A Hamiltonian system based approach for the computation of the maximal rank-minimizing solution of the LMI arising from a singular LQR problem

Imrul Qais[1], Chayan Bhawal[2], and Debasattam Pal[1]

***Abstract*— In this paper we deal with the linear matrix inequality (LMI) arising from a singular linear quadratic regulator (LQR) problem. The *maximal rank-minimizing* solution $K_{\max}$ of the LMI plays a central role in obtaining a proportional-derivative feedback law for the optimal input. The optimal cost of the LQR, too, depends on this solution $K_{\max}$. In this paper, we provide a method to compute this maximal rank-minimizing solution $K_{\max}$ of the singular LQR LMI. We compute this solution using the notions of the *weakly unobservable* or the *slow space* and the *strongly reachable* or the *fast space* of the Hamiltonian system arising from the singular LQR problem. In this process, we also provide a novel characterization of the fast space in terms of the system matrices.**

I. INTRODUCTION

In this paper we deal with the LMI arising from the singular case of the infinite horizon LQR problem. Following is the problem statement of an infinite horizon LQR problem:

Problem 1.1: Consider a stabilizable system with state-space dynamics $\frac{d}{dt}x = Ax + Bu$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Then, for every initial condition x_0, find an input u that minimizes the functional

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \qquad (1)$$

with $\lim_{t\to\infty} x(t) = 0$, where $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geqslant 0$.

This is a regular LQR problem if the cost matrix $R > 0$, and a singular LQR problem when R is singular. For the regular problem, the optimal input u^* is given by a state-feedback law $u^* = -R^{-1}(S^T + B^T K_{\max})x$, where $K_{\max}$ is the *maximal* solution of the algebraic Riccati equation (ARE):

$$A^T K + KA + Q - (KB + S)R^{-1}(B^T K + S^T) = 0; \qquad (2)$$

that is, $K_{\max} - K \geqslant 0$ for any arbitrary solution K of the ARE. The singular LQR problems do not admit the ARE due to singularity of the matrix R. However, both the regular and the singular LQR problems give rise to the following LMI:

$$\mathcal{L}(K) := \begin{bmatrix} A^T K + KA + Q & KB + S \\ B^T K + S^T & R \end{bmatrix} \geqslant 0. \qquad (3)$$

For the regular case, $K_{\max}$ is the maximal *rank-minimizing* solution of the LMI (3), that is $\operatorname{rank}\mathcal{L}(K_{\max}) \leqslant \operatorname{rank}\mathcal{L}(K)$ for any arbitrary solution K of the LMI. For singular LQR problems, too, the maximal rank-minimizing solution $K_{\max}$ of the LMI (3) is the key to obtain the optimal solution [1]. Singular LQR problem has been extensively studied in the seminal paper [2]. But, a feedback solution has not been provided there. A linear implicit control law of the form $Px + Lu = 0$ has been provided in [3]. But, unfortunately, this law is not always feedback implementable. In [4] $K_{\max}$ has been used to design a proportional-derivative (P-D) feedback optimal control law for the single-input case. The solution presented in this paper is expected to play a crucial role in designing a P-D feedback law for the multi-input case. We shall pursue this design problem elsewhere in future.

In this paper we transform Problem 1.1 to an alternative formulation of the singular LQR problem, which separates the regular part from the singular part of the problem. Since $R \geqslant 0$, there exists an orthogonal matrix $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ such that $U^T R U = \mathtt{diag}(0, \widehat{R})$, where $\widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$ and $\mathtt{r} := \operatorname{rank} R$. Clearly, $\widehat{R} > 0$. Define $BU =: \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ and $SU =: \begin{bmatrix} S_1 & S_2 \end{bmatrix}$, where $B_2, S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. Then, it is easy to verify that $S_1 = 0$ ([5, Lemma 1]). Thus, without loss of generality, we can provide the following alternative formulation of the singular LQR Problem (for more details see [5, Lemma 1]):

Problem 1.2: Let $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $S_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$, and $\widehat{R} \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}$ be such that $\widehat{R} > 0$ and $\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0_{\mathtt{d},\mathtt{d}} & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0$, where $\mathtt{d} := \mathtt{m} - \mathtt{r}$. Consider a stabilizable system with state-space dynamics $\frac{d}{dt}x = Ax + B_1 u_1 + B_2 u_2$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{d}}$, and $B_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{r}}$. Then, for every initial condition x_0, find an input $u := \mathtt{col}(u_1, u_2)$ that minimizes the functional:

$$J(x_0, u) := \int_0^\infty \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix} dt \text{ with } \lim_{t\to\infty} x(t) = 0. \qquad (4)$$

