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DISSIPATIVITY OF UNCONTROLLABLE SYSTEMS, STORAGE FUNCTIONS, AND LYAPUNOV FUNCTIONS*

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Abstract. Dissipative systems have played an important role in the analysis and synthesis of dynamical systems. The commonly used definition of dissipativity often requires an assumption on the controllability of the system. In this paper we use a definition of dissipativity that is slightly different (and less often used in the literature) to study a linear, time-invariant, possibly uncontrollable dynamical system. We provide a necessary and sufficient condition for an uncontrollable system to be strictly dissipative with respect to a supply rate under the assumption that the uncontrollable poles are not "mixed"; i.e., no pair of uncontrollable poles is symmetric about the imaginary axis. This condition is known to be related to the solvability of a Lyapunov equation; we link Lyapunov functions for autonomous systems to storage functions of an uncontrollable system. The set of storage functions for a controllable system has been shown to be a convex bounded polytope in the literature. We show that for an uncontrollable system the set of storage functions is unbounded, and that the unboundedness arises precisely due to the set of Lyapunov functions for an autonomous linear system being unbounded. Further, we show that stabilizability of a system results in this unbounded set becoming bounded from below. Positivity of storage functions is known to be very important for stability considerations because the maximum stored energy that can be drawn out is bounded when the storage function is positive. In this paper we establish the link between stabilizability of an uncontrollable system and existence of positive definite storage functions. In most of the results in this paper, we assume that no pair of the uncontrollable poles of the system is symmetric about the imaginary axis; we explore the extent of necessity of this assumption and also prove some results for the case of single output systems regarding this necessity.

Key words. dissipativity, uncontrollability, storage functions, behaviors, algebraic Riccati equation, Hamiltonian matrix, Lyapunov equation

AMS subject classifications. 93D05, 15A63, 93B05, 93B07, 15A18, 15A03

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1. Introduction.

Dissipativity of dynamical systems helps in the analysis and design of control systems. Dissipativity theory allows problems like LQR, circle criterion, Popov criterion, passivity synthesis, $\mathcal{H}_\infty$ control, and Riccati inequalities to be analyzed under a common framework. An important assumption in some of these developments is that of controllability of the dynamical system. In this paper we study dissipativity of general linear time-invariant systems, possibly uncontrollable.

Uncontrollable systems arise naturally in the process of modeling dynamical systems. The inability to shape one or more system variables in an arbitrary desired fashion is frequently encountered in systems. For example, loss of controllability could happen to otherwise controllable systems when certain system parameters satisfy relevant equations arising in controllability check methods: see a simple electrical circuit below in section 5.1. Uncontrollability could also arise generically due to *structural* inabilities to influence one or more system variables. (See [14, 12] and references therein about structural controllability studies.)

In the context of synthesis of a dynamical system, one sometimes has to settle for an uncontrollable realization of a given transfer function: the case of nonminimality of

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the transformerless synthesis of a positive real transfer function is well known. This issue concerns the synthesis of a positive real transfer matrix using only resistors, capacitors, and inductors. The currently known methods (the Bott–Duffin method [3] and its variants) bring us naturally to systems that are dissipative, but are not controllable. See [6] for a recent overview about this classical problem.

Dissipativity of a system is about the absence of any source of energy within the system, and hence all interactions with the environment have to satisfy the condition that the "net energy" is directed inwards. This is made precise below in Definition 3.1. Such a property is intrinsic to the system and therefore should be independent of the question of controllability. For example, a passive electrical network made out of passive circuit elements must continue to be dissipative even if it loses controllability. In this paper we consider a general linear time-invariant system and work on a theory of dissipativity free from any controllability assumption. Our work is based on the signature characteristic of a dissipative system to store energy, i.e., existence of a storage function. An important issue that immediately arises is whether to include unobservable variables to describe this storage of energy (see [33, 5]). Our main result sorts out this issue: for the case of strict dissipativity, we show that a storage function depending only on the manifest variables suffices, and no unobservable variables are necessary (see Remark 5.1).

The present theory of dissipative systems is well-developed primarily for controllable systems because it is possible there to define dissipativity without taking recourse to the existence of a storage function. This is done using an integral inequality involving only the compactly supported trajectories allowed by the system. This definition turns out to be inadequate for a general, possibly uncontrollable, linear behavior. In order to overcome this inadequacy, there has been prior work of taking existence of storage functions satisfying a dissipation inequality as a definition of dissipativity; see [28, 5], for example. In this paper we further develop using this definition, prove results regarding existence, and relate it to the situation of controllability. The principal finding is that a certain condition on the uncontrollable poles, which we call as the *unmixing* condition, plays a key role. If no pair of the uncontrollable poles of the system is symmetric with respect to the imaginary axis, then the noncontrollability poses no hindrance to strict dissipativity, i.e., the strict dissipativities of the behavior and its controllable part are equivalent (Theorem 3.4). This result is utilized to show useful identities about positive storage functions and unboundedness of the set of storage functions for the case of uncontrollability.

The paper is structured as follows. The rest of this section has a few words about the notation we follow. Section 2 contains some preliminaries we require regarding behavioral theory. The next section (section 3) has some definitions that we need in order to state the main result of this paper and for the proofs. In this section we also present the main result: a necessary and sufficient condition for a general linear time-invariant system to be strictly dissipative with respect to a supply rate that depends on the manifest variables, under the assumption that the set of uncontrollable poles satisfies the unmixing condition. Interestingly, this unmixing condition on the uncontrollable poles is reminiscent of the solvability condition of Lyapunov equations: this is elaborated in sections 3 and 8. This paper utilizes the wealth of existing literature on Hamiltonian matrices and Riccati equations; section 4 has results about relations among Riccati equations, the Hamiltonian matrix, and dissipativity. A proof of the main result follows in section 5, together with some auxiliary results. In section 6 we present a necessary and sufficient condition for existence of positive storage functions: here we relate stabilizability to positive storage functions. In section 7 we present some insight on the nature, namely, unboundedness and convexity, of the set of all storage functions of an uncontrollable dissipative behavior. (The set of storage functions is known to be *bounded* in the case of controllability.) Section 8 explores into the extent of necessity of the unmixing property that we have assumed throughout this paper. In this section we show an interesting result about rank one symmetric matrices and the solvability of the Lyapunov equation. When one or more pairs of the uncontrollable poles have symmetry about the imaginary axis, it turns out that the solvabilities of a certain Riccati equation and the corresponding Riccati *inequality* differ significantly from the situation in the controllable case. We conclude the paper in section 9 following which is an appendix containing some proofs and peripheral results needed for the proofs.

The notation we follow is standard. $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers. The ring of polynomials in ξ with real coefficients is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ stands for the set of $\mathtt{p}\times\mathtt{w}$ matrices with entries from $\mathbb{R}[\xi]$. In the context of quadratic differential forms, we require polynomials in two indeterminates: ζ and η. The set of such polynomials with real coefficients is denoted by $\mathbb{R}[\zeta,\eta]$, and the set of $\mathtt{w}\times\mathtt{w}$ matrices with entries from $\mathbb{R}[\zeta,\eta]$ by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes its subspace of all compactly supported trajectories. We use $\bullet$ when it is unnecessary to specify a dimension. For example, $R\in\mathbb{R}^{\bullet\times\mathtt{w}}$ means R is a real matrix with $\mathtt{w}$ columns. When dealing with many variables, in order to keep track of the dimensions, we use the same letter as a generic variable w, but in typewriter font $\mathtt{w}$, to denote the number of components; for example, $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. In the context of stability, we require certain regions of the complex plane $\mathbb{C}$. The open left and right half complex planes are denoted by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. To improve readability within text, we use $\mathrm{col}(\cdot,\cdot)$ to stack its arguments into a column, i.e., $\mathrm{col}(w_1,w_2)=[w_1^T \quad w_2^T]^T$.

2. Behaviors, QDFs, and state representations. A linear differential behavior $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of the solutions to a set of ordinary linear differential equations with constant coefficients; i.e.,

$$\mathfrak{B} := \left\{ w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\},$$

where $R(\xi)$ is a polynomial matrix having $\mathtt{w}$ number of columns: $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. We shall denote the set of linear differential behaviors with $\mathtt{w}$ number of variables by $\mathfrak{L}^{\mathtt{w}}$. The linear differential behavior $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ can also be written as $\mathfrak{B}=\ker R(\frac{\mathrm{d}}{\mathrm{d}t})$. That is why this representation is called a *kernel representation* of $\mathfrak{B}$. We call w the *manifest variable*; these are the variables of interest. In this paper, w is the variable through which the system exchanges energy with the environment. It turns out that we can assume, without loss of generality, that $R(\xi)$ is of full row rank (see [17]); in this paper, a kernel representation matrix $R(\xi)$ is assumed to be of full row rank. For a behavior $\mathfrak{B}=\ker R(\frac{\mathrm{d}}{\mathrm{d}t})$, the row rank of $R(\xi)$ gives the output cardinality (number of outputs in the system). Though the variables w can often be partitioned into inputs and outputs in more than one way, the output cardinality remains the same: $\operatorname{rank} R$. Further, the cardinality does not depend on the R used to define it, but depends only on $\mathfrak{B}$. In this sense, the output cardinality is an integer invariant of $\mathfrak{B}$ and we denote it by $\mathtt{p}(\mathfrak{B})$. The number of inputs to the system, the input cardinality, is another integer invariant of $\mathfrak{B}$. This integer is denoted by $\mathtt{m}(\mathfrak{B})$ and is calculated using $\mathtt{m}(\mathfrak{B})=\mathtt{w}-\mathtt{p}(\mathfrak{B})$, where $\mathtt{w}$ is the number of components in the manifest variable w.

A concept of central importance for this paper is that of controllability. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *controllable* if for every $w', w'' \in \mathfrak{B}$, there exists a $w \in \mathfrak{B}$ and a $\tau > 0$ such that

$$\begin{aligned} w(t) &= w'(t) \quad \text{ for all } t \leqslant 0, \\ &= w''(t) \quad \text{ for all } t \geqslant \tau. \end{aligned}$$

We denote the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. A behavior $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ is controllable if and only if $R(\lambda)$ does not lose rank for any $\lambda \in \mathbb{C}$. An important characterization of controllable behaviors is that they also admit *image representations*. It was shown in [31] that $\mathfrak{B}$ is controllable if and only if it can be represented as

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\},$$

for some polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$. This representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is called an image representation. It turns out that for an image representation, without loss of generality, one can assume $M(\xi)$ to have the property that $M(\lambda)$ has full column rank for every $\lambda \in \mathbb{C}$. We call such an M satisfying this property a *right-prime* polynomial matrix. M being right-prime means that we are able to deduce the ℓ trajectory corresponding to a w trajectory satisfying the equation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$. Hence, such an M is also said to induce an *observable* image representation.

In the context of uncontrollable systems, we use the key notion of *uncontrollable poles* and *uncontrollable characteristic polynomial*. Suppose $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ and suppose $\mathfrak{B}$ is *not* controllable. Then there exist one or more complex numbers λ such that $R(\lambda)$ loses rank. These complex numbers, together with multiplicities,[1] are defined as uncontrollable poles in the definition below. Uncontrollable poles are the roots of a monic polynomial called the uncontrollable characteristic polynomial. See [32] for details.

DEFINITION 2.1. *Let $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ have full row rank and suppose $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a kernel representation for $\mathfrak{B}$. Consider a factorization of R into $R(\xi) = F(\xi)R_{\text{cont}}(\xi)$ such that $R_{\text{cont}} \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$, $R_{\text{cont}}(\lambda)$ has full row rank for every complex number λ, and* $\det F$ *is a monic polynomial. The* uncontrollable characteristic polynomial *of $\mathfrak{B}$, denoted by $\chi_{\text{un}}(\mathfrak{B})$, is defined as* $\det F$. *The set of* uncontrollable poles *is defined as* roots (χ_{un}), *and is denoted by $\Lambda_{\text{un}}(\mathfrak{B})$.*

If the behavior $\mathfrak{B}$ is clear from the context, we write just χ_{un} and Λ_{un}. Notice that if $\mathfrak{B}$ is controllable, then $\chi_{\text{un}} = 1$. When a behavior is not controllable, we often require the *controllable part* of $\mathfrak{B}$. This is the largest controllable behavior contained in $\mathfrak{B}$; the controllable part of $\mathfrak{B}$ is denoted by $\mathfrak{B}_{\text{cont}}$. Consider the above definition in which R has been factorized as described to obtain R_{cont}. A kernel representation for $\mathfrak{B}_{\text{cont}}$ is induced by R_{cont}. For a detailed exposition on behaviors, controllability, and uncontrollable characteristic polynomial, we refer the reader to [17, 32].

This paper deals with dissipativity and in this context we deal with quadratic forms in the system variables and a finite number of their derivatives. It turns out to be very natural to associate two variable polynomial matrices to such quadratic forms. Consider a two variable polynomial matrix $\Phi(\zeta, \eta) := \sum_{i,k} \Phi_{ik}\zeta^i\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, where $\Phi_{ik} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. A Quadratic Differential Form (QDF) Q_Φ induced by $\Phi(\zeta, \eta)$ is a

[1] See Remark 4.2 below.

map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{i,k} \left(\frac{\mathrm{d}^i w}{\mathrm{d}t^i}\right)^T \Phi_{ik} \left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right).$$

When dealing with quadratic forms in w and its derivatives, we can assume without loss of generality that $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$. We call such $\Phi(\zeta, \eta)$ a symmetric two-variable polynomial matrix, and we denote the set of all such symmetric two-variable polynomial matrices by $\mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. A quadratic form induced by a real symmetric constant matrix $S \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$ is a special QDF and we shall often need this in this paper. For a given $\Phi \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, we often require the one variable polynomial matrix $\Phi(-\xi, \xi)$: we shall denote this by $\partial\Phi(\xi)$. Due to the symmetry of $\Phi(\zeta, \eta)$ the one variable polynomial matrix $\partial\Phi(\xi)$ is *para-Hermitian*, i.e., $\partial\Phi(-\xi) = \partial\Phi^T(\xi)$. Notice that this property makes $\partial\Phi(j\omega)$ Hermitian for all $\omega \in \mathbb{R}$. Throughout this paper, ample use is made of the well-developed theory of QDFs; only the essential results of which are reviewed here. See [28] for a thorough and complete treatment on QDFs.

The notion of state is central to this paper due to the claim that, for uncontrollable systems, also the storage of energy is possible due to memory elements in the system. The state variable x is an auxiliary variable that relates to the memory of the system. Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with manifest variables w. A variable x of the system is called a state variable if it satisfies the system equations together with w, and has the concatenation property. More precisely, if (w', x') and (w'', x'') are two smooth trajectories allowed by the system, and $x'(0) = x''(0)$, then the new trajectory (w, x) formed by concatenating (w', x') and (w'', x'') at $t = 0$, i.e.,

$$\begin{aligned} (w, x)(t) &= (w', x')(t) \qquad \text{for all } t \leqslant 0 \\ &= (w'', x'')(t) \qquad \text{for all } t > 0, \end{aligned}$$

also satisfies the system equations in a distributional sense. A formal treatment on this is contained in [19], where it was proved that a variable x is a state variable for $\mathfrak{B}$ if and only if the behavior satisfies

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, x \in \mathfrak{C}^\infty \text{ such that } E\frac{\mathrm{d}}{\mathrm{d}t}x + Fx + Gw = 0 \right\}$$

for suitable real constant matrices E, F and G. The above first order representation is called a state representation. Such a representation is said to be minimal if it has the least number of state variables among all state representations describing the behavior. The dimension of the state space for a minimal state representation is defined to be the *McMillan degree* of the behavior, and is denoted by $\mathtt{n}(\mathfrak{B})$. It was shown in [19] that a set of state variables can be obtained through the manifest variables by a *state map*, $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, which gives the state variables by $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$. Among all state maps, if X has the minimum number of rows, then it is called a minimal state map; in this case, the number of rows equals $\mathtt{n}(\mathfrak{B})$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ has an *input/output* partition $w = \mathrm{col}(w_1, w_2)$ where w_1 is input and w_2 is output such that the transfer function from w_1 to w_2 is *proper*, then $\mathfrak{B}$ admits a minimal state representation that is more special and well known: an *input/state/output* (i/s/o) representation

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1, \quad w_2 = Cx + Dw_1,$$

with (C, A) observable. Controllability of the behavior $\mathfrak{B}$ and that of the pair (A, B) are related as shown in the following well-known result. We write $\Lambda_{\mathrm{un}}^{(A,B)}$ for the set of uncontrollable eigenvalues of the (A, B) pair (counted with multiplicities). Proposition 2.2 shows when a behavior allows a state map that gives rise to an i/s/o representation with observability properties (see [31, 19]). We state this as a proposition for easy reference later in this paper.

PROPOSITION 2.2. *Let behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given by $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$, where $R(\xi)$ has full row rank. Let (w_1, w_2) be an input/output partition of w such that the resulting transfer function matrix is proper. Suppose $\mathtt{n}$ is the McMillan degree of $\mathfrak{B}$ and $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ gives a minimal state map. Then there exist $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, such that $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$ with (C, A) observable, and $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$. Moreover, $\mathfrak{B}$ is controllable if and only if the above i/s/o representation is state-controllable. If $\mathfrak{B}$ is uncontrollable, then $\Lambda_{\mathrm{un}}(\mathfrak{B})$, the set of uncontrollable poles of $\mathfrak{B}$, is equal to $\Lambda_{\mathrm{un}}^{(A,B)}$ the set of uncontrollable eigenvalues of A (counted with multiplicities).*

3. Dissipative systems: Definition and main result.

Dissipative systems are those that have no source of energy within, and hence any energy stored within the system has to have been supplied from its environment. This intuitive physical concept was made concrete in [30, 28] using the *dissipation inequality*: the rate of increase of stored energy is at most the power supplied to the system. In this paper, the power supplied and the stored energy are both QDFs in the manifest variables w of the system. (See Remark 5.1 below regarding storage function's dependence on just the manifest variables.) In this paper we use the following definition of dissipativity; its relation to other definitions is discussed below.

