Then we show how restriction is related to the algebraic idea of intersection submodules. In the sequel, we shall make frequent use of the following notation: given a nonzero real vector $v \in \mathbb{R}^n$, L_v denotes the line spanned by v, *i.e.*,

$$L_v := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = vt, t \in \mathbb{R}\}. \tag{5}$$

Now, given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $0 \neq v \in \mathbb{R}^n$, by restriction of $\mathfrak{B}$ to the line L_v we mean the following set of 1-d trajectories

$$\mathfrak{B}|_v := \{w(vt) \mid w \in \mathfrak{B}\}.$$

Remark 3 Note that $\mathfrak{B}|_v \subseteq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ when the parameter t is treated as the independent variable. For some other $0 \neq v' \in \mathbb{R}^n$ spanning the same 1-d subspace L_v we obtain a different parametrization. Accordingly, a restricted trajectory, viewed as a 1-d trajectory in the independent variable t, will also be different. Because any v' that spans L_v must be a scalar multiple of v, it can be shown that if $v' = av$ with $0 \neq a \in \mathbb{R}$, then elements of $\mathfrak{B}_{v'}$ differ from elements in $\mathfrak{B}_v$ by a dilation in the independent variable t by the factor of $1/a$. Further, certain frequency domain quantities, like eigenvalues, points in the characteristic variety, etc., will differ by a factor of a. In this paper, we do not make an attempt to make the notion of restriction independent of the parametrization. For us, once a $v \in \mathbb{R}^n$ is given, we fix the parametrization for L_v given by Eq. (5).

When a trajectory w is restricted to a line L_v, its derivative with respect to the parameter t follows the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} w(vt) = ((v_1\partial_1 + v_2\partial_2 + \cdots + v_n\partial_n)w)(vt), \tag{6}$$

where v_i is the ith entry in the vector v defining the line L_v. We shall write $\langle v, \partial\rangle$ for the linear polynomial $\sum_{i=1}^{n} v_i \partial_i$. A straightforward extension of Eq. (6) shows that for $f(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$

$$f\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(vt) = (f(\langle v, \partial\rangle)w)(vt). \tag{7}$$

This observation brings out the fact that the action of the $\mathbb{R}$-algebra $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$ on $w(vt)$ is the same as that of the sub-algebra $\mathbb{R}[\langle v, \partial\rangle]$ of $\mathbb{R}[\partial]$ on w followed by restriction to L_v. Our main result of this section, Theorem 6, is a consequence of this observation. Like the sub-algebra $\mathbb{R}[\langle v, \partial\rangle]$ we consider the free module $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ over $\mathbb{R}[\langle v, \partial\rangle]$ to be sitting inside $\mathbb{R}[\partial]^{\mathtt{w}}$ as a subset. Equation (7) can be extended to cater for the action of $r(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}$ on $w(vt) \in \mathfrak{B}|_v$:

$$r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(vt) = (r(\langle v, \partial\rangle)w)(vt). \tag{8}$$

Given a behavior $\mathfrak{B}$ and its corresponding equation module $\mathcal{R}$, we look into the following $\mathbb{R}[\langle v, \partial\rangle]$-submodule of $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ obtained by intersecting $\mathcal{R}$ with $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$, we call this the *$v$-intersection submodule of $\mathcal{R}$* and denote it by $\mathcal{R}_v$:

$$\mathcal{R}_v := \mathcal{R} \cap \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}. \tag{9}$$

Related to the intersection submodule $\mathcal{R}_v$ is the following 1-d behavior, which plays a central role in the paper:

$$\mathfrak{B}_v := \{\widetilde{w} \in \mathfrak{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{\mathtt{w}}\right)) \mid r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\widetilde{w} = 0 \text{ for all } r(\langle v, \partial\rangle) \in \mathcal{R}_v\}.$$

We also define the following quotient module obtained by factoring $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ by its submodule $\mathcal{R}_v$:

$$\mathcal{M}_v := \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}/\mathcal{R}_v.$$

This is naturally a finitely generated module over the ring $\mathbb{R}[\langle v, \partial\rangle]$. Thus it makes sense to define the annihilator ideal of $\mathcal{M}_v$ as

$$\mathrm{ann}(\mathcal{M}_v) := \{f \in \mathbb{R}[\langle v, \partial\rangle] \mid fm = 0 \text{ for all } m \in \mathcal{M}_v\}.$$

There is another ideal of $\mathbb{R}[\langle v, \partial\rangle]$ related with $\mathcal{R}$, namely

$$(\mathrm{ann}(\mathcal{M}))_v := \mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle],$$

the v-intersection ideal of $\mathrm{ann}(\mathcal{M})$.

We sum up the definitions of all these algebraic objects and behaviors in the following definition. We list them in a tabular form for easy referencing.

Definition 4 Let $\mathfrak{B} \in \mathfrak{L}^{w}$ and $0 \neq v \in \mathbb{R}^n$ be given.

Notation	Name	Definition
$\mathfrak{B}\vert_v$	$\mathfrak{B}$ restricted to v	$\{w(vt) \mid w \in \mathfrak{B}\}$
$\mathcal{R}_v$	v-intersection submodule	$\mathcal{R} \cap \mathbb{R}[\langle v, \partial\rangle]^{w}$
$\mathfrak{B}_v$	1-d behavior associated to $\mathcal{R}_v$	$\{\widetilde{w} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w})) \mid r(\frac{\mathrm{d}}{\mathrm{d}t})\widetilde{w} = 0$ for all $r(\langle v, \partial\rangle) \in \mathcal{R}_v\}$
$\mathcal{M}_v$	quotient module of $\mathcal{R}_v$	$\mathbb{R}[\langle v, \partial\rangle]^{w}/\mathcal{R}_v$
$\mathrm{ann}(\mathcal{M}_v)$	annihilator of $\mathcal{M}_v$	$\{f \in \mathbb{R}[\langle v, \partial\rangle] \mid fm = 0$ for all $m \in \mathcal{M}_v\}$
$(\mathrm{ann}(\mathcal{M}))_v$	v-intersection ideal of $\mathrm{ann}(\mathcal{M})$	$\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$

Example 5 below demonstrates the explicit computation of the intersection submodule.

Example 5 Consider the following kernel representation matrix

$$R(\partial) = \begin{bmatrix} \partial_1^2 - 2\partial_1\partial_2 & -\partial_2 \\ \partial_2 & 1 \\ 1 & \partial_2 \end{bmatrix},$$

and the corresponding equation submodule $\mathcal{R} = \text{rowspan}(R(\partial)) \subseteq \mathbb{R}[\partial_1, \partial_2]^2$. Suppose $v = (1, 1)$ is given, so that $\langle v, \partial\rangle = \partial_1 + \partial_2$. Then $\mathcal{R}_v = \mathcal{R} \cap \mathbb{R}[\partial_1 + \partial_2]^2$ can be shown to be given by the row-span of the following matrix over $\mathbb{R}[\partial_1 + \partial_2]$:

$$R_v(\langle v, \partial\rangle) = \begin{bmatrix} (\partial_1 + \partial_2)^2 + 4 & 4(\partial_1 + \partial_2) \\ 4(\partial_1 + \partial_2) & (\partial_1 + \partial_2)^2 + 4 \end{bmatrix}.$$

The above matrix can be obtained in the following manner:
Consider the larger ring $\mathbb{R}[\partial_1, \partial_2, \partial_3]$ by introducing an auxiliary variable ∂_3, and impose the relation induced by v on the variables $\partial_1, \partial_2, \partial_3$ as $\partial_3 = \langle v, \partial\rangle = \partial_1 + \partial_2$. Now, adjoin to the rows of the kernel representation matrix $R(\partial)$ the following new generators induced by this relation: $\begin{bmatrix}\partial_3 - \partial_1 - \partial_2 & 0\end{bmatrix}$and $\begin{bmatrix}0 & \partial_3 - \partial_1 - \partial_2\end{bmatrix}$. Then, consider the row-span of this matrix over the larger ring $\mathbb{R}[\partial_1, \partial_2, \partial_3]$ to obtain the submodule $\widetilde{\mathcal{R}}$ of $\mathbb{R}[\partial_1, \partial_2, \partial_3]^2$ as

$$\widetilde{\mathcal{R}} = \text{rowspan} \left[\begin{array}{cc} \partial_1^2 - 2\partial_1\partial_2 & -\partial_2 \\ \partial_2 & 1 \\ 1 & \partial_2 \\ \hline \partial_3 - \partial_1 - \partial_2 & 0 \\ 0 & \partial_3 - \partial_1 - \partial_2 \end{array}\right] \subseteq \mathbb{R}[\partial_1, \partial_2, \partial_3]^2$$

Now elimination (see Cox et al. 1998) of the variables ∂_1 and ∂_2 yields

$$\widetilde{\mathcal{R}} \cap \mathbb{R}[\partial_3]^2 = \begin{bmatrix} \partial_3^2 + 4 & 4\partial_3 \\ 4\partial_3 & \partial_3^2 + 4 \end{bmatrix}.$$

Substituting back $\partial_3 = \partial_1 + \partial_2$ gives the desired $R_v(\langle v, \partial\rangle)$.

Correspondingly, the 1-d behavior associated to $\mathcal{R}_v$ is given by the kernel representation

$$\mathfrak{B}_v = \ker \begin{bmatrix} \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 4 & 4\frac{\mathrm{d}}{\mathrm{d}t} \\ 4\frac{\mathrm{d}}{\mathrm{d}t} & \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 4 \end{bmatrix}.$$

Our first main result (Theorem 6 below) brings out two crucial relations amongst the various objects in Definition 4.

Theorem 6 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ and its corresponding equation submodule be $\mathcal{R}$. Let $0 \neq v \in \mathbb{R}^n$ be given. Further, let $\mathfrak{B}|_v$, $\mathfrak{B}_v$, $\mathcal{M}_v$, $\text{ann}(\mathcal{M}_v)$ and $(\text{ann}(\mathcal{M}))_v$ be as defined in Definition 4. Then the following hold:*

1. $\mathfrak{B}|_v \subseteq \mathfrak{B}_v$.
2. *If $\mathfrak{B}$ is autonomous then* $\text{ann}(\mathcal{M}_v) = (\text{ann}(\mathcal{M}))_v$.

Proof 1. Suppose $r(\langle v, \partial\rangle) \in \mathcal{R}_v$ and $w(vt) \in \mathfrak{B}|_v$ for some $w \in \mathfrak{B}$. By Eq. (8), we have

$$r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(vt) = (r(\langle v, \partial\rangle)w)(vt).$$

But, since $r(\langle v, \partial\rangle) \in \mathcal{R}_v$, $r(\langle v, \partial\rangle)$ is in $\mathcal{R}$ too. Therefore $r(\langle v, \partial\rangle)w$ is the zero trajectory. In particular, $(r(\langle v, \partial\rangle)w)(vt) = 0$ for all t. This means that $r(\frac{\mathrm{d}}{\mathrm{d}t})w(vt) = 0$ for all $r(\langle v, \partial\rangle) \in \mathcal{R}_v$. Hence, it follows from the definition of $\mathfrak{B}_v$ (Definition 4) that $w(vt) \in \mathfrak{B}_v$. Thus $\mathfrak{B}|_v \subseteq \mathfrak{B}_v$.

2. For the second part we have to show that the v-intersection of the annihilator ideal, that is, $(\mathrm{ann}(\mathcal{M}))_v$, is equal to the annihilator ideal of the quotient module $\mathcal{M}_v$. We first show that $\mathrm{ann}(\mathcal{M}_v) \supseteq (\mathrm{ann}(\mathcal{M}))_v$. Let $f \in (\mathrm{ann}(\mathcal{M}))_v$. So $f \in \mathrm{ann}(\mathcal{M})$, which means that for any $r \in \mathbb{R}[\partial]^{\mathtt{w}}$, $fr \in \mathcal{R}$. In other words, the row span over $\mathbb{R}[\partial]$ of the $(\mathtt{w} \times \mathtt{w})$ matrix $fI_{\mathtt{w}}$ is contained in $\mathcal{R}$. But, since f also belongs to $\mathbb{R}[\langle v, \partial\rangle]$ (which is a subalgebra of $\mathbb{R}[\partial]$) the row span of $fI_{\mathtt{w}}$ over $\mathbb{R}[\langle v, \partial\rangle]$ is contained in $\mathcal{R} \cap \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}} = \mathcal{R}_v$, which means $f \in \mathrm{ann}(\mathcal{M}_v)$.