Under this transformation LMI (3) takes the form:

$$\mathcal{L}_{\mathtt{t}}(K) := \begin{bmatrix} A^T K + KA + Q & KB_1 & KB_2 + S_2 \\ B_1^T K & 0 & 0 \\ B_2^T K + S_2^T & 0 & \widehat{R} \end{bmatrix} \geqslant 0. \qquad (5)$$

It is easy to verify that $K_{\max}$ is the maximal rank-minimizing solution of LMI (3) if and only if $K_{\max}$ is the maximal rank-minimizing solution of LMI (5).

In [6], [7], [8], it has been shown that $K_{\max}$ can be found by solving the following set of equations known as constrained generalized continuous ARE (CGCARE):

$$A^T K + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^T K + S_2^T) = 0 \text{ and } KB_1 = 0.$$

But, [5] shows that such equations are generically unsolvable. For a regular LQR problem, the maximal rank-minimizing solution of the LQR LMI is given by the maximal solution of the corresponding ARE. There are numerous methods to compute the maximal solution of an ARE: see [9] for different methods. However, these methods cannot be used to compute the maximal rank-minimizing solution of an LQR LMI for the singular case primarily due to the singularity of R matrix. In this paper, we provide a novel method of computing the maximal rank-minimizing solution of the singular LQR LMI. This method, in principle, is an extension

[1] The authors are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai, India (e-mail: imrul@ee.iitb.ac.in, debasattam@ee.iitb.ac.in).

[2] The author is with the Department of Electronics and Electrical Engineering, Indian Institute of Technology Guwahati, Guwahati, India (e-mail: bhawal@iitg.ac.in).

of the Hamiltonian matrix based method prevalently used for the regular case. However, one crucial distinction in our approach is the substitution of the eigenspace of the Hamiltonian matrix in the regular case by the *weakly unobservable (slow) subspace* of the corresponding Hamiltonian system and the *strongly reachable (fast) subspace* of the primal. This approach substantially helps in the design of the P-D feedback for the optimal input ([4]).

II. Notation and Preliminaries

A. Notation

The symbols $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ are used for the sets of real numbers, complex numbers, and natural numbers, respectively. $\mathbb{R}_+$ and $\mathbb{C}_-$ denote the sets of positive real numbers and complex numbers with negative real parts, respectively. $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}$. We use the symbol $I_{\mathtt{n}}$ for an $\mathtt{n}\times\mathtt{n}$ identity matrix and the symbol $0_{\mathtt{n},\mathtt{m}}$ for an $\mathtt{n}\times\mathtt{m}$ matrix with all entries zero. $\mathtt{col}(B_1, B_2,\dots,B_{\mathtt{n}})$ represents a matrix of the form $\left[\, B_1^T \; B_2^T \; \cdots \; B_{\mathtt{n}}^T \,\right]^T$. By $\mathtt{img}\, A$ and $\mathtt{ker}\, A$ we denote the image and nullspace of a matrix A, respectively. The symbol $\mathtt{rank}\, A$ denotes the rank of a matrix A. $\mathtt{det}(A)$ represents the determinant of a square matrix A. The symbols $\mathtt{deg}(p(s))$ and $\mathtt{roots}(p(s))$ denote the degree and the set of roots (over complex numbers) of a polynomial $p(s)$ with real or complex coefficients (counted with multiplicity), respectively. The symbol $\mathtt{num}(p(s))$ is used to denote the numerator of a rational function $p(s)$. By $\mathtt{degdet}(A(s))$ we denote the degree of the determinant of a polynomial matrix $A(s)$ and by $\mathtt{numdet}(A(s))$ we denote the numerator of the determinant of a rational function matrix $A(s)$. The symbol $\sigma(A)$ denotes the set of eigenvalues of a square matrix A (counted with multiplicity). The symbol $|\Gamma|$ denotes the cardinality of a set Γ. We use the symbol $\sigma(A|_{\mathcal{S}})$ to represent the set of eigenvalues of A restricted to an A-invariant subspace $\mathcal{S}$. We use the symbol $\mathtt{dim}\,(\mathcal{S})$ to denote the dimension of a space $\mathcal{S}$. The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ is represented by the symbol $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})$, while $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})|_{\mathbb{R}_+}$ represents the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}^{\mathtt{n}}$ that are restrictions of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})$ functions to $\mathbb{R}_+$. δ represents the Dirac delta impulse distribution and $\delta^{(i)}$ represents the i-th distributional derivative of δ with respect to t.