DEFINITION 3.1. *A linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be* dissipative *with respect to supply rate $S \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$ if there exists a quadratic differential form $Q_\Psi(w)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_S(w) \text{ for all } w \in \mathfrak{B}. \tag{3.1}$$

The quadratic differential form Q_Ψ is called a storage function *for $\mathfrak{B}$ with respect to the supply rate S.*

The inequality (3.1) above is called the dissipation inequality. In some control problems like in LQR and the suboptimal $\mathcal{H}_\infty$ control, a stricter notion of dissipativity plays a key role. In this paper we shall deal primarily with strict dissipativity, although many of our results are valid for just dissipativity also (see Remark 5.7 below). We define strict dissipativity as follows.

DEFINITION 3.2. *A linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be* strictly dissipative *with respect to $S \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$ if there exists an $\epsilon > 0$ and a storage function $Q_\Psi(w)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_S(w) - \epsilon |w|^2 \quad \text{for all } w \in \mathfrak{B}.$$

Because the above definitions require the existence of a hitherto unknown storage function, it has been common to use an equivalent statement for the definition of (strict) dissipativity when dealing with *controllable* systems. The following result from [28] shows the equivalence.

PROPOSITION 3.3. *Let $\mathfrak{B} \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ and $S \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Then the following statements are equivalent.*

1. *There exists a storage function $Q_\Psi(w)$ such that $\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_S(w) - \epsilon |w|^2$ for all $w \in \mathfrak{B}$.*

2. *For all $w \in \mathfrak{B} \cap \mathfrak{D}$ the integral inequality $\int_{\mathbb{R}} Q_S(w)\mathrm{d}t \geqslant \epsilon \int_{\mathbb{R}} |w|^2 \mathrm{d}t$ is satisfied.*

The above proposition shows that the existence of a storage function satisfying the dissipation inequality is equivalent to saying that the total energy transferred into the system is strictly positive whenever we start the system from rest and bring the system back to rest. Statement 2 was used as the definition of strict dissipativity in [28]. With $\epsilon = 0$, we get the definition of nonstrict dissipativity given in [28, 29]. It is important to note here that the second statement above holds over only compactly supported trajectories in $\mathfrak{B}$, while the first holds for all $w \in \mathfrak{B}$. Controllability of $\mathfrak{B}$ is crucial for the compactly supported trajectories in $\mathfrak{B}$ to be representative enough of the whole behavior for the above equivalence to hold (see [16]). Definitions using an integral over a finite interval (which means energy supplied for a finite period of time) ends up having initial and final storage function values in the defining inequality. Use of compactly supported trajectories makes the integrals over the whole of $\mathbb{R}$ well-defined and also makes the defining integral inequality free from the initial and final storage function values. However, for an uncontrollable behavior, Statement 2 of Proposition 3.3 puts no restrictions on the trajectories in the behavior which are outside the controllable part (see [16]), and hence this cannot be used as a definition of dissipativity.

We define signature of a real symmetric nonsingular matrix S, denoted by $\sigma(S)$ as the pair of integers $\sigma(S) = (\sigma_-(S), \sigma_+(S))$, where $\sigma_-(S)$ and $\sigma_+(S)$ are the number of negative and positive eigenvalues of S, respectively. In this paper we shall deal only with the case when the positive signature $\sigma_+(S)$ equals the input cardinality $\mathtt{m}(\mathfrak{B})$ of the behavior $\mathfrak{B}$. Dissipativity with respect to this matrix is required for the $\mathcal{H}_\infty$ norm of a corresponding transfer matrix to be at most one. Further, as shown in [29, Proposition 2, Part I], $\sigma_+(S) = \mathtt{m}(\mathfrak{B})$ means that the behavior has as high an input cardinality as S-dissipativity allows; we call this condition the *maximum input cardinality condition.*

We are now ready to state one of the main results of this paper. The following theorem tells that if a certain unmixing condition is satisfied for the uncontrollable poles, then the controllable part of a behavior being strictly dissipative is equivalent to the existence of a storage function for the whole behavior's strict dissipativity. Recall from Definition 2.1 that the uncontrollable characteristic polynomial χ_{un} of $\mathfrak{B}$ is the monic polynomial whose roots (with suitable multiplicities) are those complex numbers where $R(\xi)$ loses rank. Theorem 3.4 below states that if the uncontrollable poles are such that no pair of the uncontrollable poles is symmetric with respect to the imaginary axis, then noncontrollability of $\mathfrak{B}$ poses no hindrance to strict dissipativity of $\mathfrak{B}$; i.e., strict dissipativities of $\mathfrak{B}$ and $\mathfrak{B}_{\mathrm{cont}}$ are equivalent.

THEOREM 3.4. *Consider a linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and a nonsingular $S \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$ with the input cardinality of $\mathfrak{B}$ equal to the positive signature of S: $\mathtt{m}(\mathfrak{B}) = \sigma_+(S)$. Assume that the uncontrollable characteristic polynomial of $\mathfrak{B}$, χ_{un}, is such that $\chi_{\mathrm{un}}(\xi)$ and $\chi_{\mathrm{un}}(-\xi)$ are coprime. Then, $\mathfrak{B}$ is strictly S-dissipative if and only if its controllable part $\mathfrak{B}_{\mathrm{cont}}$ is strictly S-dissipative.*

We call the condition of coprimeness of $\chi_{\mathrm{un}}(\xi)$ and $\chi_{\mathrm{un}}(-\xi)$ the *unmixing condition.* In the context of autonomous systems, it is well known (see [34], for example) that the unmixing condition is a necessary and sufficient condition for the existence of a unique solution to the Lyapunov equation. In section 8 we explore the extent of necessity of this condition. For some autonomous behaviors (and hence some uncontrollable behaviors), we show that the unmixing condition is not necessary (section 8).

Throughout this paper, we shall assume S has the following form:

$$\Sigma := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}. \tag{3.2}$$

Lemma 3.5 below shows that, for dissipativity considerations, taking Σ as in (3.2) is without any loss of generality. Dissipativity with respect to a different constant matrix S can be easily treated by modifying the behavior suitably, as shown in the following lemma.

LEMMA 3.5. *Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and a real symmetric nonsingular matrix $S \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$. Let $S = T^T \Sigma T$, with $T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ nonsingular, be a symmetric factorization of S. Define $\widetilde{\mathfrak{B}} := T\mathfrak{B}$. Then $\mathfrak{B}$ is S-dissipative if and only if $\widetilde{\mathfrak{B}}$ is Σ-dissipative.*

Before we continue with other preliminaries, results, and proofs, we list the assumptions we make in the rest of this paper that are without loss of any generality. While we do write these standing assumptions explicitly in some results, they are sometimes skipped for brevity. For a kernel representation, the polynomial matrix $R(\xi)$ is assumed to be full row rank, and the matrix $M(\xi)$ in an image representation is assumed to be right-prime. Further, when we start with an i/s/o representation of a behavior, we assume this to be minimal. In the context of dissipativity, Σ is used for the supply rate. We assume Σ to be symmetric and nonsingular.

4. Dissipativity, Riccati equation, and Hamiltonian matrix. In this section we first briefly review existing literature about how the dissipation inequality gives us a well-studied Linear Matrix Inequality (LMI). We bring out the connection between a certain para-Hermitian polynomial matrix related to the behavior and an associated Hamiltonian matrix. See [8, 13, 25] for related results.

For the case that the input cardinality of the behavior is equal to the positive signature of Σ, a necessary condition for dissipativity of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is that a partition of $w = (w_1, w_2)$ corresponding to the matrix Σ (see (3.2)) results in an input/output partition for $\mathfrak{B}$ such that w_1 is input and w_2 is output (see [28, Remark 5.11]). Note that the above partition means $Q_\Sigma(w) = |w_1|^2 - |w_2|^2$. Further, due to the dissipativity, the transfer function from w_1 to w_2 turns out to be *proper*. This implies that $\mathfrak{B}$ allows an i/s/o representation as

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1, \quad w_2 = Cx + Dw_1, \tag{4.1}$$

with (C, A) observable (see Proposition 2.2). The link among dissipativity of a controllable behavior, storage functions, and LMIs is the subject of [30, 21, 4]; we state this result below for easy reference.

PROPOSITION 4.1. *$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is Σ-dissipative if and only if there exists a solution $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ for the following LMI*

$$\begin{bmatrix} \left(C^TC + A^TK + KA\right) & \left(KB + C^TD\right) \\ \left(B^TK + D^TC\right) & -\left(I_{\mathtt{m}} - D^TD\right) \end{bmatrix} \leqslant 0, \tag{4.2}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$ and $w_2 = Cx + Dw_1$ is a state-controllable and state-observable i/s/o representation of $\mathfrak{B}$ with the input/output partition induced by the block matrix Σ.

In the above proposition, the memoryless state function x^TKx acts as a storage function for the controllable behavior $\mathfrak{B}$. See Corollary 5.6 below for our result regarding uncontrollable behaviors. The above LMI is the well-known *bounded-real*

LMI. Assume $(I_{\mathtt{m}} - D^T D) > 0$ (some implications of this assumption will be clarified in the next section). The Schur complement of $(I_{\mathtt{m}} - D^T D)$ in inequality (4.2) gives the Algebraic Riccati Inequality (ARI)[2]

$$\begin{aligned}&\left(A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C\right)^T K + K\left(A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C\right)\\ &\quad + C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1} C + KB\left(I_{\mathtt{m}} - D^T D\right)^{-1} B^T K \leqslant 0.\end{aligned} \tag{4.3}$$

Note that K satisfies the ARI if and only if K satisfies the above LMI.

The corresponding equation is the Algebraic Riccati Equation (ARE), and we use properties of this equation for various results in this paper. Interestingly, the solution to the ARE can be found from certain $\mathtt{n}$-dimensional invariant subspaces of a $2\mathtt{n} \times 2\mathtt{n}$ matrix known as the *Hamiltonian* matrix. This paper uses properties of the Hamiltonian matrix to relate to dissipativity. The procedure of constructing a solution to the ARE from an $\mathtt{n}$-dimensional eigenspace of the Hamiltonian matrix comes in several texts, for example, [7, 10]. For easy reference we present this result as Proposition 4.4 below. We first define *Lambda-sets* of the roots of an even polynomial $p(\xi)$ having no roots on the imaginary axis, for it will be of importance in the sequel.

Remark 4.2. As a convention in this paper, a set of roots of a polynomial (or that of eigenvalues of a real constant matrix) has every element appearing as many number of times as its multiplicity (algebraic multiplicity in case of eigenvalues), and therefore equality of such sets means equality with the multiplicities counted. This helps avoid writing certain polynomials are equal after ensuring monicity.

The definition of a Lambda-set plays an important role in the partition of a set of complex numbers which are symmetric with respect to the imaginary axis. This notion is similar to that of an S-set [20]. $\bar{\Lambda}$ below denotes the set of complex conjugates of the elements in Λ.

Definition 4.3. *Let $p(\xi)$ be a nonzero even polynomial in ξ with no roots on the imaginary axis. A set of complex numbers $\Lambda \subset$* roots $(p(\xi))$ *is said to be a* Lambda-set *of* roots $(p(\xi))$ *if it satisfies the following properties:*

1. $\Lambda = \bar{\Lambda}$,
2. $\Lambda \cap (-\Lambda) = \phi$, *and*
3. $\Lambda \cup (-\Lambda) =$ roots $(p(\xi))$ *(counted with multiplicity).*

The disjointness condition (condition 2) requires that p has no roots on the imaginary axis. We called this the unmixing condition in the remark following Theorem 3.4. Proposition 4.4 below is well known; it relates ARE solutions to the *Hamiltonian matrix* H, defined below.

Proposition 4.4. *Consider the ARE: $A^T K + KA + C^T C + KBB^T K = 0$ in the unknown real symmetric matrix $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Corresponding to this ARE, construct the Hamiltonian matrix, $H := \begin{bmatrix} A & BB^T \\ -C^T C & -A^T \end{bmatrix}$. Assume H does not have eigenvalues on the imaginary axis and let Λ be a Lambda-set of* $\operatorname{spec}(H)$. *Suppose the $\mathtt{n}$-dimensional H-invariant subspace corresponding to Λ is given by*

$$\mathcal{X}_\Lambda(H) := \operatorname{im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{4.4}$$

where $X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. A real symmetric solution K to the ARE satisfying $\operatorname{spec}(A + BB^T K) = \Lambda$ *exists if and only if X_1 is nonsingular.*

[2] We used the fact that positive definiteness of $I_{\mathtt{m}} - D^T D$ and $I_{\mathtt{p}} - DD^T$ are equivalent.

If X_1 as defined above is nonsingular, then $K := X_2 X_1^{-1}$ is a solution to the ARE. Thus the solvability of the ARE through suitable $\mathtt{n}$-dimensional invariant subspaces of the Hamiltonian matrix gives a condition for the existence of a storage function (state function) that satisfies the dissipation inequality. Note that the above result is independent of any controllability assumption. For controllable behaviors that have an i/s/o representation given by (4.1) such that (A, B) is controllable and (C, A) is observable, it turns out that the eigenvalues of the Hamiltonian matrix are exactly equal to the roots of the determinant of the corresponding para-Hermitian matrix $\partial\Phi(\xi)$ coming from the image representation matrix of the behavior and Σ. We state this result as a lemma below. The result is quite expected, and we prove it (in Appendix A) for the sake of completeness, and since we shall extend it to the case of uncontrollability in Theorem 5.4.

LEMMA 4.5. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ have an i/s/o representation, $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$ with (A, B) controllable and (C, A) observable and $\Sigma := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}$. Assume $(I_{\mathtt{m}} - D^T D) > 0$. Define the real $2\mathtt{n} \times 2\mathtt{n}$ Hamiltonian matrix*

$$H := \begin{bmatrix} A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C & B\left(I_{\mathtt{m}} - D^T D\right)^{-1} B^T \\ -C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1} C & -\left(A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C\right)^T \end{bmatrix}.$$

Suppose $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$; $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ is an observable image representation of $\mathfrak{B}$ and consider $\partial\Phi(\xi) := M^T(-\xi)\Sigma M(\xi)$. Then, the Hamiltonian matrix eigenvalues are same as the zeros of $\partial\Phi(\xi)$, counted with multiplicities, i.e., $\operatorname{spec}(H) = \operatorname{roots}(\det \partial\Phi(\xi))$.

Proof. See Appendix A. ∎

The para-Hermitian matrix $\partial\Phi(\xi)$ comes from the image representation of the behavior and Σ, whereas the Hamiltonian matrix is formed from the i/s/o representation with the i/o partition induced by Σ. Lemma 4.5 above nicely brings out a relation between these two matrices and helps in establishing a system theoretic meaning to the Hamiltonian matrix (see [15]). We shall make use of this lemma in proving our main result: Theorem 3.4.

For the special case that (A, B) is controllable, the result in Proposition 4.4 can be further extended. It has been shown in [7, 10] that if (A, B) is controllable and if H has no eigenvalues on the imaginary axis, then H gives a solution to the ARE. We state this result as a proposition below. We shall make use of this result within some proofs in the sequel to infer about the existence of a solution to an ARE coming from a strict dissipation inequality

PROPOSITION 4.6. *Consider the Hamiltonian matrix given by $H := \begin{bmatrix} A & BB^T \\ -C^T C & -A^T \end{bmatrix}$. If (A, B) is controllable and H has no roots on the imaginary axis, then there exists a real symmetric solution to the ARE: $A^T K + KA + C^T C + KBB^T K = 0$.*

5. Dissipativity of uncontrollable behaviors.

In this section we prove Theorem 3.4 using the results presented in the last section and some more presented here. We first consider an example of a simple electrical circuit as shown in Figure 5.1. Under the condition $R_1 C \neq L/R_2$, the port variables (manifest variables) (v, i) satisfy the following differential equation:

$$\left[\left(LC\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (R_1 + R_2)C\frac{\mathrm{d}}{\mathrm{d}t} + 1\right) - \left(R_1 LC\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (R_1 R_2 C + L)\frac{\mathrm{d}}{\mathrm{d}t} + R_2\right)\right]\begin{bmatrix} v \\ i \end{bmatrix} = 0.$$

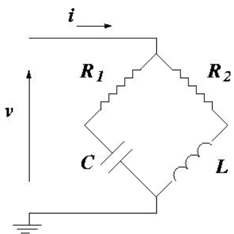

FIG. 5.1. *An LCR circuit.*

For the case that $R_1C = L/R_2$, and $R_1 = R_2$, the system becomes uncontrollable. The corresponding kernel representation is

$$\left[\left(R_2C\frac{\mathrm{d}}{\mathrm{d}t}+1\right) - \left(L\frac{\mathrm{d}}{\mathrm{d}t}+R_2\right)\right]\begin{bmatrix} v \\ i \end{bmatrix} = 0.$$

If the voltage across the capacitor v_{C} and current through the inductor i_{L} are considered as internal system variables, then we can write the following dissipation inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Cv_{\mathrm{C}}^2 + Li_{\mathrm{L}}^2\right) \leqslant \begin{bmatrix} v & i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix}.$$

However, it turns out that the latent variables $(v_{\mathrm{C}}, i_{\mathrm{L}})$ are not observable from (v, i), and so the storage function in the left-hand side of above inequality cannot be written in terms of a QDF in just the manifest variables (see Remark 5.1 below). We ask the question: is it possible to find a storage function in terms of the manifest variables, or do we have to have, for some cases, storage functions in terms of "hidden" variables only (variables that are unobservable from the manifest variables are also said to be hidden)? Our main result Theorem 3.4 addresses this issue under the unmixing assumption, and gives a necessary and sufficient condition for the existence of a storage function in terms of manifest variables. Thus Theorem 3.4 rules out the necessity of hidden variables to construct storage functions.

