2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Stability Analysis of Discrete 2-D Autonomous Systems

Chirayu D. Athalye, Debasattam Pal, and Harish K. Pillai

***Abstract*— In this paper, we analyze the ℓ^2-stability and the ℓ^∞-stability of general discrete 2-D autonomous systems. The problem of ℓ^∞-stability has not been studied before for general 2-D autonomous systems. We give a necessary condition for the ℓ^∞-stability of discrete 2-D autonomous systems. We also give sufficient conditions for the ℓ^2-stability and the ℓ^∞-stability of discrete 2-D autonomous systems, which are easy to check.**

***Index Terms*— 2-D systems, ℓ^2-stability, ℓ^∞-stability.**

I. INTRODUCTION

As a result of applications in wide variety of areas like image processing, string stability, circuits, control, signal processing, seismology etc., research interest in 2-D systems and n-D systems in general has been growing and maturing for the past three to four decades. Quite a few new approaches have emerged into the literature over the time to represent and analyze 2-D systems. The classical literature on n-D systems is collated in [17], [16] and [15].

Behavioral theory developed by Willems (see [11] and [12]) for 1-D systems has turned out to be an appropriate framework to represent and analyze n-D systems. See among others [20], [10], [21] and [14] for the behavioral approach applied to n-D systems. We now briefly review the literature on stability analysis of n-D systems; we will use the terms behavior and system interchangeably.

Stability analysis of multidimensional systems is a topic of crucial importance because of their diverse applications which are of theoretical and/or practical interest. The internal stability of Fornasini-Marchesini input/state/output models for 2-D systems was studied in [8]. In [10], a notion of directional stability for n-D behaviors is proposed which is a generalization of BIBO stability of 1-D behaviors. In [19], stability properties of discrete 2-D autonomous behaviors are studied with respect to its characteristic cones. In [13], $\mathscr{L}_2$-stability of continuous n-D systems described by linear constant coefficient PDEs is discussed. Stability of strongly autonomous continuous n-D systems is investigated in [5]. Stability of time-relevant discrete 2-D autonomous systems is studied in [3]; it also gives an LMI characterization of time-relevant stability.

A. Objective and Overview

In [6], a certain representation formula is provided for discrete 2-D autonomous systems. With the help of this representation formula, solutions of discrete 2-D autonomous systems can be written in terms of powers of an appropriate Laurent polynomial matrix $A_1(\sigma_1, \sigma_1^{-1})$ acting on suitable 1-D trajectories, which are called *initial conditions*. In this paper, we make use of this representation formula to analyze the problem of ℓ^∞-stability for general discrete 2-D autonomous systems. For a specific 2-D autonomous system of platoon of vehicles, where the dynamics is governed by the relative position of a vehicle w.r.t. its immediate neighbors, the ℓ^∞-stability problem has been analyzed in [1] and [7]. However to the authors' knowledge, the problem of ℓ^∞-stability has not been studied for general 2-D autonomous systems. We give necessary condition and sufficient condition for the ℓ^∞-stability of discrete 2-D autonomous systems. We also give sufficient conditions for the ℓ^2-stability of discrete 2-D autonomous systems. Some of the results obtained on the ℓ^2-stability would be utilized to analyze the ℓ^∞-stability.

B. Organization of The Paper

In section-II, we explain background and problem statement. Section-III deals with mathematical preliminaries. In section-IV and section-V, we state our results on the ℓ^2-stability and the ℓ^∞-stability of discrete 2-D autonomous systems. Finally paper is concluded with future work in section-VI.

C. Notation

All norms on $\mathbb{R}^n$ are equivalent. Therefore, we denote a norm on $\mathbb{R}^n$ by just $\|\cdot\|$, when the explicit norm under consideration is irrelevant. For a linear operator $F : \mathbb{R}^n \to \mathbb{R}^n$, *the spectrum* $\Lambda(F)$, *the spectral radius* $\rho(F)$, *the induced* 2-*norm* $\|F\|_2$ and *the induced* ∞-*norm* $\|F\|_\infty$ are defined as:

$$\Lambda(F) := \{\lambda \in \mathbb{C} \,:\, \det(\lambda I - F) = 0\} \tag{1a}$$

$$\rho(F) := \max\{|\lambda| \,:\, \lambda \in \Lambda(F)\} \tag{1b}$$

$$\|F\|_2 := \max\{\|F\mathbf{x}\|_2 \,:\, \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_2 \leq 1\} \tag{1c}$$

$$\|F\|_\infty := \max\{\|F\mathbf{x}\|_\infty \,:\, \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_\infty \leq 1\} \tag{1d}$$

We use $\{\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_n\}$ to denote the standard basis of $\mathbb{R}^n$; and $\mathbf{1}$ to denote the vector of all ones, i.e. $\mathbf{1} = \sum_{j=1}^{n} \hat{\mathbf{e}}_j$. Transpose of a vector $\mathbf{v}$ (a matrix B) is denoted by $\mathbf{v}'$ (B').

$\mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$ is used to denote the space of vector valued bidirectional sequences; i.e. $\mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n) := \{\mathbf{a} : \mathbb{Z} \to \mathbb{R}^n\}$. $\mathbb{R}^\infty(\mathbb{Z}^2, \mathbb{R}^n)$ is used to denote the space of 2-D sequences. To denote the zero element in $\mathbb{R}^n$, $\mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$ and $\mathbb{R}^\infty(\mathbb{Z}^2, \mathbb{R}^n)$ we use boldface $\mathbf{0}$; and we expect it to become clear from the context. For $\mathbf{x} \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$, $\mathbf{x}(j)$ is used to denote the value of $\mathbf{x}$ at $j \in \mathbb{Z}$; therefore, $\mathbf{x}(j) \in \mathbb{R}^n$, $\forall\, j \in \mathbb{Z}$.

Laurent polynomial ring in two indeterminates σ_1, σ_2 with real coefficients is denoted as $\mathbb{R}[\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}]$. Laurent

Chirayu D. Athalye, Debasattam Pal, and Harish K. Pillai are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. chirayu@ee.iitb.ac.in, debasattam@ee.iitb.ac.in, hp@ee.iitb.ac.in

polynomial ring in one indeterminate is denoted analogously. We use i to denote $\sqrt{-1}$, unless specified otherwise. We use $S(0,1)$ to denote the unit circle in $\mathbb{C}$ centered at origin; i.e., $S(0,1) := \{z \in \mathbb{C} \,:\, |z| = 1\}$.