Conversely, suppose $f \in \mathrm{ann}(\mathcal{M}_v)$. Then, once again following the same logic, the row span of $fI_{\mathtt{w}}$ over $\mathbb{R}[\langle v, \partial\rangle]$ is contained in $\mathcal{R}_v$. We want to show that the row span of this matrix $fI_{\mathtt{w}}$ over $\mathbb{R}[\partial]$ is contained in $\mathcal{R}$. Since the row span over $\mathbb{R}[\langle v, \partial\rangle]$ of $fI_{\mathtt{w}}$ is contained in $\mathcal{R}_v$, it follows that, in particular, each of the rows of $fI_{\mathtt{w}}$ is in $\mathcal{R}_v$, and hence, is also in $\mathcal{R}$ because $\mathcal{R}_v \subseteq \mathcal{R}$. Therefore, if we let $R \in \mathbb{R}[\partial]^{\mathtt{g} \times \mathtt{w}}$ be a matrix whose rows span $\mathcal{R}$, then there exists another matrix $E \in \mathbb{R}[\partial]^{\mathtt{w} \times \mathtt{g}}$ such that

$$fI_{\mathtt{w}} = ER.$$

It follows that the row span of $fI_{\mathtt{w}}$ over $\mathbb{R}[\partial]$ is contained in $\mathcal{R}$. In other words, $f \in \mathrm{ann}(\mathcal{M})$. Also, by assumption, $f \in \mathbb{R}[\langle v, \partial\rangle]$. Thus $f \in (\mathrm{ann}(\mathcal{M}))_v$. □

Remark 7 Unlike the situation in the discrete case as shown in Avelli and Rocha (2010), here it is not a priori clear whether $\mathfrak{B}_v$, the 1-d behavior associated to $\mathcal{R}_v$, is the smallest 1-d behavior containing the restriction $\mathfrak{B}|_v$. Consider the equation module of $\mathfrak{B}|_v$, that is, $\mathcal{R}(\mathfrak{B}|_v) \subseteq \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}$. $\mathfrak{B}_v$ would indeed be the smallest behavior containing $\mathfrak{B}|_v$ if for all $r(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathcal{R}(\mathfrak{B}|_v)$, $r(\langle v, \partial\rangle) \in \mathcal{R}_v$. However, this may not always be the case because of the following subtlety. Suppose that $r(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathcal{R}(\mathfrak{B}|_v)$, then $(r(\langle v, \partial\rangle)w)(vt) = 0$. This means that the trajectory $r(\langle v, \partial\rangle)w$ is zero when restricted to L_v. But, this does not imply that $r(\langle v, \partial\rangle)w$ is the zero trajectory, and thus we cannot infer that $r(\langle v, \partial\rangle) \in \mathcal{R}_v$. However, we shall see in the next section (Sect. 3) that for certain types of autonomous systems, $\mathfrak{B}_v$ is not only the smallest behavior containing $\mathfrak{B}|_v$, but is in fact equal to it.

3 Restrictions of strongly autonomous systems

One of the major distinctions between 1-d and n-d systems comes from the geometry of the characteristic varieties. For 1-d autonomous systems the characteristic variety is always a discrete set of finitely many complex numbers, whereas for an n-d autonomous system the characteristic variety can be of nonzero dimension. In fact, it is this nonzero dimension of the variety that is responsible for making the solution set of a general n-d autonomous system infinite dimensional over $\mathbb{R}$. However, there is one special case when the affine variety $\mathbb{V}(\mathfrak{B})$ is a finite set of discrete points in the affine space $\mathbb{C}^n$; in this case $\mathfrak{B}$ is said to be *strongly autonomous* (see Pillai and Shankar 1998). This is drastically different from the other possible cases. Here, like in 1-d, the solution set turns out to be a finite dimensional vector space over $\mathbb{R}$. Mimicking the 1-d situation, in this case, one can obtain a first order (state) representation, although there is one inevitable distinction: here there will be n state matrices accounting for the n first order partial derivatives. This observation is not new, for the case when $n = 2$, this has been shown in Fornasini et al. (1993), while in Rocha and Willems (2006) it has been shown for general n. In this section, we first provide a proof, similar in technique to the 2-d discrete case of Fornasini et al. (1993), of this result for general n. And then we use this result to bring out a relation between a state representation of the 1-d behavior $\mathfrak{B}_v$ and the first order representation of the original n-d behavior $\mathfrak{B}$. We make crucial use of the following result from commutative algebra (see Cox et al. 1998) which asserts that $\mathfrak{B}$ is strongly autonomous if and only if the quotient module is a finite dimensional $\mathbb{R}$-vector space.

Proposition 8 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with the corresponding quotient module $\mathcal{M}$. Then the following are equivalent:*

1. *$\mathfrak{B}$ is strongly autonomous.*
2. *$\mathbb{V}(\mathfrak{B})$ is a finite set.*
3. *$\mathcal{I}$ and $\mathrm{ann}(\mathcal{M})$ are zero dimensional ideals.*
4. *$\mathcal{M}$ can be viewed as a finite dimensional vector space over $\mathbb{R}$.*

Now, for $m(\partial) \in \mathbb{R}[\partial]^{\mathtt{w}}$ let $\overline{m(\partial)}$ denote its image under the map $\mathbb{R}[\partial]^{\mathtt{w}} \twoheadrightarrow \mathcal{M}$. Now, for each of the partial derivatives ∂_j, the following map, multiplication by ∂_j in $\mathcal{M}$: $\overline{m(\partial)} \mapsto \overline{\partial_j m(\partial)}$, is an $\mathbb{R}[\partial]$-module morphism of $\mathcal{M}$ onto itself. In particular, this map is $\mathbb{R}$-linear. Moreover, since $\mathcal{M}$ is a finite dimensional $\mathbb{R}$-vector space (Proposition 8), this map is in fact a linear map between finite dimensional $\mathbb{R}$-vector spaces. So, by fixing a basis of $\mathcal{M}$, this linear map can be written as a real square matrix. The matrices, say $\{A_1, A_2, \ldots, A_n\}$, which are representations of multiplications by $\{\partial_1, \partial_2, \ldots, \partial_n\}$, respectively, are called *companion matrices* (see Cox et al. 1998). More precisely, let $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}\}$ be an $\mathbb{R}$-basis of $\mathcal{M}$, where $\gamma = \dim_{\mathbb{R}} \mathcal{M}$. Then $\mathcal{M}$ can be identified with $\mathbb{R}^{\gamma}_{\mathrm{row}}$ by the following mapping:

$$\overline{e_i(\partial)} \mapsto \begin{bmatrix} 0 \; 0 \cdots \underset{\substack{\uparrow \\ i\text{th position}}}{1} \cdots 0 \end{bmatrix}. \tag{10}$$

Now suppose multiplication by ∂_j in $\mathcal{M}$ satisfies the equation $\partial_j \overline{e_i(\partial)} = \sum_{k=1}^{\gamma} a_{ik,j} \overline{e_k(\partial)}$. The jth companion matrix A_j then gets defined as

$$A_j := \begin{bmatrix} a_{11,j} & a_{12,j} & \cdots & a_{1\gamma,j} \\ a_{21,j} & a_{22,j} & \cdots & a_{2\gamma,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\gamma 1,j} & a_{\gamma 2,j} & \cdots & a_{\gamma\gamma,j} \end{bmatrix}. \tag{11}$$

Then it follows from Eq. (11) that for any $\overline{m(\partial)} \in \mathcal{M}$ represented in the chosen basis as $\overline{m(\partial)} = \sum_{i=1}^{\gamma} m_i \overline{e_i(\partial)}$ with $m_1, m_2, \ldots, m_\gamma \in \mathbb{R}$ and for $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$ we have

$$\overline{\partial^\nu m(\partial)} = \overline{\partial^\nu} \begin{bmatrix} m_1 & m_2 & \cdots & m_\gamma \end{bmatrix} \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix} = \begin{bmatrix} m_1 & m_2 & \cdots & m_\gamma \end{bmatrix} \prod_{i=1}^{n} A_i^{\nu_i} \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix}. \tag{12}$$

Further, let $\{s_1, s_2, \ldots, s_{\mathtt{w}}\} \subseteq \mathbb{R}^{\mathtt{w}}_{\mathrm{row}}$ denote the standard basis row-vectors that generate the free module $\mathbb{R}[\partial]^{\mathtt{w}}$. Suppose for each $1 \leqslant j \leqslant \mathtt{w}$, $\overline{s_j} = \sum_{i=1}^{\gamma} c_{ji} \overline{e_i(\partial)}$. Denote by C the matrix

$$C := \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1\gamma} \\ c_{21} & c_{22} & \cdots & c_{2\gamma} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\mathrm{w}1} & c_{\mathrm{w}2} & \cdots & c_{\mathrm{w}\gamma} \end{bmatrix} \in \mathbb{R}^{\mathrm{w}\times\gamma}. \tag{13}$$

Then with this definition of C, it follows that for any $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{\mathrm{w}} \end{bmatrix} \in \mathbb{R}^{\mathrm{w}}_{\mathrm{row}}$, its image in $\mathcal{M}$ satisfies

$$\overline{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{\mathrm{w}} \end{bmatrix} C \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix}, \tag{14}$$

In particular, $\overline{I_{\mathrm{w}}} = C\mathrm{col}(\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)})$. Equation (14) above plays a crucial role in obtaining Lemma 9 below. This lemma will be used in proving Theorem 10, which provides a first order representation of a strongly autonomous behavior. The lemma tells us that a given $r(\partial) \in \mathbb{R}[\partial]^{\mathrm{w}}$ is in the submodule $\mathcal{R}$ if and only if a certain linear equation involving the companion matrices is satisfied. Recall from Eq. (3) that every element $r(\partial) \in \mathbb{R}[\partial]^{\mathrm{w}}$ can be written as a finite linear combination of monomials ∂^ν with coefficients from $\mathbb{R}^{\mathrm{w}}_{\mathrm{row}}$ as $r(\partial) = \sum_{\nu\in\mathbb{N}^n} \alpha_\nu \partial^\nu$. Now note that for a single term of the form $\alpha_\nu \partial_1^{\nu_1} \partial_1^{\nu_2} \cdots \partial_n^{\nu_n}$, it follows from Eqs. (12) and (14) that the image of this term in $\mathcal{M}$ satisfies

$$\overline{\alpha_\nu \partial^\nu} = \alpha_\nu C \prod_{i=1}^{n} A_i^{\nu_i} \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix}. \tag{15}$$

Applying Eq. (15) to each term in the finite sum $r(\partial) = \sum_{\nu\in\mathbb{N}^n} \alpha_\nu \partial^\nu$ we get the result of Lemma 9 below.

Lemma 9 *Let $\mathcal{R} \subseteq \mathbb{R}[\partial]^{\mathrm{w}}$ be a submodule and $\mathcal{M} = \mathbb{R}[\partial]^{\mathrm{w}}/\mathcal{R}$ be the corresponding quotient module, such that $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$. Let $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}\}$ be a basis for $\mathcal{M}$ with $\gamma = \dim_{\mathbb{R}}\mathcal{M}$. Further, let $\{s_1, s_2, \ldots, s_{\mathrm{w}}\} \subseteq \mathbb{R}^{\mathrm{w}}_{\mathrm{row}}$ denote the standard basis row-vectors that generate the free module $\mathbb{R}[\partial]^{\mathrm{w}}$, and correspondingly, let $C \in \mathbb{R}^{\mathrm{w}\times\gamma}$ be the matrix defined by Eq. (13). Let $\{A_1, A_2, \ldots, A_n\}$ be the companion matrices in the chosen basis. Suppose $r(\partial) \in \mathbb{R}[\partial]^{\mathrm{w}}$ is given by*

$$r(\partial) = \sum_{\nu\in\mathbb{N}^n} \partial^\nu \alpha_\nu, \quad \alpha_\nu \in \mathbb{R}^{\mathrm{w}}_{\mathrm{row}}$$

with all but finitely many $\alpha_\nu = 0$. Then $r(\partial) \in \mathcal{R}$ if and only if

$$\sum_{\nu\in\mathbb{N}^n} \alpha_\nu C \prod_{i=1}^{n} A_i^{\nu_i} = 0 \in \mathbb{R}^{\gamma}_{\mathrm{row}}, \tag{16}$$

where $\nu = (\nu_1, \nu_2 \ldots, \nu_n)$.

We now state and prove Theorem 10 which shows how the companion matrices can be used to give an equivalent first order representation of a given strongly autonomous behavior.

Theorem 10 *Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is a strongly autonomous behavior with the corresponding equation module $\mathcal{R}$ and quotient module $\mathcal{M} = \mathbb{R}[\partial]^{\mathtt{w}}/\mathcal{R}$. Let $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}\}$ be an $\mathbb{R}$-basis for $\mathcal{M}$ with $\gamma = \dim_{\mathbb{R}} \mathcal{M}$. Let $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}^{\gamma\times\gamma}$ be the companion matrices as defined in Eq. (11), and let $C \in \mathbb{R}^{\mathtt{w}\times\gamma}$ be as defined in Eq. (13). Then $\mathfrak{B}$ admits the following first order representation:*

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}\right) \mid \exists z \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}^\gamma\right) \text{ such that } \begin{bmatrix} \partial_1 I - A_1 \\ \partial_2 I - A_2 \\ \vdots \\ \partial_n I - A_n \end{bmatrix} z = 0,\ \ w = Cz\}. \tag{17}$$

Proof We first show that for all $w \in \mathfrak{B}$ one can find a suitable $z \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\gamma)$ which satisfies $(\partial_j I - A_j)z = 0$ for all $1 \leqslant j \leqslant n$ and $w = Cz$. For this purpose, let us define $z := \text{col}[\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}]w$. As we have noted before, since $w \in \mathfrak{B}$, the action of the images $\overline{e_i(\partial)}$ on w are well-defined. Now for $1 \leqslant i, j \leqslant n$, we get from the defining equations of companion matrices (Eq. (11))

$$\partial_j z_i = \partial_j \overline{e_i(\partial)} w = \overline{\partial_j e_i(\partial)} w = \sum_{k=1}^{\gamma} a_{ik,j} \overline{e_k(\partial)} w = A_j(i,:) \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix} w = A_j(i,:)z,$$

where $A_j(i,:)$ denotes the ith row of A_j. In matrix form: $\partial_j z = A_j z$. Moreover, it follows from Eq. (14) that $w = (C\text{col}[\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}])w = Cz$.