For the case of the above example, as derived in [33], $q(v - R_1 i)^2$ with any $q > 0$ is a storage function, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}q(v - R_1 i)^2 \leqslant vi,$$

which is a dissipation inequality in just the manifest variables. The fact that this storage function has no apparent interpretation as physical energy is discussed in Remark 5.1 below. Further, $q > 0$ makes the set of storage functions unbounded for this case: in section 7 we shall prove the unboundedness for general uncontrollable behaviors.

Remark 5.1. The question of whether to allow unobservable variables into the storage function has been an issue in [33, 5]. We call a storage function *observable* if it can be expressed as a function of the manifest variables w and its derivatives. In this paper a storage function is observable by definition. In certain physical systems, like the electrical circuit above, one is able to construct a storage function from the configuration of the individual elements within the system. However, as noted in above references, the situation that the internal system variables may not be observable from the manifest variables (for example, in the above circuit, (v,i), through which energy is exchanged with the environment) raises the issue of whether to allow a storage function to depend on unobservable system variables also. An important contribution of this paper is that we have resolved this issue at least for the case of strict dissipativity. Under the unmixing and the maximum input cardinality conditions, we have obtained an observable storage function for strictly dissipative behaviors. However, it may turn out for some cases, like in the circuit above, that the observable storage function we obtain has no physical energy interpretation. In general for a network consisting of an interconnection of passive elements, which are either lossless elements with memory and strictly dissipative elements without memory (see [30, p. 336]), the sum of energies stored in the individual passive elements gives a natural storage function. The fact that this property is not presently captured in the definition is perhaps causing the lack of physical energy interpretation of the observable storage function.

The following example shows that we have improved one of the main results (Theorem 2) in [5] regarding observable storage functions. Consider the uncontrollable behavior, $\mathfrak{B} \in \mathfrak{L}^2$, given by the kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, where $R(\xi) = [2(\xi^2+3\xi+2) \; -(2\xi^2+3\xi+1)]$. The QDF induced by $X^T(\zeta)KX(\eta)$ with

$$X(\xi) := \begin{bmatrix} (2\xi+6) & -(2\xi+3) \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad K := \frac{1}{4}\begin{bmatrix} 0.118 & -0.014 \\ -0.014 & 0.472 \end{bmatrix}$$

serves as an observable storage function with respect to the supply rate $S = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Before the proof of Theorem 3.4 we state and prove the following theorem, which is an important result in its own right and will also be of importance for proving Theorem 3.4. We show that for a controllable behavior, though the definition of strict dissipativity is existential in ϵ, it is equivalent to a pair of conditions that are verifiable without ϵ. The second condition ensures that $\partial\Phi(\xi)$ has no zeros on the imaginary axis while the first condition, loosely speaking, rules out the existence of zeros of $\partial\Phi(\xi)$ at infinity.

THEOREM 5.2. *Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ that has an observable image representation $\mathfrak{B} = \operatorname{im} M(\frac{\mathrm{d}}{\mathrm{d}t})$. Define $\partial\Phi(\xi) = M^T(-\xi)\Sigma M(\xi)$. Let $\mathtt{n}$ be the McMillan degree of $\mathfrak{B}$. Then $\mathfrak{B}$ is strictly dissipative with respect to Σ if and only if the following are satisfied:*

1. $\deg\,(\det \partial\Phi(\xi)) = 2\mathtt{n}$,
2. $\partial\Phi(j\omega) > 0$ *for all* $\omega \in \mathbb{R}$.

In order to prove the above theorem, we use the following lemma, whose proof is in the appendix. Notice that in both, the theorem above and the lemma below, condition 2 does *not* imply condition 1. Strictness of the dissipativity (i.e., existence of an $\epsilon > 0$) plays a role in this implication.

LEMMA 5.3. *Let a controllable behavior $\mathfrak{B} = \operatorname{im} M(\frac{\mathrm{d}}{\mathrm{d}t})$ have i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$ and $w_2 = Cx + Dw_1$. Suppose $\mathtt{n}$ is the McMillan degree of $\mathfrak{B}$. Define $\partial\Phi(\xi) = M^T(-\xi)\Sigma M(\xi)$. Then $(I_{\mathtt{m}} - D^TD) > 0$ if the following conditions are satisfied:*

1. $\deg\,(\det \partial\Phi(\xi)) = 2\mathtt{n}$,
2. $\partial\Phi(j\omega) > 0$ *for all* $\omega \in \mathbb{R}$.

Proof. See Appendix A. ∎

Proof of Theorem 5.2. **(If)** Assuming both conditions 1 and 2 are true, we shall show that the behavior is strictly Σ-dissipative. It follows from [28, Proposition 5.2] that the second condition, $\partial\Phi(j\omega) > 0$ for all real ω, implies that $\mathfrak{B}$ is guaranteed to be dissipative with respect to Σ. Therefore, it follows from Proposition 4.1 that $\mathfrak{B}$

allows an i/s/o representation (4.1) and there exists $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that the following LMI is satisfied:

$$\begin{bmatrix} (C^TC + A^TK + KA) & (KB + C^TD) \\ (B^TK + D^TC) & -(I_{\mathtt{m}} - D^TD) \end{bmatrix} \leqslant 0.$$

Using Lemma 5.3, it follows that conditions 1 and 2 of Theorem 5.2 imply $(I_{\mathtt{m}} - D^TD) > 0$. Therefore, the LMI has rank-minimizing solutions coming from the following ARE:

$$\left(A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC\right)^T K + K\left(A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC\right) \\ + C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1} C + KB\left(I_{\mathtt{m}} - D^TD\right)^{-1} B^TK = 0.$$

So from Proposition 4.4 there exists a Hamiltonian matrix H given below corresponding to the above ARE such that its solutions come from the $\mathtt{n}$-dimensional (generalized) eigenspaces of

$$H := \begin{bmatrix} A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC & B\left(I_{\mathtt{m}} - D^TD\right)^{-1} B^T \\ -C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1} C & -\left(A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC\right)^T \end{bmatrix}.$$

It follows from Lemma 4.5 that $\operatorname{spec}(H) = \operatorname{roots}(\det \partial\Phi(\xi))$. Since $\partial\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$ roots $(\det \partial\Phi(\xi)) \cap j\mathbb{R} = \phi$. Therefore, due to Lemma 4.5, H does not have any purely imaginary eigenvalues. So from the continuity of eigenvalues (see [11]) there exists $\epsilon \in \mathbb{R}$ small enough such that the following matrix

$$H_\epsilon = H - \epsilon \begin{bmatrix} 0 & 0 \\ C^TC & 0 \end{bmatrix}$$

also has no eigenvalues on the imaginary axis. Note that from Proposition A.1 in the appendix, (A, B) controllable implies so is $[(A + B(I_{\mathtt{m}} - D^TD)^{-1}D^TC), B(I_{\mathtt{m}} - D^TD)^{-\frac{1}{2}}]$.

Hence it follows from Proposition 4.6 that H_ϵ gives a solution K to the corresponding ARE

$$\left(A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC\right)^T K + K\left(A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC\right) \\ + C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1} C + \epsilon C^TC + KB\left(I_{\mathtt{m}} - D^TD\right)^{-1} B^TK = 0,$$

which implies, from Proposition 4.1, that $\mathfrak{B}$ is dissipative with respect to $\begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -(1+\epsilon)I_{\mathtt{p}} \end{bmatrix}$ for some $\epsilon > 0$.

Utilizing the Lemma A.2 in the appendix, we conclude that $\mathfrak{B}$ being dissipative with respect to $\begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -(1+\epsilon)I_{\mathtt{p}} \end{bmatrix}$ implies $\mathfrak{B}$ is strictly Σ-dissipative, and this completes the "if" part of Theorem 5.2.

(Only if) First we show that $\mathfrak{B}$ being strictly Σ-dissipative implies condition 2 holds. Then we shall further show that assuming condition 2 holds, $\mathfrak{B}$ being strictly Σ-dissipative implies that condition 1 holds. Assume that $\mathfrak{B}$ is strictly Σ-dissipative. Definition 3.2 implies there exists $\epsilon > 0$ such that $\mathfrak{B}$ is dissipative with respect to $\Sigma - \epsilon I_{\mathtt{m}}$. It then follows from Proposition 3.3 and [28, Proposition 5.2] that

$$\partial\Phi(j\omega) \geqslant \epsilon M^T(-j\omega)M(j\omega). \tag{5.1}$$

Note that $M^T(-j\omega)M(j\omega) \geqslant 0$ for all real ω. Since $M(\xi)$ is right-prime, it follows from Lemma A.5 that $M^T(-j\omega)M(j\omega) > 0$ for all $\omega \in \mathbb{R}$. This implies, from inequality (5.1) that $\partial\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$. This shows that if $\mathfrak{B}$ is strictly Σ-dissipative, then condition 2 holds.

Now we show that $\mathfrak{B}$ being strictly Σ-dissipative together with $\partial\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$ implies that deg (det $\partial\Phi(\xi)) = 2\mathtt{n}$. Consider a spectral factorization of $\partial\Phi(\xi) = N^T(-\xi)N(\xi)$ (existence of $N \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ is guaranteed due to the inequality $\partial\Phi(j\omega) > 0$; see [18]). Again, since $M(\xi)$ is right-prime, $M^T(-j\omega)M(j\omega) > 0$ for all $\omega \in \mathbb{R}$ (Lemma A.5). So $M^T(-\xi)M(\xi)$ also allows a spectral factorization

$$M^T(-\xi)M(\xi) = D^T(-\xi)D(\xi); \quad D \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}(\xi).$$

By Lemma A.5 in the appendix, $M(\xi)$ being a right-prime polynomial matrix implies that deg (det $(M^T(-\xi)M(\xi))) = 2\mathtt{n}$. Therefore deg (det $D(\xi)) = \mathtt{n}$. Since $\mathfrak{B}$ is strictly Σ-dissipative, inequality (5.1) holds, which we can now rewrite in terms of the spectral factors of $\partial\Phi(\xi)$ and $M^T(-\xi)M(\xi)$ as

$$\begin{aligned} & N^T(-j\omega)N(j\omega) \geqslant \epsilon D^T(-j\omega)D(j\omega) \quad \forall\omega \in \mathbb{R} \\ \Rightarrow & \left[N(-j\omega)D^{-1}(-j\omega)\right]^T \left[N(j\omega)D^{-1}(j\omega)\right] \geqslant \epsilon I_{\mathtt{m}} \quad \forall\omega \in \mathbb{R}. \end{aligned} \tag{5.2}$$

The above inequality (5.2) implies that there exists a real $\epsilon > 0$ such that the minimum singular value of the rational function matrix $N(j\omega)D^{-1}(j\omega)$ is at least ϵ for all real ω. This further implies that det $(N(\xi)D^{-1}(\xi))$ is a biproper rational function, which means deg (det $N(\xi)) = \mathtt{n}$ and hence deg (det $\partial\Phi(\xi)) = 2\mathtt{n}$. ∎

The following result will be important in order to prove the main result: Theorem 3.4. It is an extension of Lemma 4.5, where we saw that for a controllable behavior $\mathfrak{B}$ the set of eigenvalues of the Hamiltonian matrix is equal to the set of roots of the determinant of the para-Hermitian matrix $\partial\Phi(\xi)$. For the case of uncontrollability we show that every uncontrollable pole λ of $\mathfrak{B}$, together with $-\lambda$, is an eigenvalue of H, in addition to those coming from the controllable part of $\mathfrak{B}$ like in Lemma 4.5.

THEOREM 5.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and let χ_{un} be its uncontrollable characteristic polynomial. Assume $\mathfrak{B}$ has an observable i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$, and suppose $I_{\mathtt{m}} - D^TD$ is invertible. Let $\mathfrak{B}_{\text{cont}} = \text{im } M(\frac{\mathrm{d}}{\mathrm{d}t})$. Define $\partial\Phi(\xi) = M^T(-\xi)\Sigma M(\xi)$. Construct the Hamiltonian matrix*

$$H := \begin{bmatrix} A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1}D^TC & B\left(I_{\mathtt{m}} - D^TD\right)^{-1}B^T \\ -C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1}C & -\left(A + B\left(I_{\mathtt{m}} - D^TD\right)^{-1}D^TC\right)^T \end{bmatrix},$$

Then, the Hamiltonian matrix eigenvalues, the zeros of $\partial\Phi(\xi)$, and the uncontrollable poles of $\mathfrak{B}$ are related by: spec(H) = roots [det $\partial\Phi(\xi)\ \chi_{\text{un}}(\xi)\ \chi_{\text{un}}(-\xi)$], *counted with multiplicities.*

Proof. See Appendix A. ∎

The next result is one of the main results of this paper and it is pivotal for proving Theorem 3.4. It brings out an important property about certain $\mathtt{n}$-dimensional invariant subspaces of the Hamiltonian matrix. We already know from Proposition 4.4 that statement 2 in the theorem below is equivalent to existence of a solution to the ARE. In this sense, Theorem 5.5 below is an important extension to Proposition 4.6. The theorem shows that a given Lambda set results in a solution to the ARE if and only if this Lambda set contains the uncontrollable poles of the system. The proof comes following a line of argument similar to the one used in proving [10, Theorem 7.2].

THEOREM 5.5. *Consider the Hamiltonian matrix,* $H = \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix}$. *Define* $\Lambda_{\mathrm{un}}^{(A,B)}$ *to be the set of uncontrollable eigenvalues of* (A, B) *pair and let* Λ *be a Lambda-set of* $\operatorname{spec}(H)$. *Then the following are equivalent.*

1. $\Lambda \supseteq \Lambda_{\mathrm{un}}^{(A,B)}$.
2. *The* $\mathtt{n}$*-dimensional invariant subspace of* H *corresponding to* Λ *is complementary to* $\operatorname{im} \begin{bmatrix} 0_{\mathtt{n}} \\ I_{\mathtt{n}} \end{bmatrix}$.

Proof. **(1 ⇒ 2)** Denote by $\mathcal{X}_\Lambda(H)$ the invariant subspace of H corresponding to a Lambda-set Λ of $\operatorname{spec}(H)$. Since Λ is a Lambda-set of $\operatorname{spec}(H)$ $\Lambda \cap (-\Lambda) = \phi$ and $\Lambda \cup (-\Lambda) = \operatorname{spec}(H)$, and therefore $\dim(\mathcal{X}_\Lambda(H)) = \mathtt{n}$. Let $\mathcal{X}_\Lambda(H)$ be given by

$$\mathcal{X}_\Lambda(H) = \operatorname{im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \tag{5.3}$$

where $X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Since $\mathcal{X}_\Lambda(H)$ is an invariant subspace corresponding to Λ, we have the following equality

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H_\Lambda, \tag{5.4}$$

where $H_\Lambda \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is such that $\operatorname{spec}(H_\Lambda) = \Lambda$.

In order to prove that $\mathcal{X}_\Lambda(H)$ is complementary to $\operatorname{im} \begin{bmatrix} 0_{\mathtt{n}} \\ I_{\mathtt{n}} \end{bmatrix}$, we have to show that $\ker X_1 = \{0\}$. Assume to the contrary that $\ker X_1$ is nontrivial; we shall show that this will lead to a contradiction to $\Lambda \supseteq \Lambda_{\mathrm{un}}^{(A,B)}$.

We may assume without loss of generality that$\mathcal{X}_\Lambda(H)$ is a generalized (right) eigenspace of H with respect to Λ. Using Lemma A.4 from the appendix we get, $\operatorname{im} \begin{bmatrix} X_2 \\ -X_1 \end{bmatrix}$ is a generalized left-eigenspace of H corresponding to $-\Lambda$. Since Λ is a Lambda-set, $\Lambda \cap (-\Lambda) = \phi$, so the two generalized eigenspaces are orthogonal to each other, i.e.,

$$\begin{bmatrix} X_2^T & -X_1^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0.$$

Because $\mathcal{X}_\Lambda(H)$ is H-invariant, the last equation leads to

$$\begin{bmatrix} X_2^T & -X_1^T \end{bmatrix} \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$
$$\Rightarrow X_2^TAX_1 + X_1^TA^TX_2 + X_1^TC^TCX_1 + X_2^TBB^TX_2 = 0. \tag{5.5}$$

Let $x \in \ker X_1$. Pre- and postmultiplying (5.5) by x^T and x, respectively, we get $x^TX_2^TBB^TX_2x = 0$, which implies that $B^TX_2x = 0$. Consider (5.4), which gives $AX_1 + BB^TX_2 = X_1H_\Lambda$. After postmultiplying by x we get

$$\begin{aligned} AX_1x + BB^TX_2x &= X_1H_\Lambda x \\ \Rightarrow X_1H_\Lambda x &= 0 \quad \Rightarrow H_\Lambda x \in \ker X_1. \end{aligned}$$

This implies $\ker X_1$ is H_Λ-invariant. Therefore, there exists $v \neq 0$ an eigenvector of H_Λ such that $X_1v = 0$. Let the eigenvalue corresponding to v be λ. Since $\operatorname{spec}(H_\Lambda) = \Lambda$, $\lambda \in \Lambda$. Now, from (5.4) we can write

$$-C^TCX_1 - A^TX_2 = X_2H_\Lambda. \tag{5.6}$$

Postmultiplying (5.6) by v we get

$$\begin{aligned}&-C^TCX_1v - A^TX_2v = X_2H_\Lambda v\\ &\Rightarrow -A^TX_2v = \lambda X_2v.\end{aligned}$$

But this means X_2v is a left eigenvector of A with eigenvalue $-\lambda$. Moreover $-\lambda \notin \Lambda$ (because $\Lambda \cap (-\Lambda) = \phi$) and $B^TX_2v = 0$. This together means that $-\lambda$ is an uncontrollable eigenvalue of A, which is a contradiction to $\Lambda \supseteq \Lambda_{\text{un}}^{(A,B)}$. Thus ker X_1 cannot be nontrivial, and therefore $\mathcal{X}_\Lambda$ is complementary to im $\begin{bmatrix} 0_{\mathtt{n}} \\ I_{\mathtt{n}} \end{bmatrix}$.