II. Background Development and Problem Formulation

In this section, we first briefly summarize relevant results from [6], and then we explain how the problem studied in this paper is equivalent to the stability analysis of general discrete 2-D autonomous systems with the representation given in [6]. There are quite a few equivalent definitions of 2-D autonomous behaviors; we refer the reader to [19], [10], [3] and [6] for the same. In this paper, we use following definition of discrete 2-D autonomous behavior, which is as per [19] and [3].

Definition 2.1: Discrete 2-D behavior is said to be autonomous if it is given by the kernel of a full column rank Laurent polynomial matrix in two shift operators.

Let $\mathfrak{B}$ be a discrete 2-D autonomous behavior with $\mathtt{w}$ manifest variables; i.e. $\mathfrak{B}$ is given by the kernel of a full column rank Laurent polynomial matrix $R(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1}) \in \mathbb{R}^{g\times \mathtt{w}}[\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1}]$. The shift operators $\sigma_j : \mathbb{R}^\infty(\mathbb{Z}^2,\mathbb{R}^n) \to \mathbb{R}^\infty(\mathbb{Z}^2,\mathbb{R}^n)$ for $j=1,2$ are defined as follows:

$$\sigma_1\left(w(h,k)\right) := w(h+1,k) \quad \text{and} \quad \sigma_2\left(w(h,k)\right) := w(h,k+1).$$

The inverse shift operators σ_j^{-1} for $j=1,2$ are defined analogously. Let $\mathscr{R}$ denote the equation module of $\mathfrak{B}$; i.e. $\mathscr{R} := \operatorname{rowspan}\left(R(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})\right)$ over the ring $\mathbb{R}[\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1}]$. The quotient module $\mathscr{M}$ of $\mathfrak{B}$ is defined as,

$$\mathscr{M} := \mathbb{R}^{1\times\mathtt{w}}[\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1}]/\mathscr{R}. \tag{2}$$

Definition 2.2: An autonomous behavior $\mathfrak{B}$ with equation module $\mathscr{R}$ is said to be *strongly σ_2-relevant* if the quotient module $\mathscr{M}$ is a finitely generated module over the ring $\mathbb{R}[\sigma_1,\sigma_1^{-1}]$.

We give below Theorem 3.7 from [6] for strongly σ_2-relevant autonomous behaviors.

Theorem 2.1: Let $\mathfrak{B}$ be a strongly σ_2-relevant autonomous behavior. Then there exist positive integers g_1, n_1, and the following one variable Laurent polynomial matrices,

1) $R_1(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{g_1\times n_1}[\sigma_1,\sigma_1^{-1}]$
2) $C_1(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{\mathtt{w}\times n_1}[\sigma_1,\sigma_1^{-1}]$
3) $A_1(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n_1\times n_1}[\sigma_1,\sigma_1^{-1}]$

with $A_1(\sigma_1,\sigma_1^{-1})$ invertible in $\mathbb{R}^{n_1\times n_1}[\sigma_1,\sigma_1^{-1}]$, such that: $\mathbf{w}\in\mathfrak{B}$ if and only if there exists $\mathbf{y}_0 \in \mathbb{R}^\infty(\mathbb{Z},\mathbb{R}^{n_1})$ which satisfies $R_1(\sigma_1,\sigma_1^{-1})\mathbf{y}_0 = 0$ and

$$\mathbf{w}(h,k) = \left(C_1(\sigma_1,\sigma_1^{-1})A_1(\sigma_1,\sigma_1^{-1})^k\mathbf{y}_0\right)(h), \tag{3}$$

for all $(h,k)\in\mathbb{Z}^2$.

It has been shown in [6] (see Theorem 5.3 in [6]) with the help of extension of Noether's normalization lemma that, every autonomous behavior can be converted into a strongly σ_2-relevant autonomous behavior by a suitable coordinate transformation on the domain $\mathbb{Z}^2$. Therefore, without loss of generality, we can assume that, the given discrete 2-D autonomous behavior is a strongly σ_2-relevant autonomous behavior.

Remark 2.1: It is important to note here that, due to the coordinate transformation required to carry out Noether's normalization, the new coordinate variables may not have the same physical meaning as the old ones. For example, in the original system the independent variables, $(h,k)\in\mathbb{Z}^2$, may be time and space respectively, whereas after the coordinate change, the independent variables might turn out to be some combinations of space and time.

Let $\mathfrak{X}$ denote the 1-D behavior given by the kernel of $R_1(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{g_1\times n_1}[\sigma_1,\sigma_1^{-1}]$. As $\mathfrak{X}$ is a 1-D behavior, it can be written as a direct sum of its controllable part $\mathfrak{X}_{cont}$ and autonomous part $\mathfrak{X}_{aut}$; i.e. $\mathfrak{X} = \mathfrak{X}_{cont}\oplus\mathfrak{X}_{aut}$.

Definition 2.3: Let $\mathfrak{B} := \{\mathbf{w} \in \mathbb{R}^\infty(\mathbb{Z}^2,\mathbb{R}^{\mathtt{w}}) : R(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})\mathbf{w} = 0\}$ be a strongly σ_2-relevant autonomous behavior.

- $\mathfrak{B}$ is said to be ℓ^2-stable if $\lim_{k\to\infty}\|\mathbf{w}(\cdot,k)\|_2 = 0$, $\forall\,\mathbf{y}_0 \in \mathfrak{X}\bigcap\ell^2(\mathbb{Z},\mathbb{R}^{n_1})$.
- $\mathfrak{B}$ is said to be ℓ^∞-stable if $\lim_{k\to\infty}\|\mathbf{w}(\cdot,k)\|_\infty = 0$, $\forall\,\mathbf{y}_0 \in \mathfrak{X}\bigcap\ell^\infty(\mathbb{Z},\mathbb{R}^{n_1})$.