For the converse we have to show that if there is $z \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\gamma)$ satisfying $(\partial_j I - A_j)z = 0$, then $w := Cz \in \mathfrak{B}$. Since $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ is a large injective cogenerator, in order to show $Cz \in \mathfrak{B}$ it suffices to show that for all $r(\partial) \in \mathcal{R}$ we have $r(\partial)Cz = 0$. As in Eq. (3), let $r(\partial) \in \mathcal{R}$ be given by the finite sum $r(\partial) = \sum_{\nu\in\mathbb{N}^n} \partial^\nu \alpha_\nu$, $\alpha_\nu \in \mathbb{R}^{\mathtt{w}}_{\text{row}}$. Then the action of $r(\partial)$ on Cz can be written as $r(\partial)Cz = \sum_{\nu\in\mathbb{N}^n} \partial^\nu \alpha_\nu Cz = \sum_{\nu\in\mathbb{N}^n} \alpha_\nu C(\partial^\nu z)$. But z satisfies $\partial_j z = A_j z$ for all $1 \leqslant j \leqslant n$. Hence for any $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$ we have $\partial^\nu z = \prod_{i=1}^n A_i^{\nu_i} z$. Thus the expression for $r(\partial)Cz$ becomes

$$r(\partial)Cz = \sum_{\nu\in\mathbb{N}^n} \alpha_\nu C \prod_{i=1}^n A_i^{\nu_i} z. \tag{18}$$

It now follows from Lemma 9 that the right-hand side of Eq. (18) above is zero because $r(\partial) \in \mathcal{R}$. Thus $r(\partial)Cz = 0$ and the proof is complete. □

The auxiliary variables z defined by Eq. (17) in the last theorem are like *state variables* used predominantly in 1-d systems theory. Our construction of z in the proof shows that z is observable from w (see Polderman and Willems 1998; Pommaret and Quadrat 1999; Pillai and Shankar 1998 for details about observability). In fact, the matrix $\text{col}[e_1(\partial), e_2(\partial), \ldots, e_\gamma(\partial)] \in \mathbb{R}[\partial]^{\gamma\times\mathtt{w}}$ is like a *state map*. We illustrate the result in the last theorem by the following example.

Example 11 Consider the following system of PDEs in 2-d

$$\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y} + w = 0, \quad \frac{\partial^2 w}{\partial y^2} + w = 0.$$

Denoting $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by ∂_1 and ∂_2, respectively, we get the equation ideal to be $\mathcal{I} =< \partial_1^2 - 2\partial_1\partial_2 + 1, \partial_2^2 + 1 >$. It can be shown that $\mathcal{I}$ is a zero dimensional ideal. Further, with lexicographic term ordering $\partial_1 \prec \partial_2$, the initial ideal of $\mathcal{I}$ turns out to be $< \partial_1^2, \partial_2^2 >$, and accordingly, the two generators turn out to be the the reduced Gröbner basis of $\mathcal{I}$. It follows that the monomials *not* in the initial ideal of $\mathcal{I}$ are precisely $\{1, \partial_1, \partial_2, \partial_1\partial_2\}$. Thus, the respective equivalence classes of these monomials generate the quotient ring $\mathcal{M} := \mathbb{R}[\partial_1, \partial_2]/\mathcal{I}$ as a vector space over $\mathbb{R}$. We identify the elements of this basis with the standard basis row-vectors of $\mathbb{R}^4_{\text{row}}$. Then it follows from division algorithm by Gröbner basis that multiplications by ∂_1 and ∂_2 in $\mathcal{M}$ are represented by right multiplications by the following two matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

respectively. We can record this equivalently in the notation used in the proof of Theorem 10 as

$$\partial_1 \begin{bmatrix} \overline{1} \\ \overline{\partial_1} \\ \overline{\partial_2} \\ \overline{\partial_1\partial_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{1} \\ \overline{\partial_1} \\ \overline{\partial_2} \\ \overline{\partial_1\partial_2} \end{bmatrix},$$

$$\partial_2 \begin{bmatrix} \overline{1} \\ \overline{\partial_1} \\ \overline{\partial_2} \\ \overline{\partial_1\partial_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{1} \\ \overline{\partial_1} \\ \overline{\partial_2} \\ \overline{\partial_1\partial_2} \end{bmatrix}.$$

We define 4 state variables $z = (z_1, z_2, z_3, z_4) := (w, \partial_1 w, \partial_2 w, \partial_1\partial_2 w)$. It then follows that

$$\partial_1 z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = A_1 z.$$

Similarly, $\partial_2 z = A_2 z$.

In Sect. 2 we defined the 1-d behavior $\mathfrak{B}_v$ related to the v-intersection submodule $\mathcal{R}_v$. Now, once we have obtained a first order representation of a strongly autonomous system $\mathfrak{B}$ *a là* Theorem 10, a natural question that arises is: how is this representation related with a possible first order representation of $\mathfrak{B}_v$? In Theorems 15 and 16 we show that a first order representation of $\mathfrak{B}_v$ can be obtained as a linear combination of these companion matrices. Utilizing this result, we show that for the strongly autonomous case, $\mathfrak{B}_v$ is always equal to the restriction $\mathfrak{B}|_v$.

As a first step towards relating the first order representation of $\mathfrak{B}$ with that of $\mathfrak{B}_v$, in the following lemma, we bring out a connection between the companion matrices and the v-intersection ideal of $\operatorname{ann}(\mathcal{M})$ (*i.e.*, $(\operatorname{ann}(\mathcal{M}))_v = \operatorname{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$, see Definition 4). First, note that the polynomial $\langle v, \partial\rangle$ is transcendental over $\mathbb{R}$. Therefore, the $\mathbb{R}$-algebra $\mathbb{R}[\langle v, \partial\rangle]$ is in fact isomorphic to the polynomial ring in one variable. Hence $\mathbb{R}[\langle v, \partial\rangle]$ is a PID, and therefore, every ideal in $\mathbb{R}[\langle v, \partial\rangle]$ is generated by a single polynomial in $\mathbb{R}[\langle v, \partial\rangle]$.Thus, if the v-intersection ideal $(\operatorname{ann}(\mathcal{M}))_v$ is nonzero, then it has a unique monic generator.

Lemma 12 *Let $\mathfrak{B}$ be a strongly autonomous behavior. Let $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}^{\gamma \times \gamma}$ be as in Theorem 10. Further, let $v = \mathrm{col}[v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$ be nonzero. Then the following hold.*

1. *The v-intersection ideal $(\mathrm{ann}(\mathcal{M}))_v$ is nonzero, and thus, has a unique monic generator $\mu_v(\langle v, \partial\rangle) \in \mathbb{R}[\langle v, \partial\rangle]$.*
2. *The eigenvalues (without counting multiplicities) of the matrix $\sum_{i=1}^{n} v_i A_i$ are given by the roots of $\mu_v(\lambda) \in \mathbb{R}[\lambda]$.*

Proof 1. Recall that the maps given by the actions of $\{\partial_1, \partial_2, \ldots, \partial_n\}$ in $\mathcal{M}$ are represented by companion matrices $\{A_1, A_2, \ldots, A_n\}$, respectively. An extension of this idea shows that action of a polynomial $f(\partial) \in \mathbb{R}[\partial]$ in $\mathcal{M}$ is similarly represented by the matrix polynomial $f(A_1, A_2, \ldots, A_n)$. (The companion matrices commute with each other, and thus it makes sense to talk about the matrix polynomial $f(A_1, A_2, \ldots, A_n)$.) Now, suppose $f(\partial) \in \mathbb{R}[\partial]$ is a nonzero polynomial such that the corresponding matrix polynomial $f(A_1, A_2, \ldots, A_n)$ is the zero matrix. It then follows that the map given by action of $f(\partial)$ in $\mathcal{M}$ is the zero map. In other words, for all $\overline{m(\partial)} \in \mathcal{M}$, we have $\overline{f(\partial)m(\partial)} = 0$ meaning $f(\partial) \in \mathrm{ann}(\mathcal{M})$. We now define $A := \sum_{i=1}^{n} v_i A_i$ and consider the minimal polynomial of A, say $\mu(\lambda) \in \mathbb{R}[\lambda]$. Note that every real square matrix has a nonzero monic minimal polynomial. Therefore, $\mu(\lambda) \neq 0$. Since $\mu(A)$ is the zero matrix, by putting $\sum_{i=1}^{n} v_i A_i$ for A we get $\mu(\sum_{i=1}^{n} v_i A_i)$ to be equal to the zero matrix. It then follows from the above discussion that the polynomial $\mu(\sum_{i=1}^{n} v_i \partial_i) = \mu(\langle v, \partial\rangle) \in \mathrm{ann}(\mathcal{M})$. Therefore, $\mu(\langle v, \partial\rangle) \in \mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle] = (\mathrm{ann}(\mathcal{M}))_v$. Clearly, μ is a nonzero polynomial.Therefore $(\mathrm{ann}(\mathcal{M}))_v$ contains a nonzero polynomial $\mu(\langle v, \partial\rangle)$, and thus, is a nonzero ideal. Since $\mathbb{R}[\langle v, \partial\rangle]$ is PID, the nonzero ideal $(\mathrm{ann}(\mathcal{M}))_v$ is a principal ideal generated by a unique nonzero monic polynomial. We call this polynomial $\mu_v(\langle v, \partial\rangle)$.

2. Part 1 of this proof actually shows that the minimal polynomial of the matrix $A := \sum_{i=1} v_i A_i$, $\mu(\lambda)$, is such that $\mu(\langle v, \partial\rangle) \in (\mathrm{ann}(\mathcal{M}))_v$. This means that if $(\mathrm{ann}(\mathcal{M}))_v$ is generated by a monic polynomial $\mu_v(\langle v, \partial\rangle)$, then $\mu_v(\lambda)$ divides $\mu(\lambda)$.

On the other hand, let $p(\langle v, \partial\rangle)$ be a nonzero polynomial in $(\mathrm{ann}(\mathcal{M}))_v$. This means that the map $\mathcal{M} \ni \overline{m(\partial)} \mapsto \overline{p(\langle v, \partial\rangle)m(\partial)}$, is the zero map. Therefore, the matrix polynomial $p(\sum_{i=1}^{n} v_i A_i) = p(A)$ is the zero matrix. Thus the minimal polynomial $\mu(\lambda)$ divides $p(\lambda)$. In particular, when $p(\langle v, \partial\rangle) = \mu_v(\langle v, \partial\rangle)$, the monic generator of $(\mathrm{ann}(\mathcal{M}))_v$, then $\mu(\lambda)$ divides $\mu_v(\lambda)$.

Thus we arrive at $\mu(\lambda) = \mu_v(\lambda)$. But the eigenvalues of A (without counting multiplicities) are given by the roots of $\mu(\lambda)$. It then follows that the eigenvalues of A are given by the roots of $\mu_v(\lambda)$. □

An immediate corollary to the above lemma follows from looking at the characteristic variety. In classical algebraic geometry, the space $\mathbb{C}^n$ is given a topology called the *Zariski topology*. Here open sets are by definition complements of zero sets of polynomial equations (see Cox et al. 2007; Hartshorne 2009). With this topology, intersection ideals can be given a nice geometric interpretation. We state this result as a proposition below. For this result, and later, we require the following set of complex numbers associated to an affine variety $\mathbb{V}(\mathcal{I})$. Given $0 \neq v \in \mathbb{R}^n$, define

$$\Pi_v(\mathbb{V}(\mathcal{I})) := \{v^{\mathrm{T}}\xi \mid \xi \in \mathbb{V}(\mathcal{I})\} \subseteq \mathbb{C}, \tag{19}$$

which we call the *projection of* $\mathbb{V}(\mathcal{I})$ on the complex 1-d subspace $L_v^{\mathbb{C}} := \{\xi \in \mathbb{C}^n \mid \xi = v\tau, \tau \in \mathbb{C}\}$. Note that, projection is usually defined independent of the parametrization of the line L_v. However, in our case, the definition of projection is dependent on the spanning vector v. See Remark 3.