B. Regular matrix pencils

Consider a regular matrix pencil $(sU_1 - U_2) \in \mathbb{R}[s]^{\mathtt{n}\times\mathtt{n}}$, i.e., $\mathtt{det}(sU_1 - U_2) \not\equiv 0$. Let $\lambda \in \mathtt{roots}\,(\mathtt{det}(sU_1 - U_2))$. Then λ is called an *eigenvalue* of (U_1, U_2) and every nonzero vector $v \in \mathtt{ker}\,(\lambda U_1 - U_2)$ is called an *eigenvector* of the matrix pair (U_1, U_2) corresponding to the eigenvalue λ. Further, every nonzero vector $\tilde{v} \in \mathtt{ker}\,(\lambda U_1 - U_2)^i$, where $i \in \{2, 3, \dots\}$, is called a *generalized eigenvector* of the matrix pair (U_1, U_2) corresponding to the eigenvalue λ. We use the symbol $\sigma(U_1, U_2)$ to denote the set of eigenvalues of (U_1, U_2) (with $\lambda \in \sigma(U_1, U_2)$ included in the set as many times as its algebraic multiplicity).

C. (A, B)-invariant subspaces

Definition 2.1: Consider $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. A subspace $\mathcal{S} \subseteq \mathbb{R}^{\mathtt{n}}$ is said to be *(A, B)-invariant* if there exists a matrix $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF)\mathcal{S} \subseteq \mathcal{S}$.

We use the symbol $\mathfrak{I}(A, B)$ for the family of all (A, B)-invariant subspaces for a given (A, B) pair. The notation $\mathfrak{I}(A, B; \mathtt{ker}\, C)$ denotes the family of (A, B)-invariant subspaces that are contained in $\mathtt{ker}\, C$, where $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$. It is known in the literature that the set $\mathfrak{I}(A, B; \mathtt{ker}\, C)$ admits a supremal element [10, Lemma 4.4], and we represent it by the symbol $\mathtt{sup}\,\mathfrak{I}(A, B; \mathtt{ker}\, C)$. Formally this means that for all $\mathcal{S} \in \mathfrak{I}(A, B; \mathtt{ker}\, C)$, we must have $\mathcal{S} \subseteq \mathtt{sup}\,\mathfrak{I}(A, B; \mathtt{ker}\, C)$. The notation $\mathbf{F}(\mathcal{S})$ is used for the collection of matrices $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF)\mathcal{S} \subseteq \mathcal{S}$.

Another important concept is the notion of *good* (A, B)-invariant subspaces. We explain this notion next. Define

$$\mathcal{B} := \{\mathcal{S} \in \mathfrak{I}(A, B, \mathtt{ker}\, C) \mid \exists F \in \mathbf{F}(\mathcal{S}) \text{ such that } \sigma\left((A+BF)|_{\mathcal{S}}\right) \subsetneq \mathbb{C}_- \},$$

We call subspaces in $\mathcal{B}$ *good (A, B)-invariant subspaces inside* $\mathtt{ker}\, C$. As shown in [10, Lemma 5.8], the set $\mathcal{B}$ admits a supremal element defined as $\mathcal{S}_{\mathtt{g}}^* := \mathtt{sup}\,\mathcal{B}$, i.e., for all elements $\mathcal{S} \in \mathcal{B}, \mathcal{S} \subseteq \mathcal{S}_{\mathtt{g}}^*$. Hence, $\mathcal{S}_{\mathtt{g}}^*$ is called the *largest good (A, B)-invariant subspace inside* $\mathtt{ker}\, C$.

D. Weakly unobservable and strongly reachable subspaces

Consider the system with an input-state-output (i/s/o) representation $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + Du$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$. Associated with such a system are two important subspaces called the weakly unobservable subspace and the strongly reachable subspace. Before we delve into the definitions of these subspaces, we need to define the space of impulsive-smooth distributions (see [2], [11]).