(2 ⇒ 1) We assume that $\mathcal{X}_\Lambda(H)$, the generalized eigenspace of H corresponding to Λ (a Lambda-set of spec(H)), is complementary to im $\begin{bmatrix} 0_{\mathtt{n}} \\ I_{\mathtt{n}} \end{bmatrix}$, and we show that $\Lambda \supseteq \Lambda_{\text{un}}^{(A,B)}$. Suppose $\Lambda \not\supseteq \Lambda_{\text{un}}^{(A,B)}$. Then there exists $\lambda \in \Lambda_{\text{un}}^{(A,B)}$ but $\lambda \notin \Lambda$. Since Λ is a Lambda-set of spec(H), $\Lambda \cup (-\Lambda) = \text{spec}(H)$. Also $\Lambda_{\text{un}}^{(A,B)} \subset \text{spec}(H)$ (see Theorem 5.4). These two facts together imply that $-\lambda \in \Lambda$. Now $\lambda \in \Lambda_{\text{un}}^{(A,B)}$ implies that there exists a nonzero vector $v \in \mathbb{C}^{\mathtt{n}}$ such that $v^TA = \lambda v^T$ and $v^TB = 0$. Therefore the $2\mathtt{n}$ vector constructed as $w = \text{col}(0_{\mathtt{n}}, v)$ satisfies $Hw = -\lambda w$; i.e., w is an eigenvector of H with eigenvalue $-\lambda$. Since $-\lambda \in \Lambda$, the last w being an eigenvector of H with eigenvalue $-\lambda$ means $\mathcal{X}_\Lambda$ is *not* complementary to im $\begin{bmatrix} 0_{\mathtt{n}} \\ I_{\mathtt{n}} \end{bmatrix}$ because w has the upper $\mathtt{n}$ entries zero. Thus $\Lambda \not\supseteq \Lambda_{\text{un}}^{(A,B)}$ leads to a contradiction to statement 2. This proves "2 ⇒ 1." □

With the above results we are now in a position to prove our main result, Theorem 3.4. The "if" part requires construction of a storage function for the uncontrollable behavior $\mathfrak{B}$ to show strict dissipativity. We use the Hamiltonian matrix properties proved above, combined with some perturbation arguments, to show the Riccati equality and inequality have solutions, and then construct a storage function for proving the strict dissipativity. The unmixing property plays a key role in this construction.

Proof of Theorem 3.4. **(Only if)** We assume that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is strictly Σ-dissipative, and we show that this implies $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative. Since $\mathfrak{B}$ is strictly Σ-dissipative, according to Definition 3.2 there exists $\Psi(\zeta,\eta) \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and a real number $\epsilon > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_\Psi(w) \leqslant Q_\Sigma(w) - \epsilon|w|^2 \quad \text{for all } w \in \mathfrak{B}.$$

Integrating both sides of the above inequality, considering only those trajectories in $\mathfrak{B} \cap \mathfrak{D}$ we get

$$\int_{-\infty}^{\infty} Q_\Sigma(w)dt \geqslant \epsilon \int_{-\infty}^{\infty} |w|^2 dt \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Since $\mathfrak{B} \cap \mathfrak{D} = \mathfrak{B}_{\text{cont}} \cap \mathfrak{D}$ (see [16]), the above inequality implies that $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative.

(If) We assume that the controllable part $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative and that the unmixing condition on the uncontrollable poles holds. We show that $\mathfrak{B}$ too is strictly Σ-dissipative. A necessary condition for $\mathfrak{B}_{\text{cont}}$ to be strictly Σ-dissipative is that the transfer function for $\mathfrak{B}_{\text{cont}}$ with the i/o partition induced by Σ is proper. Therefore, from Proposition 2.2 above, $\mathfrak{B}$ has a state map $X(\xi)$ and an i/s/o representation with $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$ such that $\frac{\mathrm{d}}{\mathrm{d}t}x = \widetilde{A}x + \widetilde{B}w_1$, $w_2 = \widetilde{C}x + \widetilde{D}w_1$ with $(\widetilde{C}, \widetilde{A})$ pair observable and $(\widetilde{A}, \widetilde{B})$ pair uncontrollable with $\Lambda_{\text{un}}^{(\widetilde{A},\widetilde{B})} = \Lambda_{\text{un}}(\mathfrak{B})$, counting multiplicities. Let $\mathfrak{B}_{\text{cont}}$ have an image representation

$$\mathfrak{B}_{\text{cont}} = \text{im} \begin{bmatrix} W_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ W_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}; \quad W_1 \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi], W_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi].$$

Since $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative, from Theorem 5.2,

$$\begin{aligned} \deg\ \left(\det\left[W_1^T(-\xi)W_1(\xi) - W_2^T(-\xi)W_2(\xi)\right]\right) = 2 \times \mathtt{n}(\mathfrak{B}_{\text{cont}}) \text{ and} \\ W_1^T(-j\omega)W_1(j\omega) - W_2^T(-j\omega)W_2(j\omega) > 0 \quad \forall \omega \in \mathbb{R}. \end{aligned}$$

Again, from Lemma 5.3, these two facts together imply that $I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D} > 0$. This implies that there exists a Hamiltonian matrix given by

$$H = \begin{bmatrix} \widetilde{A} + \widetilde{B}\left(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}\right)^{-1}\widetilde{D}^T\widetilde{C} & \widetilde{B}\left(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}\right)^{-1}\widetilde{B}^T \\ -\widetilde{C}^T\left(I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T\right)^{-1}\widetilde{C} & -\left(\widetilde{A} + \widetilde{B}\left(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}\right)^{-1}\widetilde{D}^T\widetilde{C}\right)^T \end{bmatrix}.$$

Define $A := \widetilde{A} + \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-1}\widetilde{D}^T\widetilde{C}$, $B := \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-\frac{1}{2}}$, and $C := (I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T)^{-\frac{1}{2}}\widetilde{C}$. Using Theorem 5.4 we get $\text{spec}(H) = \text{roots}\ [\det \partial\Phi(\xi)\ \chi_{\text{un}}(\xi)\ \chi_{\text{un}}(-\xi)]$, where $\partial\Phi(\xi) := W_1^T(-\xi)W_1(\xi) - W_2^T(-\xi)W_2(\xi)$. Since $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative, Theorem 5.2 implies that $\text{roots}\ (\det \partial\Phi(\xi)) \cap j\mathbb{R} = \phi$. By assumption $\chi_{\text{un}}(\xi)$ and $\chi_{\text{un}}(-\xi)$ are coprime, i.e., $\Lambda_{\text{un}} \cap (-\Lambda_{\text{un}}) = \phi$. Thus H has no eigenvalues on the imaginary axis and so $\text{spec}(H)$ allows a Lambda-set Λ. Moreover, we can construct Λ in such a way that $\Lambda_{\text{un}} \subseteq \Lambda$. From Proposition 2.2 it follows that the set of uncontrollable eigenvalues of (A, B) pair is exactly equal to Λ_{un}. Hence from Theorem 5.5 a generalized eigenspace of H corresponding to Λ, $\mathcal{X}_\Lambda := \text{im} \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right]$, with $X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, is complementary to $\text{im} \left[\begin{smallmatrix} 0_{\mathtt{n}} \\ I_{\mathtt{n}} \end{smallmatrix}\right]$, which implies X_1 is nonsingular allowing us to define $K = X_2X_1^{-1} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Again since $\Lambda \cap (-\Lambda) = \phi$, applying Lemma A.4 we get

$$\begin{bmatrix} X_2^T & -X_1^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0, \text{ which implies that } \left(X_2X_1^{-1}\right)^T = X_2X_1^{-1}.$$

Thus $K = K^T$ is a symmetric solution to the ARE: $A^TK + KA + C^TC + KBB^TK = 0$.

In order to complete the proof we have to show that there exists a storage function for the strict dissipation inequality of Definition 3.2. For this we make use of Lemma A.2 of Appendix A to infer that the strict dissipation inequality is equivalent to the following LMI

$$\begin{bmatrix} -\left(\widetilde{C}^T\widetilde{C} + \widetilde{A}^TK + K\widetilde{A}\right) & -\left(K\widetilde{B} + \widetilde{C}^T\widetilde{D}\right) \\ -\left(\widetilde{B}^TK + \widetilde{D}^T\widetilde{C}\right) & sI_{\mathtt{m}} - \widetilde{D}^T\widetilde{D} \end{bmatrix} - \epsilon \begin{bmatrix} \widetilde{C}^T\widetilde{C} & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0, \tag{5.7}$$

for some $\epsilon > 0$. The corresponding Hamiltonian matrix turns out to be

$$\begin{aligned} H_\epsilon &:= \begin{bmatrix} \widetilde{A} + \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-1}\widetilde{D}^T\widetilde{C} & \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-1}\widetilde{B}^T \\ -\widetilde{C}^T(I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T)^{-1}\widetilde{C} & -(\widetilde{A} + \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-1}\widetilde{D}^T\widetilde{C})^T \end{bmatrix} - \epsilon \begin{bmatrix} 0 & 0 \\ \widetilde{C}^T\widetilde{C} & 0 \end{bmatrix} \\ &= H - \epsilon \begin{bmatrix} 0 & 0 \\ \widetilde{C}^T\widetilde{C} & 0 \end{bmatrix}. \end{aligned}$$

Observe that the kind of perturbation that takes H to H_ϵ is such that $\mathrm{spec}(H_\epsilon) \supset \Lambda_{\mathrm{un}}$. Further, since the perturbation is analytic, according to [11], there exists an $\epsilon_1 > 0$ small enough such that the following property of H holds for H_{ϵ_1} also:

$$\mathrm{spec}(H_{\epsilon_1}) \cap j\mathbb{R} = \phi.$$

Therefore, $\mathrm{spec}(H_{\epsilon_1})$ also allows a Lambda-set Λ_{ϵ_1} such that $\Lambda_{\epsilon_1} \supseteq \Lambda_{\mathrm{un}}$. It follows from Theorem 5.5 that there exists $K_{\epsilon_1} = K_{\epsilon_1}^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, which is a rank-minimizing solution to the LMI (5.7) with $\epsilon = \epsilon_1$.

The observable i/s/o representation is obtained from the manifest variables through a state map as $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$. Thus, $X(\xi)$ and K_{ϵ_1} give $\Psi(\zeta,\eta) := X^T(\zeta)K_{\epsilon_1}X(\eta)$ that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_\Psi(w) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w\right)^T K_{\epsilon_1} X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w\right] \leqslant Q_\Sigma(w) - \epsilon_1|w|^2 \quad \forall w \in \mathfrak{B},$$

which from Definition 3.2 means that $\mathfrak{B}$ is strictly Σ-dissipative. □

It is evident from the above proof that the storage function we construct to show strict dissipativity is, in fact, a memoryless quadratic function of the state of the system. More concretely, under the assumption that the uncontrollable poles are unmixed, such a storage function, which is a memoryless state function, exists if the behavior is strictly dissipative. We state this important consequence as a corollary below.

COROLLARY 5.6. *Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ has uncontrollable poles satisfying $\Lambda_{\mathrm{un}} \cap (-\Lambda_{\mathrm{un}}) = \phi$, and let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ give a minimal state map for $\mathfrak{B}$. Consider a nonsingular $\Sigma \in \mathbb{R}_{\mathrm{s}}^{\mathtt{w}\times\mathtt{w}}$ with $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then, the following are equivalent.*

1. *$\mathfrak{B}$ is strictly Σ-dissipative.*
2. *There exists a $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $\epsilon > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}[(X(\frac{\mathrm{d}}{\mathrm{d}t})w)^T KX(\frac{\mathrm{d}}{\mathrm{d}t})w] \leqslant w^T(\Sigma - \epsilon I)w$ for all $w \in \mathfrak{B}$.*

The above important and intuitive result that storage of energy requires memory elements, namely states, was proved formally for the controllable case in [27]. Note that for the special case that the behavior is controllable we have provided a new and alternative proof of the main result of [27], by showing that for a strictly dissipative behavior there exists a storage function which is a state function.

Remark 5.7. In this paper we have worked with strict dissipativity as defined in Definition 3.2. One of the main reasons to invoke strictness is that it guarantees nonsingularity of $I_{\mathtt{m}} - D^TD$ and hence the existence of the Hamiltonian matrix and the Riccati equation. It also rules out the possibility of the Hamiltonian matrix having purely imaginary eigenvalues, and thus enables us to use Lambda-set arguments. It remains to explore which of the above results are true for the case of nonstrict dissipativity.

Remark 5.8. The question of solvability of the positive-real LMI without imposing system theoretic assumptions like controllability or observability has been dealt with in [9]. However, a very restrictive assumption made there is that the whole set of eigenvalues of the system matrix A satisfies the unmixing property, i.e., $\mathrm{spec}(A) \cap (\mathrm{spec}(-A)) = \phi$. According to our main result (Theorem 3.4) this assumption is not necessary. It is sufficient that only the uncontrollable poles satisfy the unmixing property. We shall see later in section 8 the extent of necessity of this unmixing property. The following example shows how the positive-real LMI is solvable when some elements of $\mathrm{spec}(A)$ have symmetry with respect to the imaginary axis and the system is uncontrollable.

Example 5.9. Consider an i/s/o system with the following A, B, C, D matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 1.$$

Observe that $\mathrm{spec}(A) = \{1, -1\}$, which is symmetric with respect to the imaginary axis. Here $\Lambda_{\mathrm{un}} = \{-1\}$, and the other eigenvalue $(= 1)$ is controllable. An equivalent kernel representation of the manifest behavior is given by

$$\begin{bmatrix} \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\mathrm{d}}{\mathrm{d}t} - 2\right) & -\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 1\right) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.$$

We ask the question: is this i/s/o system $S := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ dissipative or equivalently, is there a real symmetric solution $K = K^T \in \mathbb{R}^{2\times 2}$ for the following LMI

$$\begin{bmatrix} -A^T K - KA & C^T - KB \\ C - B^T K & D + D^T \end{bmatrix} \geqslant 0?$$

Obviously, $\Lambda_{\mathrm{un}} = \{-1\}$ satisfies the unmixing property, and one can check that the controllable part $\mathfrak{B}_{\mathrm{cont}} = \ker \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} - 2 & \frac{\mathrm{d}}{\mathrm{d}t} + 1 \end{bmatrix}$ is strictly S-dissipative, which from Theorem 3.4 implies that $\mathfrak{B}$ is strictly S-dissipative. This can be verified by checking that the following real symmetric matrix induces a storage function that satisfies the dissipation inequality

$$K = \begin{bmatrix} -0.957 & -1.457 \\ -1.457 & -1.957 \end{bmatrix},$$

and therefore solves the LMI.

6. Positive storage functions and stabilizability. In this section we establish an important link between stabilizability of systems and positive definiteness of storage functions of strictly dissipative systems. The importance of this link lies in the fact that the energy stored in physical systems is a nonnegative quantity and dissipative physical systems satisfy an additional property that, if the system was initially discharged, then the net energy supplied into the system *upto any time instant* is nonnegative; this is called *half-line dissipativity.* We review these concepts (from [28]) below and prove similar results for uncontrollable systems in this section.

For this paper, we need half-line dissipativity for only the negative half of the real line: $\mathbb{R}_-$. A controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is said to be Σ-dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^{0} Q_\Sigma(w)\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. (Due to time invariance of $\mathfrak{B}$, it is enough to integrate up to 0.) Half-line dissipativity is related to (semi-)definiteness of the storage function. A storage function Q_Ψ is called nonnegative if $Q_\Psi(w)(t) \geqslant 0$ for all $t \in \mathbb{R}$ and $w \in \mathfrak{B}$. For controllable behaviors, it was shown in [28] that existence of a nonnegative storage function is equivalent to dissipativity of $\mathfrak{B}$ on $\mathbb{R}_-$. The importance of nonnegative storage functions is due to such functions being bounded from below (namely, by zero), because of which we expect that when the supply of energy is stopped, then the trajectories cannot become unbounded. This link to stability was made precise and proved in [29, Proposition 1, Part I].

We saw in Corollary 5.6 that, for a dissipative behavior $\mathfrak{B}$ with a minimal state map $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$, a storage function Q_Ψ is associated to a symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $Q_\Psi(w) = (X(\frac{\mathrm{d}}{\mathrm{d}t})w)^T K X(\frac{\mathrm{d}}{\mathrm{d}t})w$. Hence Q_Ψ is nonnegative if and only if $K \geqslant 0$ (see [28]). In the context of strict dissipativity, we define a *positive definite* storage function. A storage function Q_Ψ is called positive definite if $K > 0$.

The following result is one of the main results of this paper. It relates existence of positive definite storage functions to stability of the autonomous part of the uncontrollable dissipative behavior. A behavior with a stable autonomous part is nothing but a *stabilizable* behavior. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called stabilizable if for every $w \in \mathfrak{B}$, there exists a $w' \in \mathfrak{B}$ such that $w(t) = w'(t)$ for $t \leqslant 0$ and $w'(t) \to 0$ as $t \to \infty$. A behavior is stabilizable if and only if $\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$ (see [32]).

THEOREM 6.1. *Let a linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be strictly Σ-dissipative with $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then there exists a positive definite storage function if and only if the following are satisfied:*

1. *there exists $\epsilon > 0$ such that $\int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t \geqslant \int_{\mathbb{R}_-} \epsilon|w|^2\mathrm{d}t$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$ and*
2. *$\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$.*

The first condition is clearly a necessary condition for existence of a positive definite storage function; namely, the controllable part has to be strictly dissipative on $\mathbb{R}_-$. The second condition is also necessary because of the notion that the storage function behaves like a Lyapunov function for an autonomous system, and as is well known, a positive Lyapunov function exists if and only if the autonomous system is asymptotically stable. The fact that these two conditions are together *sufficient* for the existence of a positive definite storage function for the whole behavior is one of the main contributions of this paper. Also notice that $\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$ is a very special case of the unmixing condition. Thus the uncontrollability of the stabilizable behavior poses no hindrance to existence of a storage function for strict dissipativity as long as the controllable/autonomous parts allow storage/Lyapunov functions individually. As noted above, this is the principal finding of this paper.