Proposition 13 *Let $\mathcal{I} \subseteq \mathbb{R}[\partial]$ be an ideal and $\mathbb{V}(\mathcal{I}) \subseteq \mathbb{C}^n$ its variety. Define the following set of complex numbers called the variety of the v-intersection ideal $\mathcal{I}_v := \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$:*

$$\mathbb{V}(\mathcal{I}_v) := \{\tau \in \mathbb{C} \mid f(\tau) = 0 \text{ for all } f(\langle v, \partial\rangle) \in \mathcal{I}_v\}.$$

Also, let $\Pi_v(\mathbb{V}(\mathcal{I}))$ be the projection of $\mathbb{V}(\mathcal{I})$ on the complex 1-d subspace $L_v^{\mathbb{C}} := \{\xi \in \mathbb{C}^n \mid \xi = v\tau,\ \tau \in \mathbb{C}\}$. Then the variety of $\mathcal{I}_v$ is equal to the Zariski closure of $\Pi_v(\mathbb{V}(\mathcal{I}))$, i.e.,

$$\mathbb{V}(\mathcal{I}_v) = \overline{\Pi_v(\mathbb{V}(\mathcal{I}))}.$$

For the present case, since $\mathbb{V}(\mathfrak{B})$ is zero dimensional, $\Pi_v(\mathbb{V}(\mathfrak{B}))$ is already a Zariski closed set. Therefore, from Proposition 13 we have

$$\mathbb{V}((\text{ann}(\mathcal{M}))_v) = \Pi_v(\mathbb{V}(\mathfrak{B})).$$

However, Lemma 12 says $(\text{ann}(\mathcal{M}))_v$ is the ideal generated by $\mu_v(\langle v, \partial\rangle)$ over $\mathbb{R}[\langle v, \partial\rangle]$. This means $\mathbb{V}((\text{ann}(\mathcal{M}))_v)$ is equal to the set of roots of $\mu_v(\lambda)$. Combining the two facts we get that the eigenvalues of the matrix $\sum_{i=1}^n v_i A_i$ are given by the elements of $\Pi_v(\mathbb{V}(\mathfrak{B}))$. We state this observation as a corollary below.

Corollary 14 *Let $\mathfrak{B}$ be a strongly autonomous behavior with $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}^{\gamma\times\gamma}$ as in Theorem 10. Let $v = \text{col}[v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$ be nonzero. Define $\Pi_v(\mathbb{V}(\mathcal{I}))$ as in Eq. (19). Then the set of eigenvalues (without counting multiplicities) of the matrix $\sum_{i=1}^n v_i A_i$ is equal to $\Pi_v(\mathbb{V}(\mathfrak{B}))$.*

One important question raised in Remark 7 was: when is the restriction of a behavior equal to the behavior obtained from the intersection submodule? For the case when the behavior is strongly autonomous, we shall see, that this happens for every nonzero v. Our first observation is that the quotient module $\mathcal{M}_v = \mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}/\mathcal{R}_v$ can be embedded inside the original quotient module $\mathcal{M}$ as an $\mathbb{R}$-subspace. This follows from the following diagram of set-maps.

$$\begin{array}{ccc} \mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}} & \twoheadrightarrow & \mathcal{M}_v \\ \downarrow & & \downarrow \iota \\ \mathbb{R}[\partial]^{\mathrm{w}} & \twoheadrightarrow & \mathcal{M}. \end{array}$$

We define the map ι via the inclusion $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}} \hookrightarrow \mathbb{R}[\partial]^{\mathrm{w}}$: for an element in $\mathcal{M}_v$ we take a lift in $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$, consider it inside $\mathbb{R}[\partial]^{\mathrm{w}}$ by the inclusion map, and then project it onto $\mathcal{M}$. From the definitions of $\mathcal{R}_v$ and $\mathcal{M}_v$ it follows that ι is well-defined, and injective. Crucially, when $\mathcal{M}$ and $\mathcal{M}_v$ are considered as $\mathbb{R}$-vector spaces then ι becomes an $\mathbb{R}$-linear map of finite dimensional $\mathbb{R}$-vector spaces, and therefore, gives an embedding of $\mathcal{M}_v$ into $\mathcal{M}$ as a subspace. Note that by this embedding, $\mathcal{M}_v$ is identified with the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$ onto $\mathcal{M}$.

Our next result shows that the image of ι is a $(\sum_{i=1}^n v_i A_i)$-invariant subspace. In fact, it is the smallest such subspace containing the image of the matrix I_{w} under the projection $\mathbb{R}[\partial]^{\mathrm{w}} \twoheadrightarrow \mathcal{M}$. This observation constitutes the following theorem. From now on, we are going to omit the use of ι and consider $\mathcal{M}_v$ to be a subspace of $\mathcal{M}$. Once again, we are going to identify $\mathcal{M}$ with $\mathbb{R}^{\gamma}_{\text{row}}$, by identifying the basis vectors of $\mathcal{M}$ with the standard basis vectors of $\mathbb{R}^{\gamma}_{\text{row}}$ as done in Eq. (10) above. Recall that with this identification, the image of I_{w} under the map $\mathbb{R}[\partial]^{\mathrm{w}} \twoheadrightarrow \mathcal{M}$ is given by the C matrix defined in Eq. (13). That is: with $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}\}$ being a basis for $\mathcal{M}$ as a $\mathbb{R}$-vector space and $\{s_1, s_2, \ldots, s_{\mathrm{w}}\}$ denoting the standard basis row-vectors that generate the free module $\mathbb{R}[\partial]^{\mathrm{w}}$ we have

$$\begin{bmatrix} \overline{s_1} \\ \overline{s_2} \\ \vdots \\ \overline{s_{\mathrm{w}}} \end{bmatrix} = C \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix}. \tag{20}$$

Theorem 15 *Let $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, \ldots, \overline{e_\gamma(\partial)}\}$ be a basis for $\mathcal{M}$ as an $\mathbb{R}$-vector space,where $\gamma := \dim_{\mathbb{R}}(\mathcal{M})$. Identify $\mathcal{M}$ with $\mathbb{R}^{\gamma}_{\mathrm{row}}$ as in Eq. (10). Let $C \in \mathbb{R}^{\mathrm{w}\times\gamma}$ be as defined by Eq. (20) above and define $A := \sum_{i=1}^{n} v_i A_i$. Then, with the above mentioned identification of $\mathcal{M}$ with $\mathbb{R}^{\gamma}_{\mathrm{row}}$, we have*

$$\mathcal{V}_v := \mathrm{rowspan}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\gamma-1} \end{bmatrix} = \mathcal{M}_v.$$

Proof First, recall that $\mathcal{M}_v$, considered as a subspace of $\mathcal{M}$ via the inclusion $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}} \hookrightarrow \mathbb{R}[\partial]^{\mathrm{w}}$, is equal to the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$ in $\mathcal{M}$. Now, under the identification of $\mathcal{M}$ with $\mathbb{R}^{\gamma}_{\mathrm{row}}$, the $\mathbb{R}$-linear row-span of $\begin{bmatrix} \overline{s_1} \\ \overline{s_2} \\ \vdots \\ \overline{s_{\mathrm{w}}} \end{bmatrix}$ goes to rowspan(C). But, clearly, each of the vectors $\{s_1, s_2, \ldots, s_{\mathrm{w}}\}$ is contained in $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$. Therefore, $\mathcal{M}_v$, which is the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$ in $\mathcal{M}$, contains rowspan(C). Moreover, $\mathcal{M}_v$ is also right-A-invariant, that is, for all $b = \begin{bmatrix} b_1 & b_2 & \cdots & b_\gamma \end{bmatrix} \in \mathcal{M}_v$ we have $bA \in \mathcal{M}_v$. This is because multiplication from the right by A in $\mathcal{M}$ represents multiplication by $\langle v, \partial\rangle$, but $\mathcal{M}_v$ is the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$ in $\mathcal{M}$ and $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$ is invariant under multiplication by $\langle v, \partial\rangle$. Now, $\mathcal{V}_v$ by definition is the smallest right-A-invariant subspace containing rowspan(C). Since $\mathcal{M}_v$ is right-A-invariant and contains rowspan(C), it follows that $\mathcal{M}_v \supseteq \mathcal{V}_v$.

Conversely, any element in $\mathcal{M}_v$, say $\overline{m(\langle v, \partial\rangle)}$, when lifted to $\mathbb{R}[\langle v, \partial\rangle]^{\mathrm{w}}$ looks like

$$m(\langle v, \partial\rangle) = \begin{bmatrix} f_1(\langle v, \partial\rangle) & f_2(\langle v, \partial\rangle) & \cdots & f_{\mathrm{w}}(\langle v, \partial\rangle) \end{bmatrix}, \tag{21}$$

where $f_1(\langle v, \partial\rangle), f_2(\langle v, \partial\rangle), \ldots, f_{\mathrm{w}}(\langle v, \partial\rangle) \in \mathbb{R}[\langle v, \partial\rangle]$. This polynomial vector can be expanded according to ascending degrees of $\langle v, \partial\rangle$ as

$$\begin{bmatrix} f_1(\langle v, \partial\rangle) & f_2(\langle v, \partial\rangle) & \cdots & f_{\mathrm{w}}(\langle v, \partial\rangle) \end{bmatrix} = \left(\sum_{i=1}^{k} \langle v, \partial\rangle^i \alpha_i\right),$$

for some $k \in \mathbb{N}$ with $\alpha_i \in \mathbb{R}^{\mathrm{w}}_{\mathrm{row}}$. As in the proof of Lemma 9 we project $m(\langle v, \partial\rangle)$ to $\mathcal{M}$ and make use of Eq. (14) to get the following:

$$\overline{m(\langle v, \partial\rangle)} = \sum_{i=1}^{k} \overline{\langle v, \partial\rangle^i}\,\overline{\alpha_i} = \left(\sum_{i=1}^{k} \overline{\langle v, \partial\rangle^i} \alpha_i C\right) \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix}.$$

Then by Eq. (12), for each i multiplication to $\alpha_i C$ by each $\overline{\langle v, \partial\rangle^i}$ can be replaced by multiplication from the right by A^i:

$$\overline{m(\langle v, \partial\rangle)} = \left(\sum_{i=1}^{k} \alpha_i C A^i\right) \begin{bmatrix} \overline{e_1(\partial)} \\ \overline{e_2(\partial)} \\ \vdots \\ \overline{e_\gamma(\partial)} \end{bmatrix}. \tag{22}$$

When the identification of $\mathcal{M}$ with $\mathbb{R}^{\gamma}_{\text{row}}$ is done, the right-hand side of the above equation lies in

$$\text{rowspan}(C) + \text{rowspan}(CA) + \cdots$$

Now, by Cayley–Hamilton theorem

$$\text{rowspan}(C) + \text{rowspan}(CA) + \cdots = \text{rowspan}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\gamma-1} \end{bmatrix}$$

because $\dim_{\mathbb{R}}\mathcal{M} = \gamma$. It follows that

$$\overline{m(\langle v, \partial\rangle)} \in \text{rowspan}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\gamma-1} \end{bmatrix} = \mathcal{V}_v.$$

This proves that $\mathcal{M}_v \subseteq \mathcal{V}_v$. □

We now use Theorem 15 to show that when $\mathfrak{B}$ is strongly autonomous, its restriction to L_v is always equal to the behavior $\mathfrak{B}_v$ associated to the v-intersection submodule $\mathcal{R}_v = \mathcal{R} \cap \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$. The one-to-one correspondence between behaviors and submodules, discussed in Sect. 1, tells us that the quotient module corresponding to $\mathfrak{B}_v$ is nothing but $\mathcal{M}_v$. To see this consider the following commutative diagram:

$$\begin{array}{llll} \langle v, \partial\rangle s_j & \mapsto & \frac{\mathrm{d}}{\mathrm{d}t} s_j & \forall j \in \{1, 2, \ldots, \mathtt{w}\} \\ & & & \\ \varphi : \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}} & \longrightarrow & \mathbb{R}\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]^{\mathtt{w}} & \\ \quad\downarrow & & \quad\downarrow & \\ \widetilde{\varphi} : \mathcal{M}_v & \longrightarrow & \mathbb{R}\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]^{\mathtt{w}} / \varphi(\mathcal{R}_v), & \end{array}$$

where s_j is the standard jth basis vector in $\mathbb{R}[\partial]^{\mathtt{w}}$. Because $\langle v, \partial\rangle$ is transcendental over $\mathbb{R}$, the map φ in the above diagram is an isomorphism. Moreover, from the definition of the the behavior $\mathfrak{B}_v$, the equation module of $\mathfrak{B}_v$ is equal to this $\varphi(\mathcal{R}_v)$. Now, observe that $\widetilde{\varphi}$ defined via φ by taking lifts in $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ is well-defined, and not only that, it is in fact an isomorphism of modules over 1-variable polynomial rings. Thus the quotient module corresponding to $\mathfrak{B}_v$ can be identified with $\mathcal{M}_v$, with the role of $\frac{\mathrm{d}}{\mathrm{d}t}$ played by $\langle v, \partial\rangle$. Since multiplication by $\langle v, \partial\rangle$ in $\mathcal{M}$ is represented by the matrix $A := \sum_{i=1}^{n} v_i A_i$, and $\mathcal{M}_v$ is A-invariant, the restriction of A to this invariant subspace, $A|_{\mathcal{M}_v}$, must be the representation of multiplication by $\langle v, \partial\rangle$. But we just showed that $\mathcal{M}_v$ is isomorphic to $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}/\varphi(\mathcal{R}_v)$, therefore, it follows that multiplication by $\frac{\mathrm{d}}{\mathrm{d}t}$ in $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathrm{w}}/\varphi(\mathcal{R}_v)$ is represented by $A|_{\mathcal{M}_v}$. By following exactly the same line of arguments as in the proof of Theorem 10, it can be concluded that a state representation of $\mathfrak{B}_v$ is given by the matrix $A|_{\mathcal{M}_v}$. Our next result makes use of this observation to infer that $\mathfrak{B}$'s restriction $\mathfrak{B}|_v$ is equal to $\mathfrak{B}_v$.