Definition 2.2: The set of impulsive-smooth distributions $\mathfrak{C}^{\mathtt{w}}_{\mathrm{imp}}$ is defined as:

$$\mathfrak{C}^{\mathtt{w}}_{\mathrm{imp}} := \Big\{ f = f_{\mathtt{reg}} + f_{\mathtt{imp}} \mid f_{\mathtt{reg}} \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})|_{\mathbb{R}_+} \text{ and } f_{\mathtt{imp}} = \sum_{i=0}^{k} a_i \delta^{(i)}, \text{ with } a_i \in \mathbb{R}^{\mathtt{w}}, k \in \mathbb{N} \Big\}.$$

In what follows, we denote the state-trajectory x and output-trajectory y of the system Σ, that result from initial condition x_0 and input u, using the symbols $x(x_0, u)$ and $y(x_0, u)$, respectively. The symbol $x(0^+; x_0, u)$ denotes the value of the state-trajectory that can be reached from x_0 instantaneously on application of the input u at $t = 0$.

Definition 2.3: A state $x_0 \in \mathbb{R}^{\mathtt{n}}$ is called *weakly unobservable* if there exists an input $u \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})|_{\mathbb{R}_+}$ such that $y(x_0, u) \equiv 0$ for all $t \geqslant 0$. The collection of all such weakly unobservable states is called the *weakly unobservable subspace* or the *slow space* and is denoted by $\mathcal{O}_w$.

Proposition 2.4: The slow space $\mathcal{O}_w$ is the largest subspace $\mathcal{V}$ of the state-space for which there exists a feedback $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C + DF)\mathcal{V} = 0$. In other words, $\mathcal{O}_w$ satisfies the above condition; and $\mathcal{V} \subseteq \mathcal{O}_w$ for any subspace $\mathcal{V}$ satisfying the given condition.

Proposition 2.4 tells that $\mathcal{O}_w$ is the largest (A, B)-invariant subspace inside $\mathtt{ker}\,(C+DF)$ over all $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. Therefore, such a subspace also admits largest *good* (A, B)-invariant subspace inside $\mathtt{ker}\,(C + DF)$ over all $F \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. We call such a space the *good slow space* of the system.

Definition 2.5: A state $x_1 \in \mathbb{R}^{\mathtt{n}}$ is called *strongly reachable* (from the origin) if there exists an input $u \in \mathfrak{C}^{\mathtt{m}}_{\mathrm{imp}}$ such that $x(0^+; 0, u) = x_1$ and $y(0, u) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{p}})|_{\mathbb{R}_+}$. The collection of all such strongly reachable states is called the *strongly reachable subspace* or the *fast space* and is denoted by $\mathcal{R}_s$.

Proposition 2.6: The fast space $\mathcal{R}_s$ is the smallest subspace $\mathcal{W}$ of the state-space for which there exists $G \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ such that $(A+GC)\mathcal{W} \subseteq \mathcal{W}$ and $\mathtt{img}\,(B+GD) \subseteq \mathcal{W}$. In other words, $\mathcal{R}_s$ satisfies the above condition; and $\mathcal{R}_s \subseteq \mathcal{W}$ for any subspace $\mathcal{W}$ satisfying the given condition.

E. The primal and the Hamiltonian

Suppose $\mathtt{p} := \mathtt{rank}\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{bmatrix}$. Since $\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{bmatrix} \geqslant 0$, it admits a factorization given by $\begin{bmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{bmatrix} = \begin{bmatrix} C^T \\ 0 \\ D_2^T\end{bmatrix}\begin{bmatrix} C & 0 & D_2\end{bmatrix}$, where $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, and $D_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{r}}$. Using this factorization in equation (4), it can be easily seen that the singular LQR Problem 1.2 can be viewed as an output energy minimization problem of the system Σ defined as follows:

$$\frac{d}{dt}x = Ax + B_1u_1 + B_2u_2 \text{ and } y = Cx + D_2u_2. \tag{6}$$

We call the system Σ the *primal* for the given LQR Problem.