The 1-D behavior $\mathfrak{X}$ is $A_1(\sigma_1,\sigma_1^{-1})$ invariant (see Remark 3.11 in [6]). Now for $k\in\mathbb{N}$, let us define $\mathbf{y}_k\in\mathfrak{X}$ as,

$$\mathbf{y}_k(\cdot) := A_1(\sigma_1,\sigma_1^{-1})^k\,\mathbf{y}_0(\cdot). \tag{4}$$

It follows from the construction of $C_1(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{\mathtt{w}\times n_1}[\sigma_1,\sigma_1^{-1}]$ and $A_1(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n_1\times n_1}[\sigma_1,\sigma_1^{-1}]$ (refer section-3 in [6]) that, the pair $\left(C_1(\sigma_1,\sigma_1^{-1}),A_1(\sigma_1,\sigma_1^{-1})\right)$ is observable; which is equivalent to saying that, the two variable Laurent polynomial matrix $\begin{bmatrix} C_1(\xi_1,\xi_1^{-1}) \\ \xi_2 I - A(\xi_1,\xi_1^{-1}) \end{bmatrix}$ is zero right prime. Therefore, we have the following equivalence w.r.t. (3) and (4):

$$\lim_{k\to\infty}\|\mathbf{w}(\cdot,k)\|_2 = 0 \iff \lim_{k\to\infty}\|\mathbf{y}_k\|_2 = 0 \tag{5a}$$

$$\lim_{k\to\infty}\|\mathbf{w}(\cdot,k)\|_\infty = 0 \iff \lim_{k\to\infty}\|\mathbf{y}_k\|_\infty = 0 \tag{5b}$$

Remark 2.2: Note that,

1) $\mathfrak{X}\bigcap\ell^2(\mathbb{Z},\mathbb{R}^{n_1}) = \mathfrak{X}_{cont}\bigcap\ell^2(\mathbb{Z},\mathbb{R}^{n_1})$.
2) $\mathfrak{X}\bigcap\ell^\infty(\mathbb{Z},\mathbb{R}^{n_1}) = \mathfrak{X}_{cont}\bigcap\ell^\infty(\mathbb{Z},\mathbb{R}^{n_1})$, when $\mathfrak{X}_{aut}$ does not have any characteristic value on $S(0,1)$.

Remark 2.3: Note that, $A_1(\sigma_1,\sigma_1^{-1})$ is invertible in $\mathbb{R}^{n_1\times n_1}[\sigma_1,\sigma_1^{-1}]$, so we can also talk about stability of strongly σ_2-relevant autonomous behavior $\mathfrak{B}$ as k tends to $(-\infty)$. The analysis would be exactly analogous to the case considered here and can be done by just replacing $A_1(\sigma_1,\sigma_1^{-1})$ by $A_1^{-1}(\sigma_1,\sigma_1^{-1})$.

Let $M(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n_1\times n}[\sigma_1,\sigma_1^{-1}]$ be an observable image representation matrix for $\mathfrak{X}_{cont}$; and let $M^{\dagger}(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n\times n_1}[\sigma_1,\sigma_1^{-1}]$ denote its left inverse. We define $A(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n\times n}[\sigma_1,\sigma_1^{-1}]$ as,

$$A(\sigma_1,\sigma_1^{-1}) := M^{\dagger}(\sigma_1,\sigma_1^{-1})A_1(\sigma_1,\sigma_1^{-1})M(\sigma_1,\sigma_1^{-1}) \quad (6)$$

For every $\mathbf{y} \in \mathfrak{X}_{cont}$, there exists $\mathbf{x} \in \mathbb{R}^{\infty}(\mathbb{Z},\mathbb{R}^n)$ such that, $\mathbf{y} = M(\sigma_1,\sigma_1^{-1})\mathbf{x}$; so for a given $\mathbf{y}_0 \in \mathfrak{X}_{cont}$, let $\mathbf{x}_0 \in \mathbb{R}^{\infty}(\mathbb{Z},\mathbb{R}^n)$ be such that, $\mathbf{y}_0 = M(\sigma_1,\sigma_1^{-1})\mathbf{x}_0$. Now for $k \in \mathbb{N}$, let us define $\mathbf{x}_k \in \mathbb{R}^{\infty}(\mathbb{Z},\mathbb{R}^n)$ as,

$$\mathbf{x}_k(\cdot) := A(\sigma_1,\sigma_1^{-1})^k\,\mathbf{x}_0(\cdot). \quad (7)$$

Lemma 2.1: 1) Let $\mathbf{y}_0 \in \mathfrak{X}_{cont} \bigcap \ell^2(\mathbb{Z},\mathbb{R}^{n_1})$ and let $\mathbf{y}_k$ be defined as in (4). Suppose for every $k \in \mathbb{N}$, $\mathbf{x}_k \in \ell^2(\mathbb{Z},\mathbb{R}^n)$ be such that, $\mathbf{y}_k(\cdot) = M(\sigma_1,\sigma_1^{-1})\mathbf{x}_k(\cdot)$. Then, $\mathbf{x}_k$ satisfies (7), for every $k \in \mathbb{N}$.

2) Suppose $\mathfrak{X}_{aut}$ does not have any characteristic value on $S(0,1)$. Let $\mathbf{y}_0 \in \mathfrak{X}_{cont} \bigcap \ell^{\infty}(\mathbb{Z},\mathbb{R}^{n_1})$ and let $\mathbf{y}_k$ be defined as in (4). Suppose for every $k \in \mathbb{N}$, $\mathbf{x}_k \in \ell^{\infty}(\mathbb{Z},\mathbb{R}^n)$ be such that, $\mathbf{y}_k(\cdot) = M(\sigma_1,\sigma_1^{-1})\mathbf{x}_k(\cdot)$. Then, $\mathbf{x}_k$ satisfies (7), for every $k \in \mathbb{N}$.

Proof:

1) It follows from Remark 2.2 that, $\mathfrak{X}_{cont} \bigcap \ell^2(\mathbb{Z},\mathbb{R}^{n_1})$ is $A_1(\sigma_1,\sigma_1^{-1})$ invariant. The proof then follows by induction on k.
2) When $\mathfrak{X}_{aut}$ does not have any characteristic value on $S(0,1)$, it follows from Remark 2.2 that, $\mathfrak{X}_{cont} \bigcap \ell^{\infty}(\mathbb{Z},\mathbb{R}^{n_1})$ is $A_1(\sigma_1,\sigma_1^{-1})$ invariant. The proof then follows by induction on k.

■

Written below are the consequences of Lemma 2.1, Remark 2.2, and the fact that $M(\sigma_1,\sigma_1^{-1})$ is an observable image representation of $\mathfrak{X}_{cont}$.