Theorem 16 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ be strongly autonomous and let $0 \neq v \in \mathbb{R}^n$. Then the restriction of $\mathfrak{B}$ to the line L_v is equal to $\mathfrak{B}_v$, that is,*

$$\mathfrak{B}|_v = \mathfrak{B}_v.$$

Proof Let $C \in \mathbb{R}^{\mathrm{w}\times\gamma}$ be the matrix as in Eq. (20). By Theorem 10, we have a first order representation for $\mathfrak{B}$ given by

$$\begin{bmatrix} \partial_1 I - A_1 \\ \partial_2 I - A_2 \\ \vdots \\ \partial_n I - A_n \end{bmatrix} z = 0, \quad w = Cz. \tag{23}$$

Recall, by Theorem 15, $\mathcal{M}_v$ is the smallest right-A-invariant subspace containing rowspan(C), where $A := \sum_{i=1}^n v_i A_i$. Let $\gamma_1 := \dim_{\mathbb{R}}(\mathcal{M}_v)$. Then, if we take a basis of $\mathcal{M}_v$ and extend it to a basis of $\mathcal{M}$, in this new basis the matrices C and A will look like:

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $C_1 \in \mathbb{R}^{\mathrm{w}\times\gamma_1}$, $A_{1,1} \in \mathbb{R}^{\gamma_1\times\gamma_1}$, $A_{2,1} \in \mathbb{R}^{(\gamma-\gamma_1)\times\gamma_1}$, $A_{2,2} \in \mathbb{R}^{(\gamma-\gamma_1)\times(\gamma-\gamma_1)}$. The structure of C, in this new basis, is as above because rowspan$(C) \subseteq \mathcal{M}_v$, while that of A is due to the fact that $\mathcal{M}_v$ is right-A-invariant. Notice that in this new basis $A|_{\mathcal{M}_v} = A_{1,1}$. Moreover, in the new basis, the images of the standard basis vectors $\{s_1, s_2, \ldots, s_{\mathrm{w}}\}$ of $\mathbb{R}[\partial]^{\mathrm{w}}$ in $\mathcal{M}_v$ is given by rowspan(C_1). It then follows from the discussion preceding the statement of the theorem that $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{z} = A_{1,1}\widetilde{z}$, $\widetilde{w} = C_1\widetilde{z}$ is a state representation for $\mathfrak{B}_v$. So every trajectory in $\mathfrak{B}_v$ can be obtained as

$$\widetilde{w}(t) = C_1\exp(A_{1,1}t)\widetilde{z}(0).$$

On the other hand every solution in $\mathfrak{B}|_v$ looks like

$$w(vt) = C\exp\left(\sum_{i=1}^n v_i A_i t\right) z(\mathbf{0}) = C\exp(At)z(\mathbf{0}),$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^n$. It easily follows from the structures of C and A that

$$w(vt) = C\exp(At)z(\mathbf{0}) = \begin{bmatrix} C_1\exp(A_{1,1}t) & 0 \end{bmatrix} z(\mathbf{0}).$$

Therefore,by choosing $z(\mathbf{0}) = \begin{bmatrix} \widetilde{z}(0) \\ * \end{bmatrix}$, $* \in \mathbb{R}^{\gamma-\gamma_1}$ being arbitrary, we get

$$w(vt) = C_1\exp(A_{1,1}t)\widetilde{z}(0) = \widetilde{w}(t).$$

Hence we conclude that $\mathfrak{B}_v \subseteq \mathfrak{B}|_v$. That $\mathfrak{B}_v \supseteq \mathfrak{B}|_v$ has already been proved in Theorem 6. Thus equality follows. □

4 Restrictions of general autonomous systems

In this section we look into restrictions of general autonomous systems, which are not necessarily strongly autonomous. We have already seen in Sect. 3 that for the strongly autonomous case, restriction to the line L_v always turns out to be an autonomous system given by ODEs, namely, the system associated to the v-intersection submodule. For a general autonomous system, however, the situation is much different. As we shall see shortly, for a general autonomous system, a given direction may turn out to be *free*, in the sense that *every* possible 1-d trajectory can be obtained by restricting trajectories in the autonomous system. Interestingly, it turns out that for a given autonomous system, a direction is *either free or the restriction of the system is given by the solution set of ODEs* (see Theorem 29).

We have seen in the strongly autonomous case that the quotient module turns out to be a finite dimensional vector space. As a consequence a strongly autonomous behavior also happens to be a finite dimensional real vector space. This situation no longer persists for a general autonomous system that is not strongly autonomous—the quotient module ceases to be finite dimensional in this case. Consequently, the space of trajectories also loses finite dimensionality. Thus, unlike the strongly autonomous case, here there can be non-exponential type trajectories too. However, for the purpose of bringing out properties of a system restricted to a 1-d subspace it suffices to consider only exponential type solutions (see Theorem 29 and Corollary 30). In this section we are going to consider our solution space to be real analytic solutions of exponential type. A crucial benefit of this consideration is that it lets us use an algorithm (from Oberst 1990, 2006) based on Gröbner basis method (Algorithm 22 below) to carry out explicit computation of exponential solutions of a given set of PDEs.

We begin with the definition of real entire analytic solutions of exponential type.

Definition 17 We denote by $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ the set of all formal power series in n variables

$$w(\mathbf{x}) = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu,$$

where

1. $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$ is a multi-index,
2. $\mathbf{x}^\nu$ means the monomial $x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$ and
3. $\nu!$ denotes $\nu_1! \nu_2! \cdots \nu_n!$

with the sequence of real numbers $\{w_\nu\}_{\nu \in \mathbb{N}^n}$ being such that w is convergent everywhere, that is, $w(\mathbf{a}) \in \mathbb{R}$ for all $\mathbf{a} \in \mathbb{R}^n$.

Note that, like before, $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ is endowed with an $\mathbb{R}[\partial]$-module structure by defining multiplication by an element from $\mathbb{R}[\partial]$ as differentiation,

Remark 18 In Oberst (2006), Oberst and Pauer (2001), it has been shown that the set of exponential trajectories, $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$, is a large injective cogenerator. This means that the module-behavior correspondence that we have been exploiting so far is applicable to this case too.

With this idea of exponential solutions, we now define the notion of a free direction of an autonomous behavior. Recall the definition (Definition 4) of restriction of a behavior to a line spanned by a nonzero vector.

Definition 19 Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be an autonomous exponential behavior and let $0 \neq v \in \mathbb{R}^n$ define the line L_v. Then v is said to be a *free direction of* $\mathfrak{B}$ if the restriction of the behavior to L_v is the whole of 1-d exponential trajectories, *i.e.*,

$$\mathfrak{B}|_v = \mathfrak{Exp}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}).$$

Example 20 As an example of free directions consider the following scalar system of PDEs:

$$\mathfrak{B} = \ker \begin{bmatrix} \partial_2^2 \\ \partial_3^2 \\ \partial_1\partial_3 - \partial_2 \end{bmatrix}.$$

Clearly, any exponential trajectory of the form $w(x_1, x_2, x_3) = p(x_1)e^{\alpha x_1}$ with $p(x_1) \in \mathbb{R}[x_1]$ is a solution to the above system of equations. Every 1-d exponential function is of the above form. So, indeed, x_1-axis is a free direction.

4.1 Characterization of free directions of a scalar autonomous system

By a scalar autonomous system we mean that there is only one manifest variable, *i.e.*, $\mathtt{w} = 1$. In this case the equation module $\mathcal{R}$ is equal to the characteristic ideal $\mathcal{I}$, which in turn is equal to the annihilator ideal $\mathrm{ann}(\mathcal{M})$. The next result gives necessary and sufficient conditions for a given direction in $\mathbb{R}^n$ to be a free direction of a scalar autonomous system. We shall see here that v is a free direction if and only if the v-intersection ideal $\mathcal{I}_v := \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$ is the zero ideal. Equivalently, by Proposition 13, v is a free direction if and only if the projection of $\mathbb{V}(\mathfrak{B})$ onto the complex 1-d subspace $L_v^{\mathbb{C}} = \{\xi \in \mathbb{C}^n \mid \xi = v\tau, \tau \in \mathbb{C}\}$ is Zariski dense.

Theorem 21 *Let $\mathfrak{B}$ be a scalar autonomous behavior defined by the equation ideal $\mathcal{I} \subseteq \mathbb{R}[\partial]$ and let $0 \neq v \in \mathbb{R}^n$. Then the following conditions are equivalent:*

1. *v is a free direction of $\mathfrak{B}$.*
2. *The intersection ideal $\mathcal{I}_v := \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$ is the zero ideal.*
3. *Let $\Pi_v(\mathbb{V}(\mathfrak{B}))$ define the projection of the characteristic variety $\mathbb{V}(\mathfrak{B})$ on to the complex line $L_v^{\mathbb{C}}$ as in Eq. (19). Then*

$$\overline{\Pi_v(\mathbb{V}(\mathfrak{B}))} = \mathbb{C}.$$

4. *The $\mathbb{R}$-algebra homomorphism φ in the following commutative diagram is an injection.*

$$\begin{array}{ccc} \mathbb{R}[\partial] & \twoheadrightarrow & \mathbb{R}[\partial]/\mathcal{I} \\ \uparrow & \nearrow\varphi & \\ \mathbb{R}[\langle v, \partial\rangle] & & \end{array}.$$

Before we embark on proving Theorem 21, we first have a look at a Gröbner basis method of obtaining exponential type solutions of PDEs; our proof crucially relies on this method. In Oberst (1990, 2006) Oberst elaborated this method extensively and showed how it can be utilized to construct power series solutions to the Cauchy problems in PDEs. Algorithm 22 is a short description of this Gröbner basis method for formal integration of PDEs. We describe the method for the scalar case, *i.e.*, $\mathtt{w} = 1$ (see Oberst 2006 for the general case). In this method, first a Gröbner basis, $\mathcal{G}$ of the equation ideal $\mathcal{I}$ is computed for some fixed term ordering, say $\prec$. We denote by $\mathrm{in}_\prec(\mathcal{I})$ the initial ideal of $\mathcal{I}$ with respect to the term ordering $\prec$. We call the monomials *not* belonging to $\mathrm{in}_\prec(\mathcal{I})$ the *standard monomials*, and denote the set of standard monomials by $\Gamma_\prec(\mathcal{I})$. (Note that there is a bijection between monomials in $\mathbb{R}[\partial]$ and the lattice of non-negative integers $\mathbb{N}^n$. We often consider $\Gamma_\prec(\mathcal{I}) \subseteq \mathbb{N}^n$ without explicitly mentioning it since there is no risk of ambiguity.) The idea behind Algorithm 22 stems from the algebraic fact that each element in $\mathbb{R}[\partial]$, modulo the ideal $\mathcal{I}$, can be written as a unique $\mathbb{R}$-linear combination of the standard monomials (see Cox et al. 1998; Sturmfels 2002). Recall from Definition 17 that every exponential solution can be written as a convergent power series $w = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu$. Now note that in order for w to be a solution to the given set of PDEs it must satisfy the following:

$$\text{for all } \nu \in \mathbb{N}^n, \; w_\nu = \sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} w_{\nu'},$$

where the monomial ∂^ν upon division by the Gröbner basis $\mathcal{G}$ reduces to $\sum_{\nu' \in \Gamma_\prec(\mathcal{I})} \alpha_{\nu'} \partial^{\nu'}$. In the sequel, for notational convenience, we use just Γ to denote $\Gamma_\prec(\mathcal{I})$ when the ideal and the term ordering are clear from the context.

Algorithm 22 [Oberst-Riquier]

Level-1
Input: A set of PDEs $f_1(\partial)w = 0$, $f_2(\partial)w = 0$, …, $f_r(\partial)w = 0$.
Computation:

- Fix a term ordering $\prec$ in $\mathbb{R}[\partial]$.
- Compute a Gröbner basis $\mathcal{G}$ of the ideal $\mathcal{I} := < f_1, f_2, \ldots, f_r >$.
- Construct the set of standard monomials $\Gamma := \{\nu \in \mathbb{N}^n \mid \partial^\nu \notin \text{in}_\prec(\mathcal{I})\}$.

Output: *Standard monomial set* Γ.
Level-2
Input: *Initial data*: $\{w_\nu \in \mathbb{R}\}_{\nu \in \Gamma}$.
Computation:
for $\nu \notin \Gamma$

- Compute by division algorithm by $\mathcal{G}$ to obtain

$$\partial^\nu \equiv \sum_{i=1, \nu_i \in \Gamma}^{k<\infty} \alpha_i \partial^{\nu_i} \text{ modulo } \mathcal{I}.$$

- Set $w_\nu = \sum_{i=1}^{k} \alpha_i w_{\nu_i}$.

end
Output The sequence $w := \{w_\nu\}_{\nu \in \mathbb{N}^n}$.

In Oberst (1990, 2006) Oberst shows that the output of the above algorithm, when written in the power series form as $w = \sum_{\nu \in \mathbb{N}^n} \frac{w_\nu}{\nu!} \mathbf{x}^\nu$, is indeed a solution to the given set of PDEs, $f_i(\partial)w = 0$ for all $i \in \{1, 2, \ldots, r\}$. Conversely, every formal power series solution is obtained from this algorithm by giving different initial conditions $\{w_\nu\}_{\nu \in \Gamma}$, where Γ is the standard monomial set computed in Level-1 of Algorithm 22. However, note that Algorithm 22 says nothing about *convergence* of the solution. Importantly, in Oberst (2006); Oberst and Pauer (2001), it was proved that if the initial data itself is an exponential trajectory then the solution obtained following Algorithm 22 is guaranteed to be an exponential one. We paraphrase this result in the following proposition; this will be crucial for us while proving Theorem 21.