Proof. **(If)** In this part of the proof we show that the existence of $\epsilon > 0$ such that $\int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t \geqslant \int_{\mathbb{R}_-} \epsilon|w|^2\mathrm{d}t$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$ and $\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$ together imply that $\mathfrak{B}$ has a positive definite storage function. First, let the controllable part be given by an observable image representation

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} W_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ W_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell; \quad \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}),$$

where $W_1 \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ and $W_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$. Since $\mathfrak{B}\cap\mathfrak{D} = \mathfrak{B}_{\mathrm{cont}}\cap\mathfrak{D}$, $\int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$ implies that the QDF induced by the two-variable polynomial matrix $\Phi(\zeta, \eta) := [W_1^T(\zeta) \quad W_2^T(\zeta)]\Sigma[W_1^T(\eta) \quad W_2^T(\eta)]^T$ is strictly *half-line* positive, that is $\int_{\mathbb{R}_-} Q_\Phi(\ell)\mathrm{d}t > 0$ for all nonzero $\ell \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$. This, according to Theorem 6.4 of [28], implies that $W_1(\xi)$ is Hurwitz. Since $\mathfrak{B}$ is strictly Σ-dissipative the transfer function from w_1 to w_2 is proper, and so from Proposition A.3, $\mathfrak{B}$ has a state observable i/s/o representation in Kalman decomposed form as

$$A = \begin{bmatrix} A_{\mathrm{c}} & A_{\mathrm{cp}} \\ 0 & A_{\mathrm{u}} \end{bmatrix}, B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}, C = \begin{bmatrix} C_{\mathrm{c}} & C_{\mathrm{u}} \end{bmatrix},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{c}} = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}w_1, w_2 = C_{\mathrm{c}}x + Dw_1$ gives a state controllable and state observable i/s/o representation of the controllable part $\mathfrak{B}_{\mathrm{cont}}$. This implies that roots $(\det W_1(\xi)) = \mathrm{spec}(A_{\mathrm{c}})$, and, therefore, $W_1(\xi)$ being Hurwitz implies that so is A_{c}. Once again, from Proposition A.3, the set of uncontrollable poles $\Lambda_{\mathrm{un}} = \mathrm{spec}(A_{\mathrm{u}})$. So $\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$ implies A_{u} is also Hurwitz. Hence A is Hurwitz. Because $\mathfrak{B}$ is strictly

Σ-dissipative, the following ARI has a solution

$$A^TK + KA + C^TC + \left(D^TC + B^TK\right)^T \left(I_{\mathtt{m}} - D^TD\right)^{-1} \left(D^TC + B^TK\right) \leqslant 0. \tag{6.1}$$

Since $(D^TC + B^TK)^T(I_{\mathtt{m}} - D^TD)^{-1}(D^TC + B^TK) \geqslant 0$, (C, A) being observable implies that $[C^TC+(D^TC+B^TK)^T(I_{\mathtt{m}}-D^TD)^{-1}(D^TC+B^TK), A]$ too is observable (see [34]). Hence the above inequality (6.1), when treated as a Lyapunov equation $A^TK+KA+Q \leqslant 0$, where $Q := [C^TC+(D^TC+B^TK)^T(I_{\mathtt{m}}-D^TD)^{-1}(D^TC+B^TK)]$, has (Q, A) observable and A Hurwitz, which means that all the solutions K are positive definite (see [34]). Thus a positive definite solution to (6.1) induces a positive definite storage function and this completes the proof of the "if" part.

(Only if) First we shall show that $\mathfrak{B}$ being strictly dissipative with respect to Σ with a positive definite storage function implies that there exists $\epsilon > 0$ such that $\int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t \geqslant \int_{\mathbb{R}_-} \epsilon|w|^2\mathrm{d}t$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Let Q_Ψ be a positive definite storage function that satisfies the strict dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q_\Psi(w)) \leqslant Q_\Sigma(w) - \epsilon|w|^2, \text{ for all } w \in \mathfrak{B}. \tag{6.2}$$

Considering only those trajectories in $\mathfrak{B} \cap \mathfrak{D}$ and integrating over $\mathbb{R}_-$ we get

$$\int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t - \int_{\mathbb{R}_-} \epsilon|w|^2\mathrm{d}t \geqslant Q_\Psi(w)(0) \quad \Rightarrow \quad \int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t \geqslant \int_{\mathbb{R}_-} \epsilon|w|^2\mathrm{d}t,$$

where the last implication uses $Q_\Psi(w) \geqslant 0$.

Next we show that $\mathfrak{B}$ being strictly dissipative with a positive storage function implies that $\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$. Observe that $\mathfrak{B}$ being strictly Σ-dissipative implies that the partition $w = (w_1, w_2)$ induced by Σ results in $\mathfrak{B}_{\mathrm{cont}}$ having a *proper* transfer function from w_1 to w_2. Hence it follows from Proposition 2.2 that $\mathfrak{B}$ allows an i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax+Bw_1$, $w_2 = Cx+Dw_1$, where (A, B) is state uncontrollable and (C, A) is state observable. In order to show that $\Lambda_{\mathrm{un}} \subset \mathbb{C}^-$, we shall show, in fact, that A is Hurwitz. We show this implication by contradiction, i.e., if $\lambda \notin \mathbb{C}^-$ is an eigenvalue of A, then there does not exist *any* nonnegative definite storage functions. Let $\lambda \in \mathrm{spec}(A)$ and $x_0 \neq 0$, the corresponding eigenvector of A. The following w is an element of $\mathfrak{B}$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ Cx_0e^{\lambda t} \end{bmatrix}. \tag{6.3}$$

The behavior $\mathfrak{B}$ being strictly Σ-dissipative implies that there exists a storage function $Q_\Psi(w)$ that satisfies the following strict dissipation inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q_\Psi(w)) \leqslant |w_1|^2 - |w_2|^2 - \epsilon|w|^2 \text{ for all } w \in \mathfrak{B}. \tag{6.4}$$

Putting w as in (6.3) we get, by direct differentiation of the left hand-side,

$$2\Re(\lambda)Q_\Psi(w) \leqslant -x_0^TC^TCx_0e^{2\Re(\lambda)t} - \epsilon|w|^2. \tag{6.5}$$

Consider the case when $\lambda \in \mathbb{C}^+$. The right-hand side of the above inequality is negative definite, which implies that $Q_\Psi(w) \ngeqslant 0$ for this $w \in \mathfrak{B}$. This proves the contradiction for the case $\Re(\lambda) > 0$. Next we consider the case when $\lambda \in j\mathbb{R}$. In that case $\Re(\lambda) = 0$, which makes inequality (6.5) impossible. Thus we have shown that

$\lambda \notin \mathbb{C}^-$ eliminates nonnegativity of *any* storage function. Hence in order to have a positive definite storage function it is necessary that $\Lambda_{\text{un}} \subset \mathbb{C}^-$. This completes the proof that existence of a nonnegative storage function implies $\Lambda_{\text{un}} \subset \mathbb{C}^-$. ∎

Within the above proof, we have, in fact, shown that *every* storage function for the behavior is positive definite. This has been shown for the controllable case in [28, Theorem 6.4]. Intuitively, storage functions being positive is closer to their interpretation as energy-like functions. Also, the meaning of dissipativity that there is no source of energy in the system appeals to both positive definite storage functions and the stabilizability of the system. In the following section we explore other properties of the set of storage functions, like (un)boundedness of this set.

7. Set of all storage functions for an uncontrollable system. Another important topic of interest is the set of all storage functions of a dissipative behavior. For LQR/LQG theory and $\mathcal{H}_\infty$ control, certain extremum storage functions give stabilizing controllers. In this section we show that the set of storage functions is unbounded for uncontrollable dissipative systems and that for stabilizable systems, this set is bounded from below.

We have seen before how the solutions to an ARI give storage functions as state functions. Thus to further explore the set of all storage functions we shall look into the set of solutions of a related ARI. It has been shown (in [27, 28], for example) that with respect to a given state space representation i.e., with respect to a given state map $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$, there is a one to one correspondence between storage functions and solutions to the ARI. Moreover, it is known that the set of storage functions for a controllable dissipative behavior is a bounded convex polytope with its vertices given by the storage functions coming from so-called spectral factorizations of $\partial\Phi(\xi)$. These storage functions correspond to the algebraic Riccati *equality* solutions. However, this set of solutions to the ARI turns out to lose the boundedness property when the behavior loses controllability. This constitutes the theorem below, one of the main results of this paper.

THEOREM 7.1. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be uncontrollable, and suppose the set of its uncontrollable poles Λ_{un} satisfies the unmixing property, i.e., $\Lambda_{\text{un}} \cap (-\Lambda_{\text{un}}) = \phi$. Further, let $\mathfrak{B}$ be strictly Σ-dissipative. Then the set of all storage functions is an unbounded convex set.*

Proof. By Proposition 2.2, $\mathfrak{B}$ allows an i/s/o representation as $\frac{\mathrm{d}}{\mathrm{d}t}x = \widetilde{A}x + \widetilde{B}w_1$, $w_2 = \widetilde{C}x + \widetilde{D}w_1$. Since $\mathfrak{B}$ is strictly Σ-dissipative, $(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}) > 0$ and all the storage functions come from real symmetric solutions of the following ARI

$$A^TK + KA + C^TC + KBB^TK \leqslant 0,$$

where $A := \widetilde{A} + \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-1}\widetilde{D}^T\widetilde{C}$, $B := \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-\frac{1}{2}}$, and $C := (I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T)^{-\frac{1}{2}}\widetilde{C}$. That the set of solutions to this ARI is convex is well known (see [4], for example). In order to show the unboundedness of the solution set, we once again make use of the Kalman decomposition result (Proposition A.3): there exists a similarity transformation that results in A, B, C matrices in the following forms: $A = \begin{bmatrix} A_{\text{c}} & A_{\text{cp}} \\ 0 & A_{\text{u}} \end{bmatrix}$, $B = \begin{bmatrix} B_{\text{c}} \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} C_{\text{c}} & C_{\text{u}} \end{bmatrix}$. In order to show that the set of ARI solutions is unbounded, we shall show that there exists a nonzero $P \in \mathbb{R}_{\text{s}}^{\mathtt{n}\times\mathtt{n}}$ such that if $K \in \mathbb{R}_{\text{s}}^{\mathtt{n}\times\mathtt{n}}$ is a solution to the ARI, then for all $\lambda > 0$, the new real symmetric matrix defined by $\widetilde{K} := K + \lambda P$ is also a solution of the ARI. Define P

$$P := \begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix},$$

where $P_1 \in \mathbb{R}_{\mathrm{s}}^{\mathtt{n_u} \times \mathtt{n_u}}$ satisfies the following Lyapunov inequality

$$A_{\mathrm{u}}^T P_1 + P_1 A_{\mathrm{u}} \leqslant 0. \tag{7.1}$$

Since the set of uncontrollable poles satisfy the unmixing property, that is, $\Lambda_{\mathrm{un}} \cap (-\Lambda_{\mathrm{un}}) = \phi$, the Lyapunov inequality is guaranteed to have a nonzero solution. Consider the following expression in $\widetilde{K}$:

$$\begin{aligned} &A^T \widetilde{K} + \widetilde{K} A + C^T C + \widetilde{K} B B^T \widetilde{K} \\ &= A^T K + KA + C^T C + KBB^T K + \lambda\left(A^T P + PA\right) + \lambda^2 PBB^T P \\ &\quad + \lambda\left(KBB^T P + PBB^T K\right). \end{aligned}$$

Using the Kalman decomposed form of A, B, and C above and the structure of P, the above expression simplifies to

$$\left(A^T K + KA + C^T C + KBB^T K\right) + \lambda \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathrm{u}}^T P_1 + P_1 A_{\mathrm{u}} \end{bmatrix}.$$

From the fact that K is a solution to the ARI, and that P_1 satisfies the Lyapunov inequality (7.1) the above expression is negative semidefinite for all $\lambda > 0$. Thus $A^T \widetilde{K} + \widetilde{K} A + C^T C + \widetilde{K} B B^T \widetilde{K} \leqslant 0$, and $\widetilde{K}$ is also a solution of the ARI for all $\lambda > 0$. This proves that the set of solutions to the ARI is unbounded. $\square$

Notice that within the above proof we used that, if a solution K to the Riccati inequality exists, then a solution P_1 to the Lyapunov inequality (7.1) can be added to K giving solutions $\widetilde{K}$ of the ARI. Though we have demonstrated the existence of solutions to the ARI primarily under the unmixing condition on the uncontrollable poles, this method shows that whenever the Riccati inequality admits solutions, uncontrollability forces the set of storage functions to be unbounded.

The following simple example shows a pictorial representation of the set of all storage functions for a controllable behavior.

Example 7.2. Consider behavior $\mathfrak{B}$ having an i/s/o representation with $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and supply rate $S = \begin{bmatrix} 100 & 0 \\ 0 & -1 \end{bmatrix}$. Let all the symmetric solutions to the corresponding ARI be of the form $K = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$. The figure below shows the set of all ARI solutions. Clearly, the set is a bounded convex polytope.

Remark 7.3. Figure 7.1 corresponds to a controllable behavior, and hence the set is bounded. As the behavior becomes uncontrollable, all Lambda-sets no longer admit Riccati equation solutions (see Theorem 5.5). However, the set of storage functions is now unbounded along certain directions specified by the Lyapunov equation corresponding to the autonomous part of the behavior (as shown in Theorem 7.1). For some specific examples, the transition to uncontrollability causes the Riccati equation solutions corresponding to inadmissible Lambda-sets to move to infinity along the direction of the Lyapunov equation solution. Further, loosely speaking, when restricted to the controllable part, the storage functions corresponding to the Riccati equation solutions are unaffected by translation along this direction. It remains to formulate these observations concretely and prove them.

A very interesting fact about this unbounded set of all storage functions comes up for the case when the behavior is uncontrollable but stabilizable; i.e., the set of uncontrollable poles Λ_{un} is contained in the open left half of the complex plane (see the previous section for the definition and related results about stabilizability). We

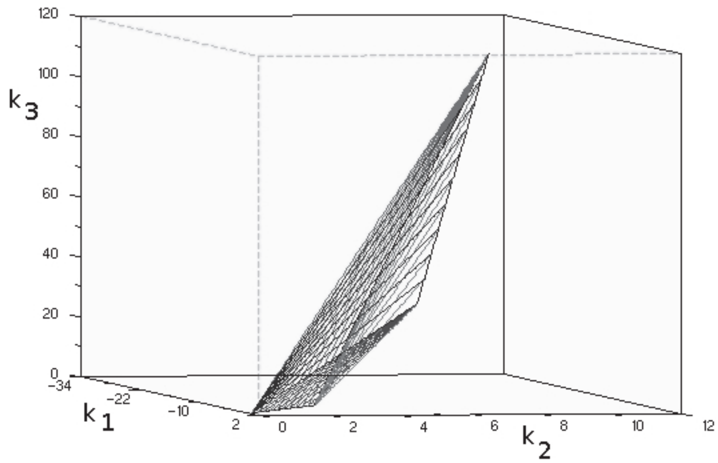

FIG. 7.1. *The set of all storage functions.*

show below that for stabilizability, the set of storage functions, though an unbounded set, is bounded from below. In other words, there exists a storage function Q_{Ψ_-} such that every storage function Q_Ψ satisfies $Q_\Psi(w) - Q_{\Psi_-}(w) \geqslant 0$ for all $w \in \mathfrak{B}$. We state this result as a theorem below.

THEOREM 7.4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be an uncontrollable, strictly Σ-dissipative behavior. Also assume that the set of uncontrollable poles $\Lambda_{\rm un} \subset \mathbb{C}^-$. Then the set of all storage functions is bounded from below; i.e., there exists a storage function $Q_{\Psi_-}(w)$ for $\mathfrak{B}$ such that for each storage function $Q_\Psi(w)$ for $\mathfrak{B}$, $Q_{\Psi_-}(w) \leqslant Q_\Psi(w)$ for all $w \in \mathfrak{B}$.*

Note the analogy of this result with that for controllable behaviors, where the set of storage functions is bounded and has a maximum and a minimum element (see [28, Theorem 5.7]). While we have shown unboundedness for the case of uncontrollability, stabilizability ensures the existence of the minimum element in this unbounded set.