According to Pontryagin's maximum principle, all the smooth optimal trajectories must necessarily be a trajectory of the following singular descriptor system:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}}_{E}\frac{d}{dt}\begin{bmatrix} x \\ z \\ u_1 \\ u_2\end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B_1 & B_2 \\ -Q & -A^T & 0 & -S_2 \\ 0 & B_1^T & 0 & 0 \\ S_2^T & B_2^T & 0 & \widehat{R}\end{bmatrix}}_{H}\begin{bmatrix} x \\ z \\ u_1 \\ u_2\end{bmatrix}, \tag{7}$$

where $\mathtt{col}(x,z)$ is the state-costate pair. The system described by equation (7) is known in the literature as the *Hamiltonian* system corresponding to the LQR Problem 1.2 and the matrix pair (E,H) is known as the Hamiltonian matrix pair. The Hamiltonian system admits an output-nulling representation given by

$$\tfrac{d}{dt}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] = \widehat{A}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] + \widehat{B}\left[\begin{smallmatrix} u_1 \\ u_2\end{smallmatrix}\right] \text{ and } 0 = \widehat{C}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] + \widehat{D}\left[\begin{smallmatrix} u_1 \\ u_2\end{smallmatrix}\right], \text{ where}$$
$$\widehat{A} := \begin{bmatrix} A & 0 \\ -Q & -A^T\end{bmatrix}, \widehat{B} := \begin{bmatrix} B_1 & B_2 \\ 0 & -S_2\end{bmatrix}, \widehat{C} := \begin{bmatrix} 0 & B_1^T \\ S_2^T & B_2^T\end{bmatrix}, \text{ and } \widehat{D} := \begin{bmatrix} 0 & 0 \\ 0 & \widehat{R}\end{bmatrix}. \tag{8}$$

It has been recently shown that not only the smooth optimal trajectories, but also the distributional ones must necessarily obey the Hamiltonian system's equation distributionally [4].

Due to non-singularity of $\widehat{R}$, we can further reduce the Hamiltonian system to obtain an equivalent system described by the following differential algebraic equations:

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0\end{bmatrix}}_{E_r}\frac{d}{dt}\begin{bmatrix} x \\ z \\ u_1\end{bmatrix} = \underbrace{\begin{bmatrix} A - B_2\widehat{R}^{-1}S_2^T & -B_2\widehat{R}^{-1}B_2^T & B_1 \\ -Q + S_2\widehat{R}^{-1}S_2^T & -(A - B_2\widehat{R}^{-1}S_2^T)^T & 0 \\ 0 & B_1^T & 0\end{bmatrix}}_{H_r}\begin{bmatrix} x \\ z \\ u_1\end{bmatrix}. \tag{9}$$

We call the system described by equation (9), the reduced Hamiltonian system, and the pair (E_r, H_r) the reduced Hamiltonian matrix pair. The reduced Hamiltonian system admits an output-nulling representation Σ_{Ham} as follows:

$$\frac{d}{dt}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] = \begin{bmatrix} \widetilde{A} & -A_z \\ -\widetilde{Q} & -\widetilde{A}^T\end{bmatrix}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right] + \begin{bmatrix} \widetilde{B} \\ 0\end{bmatrix}u_1 \text{ and } 0 = \begin{bmatrix} 0 & \widetilde{B}^T\end{bmatrix}\left[\begin{smallmatrix} x \\ z\end{smallmatrix}\right], \tag{10}$$

where $\widetilde{A} := A - B_2\widehat{R}^{-1}S_2^T$, $\widetilde{Q} := Q - S_2\widehat{R}^{-1}S_2^T$, $A_z := B_2\widehat{R}^{-1}B_2^T$, and $\widetilde{B} := B_1$.

The following lemma relates the transfer function matrices of the primal and the Hamiltonian [12, Lemma 4.4].

Lemma 2.7: Consider the primal Σ, the Hamiltonian matrix pair (E,H), the reduced Hamiltonian matrix pair (E_r,H_r), and the matrices $\widehat{A},\widehat{B},\widehat{C},\widehat{D}$ defined in equation (6), equation (7), equation (9), and equation (8), respectively. Define $G(s) := C(sI_{\mathtt{n}} - A)^{-1}\begin{bmatrix} B_1 & B_2\end{bmatrix} + \begin{bmatrix} 0 & D_2\end{bmatrix}$. Then,

1. $G(-s)^TG(s) = \widehat{C}(sI_{2\mathtt{n}} - \widehat{A})^{-1}\widehat{B} + \widehat{D}$.
2. $\mathtt{numdet}G(-s)^TG(s) = \det(sE-H) = k \times \det(sE_r - H_r)$ for some $k \in \mathbb{R}\setminus\{0\}$.