1) We have the following equivalence w.r.t. (4) and (7),

$$\begin{gathered}\lim_{k\to\infty}\|\mathbf{y}_k\|_2 = 0,\ \forall \mathbf{y}_0 \in \mathfrak{X} \bigcap \ell^2(\mathbb{Z},\mathbb{R}^{n_1}) \\ \Updownarrow \\ \lim_{k\to\infty}\|\mathbf{x}_k\|_2 = 0,\ \forall \mathbf{x}_0 \in \ell^2(\mathbb{Z},\mathbb{R}^n)\end{gathered} \quad (8)$$

2) When $\mathfrak{X}_{aut}$ does not have any characteristic value on $S(0,1)$,[1] we have the following equivalence w.r.t. (4) and (7),

$$\begin{gathered}\lim_{k\to\infty}\|\mathbf{y}_k\|_\infty = 0,\ \forall \mathbf{y}_0 \in \mathfrak{X} \bigcap \ell^\infty(\mathbb{Z},\mathbb{R}^{n_1}) \\ \Updownarrow \\ \lim_{k\to\infty}\|\mathbf{x}_k\|_\infty = 0,\ \forall \mathbf{x}_0 \in \ell^\infty(\mathbb{Z},\mathbb{R}^n)\end{gathered} \quad (9)$$

Therefore, from here onwards we shall consider the following system:

$$\mathbf{x}_{k+1}(\cdot) = A(\sigma_1,\sigma_1^{-1})\,\mathbf{x}_k(\cdot), \quad (10)$$

where $A(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n\times n}[\sigma_1,\sigma_1^{-1}]$ and $\mathbf{x}_k : \mathbb{Z} \to \mathbb{R}^n$, $\forall k \in \mathbb{N}\cup\{0\}$. It follows from (10) that,

$$\mathbf{x}_k(\cdot) = A(\sigma_1,\sigma_1^{-1})^k\,\mathbf{x}_0(\cdot). \quad (11)$$

[1]Note that, $\mathfrak{X}_{aut}$ generically won't have any characteristic value on $S(0,1)$.

Let the highest power of σ_1 and σ_1^{-1} in the Laurent polynomial matrix $A(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n\times n}[\sigma_1,\sigma_1^{-1}]$ be p and m, respectively. Then, we can write $A(\sigma_1,\sigma_1^{-1})$ as follows:

$$A(\sigma_1,\sigma_1^{-1}) = \sum_{j=-m}^{p} A_j\,\sigma_1^j, \quad (12)$$

where $A_j \in \mathbb{R}^{n\times n}$.

We analyze the problem of ℓ^2 and ℓ^∞ stability of the system given in (10). We saw that, discrete 2-D autonomous behavior $\mathfrak{B}$ being strongly σ_2-relevant is without loss of generality. Therefore, we have the following consequences from (5), (8) and (9).

- Analyzing the ℓ^2-stability of (10) is equivalent to analyzing ℓ^2-stability of discrete 2-D autonomous behavior $\mathfrak{B}$.
- In a generic case (when $\mathfrak{X}_{aut}$ does not have any characteristic values $S(0,1)$), analyzing the ℓ^∞-stability of (10) is equivalent to analyzing ℓ^∞-stability of discrete 2-D autonomous behavior $\mathfrak{B}$.

III. Mathematical Preliminaries

A. Functional Analysis

Here we briefly mention some preliminaries from functional analysis; reader can refer to [2], [9] for a detailed treatment on these topics.

We are mainly interested in the following two normed subspaces of $\mathbb{R}^{\infty}(\mathbb{Z},\mathbb{R}^n)$:

1) The space of square summable sequences $\ell^2(\mathbb{Z},\mathbb{R}^n)$ with norm $\|\cdot\|_2$. For $\mathbf{x} \in \ell^2(\mathbb{Z},\mathbb{R}^n)$,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=-\infty}^{\infty}(\|\mathbf{x}(j)\|_2)^2\right)^{1/2}. \quad (13)$$

2) The space of bounded sequences $\ell^\infty(\mathbb{Z},\mathbb{R}^n)$ with norm $\|\cdot\|_\infty$. For $\mathbf{x} \in \ell^\infty(\mathbb{Z},\mathbb{R}^n)$,

$$\|\mathbf{x}\|_\infty := \sup\{\|\mathbf{x}(j)\|_\infty : j \in \mathbb{Z}\}. \quad (14)$$

For every $\mathbf{y} \in \ell^2(\mathbb{Z},\mathbb{R}^n)$, we have $\|\mathbf{y}\|_\infty \le \|\mathbf{y}\|_2$. Therefore, for a sequence $(\mathbf{y}_m)$ in $\ell^2(\mathbb{Z},\mathbb{R}^n)$ we have the following implication:

$$\mathbf{y}_m \xrightarrow{\|\cdot\|_2} \mathbf{y} \implies \mathbf{y}_m \xrightarrow{\|\cdot\|_\infty} \mathbf{y} \quad (15)$$

However, the converse is not true. Therefore $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are not equivalent on $\ell^2(\mathbb{Z},\mathbb{R}^n)$; in fact $\|\cdot\|_2$ is stronger than $\|\cdot\|_\infty$.

Let $(X,\|\cdot\|_x)$ be any normed space. The space of bounded linear (or continuous linear) operators on X is denoted as $BL(X)$; it is a normed space with the following operator norm. For $F \in BL(X)$,

$$\|F\|_x := \max\{\|F\mathbf{y}\|_x : \mathbf{y} \in X \text{ and } \|\mathbf{y}\|_x \le 1\}. \quad (16)$$

The inequality,

$$\|F(\mathbf{y})\|_x \le \|F\|_x\,\|\mathbf{y}\|_x,\ \forall \mathbf{y} \in X \quad (17)$$

is called *the basic inequality*.

Let M be a doubly infinite matrix with entries from $\mathbb{R}$.

$M(j,k)$ is used to denote the entry in j-th row and k-th column of M. Let X be a subspace of $\mathbb{R}^\infty(\mathbb{Z},\mathbb{R})$. We say, a doubly infinite matrix M is a linear operator on X if the following conditions hold:

1) For every $\mathbf{x} \in X$ and for every $j \in \mathbb{Z}$, the series $\sum_{k=-\infty}^{\infty} M(j,k)\,\mathbf{x}(k)$ is summable in $\mathbb{R}$.
2) If for $j \in \mathbb{Z}$, we define $\mathbf{y}(j) := \sum_{k=-\infty}^{\infty} M(j,k)\,\mathbf{x}(k)$; then $\mathbf{y} \in X$.

We define α_1 and α_∞ for M as follows:

$$\alpha_1 := \sup\left\{\sum_{j=-\infty}^{\infty} |M(j,k)| \;:\; k \in \mathbb{Z}\right\} \tag{18a}$$

$$\alpha_\infty := \sup\left\{\sum_{k=-\infty}^{\infty} |M(j,k)| \;:\; j \in \mathbb{Z}\right\} \tag{18b}$$

Following is a well known result for normed spaces $\ell^2(\mathbb{Z},\mathbb{R})$ and $\ell^\infty(\mathbb{Z},\mathbb{R})$; see section-6.5 in [2].

Proposition 3.1: Let M be a doubly infinite matrix.