Proposition 23 (Theorems 24 and 26, Oberst and Pauer 2001) *Given a set of PDEs* $f_1(\partial)w = 0$, $f_2(\partial)w = 0, \ldots, f_r(\partial)w = 0$, *and a term ordering* $\prec$ *of* $\mathbb{R}[\partial]$, *let* Γ *be the set of standard* *monomials, that is, monomials that do not belong to* $\mathrm{in}_{\prec}(< f_1, f_2, \ldots, f_r >)$. *Further, let* $w_{\mathrm{in}} := \{w_\nu\}_{\nu\in\Gamma}$ *be an arbitrary sequence of real numbers indexed by* Γ. *With this* w_{in} *as the initial data, let* $\{w_\nu\}_{\nu\in\mathbb{N}^n}$ *be the output of Algorithm 22. Suppose the following formal power series*

$$\hat{w}(\mathbf{x}) := \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$$

obtained from w_{in} *converges for all* $\mathbf{x} \in \mathbb{R}^n$. *Then so does the power series*

$$w(\mathbf{x}) := \sum_{\nu\in\mathbb{N}^n} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$$

obtained from the solution of Algorithm 22. That is, $\hat{w}(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ *implies* $w(\mathbf{x}) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$.

Keeping the above result in mind, we call an initial condition w_{in} (or $\hat{w}(\mathbf{x}) = \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$) *valid* if $\hat{w} \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$.

Remark 24 Let $\widetilde{w} = \sum_{\lambda\in\mathbb{N}} \frac{\widetilde{w}_\lambda}{\lambda!} t^\lambda \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R})$ be any 1-d exponential trajectory. If we define $\Gamma_i := \{\nu \in \mathbb{N}^n \mid \nu = \lambda e_i, \lambda \in \mathbb{N}\}$, e_i being the standard ith basis vector in $\mathbb{R}^n$, and assume that for some term ordering we have $\Gamma_i \subseteq \Gamma$, then notice that the following initial condition is a valid one.

$$\hat{w}(\mathbf{x}) = \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu, \text{ where } w_\nu = \begin{cases} \widetilde{w}_\lambda & \text{if } \nu \in \Gamma_i \text{ and } \nu = \lambda e_i \\ 0 & \text{otherwise} \end{cases}.$$

This is because if we denote by x_i the ith coordinate function, then $\mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \ni \widetilde{w}(t) \mapsto w(\mathbf{x}) := \widetilde{w}(x_i) \in \mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ is an injection. Now, if indeed $\Gamma_i \subseteq \Gamma$, and w_{in} is chosen from a 1-d exponential trajectory $\widetilde{w}$, then Algorithm 22 guarantees that the corresponding solution, say w, when restricted to e_i, gives back $\widetilde{w}$. Since an initial condition can be freely chosen, it follows that for any 1-d exponential trajectory there exists a trajectory in the solution set of the PDEs whose restriction onto e_i is that 1-d trajectory. In other words, e_i is a free direction. We exploit this observation in the proof of Theorem 21.

The next result is a technical lemma required in the proof of Theorem 21. The lemma deals with the effect on the equation submodule and the behavior due to a change of basis in the domain. This is closely related to the differential geometric notion of *push-forward* of a map between two differentiable manifolds to a map between the two tangent spaces. We give a short description of this notion below; details can be found in textbooks, see for example Hörmander (1990).

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. We call the coordinate functions of the domain and the codomain spaces $\mathbf{x}$ and $\mathbf{y}$, respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are related by $\mathbf{y} = T\mathbf{x}$. This induces a map between the tangent spaces, $T^* : \mathcal{T}_{\mathbf{x}}\mathbb{R}^n \to \mathcal{T}_{\mathbf{y}}\mathbb{R}^n$, as follows. Let $\mathbf{y} \mapsto w(\mathbf{y})$ be in $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$. Define for all $1 \leqslant i \leqslant n$

$$\left(T^*\frac{\partial}{\partial x_i}\right)(w(\mathbf{y})) := \frac{\partial}{\partial x_i} w(T\mathbf{x}).$$

T^* is called the *push-forward* of the map T. For the case when T is linear T^* naturally turns out to be linear too. In fact, by making $T^*\frac{\partial}{\partial x_i}$ act on the coordinate functions y_j's, we can get an expression for $T^*\frac{\partial}{\partial x_i}$'s in terms of derivatives in $\mathbf{y}$ coordinates, *i.e.*, $\frac{\partial}{\partial y_j}$'s. Let T be given by the matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}.$$

Then it follows from the definition of T^* that

$$\left(T^*\frac{\partial}{\partial x_i}\right) y_j = \frac{\partial}{\partial x_i}\sum_{k=1}^{n} t_{jk}x_k = t_{ji}.$$

It then follows by varying j that

$$\left(T^*\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{n} t_{ji}\frac{\partial}{\partial y_j}.$$

This can be written in the matrix-vector form as

$$T^*\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} = T^{\mathrm{T}}\begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix}. \tag{24}$$

It now follows from Eq. (24) and the definition of push-forward that for $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(T\mathbf{x}) = T^*\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(\mathbf{y}) = T^{\mathrm{T}}\begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} w(\mathbf{y}). \tag{25}$$

For ease of explanation and to avoid cumbersome notation we use ∂_x and ∂_y to denote the n-tuples of partial derivatives $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ and $\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$, respectively. These partial derivatives correspond to the two coordinate functions $\mathbf{x}$ and $\mathbf{y}$, which are related by a linear coordinate transformation. In the lemma we use $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ as the signal space, but the result holds for $\mathfrak{Exp}(\mathbb{R}^n, \mathbb{R})$ since it too is a large injective cogenerator.

Lemma 25 *Let $T \in \mathbb{R}^{n\times n}$ define an invertible linear change of coordinates of $\mathbb{R}^n$ by $\mathbf{x} \mapsto T\mathbf{x} =: \mathbf{y}$. Then T induces an $\mathbb{R}$-algebra isomorphism $\psi : \mathbb{R}[\partial_x] \longrightarrow \mathbb{R}[\partial_y]$ by the linear change of variables*

$$\partial_x \mapsto T^{\mathrm{T}}\partial_y.$$

Suppose $\mathcal{I} \subseteq \mathbb{R}[\partial_x]$ is an ideal, then $\psi(\mathcal{I})$ is an ideal in $\mathbb{R}[\partial_y]$. Consider the following two behaviors

$$\mathfrak{B}_x := \{w(\mathbf{x}) \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}\right) \mid m(\partial_x)w = 0 \text{ for all } m \in \mathcal{I}\},$$
$$\mathfrak{B}_y := \{w(\mathbf{y}) \in \mathfrak{C}^\infty\left(\mathbb{R}^n, \mathbb{R}\right) \mid m(\partial_y)w = 0 \text{ for all } m \in \psi(\mathcal{I})\}.$$

Let $v_y, v_x \in \mathbb{R}^n$ be related to each other by $v_y = Tv_x$. Then there is a bijective set map between $\mathfrak{B}_x|_{v_x}$ and $\mathfrak{B}_y|_{v_y}$.

Proof That ψ is an isomorphism of n-variable polynomial algebras is clear from the fact that T^{T} is non-singular. It then follows that $\psi(\mathcal{I})$ is an ideal of $\mathbb{R}[\partial_y]$. Now notice that Eq. (25), together with the fact that T is invertible, shows that there is a set bijection between $\mathfrak{B}_x$ and $\mathfrak{B}_y$ given by $\widetilde{\psi} : \mathfrak{B}_y \to \mathfrak{B}_x$ with $\widetilde{\psi}(w(\mathbf{y})) = w(T\mathbf{x})$. This follows from the following argument. First observe that for $w \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ we have from Eq. (25)

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} w(T\mathbf{x}) = T^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix} w(\mathbf{y}).$$

More generally, for $m(\partial_x) \in \mathbb{R}[\partial_x]$

$$m(\partial_x)w(T\mathbf{x}) = \psi(m)(\partial_y)w(\mathbf{y}) = m(T^{\mathrm{T}}\partial_y)w(\mathbf{y}).$$

Hence it follows that $w(\mathbf{y}) \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ is in the kernel of $\psi(m)(\partial_y)$ if and only if $\widetilde{\psi}(w(\mathbf{y})) = w(T\mathbf{x})$ is in the kernel of $m(\partial_x)$. Thus from the one-to-one correspondence between behaviors and modules (here ideals, because the behavior is scalar) we get $w \in \mathfrak{B}_y$ if and only if $\widetilde{\psi}(w) \in \mathfrak{B}_x$.

For the restriction, observe now that for $w(\mathbf{y}) \in \mathfrak{B}_y$, we have $(\widetilde{\psi}(w))(v_x t) = w(Tv_x t) = w(v_y t) \in \mathfrak{B}_y|_{v_y}$. Thus $\widetilde{\psi}$ induces a set bijection between $\mathfrak{B}_x|_{v_x}$ and $\mathfrak{B}_y|_{v_y}$. □

We now prove Theorem 21.

Proof of Theorem 21 **(**$1 \Rightarrow 2$**):** We prove this implication by contradiction. Suppose 2 is not true, *i.e.*, the intersection ideal $\mathcal{I}_v$ is nonzero. Since $\mathbb{R}[\langle v, \partial\rangle]$ is PID, it follows that $\mathcal{I}_v$ is generated by a polynomial $g(\langle v, \partial\rangle)$. Like in Sect. 2, define the 1-d behavior

$$\mathfrak{B}_v := \left\{ \widetilde{w} \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\widetilde{w} = 0 \text{ for all } r(\langle v, \partial\rangle) \in \mathcal{I}_v \right\}.$$

Note that this 1-d behavior is a finite dimensional $\mathbb{R}$-vector space with dimension equal to the degree of the polynomial $g(\langle v, \partial\rangle)$. Therefore $\mathfrak{B}_v \subsetneq \mathfrak{Exp}(\mathbb{R}, \mathbb{R})$. By Theorem 6 it follows that $\mathfrak{B}|_v \subseteq \mathfrak{B}_v \subsetneq \mathfrak{Exp}(\mathbb{R}, \mathbb{R})$, which contradicts the claim of 1.

($2 \Leftrightarrow 3$**):** Follows from Proposition 13

($3 \Leftrightarrow 4$**):** This follows from the fact that $\ker \varphi = \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$.

($4 \Rightarrow 1$**):** In order to prove this implication we first prove a simpler case, and then we shall make use of Lemma 25, that will render the general case into the simpler one.

Case 1 ($v = e_1 = \mathrm{col}[1, 0, \ldots, 0]$**):** The problem here reduces to proving $\varphi : \mathbb{R}[\partial_1] \to \mathbb{R}[\partial]/\mathcal{I}$ being injective implies e_1 is a free direction. We claim that φ being injective implies there exists a term ordering such that the standard monomials set Γ contains $\Gamma_1 := \{\nu \in \mathbb{N}^n \mid \nu = \lambda e_1, \lambda \in \mathbb{N}\}$. Indeed, if we take a term ordering with $\partial_i \succ \partial_1$ for all $2 \leqslant i \leqslant n$, then a Gröbner basis for $\mathcal{I}$, say $\mathcal{G}$, with this term ordering will have no element which has a monomial purely in ∂_1 as the leading monomial. For if $\mathcal{G}$ had a polynomial, say $f \in \mathbb{R}[\partial]$, with leading monomial purely in ∂_1, then since ∂_1 has least priority in the term ordering, the rest of the monomials in f will also be in ∂_1 only. Thus $f \in \mathbb{R}[\partial_1] \cap \mathcal{I} = \ker \varphi$, which contradicts our assumption that φ is injective. Now since $\mathcal{G}$ has no element with leading term purely in ∂_1, the initial ideal in$_{\prec}(\mathcal{I})$, too, does not contain any monomial purely in ∂_1. In other words, the standard monomial set $\Gamma \supseteq \Gamma_1$. By Remark 24, it follows that e_1 is free.

Case 2 (general v**):** For the general case we make use of Lemma 25. First observe that since $v = \mathrm{col}[v_1, v_2, \ldots, v_n]$ is nonzero, one of its entries must be a nonzero real number.

We may assume without loss of generality that $v_1 \neq 0$. For if it is not, then we can do a permutation on the variables $\{x_1, x_2, \ldots, x_n\}$ so that v changes to $\widetilde{v}$ and $\widetilde{v}_1 \neq 0$. Such a permutation exists because v has at least one entry nonzero. (By Lemma 25 it suffices to prove that $\widetilde{v}$ is free in this transformed system.) Now we define the following $(n \times n)$ real matrix and the linear transformation defined by it. Because v_1 has been assumed to be nonzero the following matrix exists:

$$T := \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ v_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \cdots & 1 \end{bmatrix}^{-1}.$$

Note that $T^{-1}e_1 = v$, *i.e.*, $e_1 = Tv$. Also, as in Lemma 25, T induces the following $\mathbb{R}$-algebra isomorphism between $\mathbb{R}[\partial_x]$ and $\mathbb{R}[\partial_y]$ (say $\psi : \mathbb{R}[\partial_x] \to \mathbb{R}[\partial_y]$).