Remark 2.8: Due to Statement 2 we can infer that if λ is a root of $\det(sE_{\mathtt{r}} - H_{\mathtt{r}})$ (that is, $\lambda \in \sigma(E_{\mathtt{r}}, H_{\mathtt{r}})$), then $-\lambda$, too, is a root of the same. Of course, the roots also appear along with their complex conjugates. Therefore, the roots are symmetric about the origin. Consequently, $\det(sE_{\mathtt{r}} - H_{\mathtt{r}})$ is an even degree polynomial. Statement 1 and Statement 2 together imply that for a singular LQR problem $\mathtt{degdet}(sE_{\mathtt{r}} - H_{\mathtt{r}}) =: 2\mathtt{n_s}$, where $\mathtt{n_s} < \mathtt{n}$ (because $\widehat{D}$ is singular). In this paper we assume that $\det(sE_{\mathtt{r}} - H_{\mathtt{r}})$ has no root on the imaginary axis. Hence, $|\sigma(E_{\mathtt{r}}, H_{\mathtt{r}}) \cap \mathbb{C}_-| = \mathtt{n_s}$. It also implies that $G(s)$ is left-invertible as rational function matrix, that is, the primal Σ is a left-invertible system. $\square$

F. Characterization of the good slow space of the Hamiltonian

The following lemma provides us with a characterization of the good slow space of the Hamiltonian (see [13]).

Lemma 2.9: Consider the reduced Hamiltonian matrix pair (E_r, H_r) as defined in equation (9). Assume that $\sigma(E_r,H_r) \cap j\mathbb{R} = \emptyset$. Define $\mathtt{degdet}(sE_r - H_r) =: 2\mathtt{n_s}$ and $\Lambda := \sigma(E_{\mathtt{r}}, H_{\mathtt{r}}) \cap \mathbb{C}_-$ (recall from Remark 2.8 that $|\Lambda| = \mathtt{n_s}$). Let $V_{1\Lambda}$, $V_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n_s}}$ and $V_{3\Lambda} \in \mathbb{R}^{\mathtt{d}\times\mathtt{n_s}}$ be such that $\mathtt{col}(V_{1\Lambda}, V_{2\Lambda}, V_{3\Lambda})$ is full column-rank and the following holds:

$$\begin{bmatrix} \widetilde{A} & -A_z & \widetilde{B} \\ -\widetilde{Q} & -\widetilde{A}^T & 0 \\ 0 & \widetilde{B}^T & 0\end{bmatrix}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda}\end{bmatrix} = \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda} \\ V_{3\Lambda}\end{bmatrix}\Gamma, \tag{11}$$

where $\sigma(\Gamma)=\Lambda$. Then, the following are true:

1. $\left[\begin{smallmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{smallmatrix}\right]$ is full column-rank.
2. The good slow space of $\Sigma_{\text{Ham}} =: \mathcal{O}_{wg} = \mathtt{img}\left[\begin{smallmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{smallmatrix}\right]$.

The following lemma ([13, Lemma 14]) shows that the good slow space, $\mathcal{V}_g$, of the primal Σ is embedded into the good slow space, $\mathcal{O}_{wg}$, of the Hamiltonian Σ_{Ham}.

Lemma 2.10: Let $\mathcal{V}_g$ and $\mathcal{O}_{wg}$ be the good slow spaces of the primal Σ and the Hamiltonian Σ_{Ham}, respectively. Define the subspace $\mathcal{V}_{\mathtt{gHam}} := \{\left[\begin{smallmatrix} v \\ 0\end{smallmatrix}\right] \in \mathbb{R}^{2\mathtt{n}} \mid v \in \mathcal{V}_g\}$. Then, $\mathcal{V}_{\mathtt{gHam}} \subseteq \mathcal{O}_{wg}$.