1) $M \in BL(\ell^\infty(\mathbb{Z},\mathbb{R}))$ if and only if $\alpha_\infty < \infty$. Moreover, $\|M\|_\infty = \alpha_\infty$.
2) If $(\alpha_1 \times \alpha_\infty) < \infty$, then $M \in BL(\ell^2(\mathbb{Z},\mathbb{R}))$. Moreover,
$$\|M\|_2 \;\leq\; \sqrt{(\alpha_1 \times \alpha_\infty)}\,.$$

Similar results hold when M is a doubly infinite block matrix with $M(j,k) \in \mathbb{R}^{n\times n}$, $\forall (j,k) \in \mathbb{Z}^2$. In this case α_1 and α_∞ are defined as follows:

$$\alpha_1 := \sup\left\{\sum_{j=-\infty}^{\infty}\sum_{r=1}^{n} |M(j,k)(r,s)| \;:\; k \in \mathbb{Z} \text{ and } s \in \{1,\ldots,n\}\right\}$$

$$\alpha_\infty := \sup\left\{\sum_{k=-\infty}^{\infty}\sum_{s=1}^{n} |M(j,k)(r,s)| \;:\; j \in \mathbb{Z} \text{ and } r \in \{1,\ldots,n\}\right\}$$

where $M(j,k)(r,s)$ denote the entry in r-th row and s-th column of $M(j,k) \in \mathbb{R}^{n\times n}$.

B. Block Laurent Operator

Consider a Laurent polynomial matrix $A(\sigma_1,\sigma_1^{-1}) = \left(\sum_{j=-m}^{p} A_j\,\sigma_1^j\right) \in \mathbb{R}^{n\times n}[\sigma_1,\sigma_1^{-1}]$. Corresponding to each such Laurent polynomial matrix, one can associate a doubly infinite banded block Laurent operator, which we denote as L_A. For example, when $A(\sigma_1,\sigma_1^{-1}) = A_{-1}\sigma_1^{-1} + A_0 + A_1\sigma_1$, the Laurent operator L_A would be as follows:

$$L_A = \begin{array}{cc} k=0 & \\ \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_{-1} & A_0 & A_1 & 0 & 0 & \cdots \\ \cdots & 0 & A_{-1} & A_0 & A_1 & 0 & \cdots \\ \cdots & 0 & 0 & A_{-1} & A_0 & A_1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} & j=0 \end{array}.$$

We denote the (j,k)-th entry of L_A by $L_A(j,k)$. Observe that, $L_A(j,k) \in \mathbb{R}^{n\times n}$ for all $(j,k) \in \mathbb{Z}^2$. Note that, both α_1 and α_∞ are finite for L_A corresponding to $A(\sigma_1,\sigma_1^{-1}) = \sum_{j=-m}^{p} A_j\,\sigma_1^j$. Therefore by Proposition 3.1, $L_A \in BL(\ell^2(\mathbb{Z},\mathbb{R}^n)) \bigcap BL(\ell^\infty(\mathbb{Z},\mathbb{R}^n))$.

Remark 3.1: Note that, for the system given in (10) we have, $\mathbf{x}_k = L_A^k\,\mathbf{x}_0$.

IV. ℓ^2 Stability

In this section, we analyze the ℓ^2-stability of the system (10), which is defined as follows.

Definition 4.1: The system given in (10) is said to be ℓ^2-stable if

$$\lim_{k\to\infty} \|\mathbf{x}_k\|_2 \;=\; 0,\; \forall\, \mathbf{x}_0 \in \ell^2(\mathbb{Z},\mathbb{R}^n)\,. \tag{20}$$

The following Theorem gives a sufficient condition for the system given in (10) to be ℓ^2-stable. It follows from Theorem 10 in [3] (see also [4]).

Theorem 4.1: The system given in (10) is ℓ^2-stable if $\rho\left(A(e^{i\omega}, e^{-i\omega})\right) < 1,\; \forall\,\omega \in [0, 2\pi)$.

We give another sufficient condition for the ℓ^2-stability, which is simple to check. Consider the system given in (10) where $A(\sigma_1,\sigma_1^{-1}) \in \mathbb{R}^{n\times n}[\sigma_1,\sigma_1^{-1}]$ is given by (12). We define $G \in \mathbb{R}^{n\times(p+m+1)n}$ and $H \in \mathbb{R}^{(p+m+1)n\times n}$ as follows:

$$G := [\, A_{(-m)}\; A_{(-m+1)}\; \cdots\; A_0\; \cdots\; A_{p-1}\; A_p\,] \tag{21a}$$

$$H := [\, A_p'\; A_{p-1}'\; \cdots\; A_0'\; \cdots\; A_{(-m+1)}'\; A_{(-m)}'\,]' \tag{21b}$$

Theorem 4.2: If $(\|G\|_\infty \times \|H\|_1) < 1$, then the system given in (10) is ℓ^2-stable.

Proof: For the banded block Laurent operator L_A corresponding to $A(\sigma,\sigma^{-1})$, we have:

$$\begin{aligned} \alpha_1 \;&:=\; \sup\left\{\sum_{j=-\infty}^{\infty}\sum_{r=1}^{n} |L_A(j,k)(r,s)| \;:\; k \in \mathbb{Z} \text{ and } s \in \{1,\ldots,n\}\right\} \\ &=\; \|H\|_1 \end{aligned} \tag{22}$$

$$\begin{aligned} \alpha_\infty \;&:=\; \sup\left\{\sum_{k=-\infty}^{\infty}\sum_{s=1}^{n} |L_A(j,k)(r,s)| \;:\; j \in \mathbb{Z} \text{ and } r \in \{1,\ldots,n\}\right\} \\ &=\; \|G\|_\infty \end{aligned} \tag{23}$$

Recall from Proposition 3.1 that, $\|L_A\|_2 \leq \sqrt{(\alpha_1 \times \alpha_\infty)}$. Therefore, if $(\|G\|_\infty \times \|H\|_1) < 1$, then $\|L_A\|_2 \leq \sqrt{(\alpha_1 \times \alpha_\infty)} < 1$. As a consequence,

$$0 \;\leq\; \lim_{k\to\infty} \|L_A^k\|_2 \;\leq\; \lim_{k\to\infty} (\|L_A\|_2)^k \;=\; 0. \tag{24}$$

Note that, $\mathbf{x}_k = L_A^k\,\mathbf{x}_0$, where $\mathbf{x}_0 \in \ell^2(\mathbb{Z},\mathbb{R}^n)$ is an initial condition. Now using basic inequality, we get:

$$\|\mathbf{x}_k\|_2 \;\leq\; \|L_A^k\|_2\,\|\mathbf{x}_0\|_2,\; \forall\, k \in \mathbb{N}, \tag{25}$$

and for all $\mathbf{x}_0 \in \ell^2(\mathbb{Z},\mathbb{R}^n)$. Therefore,

$$(\|G\|_\infty \times \|H\|_1) < 1$$
$$\Downarrow$$
$$\lim_{k\to\infty} \|L_A^k\|_2 = 0$$
$$\Downarrow$$
$$\text{the system in (10) is } \ell^2\text{-stable.}$$

∎

V. ℓ^∞ Stability

In this section, we analyze the ℓ^∞-stability of the system (10), which is defined as follows.