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} \mapsto T^{\mathrm{T}} \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix}.$$

Now, by Lemma 25, it is enough to prove that $e_1 = Tv$ is a free direction in the autonomous system defined by the ideal $\psi(\mathcal{I})$. We claim that $\mathbb{R}[\frac{\partial}{\partial y_1}]$ injects into $\mathbb{R}[\partial_y]/\psi(\mathcal{I})$. Note that a feature of the T matrix is $\psi(\langle v, \partial_x\rangle) = \frac{\partial}{\partial y_1}$. Because of this we get the following commutative diagram.

$$\begin{array}{ccccc} \mathbb{R}[\langle v, \partial\rangle] & \hookrightarrow & \mathbb{R}[\partial_x] & \twoheadrightarrow & \mathbb{R}[\partial_x]/\mathcal{I} \\ \psi\ \downarrow\approx & & \psi\ \downarrow\approx & & \psi\ \downarrow\approx \\ \mathbb{R}[\frac{\partial}{\partial y_1}] & \hookrightarrow & \mathbb{R}[\partial_y] & \twoheadrightarrow & \mathbb{R}[\partial_y]/\psi(\mathcal{I}). \end{array}$$

It follows that $\mathbb{R}[\frac{\partial}{\partial y_1}]$ injects into $\mathbb{R}[\partial_y]/\psi(\mathcal{I})$. Thus we have reduced the general case to that of case 1, and thus the proof is complete. □

An immediate corollary to the above result is that no direction in a strongly autonomous behavior is free. By statement 3 of Theorem 21, a direction defined by a nonzero real vector v is free if and only if the projection of the variety on the line $L_v^{\mathbb{C}}$ is dense in it. But, a strongly autonomous system has only finitely many discrete set of points for its characteristic variety. Therefore, its projection on $L_v^{\mathbb{C}}$ for any $0 \neq v \in \mathbb{R}^n$ cannot be a Zariski dense set in $L_v^{\mathbb{C}}$.

Corollary 26 *If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is strongly autonomous, then no direction is a free direction.*

Statement 3 of Theorem 21 gives a nice geometric criterion for a direction to be free. We shall see presently an interesting consequence of that. So far we have considered free-ness property of a given nonzero vector $v \in \mathbb{R}^n$. However, if we look at the projection of the characteristic variety $\mathbb{V}(\mathfrak{B})$ on the complex line $L_v^{\mathbb{C}}$ (the span of v in $\mathbb{C}^n$), the nature of this projection, meaning whether it is Zariski dense or not, does not change if v is replaced by a nonzero multiple of it. Consequently, the free-ness property of v is a property of the line and not of a particular vector that spans it. This leads us to identify these directions as points in the *projective $(n-1)$-space*, denoted here by $\mathbb{RP}^{n-1}$. Our next main result, which is a consequence of Theorem 21, shows that given a scalar autonomous behavior $\mathfrak{B}$, the set of all non-free directions forms a *linear closed* set in $\mathbb{RP}^{n-1}$.

In classical algebraic geometry, the concept of irreducible varieties plays an important role. We are going to make use of this concept to obtain a characterization of the set of all non-free directions in $\mathbb{RP}^{n-1}$. The definition of an irreducible set is generally given in terms of a point set topology. A set Y which is a subset of a topological space X is said to be *irreducible* if it cannot be written as a union of two strictly smaller closed sets. Although given in topological terms, for affine varieties in $\mathbb{C}^n$ equipped with Zariski topology, irreducibility happens to have a nice algebraic characterization: *an affine variety is irreducible if and only if its ideal in the polynomial ring over* $\mathbb{C}$ *is prime* (see Hartshorne 2009). In order to make use of this fact we consider our base field to be complex numbers. However, by looking into the real part of the characteristic variety, we shall translate our results for real numbers. Our main result Theorem 28, therefore, is with the real numbers as the base field.

Suppose the characteristic variety $\mathbb{V}(\mathfrak{B})$ is irreducible. The ideal of this variety is a prime ideal, say $\mathfrak{p}$. Suppose $v \in \mathbb{R}^n$ is some nonzero real vector. By Proposition 13 the Zariski closure of the projection of $\mathbb{V}(\mathfrak{B})$ on the complex 1-d subspace $L_v^{\mathbb{C}}$ spanned by v, is given by the variety of the v-intersection ideal $\mathfrak{p} \cap \mathbb{C}[\langle v, \partial\rangle]$. It can be shown that this ideal in $\mathbb{C}[\langle v, \partial\rangle]$ is a prime ideal, which means: it is either zero or it is generated by a linear polynomial $(\langle v, \partial\rangle - \alpha)$ for some $\alpha \in \mathbb{C}$. For the latter case, which by Theorem 21 is equivalent to v being non-free, it follows from Proposition 13 that

$$\overline{\Pi_v(\mathbb{V}(\mathfrak{B}))} = \mathbb{V}(\mathfrak{p} \cap \mathbb{C}[\langle v, \partial\rangle]) = \{\alpha\}.$$

This in turn means $\Pi_v(\mathbb{V}(\mathfrak{B})) = \{\alpha\}$. But $\Pi_v(\mathbb{V}(\mathfrak{B})) = \{v^{\mathrm{T}}\xi \mid \xi \in \mathbb{V}(\mathfrak{B})\}$ by definition. It then easily follows that $\mathbb{V}(\mathfrak{B})$ is contained in the affine hyperplane $\mathcal{H}_v^{\mathbb{C}} := \{\xi \in \mathbb{C}^n \mid v^{\mathrm{T}}\xi = \alpha\}$. Because v is a real vector, this means that the real part of the characteristic variety,

$$\Pi_{\mathbb{R}^n}(\mathbb{V}(\mathfrak{B})) := \{\mathbf{x} \in \mathbb{R}^n \mid \exists\, \mathbf{y} \in \mathbb{R}^n \text{ such that } \mathbf{x} + i\mathbf{y} \in \mathbb{V}(\mathfrak{B})\},$$

is contained in the affine hyperplane $\mathcal{H}_v := \{\mathbf{x} \in \mathbb{R}^n \mid v^{\mathrm{T}}\mathbf{x} = \mathbf{Re}(\alpha)\}$. Thus we arrive at the following lemma.

Lemma 27 *Given an autonomous scalar behavior* $\mathfrak{B}$ *and* $0 \neq v \in \mathbb{R}^n$*, if the characteristic variety* $\mathbb{V}(\mathfrak{B})$ *is irreducible then* v *is a non-free direction if and only if the real part of* $\mathbb{V}(\mathfrak{B})$ *is contained in an affine hyperplane whose normal is* v.

This observation can be utilized to give a full characterization of all the non-free directions. First, note that given a subset in $\mathbb{R}^n$ there exists a unique smallest dimensional affine space containing that set, namely the *affine hull* of the points in that set. Now, in order for $v \in \mathbb{R}^n$ to be a non-free direction, there must be an affine space containing $\Pi_{\mathbb{R}^n}(\mathbb{V}(\mathfrak{B}))$ whose normal is v. This affine space clearly contains the affine hull of $\Pi_{\mathbb{R}^n}(\mathbb{V}(\mathfrak{B}))$. Therefore, v is normal to the affine hull of $\Pi_{\mathbb{R}^n}(\mathbb{V}(\mathfrak{B}))$. Now note that we can describe the affine hull of $\Pi_{\mathbb{R}^n}(\mathbb{V}(\mathfrak{B}))$ as the solution set of a linear equation:

$$\mathbf{Aff}(\Pi_{\mathbb{R}^n}(\mathbb{V}(\mathfrak{B}))) = \{\mathbf{x} \mid H\mathbf{x} = b,\ \text{for some } H \in \mathbb{R}^{g\times n}, b \in \mathbb{R}^g\}.$$

It then follows that v is a non-free direction if and only if $v \in \operatorname{colspan}(H^{\mathrm{T}})$.

We now show how the above geometric criterion for a direction to be non-free can be extended to cater for the general case when $\mathbb{V}(\mathfrak{B})$ is not irreducible. The technique is to do an irreducible decomposition of $\mathbb{V}(\mathfrak{B})$ as

$$\mathbb{V}(\mathfrak{B}) = \mathbb{V}_1 \cup \mathbb{V}_2 \cup \cdots \cup \mathbb{V}_r,$$

where each $\mathbb{V}_i$ is irreducible and $\mathbb{V}_i \not\subseteq \mathbb{V}_j$ for $i \neq j$. Such a decomposition of affine varieties always exists (see Hartshorne 2009), and is unique upto permutation. The $\mathbb{V}_i$s in such a decomposition are called *irreducible components*. It follows from the decomposition that

$$\Pi_v(\mathbb{V}(\mathfrak{B})) = \Pi_v(\mathbb{V}_1) \cup \Pi_v(\mathbb{V}_2) \cup \cdots \cup \Pi_v(\mathbb{V}_r).$$

So clearly in order for v to be a non-free direction $\Pi_v(\mathbb{V}_i)$ must be single points for each $\mathbb{V}_i$. In other words, by Lemma 27, v is non-free if and only if each $\mathbb{V}_i$ is contained in an affine hyperplane normal to v. To elaborate this further, let $\mathbf{Aff}(\Pi_{\mathbb{R}^n}(\mathbb{V}_i))$ be the affine hull of $\Pi_{\mathbb{R}^n}(\mathbb{V}_i)$. Like the previous case, these $\mathbf{Aff}(\Pi_{\mathbb{R}^n}(\mathbb{V}_i))$'s can be written as solution sets of linear equations as

$$\mathbf{Aff}(\Pi_{\mathbb{R}^n}(\mathbb{V}_i)) = \{\mathbf{x} \mid H_i\mathbf{x} = b_i, \text{ for some } H_i \in \mathbb{R}^{g\times n}, b_i \in \mathbb{R}^g\},$$

Then it follows that v is a non-free direction if and only if

$$v \in \operatorname{colspan}(H_1^{\mathrm{T}}) \cap \operatorname{colspan}(H_2^{\mathrm{T}}) \cap \cdots \cap \operatorname{colspan}(H_r^{\mathrm{T}}).$$

This gives a full characterization of all the non-free directions. An important consequence of this characterization is that the set of non-free directions are given by span of vectors, so it can be written as the zero set of homogeneous linear equations. This implies that the set of non-free directions is actually a linear closed set of $\mathbb{RP}^{n-1}$. We sum up all these observations in the following theorem.

Theorem 28 *Let $\mathfrak{B} \in \mathfrak{L}^1$ be an autonomous system with $\mathbb{V}(\mathfrak{B}) \subseteq \mathbb{C}^n$ as characteristic variety. Suppose that*

$$\mathbb{V}(\mathfrak{B}) = \mathbb{V}_1 \cup \mathbb{V}_2 \cup \cdots \cup \mathbb{V}_r$$

is the irreducible decomposition of $\mathbb{V}(\mathfrak{B})$. Further suppose that for $i \in \{1, 2, \ldots, r\}$, the real affine hull of $\Pi_{\mathbb{R}^n}(\mathbb{V}_i)$, the projection of $\mathbb{V}_i$ onto $\mathbb{R}^n$, is given by

$$\mathbf{Aff}(\Pi_{\mathbb{R}^n}(\mathbb{V}_i)) = \{\mathbf{x} \mid H_i\mathbf{x} = b_i, \textit{ for some } H_i \in \mathbb{R}^{g\times n}, b_i \in \mathbb{R}^g\}.$$

Then $0 \neq v \in \mathbb{R}^n$ is a non-free direction if and only if

$$v \in \operatorname{colspan}\left(H_1^{\mathrm{T}}\right) \cap \operatorname{colspan}\left(H_2^{\mathrm{T}}\right) \cap \cdots \cap \operatorname{colspan}\left(H_r^{\mathrm{T}}\right).$$

4.2 Restriction to non-free directions and stability

By Theorem 21, a nonzero real vector $v \in \mathbb{R}^n$ defines a non-free direction if and only if the v-intersection ideal $\mathcal{I}_v = \mathcal{I} \cap \mathbb{R}[\langle v, \partial\rangle]$ is nonzero. But this means that the 1-d behavior defined by $\mathcal{I}_v$, which we have called $\mathfrak{B}_v$, is given by a non-trivial ODE, namely the one given by the unique monic generator of $\mathcal{I}_v$. Therefore, in this case, $\mathfrak{B}_v$ is finite dimensional. Now, it is well-known that all the trajectories in such a nontrivial 1-d autonomous system are of exponential type. We are going to exploit this fact and utilize Algorithm 22 to infer that in this situation the behavior restricted to v is in fact equal to $\mathfrak{B}_v$.

Theorem 29 *Let $\mathfrak{B}$ be scalar autonomous behavior defined by equation ideal $\mathcal{I} \subseteq \mathbb{R}[\partial]$ and let $0 \neq v \in \mathbb{R}^n$. Let $\mathfrak{B}_v$ denote the 1-d exponential behavior corresponding to the v-intersection ideal $\mathcal{I}_v$,* i.e.,

$$\mathfrak{B}_v := \left\{\widetilde{w} \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\widetilde{w} = 0 \textit{ for all } r(\langle v, \partial\rangle) \in \mathcal{I}_v\right\}.$$

Then $\mathcal{I}_v \neq 0$ implies $\mathfrak{B}|_v = \mathfrak{B}_v$.