Proof. To prove the above result, we use Corollary 5.6 and look into the solutions to the corresponding ARI. Let the behavior $\mathfrak{B}$ have an i/s/o representation: $\frac{\rm d}{{\rm d}t}x = \widetilde{A}x + \widetilde{B}w_1$, $w_2 = \widetilde{C}x + \widetilde{D}w_1$. Then the set of all storage functions comes from the solution set of the following ARI

$$A^TK + KA + C^TC + KBB^TK \leqslant 0,$$

where $A := \widetilde{A} + \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-1}\widetilde{D}^T\widetilde{C}$, $\quad B := \widetilde{B}(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D})^{-\frac{1}{2}}$, and $C := (I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T)^{-\frac{1}{2}}\widetilde{C}$. Construct the corresponding Hamiltonian matrix

$$H := \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix}.$$

By assumption, $\mathfrak{B}$ is strictly Σ-dissipative, which implies $\operatorname{spec}(H) \cap j\mathbb{R} = \phi$. Also from Theorem 5.4, $\operatorname{spec}(H) \supset \Lambda_{\rm un}$ and by assumption $\Lambda_{\rm un} \subset \mathbb{C}^-$. These facts together imply that there exists a Lambda-set (say Λ) of $\operatorname{spec}(H)$ such that $\Lambda \supset \Lambda_{\rm un}$ and $\Lambda \subset \mathbb{C}^-$. By Theorem 5.5, since $\Lambda \supseteq \Lambda_{\rm un}$, there exists a real symmetric solution, K_- to the ARE such that $\operatorname{spec}(A + BB^TK_-) = \Lambda$. We shall show that this particular solution to the ARE serves as a "minimum" storage function. To show this consider any real symmetric solution K to the ARI

$$A^TK + KA + C^TC + KBB^TK \leqslant 0. \tag{7.2}$$

Also rewrite the ARE in K_- as follows

$$(7.3) \qquad \left(A + BB^T K_-\right)^T K_- + K_- \left(A + BB^T K_-\right) + C^T C - K_- BB^T K_- = 0.$$

Subtracting (7.3) from (7.2), and further adding and subtracting the terms $K_- BB^T K$ and $KBB^T K_-$ we get

(7.4)

$$\left(A + BB^T K_-\right)^T (K - K_-) + (K - K_-)\left(A + BB^T K_-\right) + (K - K_-)BB^T(K - K_-) \leqslant 0.$$

Observe that the above inequality closely resembles the Lyapunov inequality. In order to conclude that $(K - K_-) \geqslant 0$, it is enough to notice that $\operatorname{spec}(A + BB^T K_-) = \Lambda \subset \mathbb{C}^-$, and the constant-like term $(K - K_-)BB^T(K - K_-)$ is nonnegative. This proves our claim that the set of all storage functions is bounded from below. □

We saw in the previous section that, for a strictly dissipative and stabilizable behavior $\mathfrak{B}$, dissipativity on $\mathbb{R}_-$ of $\mathfrak{B}_{\text{cont}}$ assures the existence of positive definite storage functions. Combining this result with the one above, we infer that the lower bound of the set of storage functions is, in fact, positive (see discussions following Theorem 6.1). This formalizes the intuition that such a system is devoid of any energy sources within it, and hence the maximum extractable energy[3] from *any* given state is bounded.

Using a very similar argument as in the above proof, one can show that if the behavior is antistabilizable, meaning all the uncontrollable poles are unstable, i.e., $\Lambda_{\text{un}} \subset \mathbb{C}^+$, then the set of storage functions is bounded *from above*.

8. Necessity of the unmixing of uncontrollable poles. As seen in Theorem 3.4, the unmixing property of the uncontrollable poles makes strict dissipativity of the controllable part equivalent to that of the whole behavior. In this section we shall see to what extent the unmixing property is necessary. As mentioned in the introduction, the unmixing property serves as a sufficient condition for solvability of a Lyapunov equation and the corresponding Lyapunov operator becomes singular when this condition is not satisfied. We show in this section that the Lyapunov operator is onto if and only if there exists an observable rank one symmetric matrix in its image. This interesting result is utilized for exploring the extent of the unmixing condition for strict dissipativity. Theorem 8.3 below shows that for a system with single output, unmixing is necessary for existence of nonzero lossless trajectories satisfying a dissipation *equality* (see Remark 8.4 below).

When the Lyapunov operator is singular, then nonsymmetric solutions can exist even when the constant term is symmetric. This general solvability condition can be obtained from a certain eigenspace of a Hamiltonian matrix. The fact that a Lyapunov equation is a special case of an ARE with the quadratic term being zero motivates the following result. We state this as a lemma below since it will be needed later in this section. Related results on Lyapunov equation solvability can be found in [24].

LEMMA 8.1. $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *is a solution (not necessarily symmetric) to the following Lyapunov equation* $A^T K + KA + C^T C = 0$ *if and only if* $\operatorname{im}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix}$ *is an invariant space of the Hamiltonian matrix defined as*

$$H = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix}.$$

[3]This has been called *available storage* in [28].

Proof. **(If)** We first assume that $\mathcal{X}_A := \text{im}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix}$ is an $\mathtt{n}$-dimensional invariant subspace of H. We shall show that this implies K satisfies the Lyapunov equation $A^TK + KA + C^TC = 0$. Clearly $\begin{bmatrix} K & -I_{\mathtt{n}} \end{bmatrix}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix} = 0$. Since $\mathcal{X}_A$ is H-invariant the last equation can be written as $\begin{bmatrix} K & -I_{\mathtt{n}} \end{bmatrix}\begin{bmatrix} A & 0 \\ -C^TC & -A^T \end{bmatrix}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix} = 0$. Hence $A^TK + KA + C^TC = 0$.

(Only if) In this part we shall show that K being a solution to the Lyapunov equation $A^TK + KA + C^TC = 0$ implies that $\text{im}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix}$ is an $\mathtt{n}$-dimensional invariant subspace of H. Assuming K satisfies the Lyapunov equation, we can write the following equality:

$$\begin{bmatrix} A & 0 \\ -C^TC & -A^T \end{bmatrix}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix} A.$$

This implies that $\text{im}\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix}$ is an eigenspace of H corresponding to $\text{spec}(A)$. ∎

It follows from the lemma above that there is only one set of eigenvalues of the Hamiltonian matrix, which gives a solution to the Lyapunov equation: namely, the eigenvalues of A. Unlike the ARE, for the Lyapunov equation there is no choice for the set of eigenvalues of the Hamiltonian matrix. Another important fact that follows is that the set of eigenvalues, i.e., $\text{spec}(A)$ is no longer a Lambda-set when there is mixing, i.e., $\text{spec}(A) \cap \text{spec}(-A) \neq \phi$. Owing to this particular fact, the eigenspace of the Hamiltonian matrix which gives a solution is no longer guaranteed to be perpendicular to the left-eigenspace corresponding to the rest of the eigenvalues of H. However, from Lemma A.4 if $\text{im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is a (generalized) right-eigenspace of dimension $\mathtt{n}$ corresponding $\text{spec}(A)$, then $\text{im}\begin{bmatrix} X_2 \\ -X_1 \end{bmatrix}$ is an $\mathtt{n}$-dimensional (generalized) left-eigenspace corresponding to $\text{spec}(-A)$. Since the right and left eigenspaces are no longer guaranteed to be perpendicular, we do not necessarily have $X_2X_1^{-1} = X_1^{-T}X_2^T$. This implies that when the unmixing condition is not satisfied there can be nonsymmetric and nonunique solutions to the Lyapunov equation. However, the existence of a nonsymmetric solution guarantees existence of a symmetric solution. If K is a solution to the Lyapunov equation, then so are K^T and $(K + K^T)/2$. With this simple observation we now give a necessary and sufficient condition for the existence of solution to a Lyapunov equation for the special case that the constant term is of rank one. Interestingly for this case when the constant term is rank 1 the unmixing condition becomes necessary!

THEOREM 8.2. *Consider the Lyapunov equation $A^TK + KA + C^TC = 0$ with (C, A) pair observable. Assume $\text{rank}(C^TC) = 1$. Then there exists a solution K to the Lyapunov equation if and only if $\text{spec}(A) \cap \text{spec}(-A) = \phi$.*

Proof. **(If)** This implication is well known (see [26], for example). In fact, uniqueness and symmetry are guaranteed.

(Only if) Suppose A has a mixed spectrum, that is, $\text{spec}(A) \cap \text{spec}(-A) \neq \phi$. We shall show that this implies that the Lyapunov equation $A^TK + KA + C^TC = 0$ has no solution K. In Theorem 8.1 it has been shown that the eigenspace of the Hamiltonian matrix corresponding to only $\text{spec}(A)$ can give a solution to the Lyapunov equation. Hence to prove that there does not exist any solution K, it is sufficient to show that any eigenvector of the Hamiltonian matrix

$$H := \begin{bmatrix} A & 0 \\ -C^TC & -A^T \end{bmatrix},$$

corresponding to an eigenvalue $\lambda \in \operatorname{spec}(A) \cap \operatorname{spec}(-A)$ shall have all upper $\mathtt{n}$ elements zero. For that, let $v := \operatorname{col}(v_1, v_2) \in \mathbb{C}^{2\mathtt{n}}$, $v \neq 0$ be an eigenvector of H corresponding to $\lambda \in \operatorname{spec}(A) \cap \operatorname{spec}(-A^T)$. It then follows that

$$\begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

which means $Av_1 = \lambda v_1$ and $[\lambda I_{\mathtt{n}} - (-A^T)]v_2 = -C^T C v_1$. Suppose $v_1 \neq 0$, then the above implies v_1 is an eigenvector of A. Since (C, A) is observable $Cv_1 \neq 0$. Further, $\operatorname{rank}(C^T C) = 1$, $Cv_1 \neq 0$ together imply $[\lambda I_{\mathtt{n}} - (-A^T)]v_2 = -C^T C v_1$ implies[4]

$$\operatorname{im} C^T \subseteq \operatorname{im} \left[\lambda I_{\mathtt{n}} - \left(-A^T\right)\right]. \tag{8.1}$$

This gives $\ker C \supseteq \ker [\lambda I_{\mathtt{n}} - (-A)]$. Since $\lambda \in \operatorname{spec}(A) \cap \operatorname{spec}(-A)$, $-\lambda \in \operatorname{spec}(A)$. But the inclusion $\ker C \supseteq \ker [\lambda I_{\mathtt{n}} - (-A)]$ implies $\left[\begin{smallmatrix} -\lambda I_{\mathtt{n}} - A \\ C \end{smallmatrix}\right]$ loses rank, which means $-\lambda$ is unobservable. This contradicts the observability of (C, A). Hence $v_1 = 0$, and therefore, the eigenspace of H corresponding to $\operatorname{spec}(A)$ cannot have the form $\operatorname{im} \left[\begin{smallmatrix} I_{\mathtt{n}} \\ K \end{smallmatrix}\right]$. Thus, using Lemma 8.2, the Lyapunov equation is not solvable. □

It is well known that the unmixing condition is equivalent to existence and uniqueness of solution to the Lyapunov equation. In other words, the unmixing condition is equivalent to the image of the Lyapunov operator containing *all* symmetric matrices. The above theorem shows that unmixing is necessary and sufficient for the image to contain a symmetric matrix of rank one (satisfying observability conditions). However, the corresponding Lyapunov inequality can have solutions when the equation is not solvable; see Remark 8.4 below.

We have seen in the previous sections how dissipativity is related to the solvability of certain ARE/ARI. For an uncontrollable system the corresponding ARE behaves like a Lyapunov equation on certain subspaces of the state space. Following this observation we shall now utilize Theorem 8.2 to show how the unmixing becomes necessary for the solvability of the ARE. We shall make use of the fact that $C^T C$ being rank one means that the system has only one output.

THEOREM 8.3. *Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ having a single output, i.e., $\mathtt{p}(\mathfrak{B}) = 1$. Suppose $R \in \mathbb{R}^{1 \times \mathtt{w}}[\xi]$, with $R \neq 0$, gives a kernel representation for $\mathfrak{B}$. Define $\Lambda_{\text{un}} = \{\lambda \in \mathbb{C} \mid R(\lambda) = 0\}$ and let (A, B, C, D) give a minimal i/s/o representation for $\mathfrak{B}$, with (C, A) observable. Then, the following are equivalent.*

1. $I_{\mathtt{m}} - D^T D > 0$ *and there exists K satisfying*

$$A^T K + KA + C^T C + (KB + C^T D)(I_{\mathtt{m}} - D^T D)^{-1}(B^T K + D^T C) = 0, \tag{8.2}$$

with $\operatorname{spec}(A + B(I_{\mathtt{m}} - D^T D)^{-1}(B^T K + D^T C)) \cap j\mathbb{R} = \phi$.

2. $\Lambda_{\text{un}} \cap (-\Lambda_{\text{un}}) = \phi$ *and $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative.*

Proof. **(2 ⇒ 1)** From Lemma 5.3 strict dissipativity of $\mathfrak{B}_{\text{cont}}$ implies $I_{\mathtt{m}} - D^T D > 0$, so the following Hamiltonian matrix exists

$$H := \begin{bmatrix} A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C & B\left(I_{\mathtt{m}} - D^T D\right)^{-1} B^T \\ -C^T \left(I_{\mathtt{p}} - DD^T\right)^{-1} C & -\left(A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C\right)^T \end{bmatrix}. \tag{8.3}$$

[4] Existence of a solution v_2 for just one $v_1 \neq 0$ implies the inclusion (8.1) because of the rank condition.

Let the observable image representation of $\mathfrak{B}_{\text{cont}}$ induce the para-Hermitian matrix $\partial\Phi(\xi)$. Then by Theorem 5.4 we get $\text{spec}(H) = \text{roots}\,[\det \partial\Phi(\xi)\ \ \chi_{\text{un}}(\xi)\ \ \chi_{\text{un}}(-\xi)]$. Once again strict Σ-dissipativity of $\mathfrak{B}_{\text{cont}}$ implies $\partial\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$ (see Theorem 5.2) meaning roots $(\det \partial\Phi(\xi))$ has no roots on the imaginary axis. This together with the assumption $\Lambda_{\text{un}} \cap (-\Lambda_{\text{un}}) = \phi$ implies that $\text{spec}(H) \cap j\mathbb{R} = \phi$ and so it allows a Lambda-set $\Lambda \supseteq \Lambda_{\text{un}}$. It then follows from Theorem 5.5 that there exists $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that im $\begin{bmatrix} I_{\mathtt{n}} \\ K \end{bmatrix}$ is the $\mathtt{n}$-dimensional invariant subspace of H corresponding to Λ. This shows that K solves the ARE. Also this solution K has the property $\text{spec}(A + B(I_{\mathtt{m}} - D^TD)^{-1}(B^TK + D^TC)) = \Lambda$ (see Proposition 4.4) which means $\text{spec}(A + B(I_{\mathtt{m}} - D^TD)^{-1}(B^TK + D^TC)) \cap j\mathbb{R} = \phi$ because $\Lambda \subset \text{spec}(H)$ and $\text{spec}(H) \cap j\mathbb{R} = \phi$.

(1 $\Rightarrow$ 2) We shall first show that statement 1 implies that $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative. $I_{\mathtt{m}} - D^TD$ is assumed to be positive definite, and therefore the Hamiltonian matrix of equation (8.3) exists. Again, Theorem 5.4 implies $\text{spec}(H) = \text{roots}\,[\det \partial\Phi(\xi)\ \ \chi_{\text{un}}(\xi)\ \ \chi_{\text{un}}(-\xi)]$. Counting multiplicities both sides we get $\deg\,(\det \partial\Phi(\xi)) = 2\mathtt{n}(\mathfrak{B}_{\text{cont}})$. We claim that H has no eigenvalues on the imaginary axis, because otherwise any solution to the ARE, if it exists, would result in $\text{spec}(A + B(I_{\mathtt{m}} - D^TD)^{-1}(B^TK + D^TC)) \cap j\mathbb{R} \neq \phi$ which contradicts statement 1. Thus $\partial\Phi(\xi)$ also has no zeros on the imaginary axis, which implies $\partial\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$. This positive definiteness together with $\deg\,(\det \partial\Phi(\xi)) = 2\mathtt{n}_{\text{c}}$ implies that $\mathfrak{B}_{\text{cont}}$ is strictly Σ-dissipative (see Theorem 5.2).

In order to complete the proof it remains to show that the ARE (8.2) having a solution implies that Λ_{un} is unmixed. Since (C, A) is observable K is nonsingular (see [7, Lemma 3]). We have assumed that $\mathfrak{B}$ is uncontrollable; therefore, (A, B) pair is also uncontrollable and the set of uncontrollable poles Λ_{un} is exactly equal to the set of uncontrollable eigenvalues of A (this follows from Proposition 2.2). So if Λ_{un} has cardinality $\mathtt{n}_{\text{u}}$ (counting multiplicities), then there exists a full column rank matrix $T_{\text{u}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}_{\text{u}}}$ obtained from the generalized eigenvectors of A^T such that $A^TT_{\text{u}} = T_{\text{u}}A_{\text{u}}$ and $T_{\text{u}}^TB = 0$, where $A_{\text{u}} \in \mathbb{R}^{\mathtt{n}_{\text{u}}\times\mathtt{n}_{\text{u}}}$ is in Jordan form with $\text{spec}(A_{\text{u}}) = \Lambda_{\text{un}}$. Also, since K is nonsingular, there exists $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}_{\text{u}}}$ full column rank, such that $KT = T_{\text{u}}$. Then pre- and postmultiplying the ARE (8.2) by T^T and T, respectively, and making use of the fact that $A^TT_{\text{u}} = T_{\text{u}}A_{\text{u}}$ and $T_{\text{u}}^TB = 0$ we get

$$T^TKTA_{\text{u}} + A_{\text{u}}^TT^TKT + T^TC^T\left(I_{\mathtt{p}} - DD^T\right)^{-1}CT = 0.$$

Define $K_{\text{u}} := T^TKT \in \mathbb{R}_{\text{s}}^{\mathtt{n}_{\text{u}}\times\mathtt{n}_{\text{u}}}$ and $C_{\text{u}} := (I_{\mathtt{p}} - DD^T)^{-\frac{1}{2}}CT$. The above Lyapunov equation can be rewritten as $A_{\text{u}}^TK_{\text{u}} + K_{\text{u}}A_{\text{u}} + C_{\text{u}}^TC_{\text{u}} = 0$.