Definition 5.1: The system given in (10) is said to be ℓ^∞-stable if

$$\lim_{k\to\infty} \|\mathbf{x}_k\|_\infty = 0, \ \forall\, \mathbf{x}_0 \in \ell^\infty(\mathbb{Z},\mathbb{R}^n). \tag{26}$$

A. Necessary Condition

Theorem 5.1: If the system given in (10) is ℓ^∞-stable, then $\rho\left(A(e^{i\omega}, e^{-i\omega})\right) < 1, \ \forall\, \omega \in [0, 2\pi)$.

Proof: Suppose not, i.e. there exists $\psi \in [0, 2\pi)$ for which $\rho\left(A(e^{i\psi}, e^{-i\psi})\right) \geq 1$.

Let $(\lambda_1, \mathbf{v}_1)$ be an eigenpair of $A(e^{i\psi}, e^{-i\psi})$ such that, $\rho\left(A(e^{i\psi}, e^{-i\psi})\right) = |\lambda_1|$. Take $\mathbf{y}_0 \in \ell^\infty(\mathbb{Z},\mathbb{C}^n)$, which is defined as,

$$\mathbf{y}_0(j) := e^{j(i\psi)}\, \mathbf{v}_1, \ \forall\, j \in \mathbb{Z}. \tag{27}$$

From the banded block Laurent structure of L_A and the fact that $\mathbf{v}_1 \in \mathbb{C}^n$ is an eigenvector of $A(e^{i\psi}, e^{-i\psi})$ corresponding to eigenvalue λ_1, it follows that:

$$L_A\, \mathbf{y}_0 = \lambda_1\, \mathbf{y}_0. \tag{28}$$

As $|\lambda_1| \geq 1$,

$$\lim_{k\to\infty} \|L_A^k\, \mathbf{y}_0\|_\infty = \lim_{k\to\infty} |\lambda_1|^k\, \|\mathbf{v}_1\|_\infty \neq 0. \tag{29}$$

Using the real or the imaginary part of $\mathbf{y}_0 \in \ell^\infty(\mathbb{Z},\mathbb{C}^n)$ one can construct $\mathbf{x}_0 \in \ell^\infty(\mathbb{Z},\mathbb{R}^n)$ such that:

$$\lim_{k\to\infty} \|L_A^k\, \mathbf{x}_0\|_\infty \neq 0. \tag{30}$$

Hence a contradiction. ■

We give below a partial converse of Theorem 5.1; but before that, we explain some preliminaries which are required to prove the partial converse.

Preliminaries: Some important subspaces of $\ell^\infty(\mathbb{Z},\mathbb{R}^n)$ are defined below.

$$C(\mathbb{Z},\mathbb{R}^n) := \{\mathbf{x} \in \ell^\infty(\mathbb{Z},\mathbb{R}^n) \ : \ \lim_{j\to\infty} \mathbf{x}(j) = \lim_{j\to(-\infty)} \mathbf{x}(j)\}$$

$$C_0(\mathbb{Z},\mathbb{R}^n) := \{\mathbf{x} \in \ell^\infty(\mathbb{Z},\mathbb{R}^n) \ : \ \lim_{j\to\infty} \mathbf{x}(j) = \lim_{j\to(-\infty)} \mathbf{x}(j) = \mathbf{0}\}$$

Note that, $\ell^2(\mathbb{Z},\mathbb{R}^n) \subset C_0(\mathbb{Z},\mathbb{R}^n) \subset C(\mathbb{Z},\mathbb{R}^n) \subset \ell^\infty(\mathbb{Z},\mathbb{R}^n)$.

Both $(\ell^2(\mathbb{Z},\mathbb{R}^n), \|\cdot\|_2)$ and $(\ell^\infty(\mathbb{Z},\mathbb{R}^n), \|\cdot\|_\infty)$ are Banach spaces (complete normed space). As Banach spaces cannot have a denumerable[2] Hamel basis; every Hamel basis of both $\ell^2(\mathbb{Z},\mathbb{R}^n)$ and $\ell^\infty(\mathbb{Z},\mathbb{R}^n)$ is an uncountable set. Normed space $(C(\mathbb{Z},\mathbb{R}^n), \|\cdot\|_\infty)$ is also a Banach space with uncountable Hamel basis. We define elements of the set $\{\mathbf{e}_{rs} \ : \ r \in \mathbb{Z}, \ s \in \{1,...,n\}\}$ as:

$$\mathbf{e}_{rs}(k) := \begin{cases} \hat{\mathbf{e}}_s, & \text{if } k = r \\ \mathbf{0}, & \text{if } k \neq r \end{cases} \tag{31}$$

where $\hat{\mathbf{e}}_s$ denote elements of the standard basis of $\mathbb{R}^n$. The set $\{\mathbf{e}_{rs} \ : \ r \in \mathbb{Z}, \ s \in \{1,...,n\}\}$ is a Schauder basis for $\ell^2(\mathbb{Z},\mathbb{R}^n)$ and $C_0(\mathbb{Z},\mathbb{R}^n)$. Let us denote the vector valued sequence $[\cdots, \mathbf{1}', \mathbf{1}', \mathbf{1}', \cdots]'$ by $\mathbf{e}$. The set $\{\mathbf{e}\} \bigcup \{\mathbf{e}_{rs} \ : \ r \in \mathbb{Z}, \ s \in \{1,...,n\}\}$ is a Schauder basis[3] for $(C(\mathbb{Z},\mathbb{R}^n), \|\cdot\|_\infty)$. However, $\ell^\infty(\mathbb{Z},\mathbb{R}^n)$ does not have a Schauder basis, as it is not a separable metric space. We refer reader to [2] and [9] for definitions of Hamel basis, Schauder basis, and separable metric space.

Theorem 5.2: If $\rho\left(A(e^{i\omega}, e^{-i\omega})\right) < 1, \ \forall\, \omega \in [0, 2\pi)$ and the initial condition $\mathbf{x}_0 \in C(\mathbb{Z},\mathbb{R}^n)$, then $\lim_{k\to\infty} \|\mathbf{x}_k\|_\infty = 0$.