Remark 2.11: Suppose $V_{\mathtt{g}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{g}}$ be such that $V_{\mathtt{g}}$ is full column-rank and $\mathtt{img}\,V_{\mathtt{g}} = \mathcal{V}_{\mathtt{g}}$, where $\mathtt{g} := \mathtt{dim}(\mathcal{V}_{\mathtt{g}})$. Then, from Lemma 2.10 it is evident that there exist $V_{1\mathtt{e}}, V_{2\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$ such that $\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix}$ is full column-rank and $\mathtt{img}\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix} = \mathcal{O}_{wg}$. Hence, without loss of generality, in this paper we assume that $\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix} = \begin{bmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{bmatrix}$. $\square$

The following lemma ([13, Lemma 17, Theorem 18]) plays a pivotal role in the proofs of the main results of this paper.

Lemma 2.12: Consider $V_{1\Lambda}, V_{2\Lambda}$ as defined in Lemma 2.9 and $V_{\mathtt{g}}$ as defined in Remark 2.11. Let $V_{1\mathtt{e}}, V_{2\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$ be such that $\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix}$ is full column-rank and $\mathtt{img}\begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} \\ 0_{\mathtt{n},\mathtt{g}} & V_{2\mathtt{e}}\end{bmatrix} = \mathcal{O}_{wg}$. Then, the following statements are true:

1. $V_{2\mathtt{e}}$ is full column-rank.
2. $V_{1\Lambda}^TV_{2\Lambda} = V_{2\Lambda}^TV_{1\Lambda}$ (equivalently, $V_{2\mathtt{e}}^TV_{\mathtt{g}} = 0$ and $V_{1\mathtt{e}}^TV_{2\mathtt{e}} = V_{2\mathtt{e}}^TV_{1\mathtt{e}}$).
3. $V_{2\mathtt{e}}^TV_{1\mathtt{e}} > 0$.
4. $V_{1\Lambda} = \begin{bmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}}\end{bmatrix}$ is full column-rank.

III. THE FAST SPACE OF THE PRIMAL

In this section we provide a characterization for the fast space of the primal. As mentioned earlier, it is more convenient to carry out the analysis with the reduced Hamiltonian

Σ_{Ham}, which comprises of the matrices $\widetilde{A}, \widetilde{B}, \widetilde{Q}$, and A_z. Thus, we represent the fast space of the primal in terms of these matrices. The next lemma becomes useful in order to do so.

Lemma 3.1: The fast space $\mathcal{R}_s$ of the primal Σ is the same as the fast space of the system Ω defined as follows:

$$\frac{d}{dt}x = \widetilde{A}x + \widetilde{B}u \text{ and } y = \widetilde{C}x, \text{ where } \widetilde{C} := C - D_2\widehat{R}^{-1}S_2^T. \tag{12}$$

For proof of Lemma 3.1 please refer to the appendix.

Lemma 3.1 along with [14, Theorem 4.2, Theorem 4.7] enables us to write the following proposition which provides a closed-form expression as well as the dimension of the fast space of the primal Σ.

Proposition 3.2: Consider the primal Σ and the matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}$ as defined in equation (6) and equation (10), respectively. Recall from Remark 2.8 that $2\mathtt{n_s} = \mathtt{deg}\{\mathtt{numdet}(G(-s)^T G(s))\}$, where $G(s)$ is the transfer function matrix of Σ. Let $\mathcal{R}_s$ be the fast space of Σ. Then,

1. $\mathtt{dim}\,\mathcal{R}_s = \mathtt{n_f}$, where $\mathtt{n_f} := \mathtt{n} - \mathtt{n_s}$.
2. $\mathcal{R}_s = \mathtt{img}\,W$, where
$$W := \left[\, \widetilde{B} \ \widetilde{A}\widetilde{B} \ \dots \ \widetilde{A}^{\mathtt{n_f}-\mathtt{d}}\widetilde{B} \,\right] N \tag{13}$$
such that the columns of the matrix $N \in \mathbb{R}^{\mathtt{d}(\mathtt{n_f}-\mathtt{d}+1)\times \mathtt{n_f}}$ form a basis for $\ker \mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}$ with $\mathcal{M}_{\mathtt{j}}$ defined as
$$\mathcal{M}_{\mathtt{j}} := \begin{cases} 0_{\mathtt{p},\mathtt{d}} & \text{if } \mathtt{j} = 1 \\ \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \widetilde{C}\widetilde{B} \\ 0 & 0 & \dots & \widetilde{C}\widetilde{B} & \widetilde{C}\widetilde{A}\widetilde{B} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \widetilde{C}\widetilde{B} & \dots & \widetilde{C}\widetilde{A}^{\mathtt{j}-3}\widetilde{B} & \widetilde{C}\widetilde{A}^{\mathtt{j}-2}\widetilde{B} \end{bmatrix} & \text{if } \mathtt{j} \geqslant 2 \end{cases}. \tag{14}$$
3. W is full column-rank.