Proof The $\mathbb{R}$-algebra $\mathbb{R}[\langle v, \partial\rangle]$ can be thought of as $\mathbb{R}[\partial_1]$ following the arguments used in Theorem 21 and Lemma 25. Then the v-intersection ideal is given by the elimination ideal $\mathcal{I}_1 := \mathcal{I} \cap \mathbb{R}[\partial_1]$. Suppose $g(\partial_1) \in \mathcal{I}_1$ is the unique monic generator of the ideal $\mathcal{I}_1$. The behavior $\mathfrak{B}_v$ is then given by

$$\mathfrak{B}_v = \left\{ \widetilde{w} \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid g\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \widetilde{w} = 0 \right\}.$$

Let us consider the following exponential type power series to be an element in $\mathfrak{B}_v$.

$$\widetilde{w}(t) = \widetilde{w}_0 + \widetilde{w}_1 t + \frac{\widetilde{w}_2}{2!}t^2 + \frac{\widetilde{w_3}}{3!}t^3 + \cdots + \frac{\widetilde{w}_k}{k!}t^k + \cdots \in \mathfrak{B}_v. \tag{26}$$

It then follows that

$$\widetilde{w}_k = \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \widetilde{w}\right)(0) = \left(r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\widetilde{w}\right)(0),$$

where $r(\frac{\mathrm{d}}{\mathrm{d}t})$ is the remainder after division of $(\frac{\mathrm{d}}{\mathrm{d}t})^k$ by $g(\frac{\mathrm{d}}{\mathrm{d}t})$, which is clearly a polynomial of degree less than that of g. This is nothing but Algorithm 22 applied to the 1-d case. This shows that every element in $\mathfrak{B}_v$ is uniquely determined once the initial condition $\{\widetilde{w}_i\}_{0 \leqslant i \leqslant (\deg(g)-1)}$ is specified. What we are going to show next is that with a suitable choice of the term ordering, there is a Gröbner basis for $\mathcal{I}$, for which an initial condition can be constructed from the above 1-d initial condition so as to guarantee a solution in $\mathfrak{B}$ whose restriction to e_1 will be $\widetilde{w}$.

Let this $\widetilde{w}(t)$ of Eq. (26) be obtained from the initial condition

$$\{\widetilde{w}_i\}_{0 \leqslant i \leqslant (d-1)},$$

where $d := \deg(g)$. Therefore, if

$$r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = a_0 + a_1\frac{\mathrm{d}}{\mathrm{d}t} + a_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + \cdots + a_{d-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{d-1} \in \mathbb{R}\left[\frac{\mathrm{d}}{\mathrm{d}t}\right]$$

is the remainder after division of $(\frac{\mathrm{d}}{\mathrm{d}t})^k$ by $g(\frac{\mathrm{d}}{\mathrm{d}t})$, then the said solution $\widetilde{w}$ satisfies

$$\widetilde{w}_k = \left(r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\widetilde{w}\right)(0) = a_0\widetilde{w}_0 + a_1\widetilde{w}_1 + a_2\widetilde{w}_2 + \cdots + a_{d-1}\widetilde{w}_{d-1}.$$

We now fix a term ordering $\prec$ in $\mathbb{R}[\partial]$ such that for all $2 \leqslant i \leqslant n$, $\partial_1 \prec \partial_i$. If $\mathcal{G} \subseteq \mathbb{R}[\partial]$ is the monic reduced Gröbner basis of $\mathcal{I}$ with respect to the term ordering $\prec$, it then follows from standard Gröbner basis theory that $\mathcal{G}_1 := \mathcal{G} \cap \mathbb{R}[\partial_1]$ is a monic Gröbner basis for the elimination ideal $\mathcal{I}_1$ (see Cox et al. 2007, 1998). First, since $\mathcal{I}_1 \neq 0$, $\mathcal{G}_1 \neq \emptyset$. Secondly, since $\mathbb{R}[\partial_1]$ is a PID, $\mathcal{I}_1$ is principal, and so its monic Gröbner basis is nothing but the monic generator $g(\partial_1)$ of $\mathcal{I}_1$. Therefore, $\mathcal{G}_1 = \{g(\partial_1)\}$. Now, we are going to follow each step of Algorithm 22 to obtain our desired solution whose restriction to e_1 we want to be $\widetilde{w}$. Suppose $\Gamma \subseteq \mathbb{N}^n$ is the set of multi-indices corresponding to the standard monomials, and Γ_1 is the multi-indices whose every component except the first ones are zero, that is, $\Gamma_1 := \{\nu \in \mathbb{N}^n \mid \nu = ke_1\}$. Since, $\mathcal{G}_1 = \{g(\partial_1)\}$, we must have $\Gamma \cap \Gamma_1 = \{\nu \in \mathbb{N}^n \mid \nu = ke_1, 0 \leqslant k \leqslant (d-1)\}$. Let us fix the initial condition $\{w_\nu\}_{\nu\in\Gamma}$ as follows:

$$w_\nu = \begin{cases} \widetilde{w}_k & \text{if } \nu \in \Gamma \cap \Gamma_1,\ \nu = ke_1 \\ 0 & \text{otherwise} \end{cases}.$$

Such an initial condition, when written as an exponential series, $w_{\mathrm{in}} = \sum_{\nu\in\Gamma} \frac{w_\nu}{\nu!}\mathbf{x}^\nu$, is a finite sum for all $\mathbf{x} \in \mathbb{R}^n$. Hence, by Remark 24, this initial condition is valid. Now suppose

$\nu = ke_1 \in \Gamma_1$ and $r'(\partial) \in \mathbb{R}[\partial]$ is the remainder after division of ∂^ν by the Gröbner basis $\mathcal{G}$. We claim that $r'(\partial) = r(\partial_1)$, where $r(\frac{\mathrm{d}}{\mathrm{d}t})$ is the remainder after division of $(\frac{\mathrm{d}}{\mathrm{d}t})^k$ by $g(\frac{\mathrm{d}}{\mathrm{d}t})$. This is because $\partial^\nu \in \mathbb{R}[\partial_1]$, and since ∂_1 has the least weightage as per the term ordering $\prec$, ∂^ν, therefore, is divisible only by leading terms in $\mathcal{G}_1$. But we have already seen that $\mathcal{G}_1 = \{g(\partial_1)\}$. Therefore, the remainder after division by $\mathcal{G}$ is same as that after division by only g, which incidentally is $r(\partial_1)$. So, for $\nu = ke_1$ for any $k \in \mathbb{N}$, if we follow Algorithm 22, we get

$$\begin{aligned} w_\nu = (r'(\partial)w)(\mathbf{0}) &= (r(\partial_1)w)(\mathbf{0}) \\ &= \sum_{i=0}^{d-1} a_i w_{ie_1} = \sum_{i=0}^{d-1} a_i \widetilde{w}_i = \widetilde{w}_k \end{aligned}$$

Now notice that if $w(\mathbf{x})$ is the solution obtained by following Algorithm 22, then the restriction of w to the x_1 axis, that is, along the line defined by e_1, turns out to be

$$\begin{aligned} w(e_1 t) &= \sum_{\nu \in \Gamma_1} \frac{w_\nu}{\nu!}(e_1 t)^\nu \\ &= \sum_{\nu = ie_1, i \in \mathbb{N}} \frac{w_\nu}{i!} t^i \\ &= \sum_{i \in \mathbb{N}} \frac{\widetilde{w}_i}{i!} t^i = \widetilde{w}(t). \end{aligned}$$

Thus, for every exponential trajectory in $\mathfrak{B}_{e_1}$, there exists an exponential trajectory in $\mathfrak{B}$, whose restriction on e_1 is same as the trajectory in $\mathfrak{B}_{e_1}$. Hence $\mathfrak{B}_{e_1} \subseteq \mathfrak{B}|_{e_1}$.

That $\mathfrak{B}|_{e_1} \subseteq \mathfrak{B}_{e_1}$ has already been proved in Theorem 6. Thus we conclude that $\mathfrak{B}|_{e_1} = \mathfrak{B}_{e_1}$. Finally, by Lemma 25, for any nonzero vector $v \in \mathbb{R}^n$ such that $\mathcal{I}_v$ is nonzero, $\mathfrak{B}|_v = \mathfrak{B}_v$. □

One very interesting consequence of the above result is that stability in a given direction can be inferred from it. Given an autonomous scalar behavior $\mathfrak{B}$ and a nonzero vector v, by stability along v we mean that for all $w \in \mathfrak{B}$, the limit $\lim_{t\to\infty} w(vt) = 0$. Clearly, a necessary condition for stability is that v is a non-free direction. This is because otherwise any exponential 1-d trajectory can be obtained by restricting trajectories in $\mathfrak{B}$ to L_v. In particular, trajectories that are unstable in the positive half line $L_v^+ = \{vt \mid t \geqslant 0\}$, can also be obtained by restriction. Now, if indeed a direction is non-free, then our last result ensures that the restriction $\mathfrak{B}|_v$ is equal $\mathfrak{B}_v$, the behavior of the intersection ideal $\mathcal{I}_v$. Therefore, $\mathfrak{B}|_v$ has unstable trajectories in the positive half line if and only if $\mathfrak{B}_v$ is an unstable 1-d behavior. But, since $\mathfrak{B}_v$ is a 1-d behavior, governed by an ODE, its stability is reflected in the pole locations of the monic generator of $\mathcal{I}_v$. Combining these observations we obtain the following result.

Corollary 30 *Let $\mathfrak{B}$ be a scalar autonomous behavior with equation ideal $\mathcal{I}$ and let $v \in \mathbb{R}^n$ be nonzero. Then $\mathfrak{B}$ is stable on the half-line $L_v^+ := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = vt, t \geqslant 0\}$ if and only if the following two conditions hold:*

1. *v is not a free direction,*
2. *if $g(\langle v, \partial\rangle)$ is the monic generator of the v-intersection $\mathcal{I}_v$, then $g(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$ has all roots with negative real parts.*

Remark 31 The issue of stability with respect to a special type of collection of half-lines, namely a closed convex cone, was dealt with in Pillai and Shankar (1998). There, it was shown that for stability with respect to a given cone S it is necessary that the real part of the characteristic variety be strictly contained in the *polar cone* $S_<$ of S. Moreover, it was also shown that for the case when the characteristic ideal contains a polynomial with no repeated factors, this condition becomes sufficient too. For stability on the half-line generated by $0 \neq v \in \mathbb{R}^n$, this geometric condition translates to: the real part of the characteristic variety should be strictly contained in the half-space $\{\mathbf{x} \in \mathbb{R}^n \mid v^{\mathrm{T}}\mathbf{x} \leqslant 0\}$. It follows from Proposition 13 that when v is a non-free direction, this condition is equivalent to Statement 2 of Corollary 30 above. On the other hand, if v is a free direction, that is $\mathcal{I}_v = \{0\}$, then it can be shown that

$$\mathbb{C} \setminus W \subseteq \Pi_v(\mathbb{V}(\mathfrak{B})),$$

where $W \subseteq \mathbb{C}$ is *finite* [see for example (Cox et al. 2007, Theorem 3.2.3)]. This means, when v is a free direction, we can always find a point $\xi \in \mathbb{V}(\mathfrak{B})$ such that $\mathbf{Re}(v^{\mathrm{T}}\xi) > 0$. Clearly, then the real part of $\mathbb{V}(\mathfrak{B})$ cannot be contained in the half-space polar to v. Hence $\mathfrak{B}$ cannot be stable on the half-line generated by v. This way Corollary 30 strengthens the stability results in Pillai and Shankar (1998) and makes the above mentioned geometric condition necessary and sufficient for stability on a half-line.

5 Concluding remarks

In this paper, we have investigated the restriction of n-d systems to 1-d subspaces. We have brought out a strong connection between the restricted solutions and an algebraic entity, called v-intersection submodule. We have shown that the intersection submodule naturally gives rise to a 1-d system which always contains the restricted trajectories. We then looked into a special kind of autonomous systems, namely strongly autonomous systems, whose solution sets are finite dimensional vector spaces. We showed that such systems always admit a first order representation involving an n-tuple of real square matrices called companion matrices. Then we made use of this result to show that the 1-d behavior corresponding to the intersection submodule admits a state-space representation given by the restriction of a linear combination of the companion matrices to an invariant subspace. Utilizing this result we showed that, for the strongly autonomous case, the restriction of the behavior is in fact equal to the behavior of the intersection submodule. Then we looked into general autonomous systems – not necessarily strongly autonomous. Here we have shown that a given direction may turn out to be *free*: every possible 1-d trajectory can be obtained by restriction of trajectories in the system. Then we first gave a set of algebraic criteria equivalent to a given direction being free. Using this result we then gave a geometric characterization of all (non)free directions. We have shown that the set of all non-free directions is a linear closed subset of the real projective $(n-1)$-space. Finally, we proved that if a direction is non-free then restriction of the system to the corresponding 1-d subspace turns out to be equal to the 1-d behavior associated to the intersection ideal. We then made use of this fact to give an equivalent condition for stability along a given half-line.

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