Proof: The set $\{\mathbf{e}\} \bigcup \{\mathbf{e}_{rs} \ : \ r \in \mathbb{Z}, \ s \in \{1,...,n\}\}$ is a Schauder basis for $(C(\mathbb{Z},\mathbb{R}^n), \|\cdot\|_\infty)$. Therefore for an arbitrary $\mathbf{x}_0 \in C(\mathbb{Z},\mathbb{R}^n)$, there exist $\gamma, \beta_{rs} \in \mathbb{R}$ such that:

$$\mathbf{x}_0 = \gamma\mathbf{e} + \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} \beta_{rs}\, \mathbf{e}_{rs}. \tag{32}$$

- *Claim-I*: $\lim_{k\to\infty} \|L_A^k\, \mathbf{e}_{rs}\|_\infty = 0, \ \forall \ r \in \mathbb{Z}$ and $\forall s \in \{1,...,n\}$.

  The set $\{\mathbf{e}_{rs} \ : \ r \in \mathbb{Z}, \ s \in \{1,...,n\}\}$ is a subset of $\ell^2(\mathbb{Z},\mathbb{R}^n)$. As $\rho\left(A(e^{i\omega}, e^{-i\omega})\right) < 1, \ \forall\, \omega \in [0, 2\pi)$, it follows from the Theorem 4.1 that: $\lim_{k\to\infty} \|L_A^k\, \mathbf{x}_0\|_2 = 0, \ \forall\, \mathbf{x}_0 \in \ell^2(\mathbb{Z},\mathbb{R}^n)$. Therefore,

  $$\lim_{k\to\infty} \|L_A^k\, \mathbf{e}_{rs}\|_2 = 0, \ \forall\, r \in \mathbb{Z} \text{ and } \forall\, s \in \{1,...,n\}. \tag{33}$$

  On $\ell^2(\mathbb{Z},\mathbb{R}^n)$, the metric induced by $\|\cdot\|_2$ is stronger than the metric induced by $\|\cdot\|_\infty$. Therefore,

  $$\lim_{k\to\infty} \|L_A^k\, \mathbf{e}_{rs}\|_\infty = 0, \ \forall\, r \in \mathbb{Z} \text{ and } \forall\, s \in \{1,...,n\}.$$

  This proves Claim-I.

- *Claim-II*: $\lim_{k\to\infty} \|L_A^k\, \mathbf{e}\|_\infty = 0$.

  At $\omega = 0$, $A(e^{i\omega}, e^{-i\omega}) = \sum_{j=-m}^{p} A_j$. Therefore we have, $\rho\left(\sum_{j=-m}^{p} A_j\right) < 1$. Observe that, because of the banded block Laurent structure of L_A and the structure of $\mathbf{e} \in C(\mathbb{Z},\mathbb{R}^n)$, we have:

  $$L_A^k\, \mathbf{e}\,(j) = \left(\sum_{j=-m}^{p} A_j\right)^k \mathbf{1}, \quad \forall\, j \in \mathbb{Z} \text{ and } \forall\, k \in \mathbb{N}. \tag{34}$$

  Note that, $\left(\sum_{j=-m}^{p} A_j\right) \in BL(\mathbb{R}^n)$. Therefore for every norm $\|\cdot\|$ on $\mathbb{R}^n$, we have the following implication:

  $$\rho\left(\sum_{j=-m}^{p} A_j\right) < 1 \Longrightarrow \lim_{k\to\infty} \|\left(\sum_{j=-m}^{p} A_j\right)^k \mathbf{y}\| = 0, \forall\, \mathbf{y} \in \mathbb{R}^n.$$

  It follows from (34) that,

  $$\|L_A^k\, \mathbf{e}\|_\infty = \|\left(\sum_{j=-m}^{p} A_j\right)^k \mathbf{1}\|_\infty, \forall\, k \in \mathbb{N}. \tag{35}$$

[2]A set is said to be denumerable if it is in one to one correspondence with $\mathbb{N}$.

[3]Note that $\mathbf{e} \neq \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} \mathbf{e}_{rs}$. Consider a partial sum $S_m = \sum_{r=-m}^{m} \sum_{s=1}^{n} \mathbf{e}_{rs}$. Now observe that, $\|\mathbf{e} - S_m\|_\infty = 1, \ \forall\, m \in \mathbb{N}$. Therefore $\lim_{m\to\infty} \|\mathbf{e} - S_m\|_\infty = 1 \neq 0$.

Therefore we have,

$$\lim_{k\to\infty} \|L_A^k \, \mathbf{e}\|_\infty \;=\; \lim_{k\to\infty} \| \left(\sum_{j=-m}^{p} A_j \right)^k \mathbf{1} \|_\infty \;=\; 0\,. \quad (36)$$

This proves Claim-II.

As L_A is a continuous linear operator, it follows from (32) that:

$$L_A^k \, \mathbf{x}_0 = \gamma(L_A^k \, \mathbf{e}) + \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} \beta_{rs}(L_A^k \, \mathbf{e}_{rs})\,. \quad (37)$$

From Claim-II, triangle inequality and the countable sub-additivity of norms, we have:

$$\lim_{k\to\infty} \|L_A^k \, \mathbf{x}_0\|_\infty \;\leq\; \lim_{k\to\infty} \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} |\beta_{rs}| \; \|L_A^k \, \mathbf{e}_{rs}\|_\infty \quad (38)$$

- *Claim-III*: For every $\varepsilon > 0$, there exists k_0 which is independent of $\mathbf{e}_{rs}$ such that: whenever $k \geq k_0$, we have $\|L_A^k \, \mathbf{e}_{rs}\|_\infty < \varepsilon$, $\forall r \in \mathbb{Z}$ and $\forall s \in \{1,...,n\}$.

  From Claim-I; for a given $\varepsilon > 0$, there exists $k_{rs} \in \mathbb{N}$, which depends on $r \in \mathbb{Z}$ and $s \in \{1,...,n\}$, such that: when $k \geq k_{rs}$, we have $\|L_A^k \, \mathbf{e}_{rs}\|_\infty < \varepsilon$.
  Let k_{rs} be the smallest natural numbers satisfying the above property. Now observe that, because of the banded block Laurent structure of L_A, we have:

  $$k_{0s} = k_{rs},\, \forall r \in \mathbb{Z},\, \forall s \in \{1,...,n\}\,. \quad (39)$$

  Therefore, it is enough to consider just $k_{01},....,k_{0n}$. Now if we take k_0 as $\max\{k_{01},...,k_{0n}\}$, then for every $k \geq k_0$ we have:

  $$\|L_A^k \, \mathbf{e}_{rs}\|_\infty < \varepsilon,\, \forall r \in \mathbb{Z},\, \forall s \in \{1,...,n\}\,. \quad (40)$$

  This proves Claim-III.