Now we show that the new positive semidefinite matrix $C_{\text{u}}^TC_{\text{u}}$ is also rank one. In order to prove this we shall use a contradiction argument. Since T is full column rank, rank $C_{\text{u}}^TC_{\text{u}} \leqslant 1$. Assume rank $C_{\text{u}}^TC_{\text{u}} = 0$. This implies $CT = 0$ because $\mathfrak{B}$ being strictly Σ-dissipative implies $(I_{\mathtt{p}} - DD^T)^{-1} > 0$ (see footnote 2 in section 4). Postmultiplying the ARE (8.2) by T and utilizing the fact that $B^TKT = 0$ we get $A^TKT + KAT + C^T(I_{\mathtt{p}} - DD^T)^{-1}CT = 0$. Since $CT = 0$ and $A^TT_{\text{u}} = T_{\text{u}}A_{\text{u}}$ the last equation gives

$$KTA_{\text{u}} + KAT = 0. \tag{8.4}$$

Equation (8.4) along with the fact that K is nonsingular implies that im T is A-invariant. This means im T is a nontrivial A-invariant subspace contained in ker C, which is not possible due to observability of (C, A). Hence $CT \neq 0$ and so rank $(C_{\mathrm{u}}^T C_{\mathrm{u}}) = 1$. Therefore, from Theorem 8.2 it follows that $\mathrm{spec}(A_{\mathrm{u}}) = \Lambda_{\mathrm{un}}$ satisfies the unmixing property. $\square$

Remark 8.4. Statement 1 above would have been equivalent to strict dissipativity of $\mathfrak{B}$ if $\mathfrak{B}$ were controllable, but this is not the case in general. More precisely, under controllability assumptions on (A, B), the ARE not having a solution implies that the ARI also does not have a solution (see [22, 23]). However, this turns out to be untrue for an uncontrollable behavior. For example, consider the special case of an autonomous system with $A = \mathrm{diag}([1 \quad -1])$ and $C = [1 \quad 1]$. For this case the Lyapunov equation is not solvable, though the corresponding inequality admits a solution. More generally the ARE having a solution means that there exist nonzero lossless trajectories in the behavior (see [1]). For a controllable dissipative behavior, there are always nonzero lossless trajectories. The last theorem makes it clear that for an uncontrollable system with a single output, unmixing of uncontrollable poles is necessary for the existence of nontrivial lossless trajectories. We do not digress more into the notion of losslessness because it is not central to the subject of this paper. A thorough understanding of the necessity of the unmixing condition requires further investigation.

That the unmixing is not necessary in general for more than one output is quite expected. The following example gives one such simple instance.

Example 8.5. Consider the autonomous system with i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax$, $y = Cx$, where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Observe that A has "mixed" eigenvalues, i.e., $\Lambda_{\mathrm{un}} \cap (-\Lambda_{\mathrm{un}}) \neq \phi$. Σ-dissipativity of such an autonomous system together with $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$ is equivalent to existence of a real symmetric solution to the following Lyapunov inequality: $A^T K + KA + C^T C \leqslant 0$. Notice that $K = \begin{bmatrix} 2+b & a \\ a & -2-c \end{bmatrix}$ with $a, b, c \in \mathbb{R}$ and $b, c \geqslant 0$ gives a solution to the above Lyapunov inequality. This example shows that the unmixing condition of uncontrollable poles is not necessary for the system to be dissipative.

9. Concluding remarks. In this paper we studied dissipativity for a general, possibly uncontrollable, LTI system. Our starting point was a more appropriate, though less often used, definition of dissipativity in terms of a differential inequality called the dissipation inequality. With this definition we brought out an equivalence between the dissipativities of a behavior and its controllable part, under the important unmixing condition (Theorem 3.4). For the case of strict dissipativity, Theorem 3.4 also settles the issue of whether to allow unobservable variables in the defining dissipation inequality: the theorem rules out the requirement of unobservable variables. The important intuitive idea that storage of energy should take place through the state variables comes as a natural consequence of Theorem 3.4. We stated this result as a corollary.

Next we looked into the set of all storage functions for a strictly Σ-dissipative system. It is well known that this set is a bounded convex polyhedron for a controllable system. We showed that for an uncontrollable system the set loses its boundedness property. Further, this set becomes bounded from below if the system is stabilizable. If in addition the controllable part is strictly Σ-dissipative on $\mathbb{R}_-$, then we showed that this lower bound on the set is positive. We used this result to formalize the physical notion of stored energy being finite in a dissipative system that has no source of energy within: it is *not* possible to extract an indefinite amount of energy from a stabilizable system whose controllable part is strictly dissipative on $\mathbb{R}_{-}$. In this paper, though we utilized results about Hamiltonian matrices and Riccati equations, our main results pertain to the system directly, without an *apriori* input/output partition of the system variables. This is an important feature and advantage of the behavioral approach.

The unmixing condition plays a crucial role in most of the main results of this paper. In order to address the necessity of the unmixing condition we made use of the important observation that for an uncontrollable behavior a storage function acts like a Lyapunov function over certain trajectories, and we showed that unmixing is not necessary in general for existence of a Lyapunov function and therefore for dissipativity. However, an interesting situation arises when the system has only one output. In Theorem 8.2 we showed that under suitable observability conditions a singular Lyapunov operator cannot have a rank one symmetric matrix in its image. This helped us to bring out the fact that the unmixing is necessary for an uncontrollable behavior to have nonzero lossless trajectories in it. The extent of unmixing condition for a more general situation remains to be investigated.

In this paper we have dealt only with the maximum input cardinality case, i.e., the case when the number of inputs is equal to the positive signature of the supply rate function Σ. A study of the general case can also be utilized for dissipativity synthesis problems for uncontrollable systems.

Appendix A. Auxiliary results and proofs.

The following standard result from state space theory [34] is needed in the proof of Lemma 4.5; we state it for easy reference.

PROPOSITION A.1. *Consider the following state space representation of a dynamical system.* $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1, \quad w_2 = Cx + Dw_1$. *Let* $F_{\mathrm{c}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, $F_{\mathrm{o}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ *and* $G_{\mathrm{c}} \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, $G_{\mathrm{o}} \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$ *with* $G_{\mathrm{c}}, G_{\mathrm{o}}$ *nonsingular. Then,*

- $(A + BF_{\mathrm{c}}, BG_{\mathrm{c}})$ *is controllable if and only if* (A, B) *is controllable,*
- $(G_{\mathrm{o}}C, A + F_{\mathrm{o}}C)$ *is observable if and only if* (C, A) *is observable.*

Proof of Lemma 4.5. As seen earlier, the supply function matrix Σ induces an input/output partition $w = (w_1, w_2)$ where w_1 is input and w_2 is output. To prove this theorem we shall consider two cases, first the case when $\mathfrak{B}$ has a *strictly* proper transfer function from w_1 to w_2 and secondly the case when $\mathfrak{B}$ has a proper transfer function (not necessarily strictly proper) with the same input/output partition. For the second case we shall show that there exists a simple transformation that changes it to the first case and thus the proof is complete.

Case 1 ($\mathfrak{B}$ has strictly proper transfer function from w_1 to w_2) : $\mathfrak{B}$ allows an i/s/o representation with $D = 0$ as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1, \quad w_2 = Cx. \tag{A.1}$$

Σ is given as $\begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}$, so the corresponding Hamiltonian matrix is $H = \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix}$. Consider the observable image representation of $\mathfrak{B}$: $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, and partition M corresponding to $w = (w_1, w_2)$ to get

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} W_1(\frac{\mathrm{d}}{\mathrm{d}t}) \\ W_2(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell. \tag{A.2}$$

Without loss of generality we can assume that det $(W_1(\xi))$ is a monic polynomial. Now, the transfer function for $\mathfrak{B}$ is given by $G(\xi) := W_2(\xi)W_1^{-1}(\xi) = C(\xi I_{\mathtt{n}} - A)^{-1}B$.

The characteristic polynomial of H is given by $\chi(H) = \det(\xi I_{2\mathtt{n}} - H)$. Using Schur complement this can be written as

$$\begin{aligned}\chi(H) &= \det(\xi I_{\mathtt{n}} - A)\det\left[\left(\xi I_{\mathtt{n}} + A^T\right) + C^T C(\xi I_{\mathtt{n}} - A)^{-1}BB^T\right]\\ &= \det(\xi I_{\mathtt{n}} - A)\det\left(\xi I_{\mathtt{n}} + A^T\right)\det\left[I_{\mathtt{n}} + \left(\xi I_{\mathtt{n}} + A^T\right)^{-1} C^T C(\xi I_{\mathtt{n}} - A)^{-1}BB^T\right].\end{aligned}$$

Now using the identity, $\det(I + PQ) = \det(I + QP)$ we get

$$\begin{aligned}\chi(H) &= \det(\xi I_{\mathtt{n}} - A)\det\left(\xi I_{\mathtt{n}} + A^T\right)\det\left[I_{\mathtt{m}} + B^T\left(\xi I_{\mathtt{n}} + A^T\right)^{-1} C^T C(\xi I_{\mathtt{n}} - A)^{-1}B\right]\\ &= \det(\xi I_{\mathtt{n}} - A)\det\left(\xi I_{\mathtt{n}} + A^T\right)\det\left[I_{\mathtt{m}} - G^T(-\xi)G(\xi)\right]\\ &= \det(\xi I_{\mathtt{n}} - A)\det\left(\xi I_{\mathtt{n}} + A^T\right)\det\left[W_1^{-T}(-\xi)(W_1^T(-\xi)W_1(\xi)\right.\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.- W_2^T(-\xi)W_2(\xi))W_1^{-1}(\xi)\right]\\ &= \left(\frac{\det(\xi I_{\mathtt{n}} - A)\det\left(\xi I_{\mathtt{n}} + A^T\right)}{\det W_1^T(-\xi)\det W_1(\xi)}\right)\det\left[W_1^T(-\xi)W_1(\xi) - W_2^T(-\xi)W_2(\xi)\right].\end{aligned}$$

Since (A, B) is controllable and (C, A) is observable, and due to observability of the image representation (A.2), $\det W_1(\xi) = \det(\xi I_{\mathtt{n}} - A)$; therefore, the above equation simplifies to

$$\begin{aligned}\chi(H) &= -\det\left[W_1^T(-\xi)W_1(\xi) - W_2^T(-\xi)W_2(\xi)\right]\\ &= -\det \partial\Phi(\xi)\end{aligned}$$

This proves $\mathrm{spec}(H) = \mathrm{roots}\,(\det \partial\Phi(\xi))$ and hence the lemma for the strictly proper transfer matrix case.

Case 2 ($\mathfrak{B}$ has proper transfer function from w_1 to w_2): $\mathfrak{B}$ allows the following i/s/o representation, $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$ with (A, B) controllable and (C, A) observable. Since $(I_{\mathtt{m}} - D^T D) > 0$, the corresponding Hamiltonian matrix is given by

$$H := \begin{bmatrix} A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C & B\left(I_{\mathtt{m}} - D^T D\right)^{-1} B^T \\ -C^T\left(I_{\mathtt{p}} - DD^T\right)^{-1} C & -\left(A + B\left(I_{\mathtt{m}} - D^T D\right)^{-1} D^T C\right)^T \end{bmatrix}.$$

Because D is nonzero, the arguments in case 1 do not hold. As mentioned earlier, we show that there exists a transformation on the manifest variables that changes this situation to that in case 1. Consider the quadratic form $Q_\Sigma(w)$

$$\begin{aligned}Q_\Sigma(w) &= |w_1|^2 - |w_2|^2 = w_1^T w_1 - x^T C^T Cx - x^T C^T Dw_1 - w_1^T D^T Cx - w_1^T D^T Dw_1\\ &= \begin{bmatrix} w_1 \\ x \end{bmatrix}^T \Sigma_1 \begin{bmatrix} w_1 \\ x \end{bmatrix},\end{aligned}$$

where Σ_1 is defined as $\Sigma_1 := \left[\begin{smallmatrix} I_{\mathtt{m}} - D^T D & -D^T C \\ -C^T D & -C^T C \end{smallmatrix}\right]$. Notice that Σ_1 can be factorized as

(A.3)

$$\begin{bmatrix} \left(I_{\mathtt{m}} - D^T D\right)^{\frac{1}{2}} & -\left(I_{\mathtt{m}} - D^T D\right)^{-\frac{1}{2}} D^T C \\ 0 & I_{\mathtt{n}} \end{bmatrix}^T \widetilde{\Sigma} \begin{bmatrix} \left(I_{\mathtt{m}} - D^T D\right)^{\frac{1}{2}} & -\left(I_{\mathtt{m}} - D^T D\right)^{-\frac{1}{2}} D^T C \\ 0 & I_{\mathtt{n}} \end{bmatrix},$$

where $\widetilde{\Sigma} := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -C^T(I_{\mathtt{p}} - DD^T)^{-1}C \end{bmatrix}$. Define $T := \begin{bmatrix} (I_{\mathtt{m}} - D^TD)^{\frac{1}{2}} & -(I_{\mathtt{m}} - D^TD)^{-\frac{1}{2}}D^TC \\ 0 & I_{\mathtt{n}} \end{bmatrix}$. Further define $\begin{bmatrix} \tilde{w_1} \\ \tilde{x} \end{bmatrix} := T\begin{bmatrix} w_1 \\ x \end{bmatrix}$. Then from (A.3) we have

$$Q_\Sigma(w) = \begin{bmatrix} \tilde{w_1} \\ \tilde{x} \end{bmatrix}^T \widetilde{\Sigma} \begin{bmatrix} \tilde{w_1} \\ \tilde{x} \end{bmatrix}. \tag{A.4}$$

Now, since (w_1, x) satisfies the state equation, $(\tilde{w_1}, \tilde{x})$ must satisfy the following:

$$\begin{aligned} & \begin{bmatrix} -B & \frac{\mathrm{d}}{\mathrm{d}t}I_{\mathtt{n}} - A \end{bmatrix} T^{-1} \begin{bmatrix} \tilde{w_1} \\ \tilde{x} \end{bmatrix} = 0 \\ \Rightarrow & \begin{bmatrix} -B & \frac{\mathrm{d}}{\mathrm{d}t}I_{\mathtt{n}} - A \end{bmatrix} \begin{bmatrix} \left(I_{\mathtt{m}} - D^TD\right)^{-\frac{1}{2}} & \left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC \\ 0 & I_{\mathtt{n}} \end{bmatrix} \begin{bmatrix} \tilde{w_1} \\ \tilde{x} \end{bmatrix} = 0 \\ \Rightarrow & \begin{bmatrix} -B\left(I_{\mathtt{m}} - D^TD\right)^{-\frac{1}{2}} & \frac{\mathrm{d}}{\mathrm{d}t}I_{\mathtt{n}} - A - B\left(I_{\mathtt{m}} - D^TD\right)^{-1} D^TC \end{bmatrix} \begin{bmatrix} \tilde{w_1} \\ \tilde{x} \end{bmatrix} = 0. \end{aligned}$$

Therefore, $(\tilde{w_1}, \tilde{x})$ satisfies the state equation, $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = \widetilde{A}\tilde{x} + \widetilde{B}\tilde{w_1}$, where $\widetilde{A} := A + B(I_{\mathtt{m}} - D^TD)^{-1}D^TC$, $\widetilde{B} := B(I_{\mathtt{m}} - D^TD)^{-\frac{1}{2}}$. By denoting $\widetilde{C} := (I_{\mathtt{p}} - DD^T)^{-\frac{1}{2}}C$, we can define a new behavior $\widetilde{\mathfrak{B}}$ that has an i/s/o representation given by $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = \widetilde{A}\tilde{x} + \widetilde{B}\tilde{w_1}$, $\tilde{w_2} = \widetilde{C}\tilde{x}$. Now, from (A.4) we have $\tilde{w} := \mathrm{col}(\tilde{w_1}, \tilde{w_2})$ satisfies

$$Q_\Sigma(\tilde{w}) = Q_\Sigma(w). \tag{A.5}$$

In order to infer about the controllability of the new behavior $\widetilde{\mathfrak{B}}$, we make use of Proposition A.1 above; since (A, B) is controllable, so is $(\widetilde{A}, \widetilde{B})$, and similarly, since (C, A) is observable, so is $(\widetilde{C}, \widetilde{A})$. Therefore, $\widetilde{\mathfrak{B}}$ allows an observable image representation

$$\widetilde{\mathfrak{B}} = \mathrm{im}\ \begin{bmatrix} \widetilde{W_1}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ \widetilde{W_2}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}.$$

From (A.5) we have

$$\partial\widetilde{\Phi}(\xi) := \widetilde{W_1}^T(-\xi)\widetilde{W_1}(\xi) - \widetilde{W_2}^T(-\xi)\widetilde{W_2}(\xi) = \partial\Phi(\xi).$$

Also, $\widetilde{\mathfrak{B}}$ gives the Hamiltonian matrix, $\widetilde{H} = \begin{bmatrix} \widetilde{A} & \widetilde{B}\widetilde{B}^T \\ -\widetilde{C}^T\widetilde{C} & -\widetilde{A}^T \end{bmatrix} = H$. Thus for every $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with an input/output partition such that the transfer function is proper, there exists $\widetilde{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with a corresponding i/o partition that gives a strictly proper transfer function, such that $\partial\Phi(\xi)$ matrices coming from $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ are the same and so are the corresponding Hamiltonian matrices. Therefore, from case 1 it follows that

$$\mathrm{spec}\left(\widetilde{H}\right) = \mathrm{roots}\ \left(\det\ \partial\widetilde{\Phi}(\xi)\right) \quad \Rightarrow \quad \mathrm{spec}(H) = \mathrm{roots}\ \left(\det\ \partial\Phi(\xi)\right).$$

This completes the proof of Lemma 4.5. $\square$

Proof of Lemma 5.3. Let the image representation matrix have a partition as

$$M(\xi) = \begin{bmatrix} W_1(\xi) \\ W_2(\xi) \end{bmatrix}; \quad W_1 \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi], W_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi].$$

Since the image representation is observable, $W_1(\xi)$ and $W_2(\xi)$ are relatively right-prime. So the transfer function from w_1 to w_2 is given by $G(\xi) = W_2(\xi)W_1^{-1}(\xi)$. Also, since this i/o partition allows an i/s/o representation with state dimension $\mathtt{n}$, $G(\xi)$ is proper and deg (det $W_1(\xi)) = \mathtt{n}$. From the i/s/o representation we have the transfer function as $G(\xi) = C(\xi I_{\mathtt{n}} - A)^{-1}B + D$. Now the second condition can be written as

$$\begin{aligned} &\partial\Phi(j\omega) > 0 \quad \forall\omega \in \mathbb{R} \\ \Rightarrow\ & W_1^T(-j\omega)W_1(j\omega) - W_2^T(-j\omega)W_2(j\omega) > 0 \quad \forall\omega \in \mathbb{R}. \end{aligned}$$

Since deg (det $\partial\Phi(\xi)) = 2\mathtt{n}$ and deg (det $W_1(\xi)) = \mathtt{n}$, we infer the biproperness of the rational function det $[W_1^{-T}(-j\omega)[W_1^T(-j\omega)W_1(j\omega) - W_2^T(-j\omega)W_2(j\omega)]W_1^{-1}(j\omega)]$. Using the biproperness, we can see that

$$\lim_{\omega\to\infty} \left(\det\ \left[W_1^{-T}(-j\omega)\left[W_1^T(-j\omega)W_1(j\omega) - W_2^T(-j\omega)W_2(j\omega)\right]W_1^{-1}(j\omega)\right]\right) \neq 0.$$

This fact together with $W_1^T(-j\omega)W_1(j\omega) - W_2^T(-j\omega)W_2(j\omega) > 0$ for all $\omega \in \mathbb{R}$ implies that

$$\lim_{\omega\to\infty} W_1^{-T}(-j\omega)\left[W_1^T(-j\omega)W_1(j\omega) - W_2^T(-j\omega)W_2(j\omega)\right]W_1^{-1}(j\omega) > 0.$$

The expression within the above limit is nothing but $I_{\mathtt{m}} - G^T(-j\omega)G(j\omega)$, the limit of which is $I_{\mathtt{m}} - D^TD$. This proves $(I_{\mathtt{m}} - D^TD) > 0$, and completes the proof of Lemma 5.3. ☐

The following lemma related to the strict dissipation inequality was used in the proof of Theorem 5.2 above. The proof is straightforward and so omitted (see [2]).