Now, because of Claim-III and the uniform convergence Theorem for summation (integration), we can interchange limit and summation on the right hand side of inequality (38). Therefore, it follows from Claim-I that,

$$\begin{aligned} \lim_{k\to\infty} \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} |\beta_{rs}| \; \|L_A^k \, \mathbf{e}_{rs}\|_\infty \;&=\; \sum_{r=-\infty}^{\infty} \sum_{s=1}^{n} |\beta_{rs}| \; \lim_{k\to\infty} \|L_A^k \, \mathbf{e}_{rs}\|_\infty \\ &=\; 0\,. \end{aligned} \quad (41)$$

From (38) and (41), we have:

$$\lim_{k\to\infty} \|L_A^k \, \mathbf{x}_0\|_\infty \;=\; 0\,. \quad (42)$$

■

B. Sufficient Condition

We give a sufficient condition for the ℓ^∞-stability of (10) in terms of the matrix $G \in \mathbb{R}^{n\times(p+m+1)n}$ defined in (21a), which is simple to check.

Theorem 5.3: If $\|G\|_\infty < 1$, then the system given in (10) is ℓ^∞-stable.

Proof: For the banded block Laurent operator L_A corresponding to $A(\sigma,\sigma^{-1})$, $\alpha_\infty = \|G\|_\infty$. Recall from Proposition 3.1 that, $\|L_A\|_\infty = \alpha_\infty$. Therefore if $\|G\|_\infty < 1$, then $\|L_A\|_\infty < 1$. As a consequence,

$$0 \;\leq\; \lim_{k\to\infty} \|L_A^k\|_\infty \;\leq\; \lim_{k\to\infty} \left(\|L_A\|_\infty\right)^k = 0. \quad (43)$$

Note that, $\mathbf{x}_k = L_A^k \, \mathbf{x}_0$, where $\mathbf{x}_0 \in \ell^\infty(\mathbb{Z},\mathbb{R}^n)$ is an initial condition. Now using basic inequality, we get:

$$\|\mathbf{x}_k\|_\infty \;\leq\; \|L_A^k\|_\infty \, \|\mathbf{x}_0\|_\infty,\; \forall k \in \mathbb{N}\,, \quad (44)$$

and for all $\mathbf{x}_0 \in \ell^\infty(\mathbb{Z},\mathbb{R}^n)$. Therefore, if $\lim_{k\to\infty} \|L_A^k\|_\infty = 0$, then the system in (10) is ℓ^∞-stable. ■

VI. Conclusion

We have given sufficient conditions for the ℓ^2-stability of discrete 2-D autonomous systems. For the ℓ^∞-stability of discrete 2-D autonomous systems, we have given a necessary condition along with its partial converse. We have also given a sufficient condition for the ℓ^∞-stability, which is easy to check. Future work is to find equivalent conditions for the ℓ^∞-stability of discrete 2-D autonomous systems, and to analyze the ℓ^∞-stability when $\mathfrak{X}_{aut}$ has a characteristic value on the imaginary axis.

Acknowledgement: This research work was supported by Microsoft Research India.

References

[1] A. Feintuch, B. Francis, *Infinite Chains of Kinematic Points*, Automatica J. IFAC, 48 (2012), pp. 901-908.

[2] B.V. Limaye, *Functional Analysis*, 2nd ed., New Age International Publication, New Delhi, 1996.

[3] D. Napp, P. Rapisarda, P. Rocha, *Time-relevant Stability of 2D Systems*, Automatica J. IFAC, 47 (2011), pp. 2373-2382.

[4] D. Napp, P. Rapisarda, P. Rocha, *Corrigendum to: Time-relevant Stability of 2D Systems*, Automatica J. IFAC, 48 (2012), pp. 2737.

[5] D. Pal, H.K. Pillai, *Lyapunov Stability of Strongly Autonomous nD Systems*, Internat. J. Control, 84 (2011), pp. 1759-1768.

[6] D. Pal, H.K. Pilllai, *Representation Formulae for Discrete 2D Autonomous Systems*, SIAM J. Control Optim., 51 (2013), pp. 2406-2441.

[7] D. Swaroop, H.K. Hedrick, *String Stability of Interconnected Systems*, IEEE Trans. Automat. Control, 41 (1996), pp. 349-357.

[8] E. Fornasini, G. Marchesini, *Stability Analysis of 2D Systems*, IEEE Trans. Circuits Syst., 32 (1980), pp 1246-1254.

[9] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1978.

[10] H.K. Pillai, S. Shankar, *A Behavioral Approach to Control of Distributed Systems*, SIAM J. Control Optim., 37 (1998), pp. 388-408.

[11] J.C. Willems, *Paradigms and Puzzles in the Theory of Dynamical Systems*, IEEE Trans. Automat. Control, 36 (1991), pp. 259-294.

[12] J.C. Willems, H.L. Trentleman, *On Quadratic Differential Forms*, SIAM J. Control Optim., 36 (1998), pp. 1703-1749.

[13] J.C. Willems, *Stability and Quadratic Lyapunov Functions for nD Systems*, In Proc. of International Conference on Multidimensional (nD) Systems, 2007.

[14] J-F. Pommaret, A. Quadrat, *Algebraic Analysis of Linear Multidimensional Control Systems*, IMA J. Math. Control Inform., 16 (1999), pp. 275-297.

[15] J. Wood, *Modules and Behaviours in nD Systems Theory*, Multidimens. Syst. Signal Process., 11 (2000), pp. 11-48.

[16] K. Gałkowski (Ed.), *Proceedings of 2-nd International Workshop on Multidimensional Systems*, Poland, 2000.

[17] K. Gałkowski, *Lecture Notes in Control and Information Sciences 263: State-space Realizations of Linear 2-D Systems with Extensions to the General nD ($n > 2$) Case*, Springer-Verlag, London, 2001.

[18] M. Reid, *Undergraduate Commutative Algebra*, Cambridge University Press, New York, 1995.

[19] M.E. Valcher, *Characteristic Cones and Stability Properties of Two-dimensional Autonomous Behavior*, IEEE Trans. Circuits Systems I Fund. Theory Appl., 47 (2000), pp. 290-302.

[20] P. Rocha, *Structure and Representations of 2-D Systems*, Ph.D thesis, Mathematics and Natural Sciences (Systems and Control), 1990, University of Groningen, Netherlands.

[21] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math., 20 (1990), pp. 1-175.