Lemma A.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. Then $\mathfrak{B}$ is strict Σ-dissipative if and only if $\mathfrak{B}$ is dissipative with respect to* $\Sigma_1 := \left[\begin{smallmatrix} I_{\mathtt{m}} & 0 \\ 0 & -(1+\epsilon_1)I_{\mathtt{p}} \end{smallmatrix}\right]$ *for some* $\epsilon_1 > 0$.

The following proposition is regarding the standard Kalman decomposition (see [34]); we require it in the proof of Theorem 5.4, which follows after this proposition.

Proposition A.3. *Let $\mathfrak{B}$ be uncontrollable with an i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$, where $w = (w_1, w_2)$. Then there exists a nonsingular matrix $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $T^{-1}AT = \left[\begin{smallmatrix} A_{\mathrm{c}} & A_{\mathrm{cp}} \\ 0 & A_{\mathrm{u}} \end{smallmatrix}\right]$, $TB = \left[\begin{smallmatrix} B_{\mathrm{c}} \\ 0 \end{smallmatrix}\right]$, and $CT^{-1} = \left[\begin{smallmatrix} C_{\mathrm{c}} & C_{\mathrm{u}} \end{smallmatrix}\right]$. Further,*

$$\text{(A.6)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{c}} = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}w_1, \quad w_2 = C_{\mathrm{c}}x_{\mathrm{c}} + Dw_1$$

gives an i/s/o representation for $\mathfrak{B}_{\text{cont}}$.

Proof of Theorem 5.4. The last proposition enables us to consider the following i/s/o representation without loss of generality:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_{\mathrm{c}} \\ x_{\mathrm{u}} \end{bmatrix} &= \begin{bmatrix} \widetilde{A}_{\mathrm{c}} & \widetilde{A}_{\mathrm{cp}} \\ 0 & \widetilde{A}_{\mathrm{u}} \end{bmatrix}\begin{bmatrix} x_{\mathrm{c}} \\ x_{\mathrm{u}} \end{bmatrix} + \begin{bmatrix} \widetilde{B}_{\mathrm{c}} \\ 0 \end{bmatrix} w_1 \\ w_2 &= \begin{bmatrix} \widetilde{C}_{\mathrm{c}} & \widetilde{C}_{\mathrm{u}} \end{bmatrix}\begin{bmatrix} x_{\mathrm{c}} \\ x_{\mathrm{u}} \end{bmatrix} + \widetilde{D}w_1. \end{aligned}$$

Then the corresponding Hamiltonian matrix gets the following form

$$H = \begin{bmatrix} A_{\mathrm{c}} & A_{\mathrm{cp}} & B_{\mathrm{c}}B_{\mathrm{c}}^T & 0 \\ 0 & A_{\mathrm{u}} & 0 & 0 \\ -C_{\mathrm{c}}^TC_{\mathrm{c}} & -C_{\mathrm{c}}^TC_{\mathrm{u}} & -A_{\mathrm{c}}^T & 0 \\ -C_{\mathrm{u}}^TC_{\mathrm{c}} & -C_{\mathrm{u}}^TC_{\mathrm{u}} & -A_{\mathrm{cp}}^T & -A_{\mathrm{u}}^T \end{bmatrix},$$

where

$$A_{\mathrm{c}} = \widetilde{A}_{\mathrm{c}} + \widetilde{B}_{\mathrm{c}}\left(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}^T\right)^{-1}\widetilde{D}^T\widetilde{C}_{\mathrm{c}}, \qquad B_{\mathrm{c}} = \widetilde{B}_{\mathrm{c}}\left(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}\right)^{-\frac{1}{2}},$$
$$A_{\mathrm{cp}} = \widetilde{A}_{\mathrm{cp}} + \widetilde{B}_{\mathrm{c}}\left(I_{\mathtt{m}} - \widetilde{D}^T\widetilde{D}^T\right)^{-1}\widetilde{D}^T\widetilde{C}_{\mathrm{u}}, \qquad C_{\mathrm{c}} = \left(I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T\right)^{-\frac{1}{2}}\widetilde{C}_{\mathrm{c}},$$
$$A_{\mathrm{u}} = \widetilde{A}_{\mathrm{u}}, \qquad C_{\mathrm{u}} = \left(I_{\mathtt{p}} - \widetilde{D}\widetilde{D}^T\right)^{-\frac{1}{2}}\widetilde{C}_{\mathrm{u}}.$$

Once again from Proposition A.3, $\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{c}} = \widetilde{A}_{\mathrm{c}}x_{\mathrm{c}} + \widetilde{B}_{\mathrm{c}}w_1$, $w_2 = \widetilde{C}_{\mathrm{c}}x_{\mathrm{c}} + \widetilde{D}w_1$ is an i/s/o representation of $\mathfrak{B}_{\mathrm{cont}}$. So the corresponding Hamiltonian matrix for $\mathfrak{B}_{\mathrm{cont}}$ is given by $H_{\mathrm{c}} := \begin{bmatrix} A_{\mathrm{c}} & B_{\mathrm{c}}B_{\mathrm{c}}^T \\ -C_{\mathrm{c}}^TC_{\mathrm{c}} & -A_{\mathrm{c}}^T \end{bmatrix}$. Now from Proposition 2.2 we can assume the i/s/o representation for $\mathfrak{B}$ to be state observable, which implies that the controllable part is state observable; i.e., the $(\widetilde{C}_{\mathrm{c}}, \widetilde{A}_{\mathrm{c}})$ pair is observable. Again, since $(\widetilde{C}_{\mathrm{c}}, \widetilde{A}_{\mathrm{c}})$ is observable and $\mathfrak{B}_{\mathrm{cont}}$ is controllable, the i/s/o representation of $\mathfrak{B}_{\mathrm{cont}}$ is also state controllable; that is, the $(\widetilde{A}_{\mathrm{c}}, \widetilde{B}_{\mathrm{c}})$ pair is controllable. Therefore from Lemma 4.5, we get $\mathrm{spec}(H_{\mathrm{c}}) = \text{roots}\ (\det \partial\Phi(\xi))$. Again, from the Kalman-decomposed i/s/o representation of $\mathfrak{B}$, we get

$$\Lambda_{\mathrm{un}} = \mathrm{spec}\left(\widetilde{A}_{\mathrm{u}}\right) = \mathrm{spec}(A_{\mathrm{u}}).$$

Thus to prove Theorem 5.4, all we need to show is $\det\,(\xi I_{2\mathtt{n}} - H) = \det\,(\xi I - H_{\mathrm{c}})\det\,(\xi I - A_{\mathrm{u}})\det\,(\xi I + A_{\mathrm{u}})$.

To show the above equality we shall find the determinant of the polynomial matrix $(\xi I_{2\mathtt{n}} - H)$ applying some elementary transformations on it as shown below. Let $\mathtt{n}_{\mathrm{c}}$ be the number of controllable eigenvalues and $\mathtt{n}_{\mathrm{u}}$ be that of uncontrollable eigenvalues.

$$\begin{aligned}\det\,(\xi I_{2\mathtt{n}} - H) &= \det\left(\begin{bmatrix} I_{\mathtt{n}_{\mathrm{c}}} & 0 & 0 & 0 \\ 0 & 0 & I_{\mathtt{n}_{\mathrm{c}}} & 0 \\ 0 & I_{\mathtt{n}_{\mathrm{u}}} & 0 & 0 \\ 0 & 0 & 0 & I_{\mathtt{n}_{\mathrm{u}}} \end{bmatrix}(\xi I_{2\mathtt{n}} - H)\begin{bmatrix} I_{\mathtt{n}_{\mathrm{c}}} & 0 & 0 & 0 \\ 0 & 0 & I_{\mathtt{n}_{\mathrm{u}}} & 0 \\ 0 & I_{\mathtt{n}_{\mathrm{c}}} & 0 & 0 \\ 0 & 0 & 0 & I_{\mathtt{n}_{\mathrm{u}}} \end{bmatrix}\right) \\ &= \det\begin{bmatrix} \xi I_{\mathtt{n}_{\mathrm{c}}} - A_{\mathrm{c}} & -B_{\mathrm{c}}B_{\mathrm{c}}^T & -A_{\mathrm{cp}} & 0 \\ C_{\mathrm{c}}^TC_{\mathrm{c}} & \xi I_{\mathtt{n}_{\mathrm{c}}} + A_{\mathrm{c}}^T & C_{\mathrm{c}}^TC_{\mathrm{u}} & 0 \\ 0 & 0 & \xi I_{\mathtt{n}_{\mathrm{u}}} - A_{\mathrm{u}} & 0 \\ C_{\mathrm{u}}^TC_{\mathrm{c}} & A_{\mathrm{cp}}^T & C_{\mathrm{u}}^TC_{\mathrm{u}} & \xi I_{\mathtt{n}_{\mathrm{u}}} + A_{\mathrm{u}}^T \end{bmatrix}.\end{aligned}$$

Now, exploiting the block structure of the above matrix, we can find out its determinant as

$$\begin{aligned}&\det\begin{bmatrix} \xi I_{\mathtt{n}_{\mathrm{c}}} - A_{\mathrm{c}} & -B_{\mathrm{c}}B_{\mathrm{c}}^T & -A_{\mathrm{cp}} & 0 \\ C_{\mathrm{c}}^TC_{\mathrm{c}} & \xi I_{\mathtt{n}_{\mathrm{c}}} + A_{\mathrm{c}}^T & C_{\mathrm{c}}^TC_{\mathrm{u}} & 0 \\ 0 & 0 & \xi I_{\mathtt{n}_{\mathrm{u}}} - A_{\mathrm{u}} & 0 \\ C_{\mathrm{u}}^TC_{\mathrm{c}} & A_{\mathrm{cp}}^T & C_{\mathrm{u}}^TC_{\mathrm{u}} & \xi I_{\mathtt{n}_{\mathrm{u}}} + A_{\mathrm{u}}^T \end{bmatrix} \\ &= \det\,\left(\xi I_{\mathtt{n}_{\mathrm{u}}} + A_{\mathrm{u}}^T\right)\det\,(\xi I_{\mathtt{n}_{\mathrm{u}}} - A_{\mathrm{u}})\det\begin{bmatrix} \xi I_{\mathtt{n}_{\mathrm{c}}} - A_{\mathrm{c}} & -B_{\mathrm{c}}B_{\mathrm{c}}^T \\ C_{\mathrm{c}}^TC_{\mathrm{c}} & \xi I_{\mathtt{n}_{\mathrm{c}}} + A_{\mathrm{c}}^T \end{bmatrix} \\ &= \det\,\left(\xi I_{\mathtt{n}_{\mathrm{u}}} + A_{\mathrm{u}}^T\right)\det\,(\xi I_{\mathtt{n}_{\mathrm{u}}} - A_{\mathrm{u}})\det\,(\xi I_{\mathtt{n}_{\mathrm{c}}} - H_{\mathrm{c}})\end{aligned}$$

$$\Rightarrow \mathrm{spec}(H) = \text{roots}\ [\,\chi_{\mathrm{un}}(-\xi)\,\chi_{\mathrm{un}}(\xi)\partial\Phi(\xi)], \tag{A.7}$$

where the last equality follows from Lemma 4.5. ∎

The following well-known result about left and right eigenspaces of the Hamiltonian matrix was used in the proof of Theorem 5.5. This lemma can be proved by straightforward verification.

LEMMA A.4. *Let Λ be a set of eigenvalues of the Hamiltonian matrix H. If* im $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ *is the generalized* right eigenspace *of H corresponding to Λ with $X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\bullet}$, then* im $\begin{bmatrix} X_2 \\ -X_1 \end{bmatrix}$ *is the generalized* left eigenspace *of H corresponding to $-\Lambda$.*

The following lemma is used in the proof of 5.2. The result says that right-primeness is equivalent to the absence of any zeros in the complex plane for the corresponding Hermitian matrix. Statement 3 below makes a similar statement, but at the point ∞.

LEMMA A.5. *Consider $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ with M having full column rank. Suppose the behavior having the image representation $w = M(\frac{d}{dt})\ell$ has McMillan degree $\mathtt{n}$. Then, the following are equivalent.*

1. *$M(\xi)$ is right-prime,*
2. *$M^T(\bar{\lambda})M(\lambda) > 0$ for all $\lambda \in \mathbb{C}$, and*
3. deg $[\det\,(M^T(-\xi)M(\xi))] = 2\mathtt{n}$.

Proof. We shall prove the following chain of implications: $1 \Rightarrow 2$, $2 \Rightarrow 1$, $1 \Rightarrow 3$ and $3 \Rightarrow 1$.

(1 ⇒ 2) We assume right-primeness of M and show that $M^T(\bar{\lambda})M(\lambda)$ is positive definite for all $\lambda \in \mathbb{C}$. That $M^T(\bar{\lambda})M(\lambda) \geqslant 0$ for all $\lambda \in \mathbb{C}$ is obvious. Due to the right-primeness, $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and this implies the required positive definiteness for all λ.

(2 ⇒ 1) $M^T(\bar{\lambda})M(\lambda) > 0$ for all $\lambda \in \mathbb{C}$ means that $M(\lambda)$ is injective for all λ proving its full column rank property for all λ, and hence right-primeness.

(1 ⇒ 3) $M(\xi)$ can be partitioned (after a permutation of rows, if needed) into $\mathrm{col}(W_1(\xi), W_2(\xi))$ such that $W_2 W_1^{-1}$ is a proper rational matrix, and $W_1(\xi)$ has determinant of degree $\mathtt{n}$. Suppose $G(\xi) := W_2(\xi)W_1(\xi)^{-1}$. (See [19] for McMillan degree's relation to an *observable* image representation, and observability of the image representation is equivalent to right-primeness of the matrix M.) Consider $\det\,(M^T(-\xi)M(\xi))$ which equals $\det\,(W_1^T(-\xi)W_1(\xi) + W_2^T(-\xi)W_2(\xi))$

$$= \det\,(W_1(-\xi))\det\,(W_1(\xi))\det\,\left(I + \left(W_2(-\xi)W_1(-\xi)^{-1}\right)^T \left(W_2(\xi)W_1(\xi)^{-1}\right)\right)$$
$$= \det\,(W_1(-\xi))\det\,(W_1(\xi))\det\,\left(I + G^T(-\xi)G(\xi)\right).$$

In order to determine the degree of $\det\,(M^T(-\xi)M(\xi))$, we let $\xi \to \infty$ to get rid of the strictly proper part within the last term above: $\lim_{\xi\to\infty}\det\,(I + G(-\xi)^T G(\xi)) = a$ (say). Notice that $0 < a < \infty$; $a < \infty$ because of the properness of $W_2W_1^{-1} = G$, while due to the positive definiteness of $(I + \lim_{\xi\to\infty} G^T(-\xi)G(\xi))$ we obtain that its determinant $a > 0$. Thus the degree of $\det\,(M(-\xi)^T M(\xi))$ is twice the degree of $\det W_1$, and is thus $2\mathtt{n}$.

(3 ⇒ 1) In order to prove this, we assume $M(\xi)$ is not right-prime and arrive at the required contradiction. Non-right-primeness of M means that M can be factored into $M = \widetilde{M}F$ such that $\widetilde{M}$ is right-prime, and F has a nonzero and nonconstant polynomial as its determinant. This implies that $w = \widetilde{M}(\frac{d}{dt})\ell$ is an observable kernel representation for $\mathfrak{B}$ and hence $\widetilde{M}$ can be partitioned (after possibly a permutation of rows) into $\widetilde{M} = \mathrm{col}(W_1, W_2)$ such that degree of $\det W_1 = \mathtt{n}$. Notice that $\det\,(M^T(-\xi)M(\xi)) = \det F(-\xi)\,\det F(\xi)\,\det\,(\widetilde{M}(-\xi)^T\widetilde{M}(\xi))$. We now use the proof of ($1 \Rightarrow 3$) above and that $\widetilde{M}$ is right-prime to conclude that the degree of $\det\,(\widetilde{M}(-\xi)^T\widetilde{M}(\xi))$ is $2\mathtt{n}$. Hence degree of $\det M^T(-\xi)M(\xi)$ equals

$2\mathtt{n} + 2(\deg(\det F))$. Since F has determinant a nonzero, nonconstant polynomial, we obtain $\det M^T(-\xi)M(\xi) > 2\mathtt{n}$, thus obtaining the required contradiction. This proves ($3 \Rightarrow 1$) and also the lemma. □

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