We call the matrix $\mathcal{M}_{\mathtt{j}}$ the *Markov parameter matrix*. In [14, Lemma 4.1 and Theorem 4.2] it has been shown that the fast space of the primal Σ (which is the same as the fast space of Ω) having its dimension equal to $\mathtt{n_f}$ is equivalent to saying that $\mathtt{dim}(\mathtt{ker}\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}) = \mathtt{dim}(\mathtt{ker}\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+2}) = \mathtt{n_f}$.

IV. Constructive solution of the LQR LMI

In this section we provide the main results of this paper. The first of these results provides us with a method to compute a symmetric matrix $K_{\mathtt{max}}$ that satisfies the LQR LMI (5). Two subsequent results show that $K_{\mathtt{max}}$ is, indeed, the maximal rank-minimizing solution of the LMI.

In order to prove the main results we need a couple of identities that we state as lemmas next. We provide the proofs of these lemmas in the appendix.

Lemma 4.1: Let $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be an arbitrary solution of the LQR LMI (5). Then, $KW = 0$, where W is as defined in equation (13).

Lemma 4.2: Recall $V_{2\Lambda}$ and W from Lemma 2.9 and Proposition 3.2, respectively. Then, $V_{2\Lambda}^T W = 0$.

Now we provide the first main result of this paper. This result enables us to compute a solution of the LQR LMI (5). In the subsequent results we establish that this solution, indeed, is the maximal rank-minimizing solution.

Theorem 4.3: Consider the LQR LMI given by equation (5). Recall from Lemma 2.9 that the good slow space of the Hamiltonian Σ_{Ham} is given by $\mathcal{O}_{wg} = \mathtt{img}\left[\begin{smallmatrix} V_{1\Lambda} \\ V_{2\Lambda}\end{smallmatrix}\right]$. Further recall from Proposition 3.2 that the fast space of the primal Σ is given by $\mathcal{R}_s = \mathtt{img}\,W$. Define $X_\Lambda := \left[\begin{smallmatrix} V_{1\Lambda} & W \\ V_{2\Lambda} & 0\end{smallmatrix}\right] =: \left[\begin{smallmatrix} X_{1\Lambda} \\ X_{2\Lambda}\end{smallmatrix}\right]$, where $X_{1\Lambda}, X_{2\Lambda} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then, the following are true:

1. $X_{1\Lambda}$ is invertible.
2. $K_{\mathtt{max}} := X_{2\Lambda}X_{1\Lambda}^{-1}$ is symmetric.
3. $K_{\mathtt{max}}$ satisfies the LMI (5).

Proof 1. Recall from Remark 2.11 that there exists $V_{1\mathtt{e}} \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n_s}-\mathtt{g})}$ such that $V_{1\Lambda} = \left[\, V_{\mathtt{g}} \ V_{1\mathtt{e}} \,\right]$, where columns of $V_{\mathtt{g}}$ form a basis for the good slow space $\mathcal{V}_{\mathtt{g}}$ of the primal Σ and $\mathtt{g} = \mathtt{dim}\,(\mathcal{V}_{\mathtt{g}})$. To the contrary we assume that $X_{1\Lambda}$ is not invertible. So, there exist $z_1 \in \mathbb{R}^{\mathtt{g}}, z_2 \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})}, z_3 \in \mathbb{R}^{\mathtt{n_f}}$ with $\mathtt{col}\,(z_1, z_2, z_3) \neq 0$ such that

$$V_{\mathtt{g}}z_1 + V_{1\mathtt{e}}z_2 + Wz_3 = 0 \Rightarrow V_{2\mathtt{e}}^T V_{\mathtt{g}}z_1 + V_{2\mathtt{e}}^T V_{1\mathtt{e}}z_2 + V_{2\mathtt{e}}^T Wz_3 = 0, \tag{15}$$

where $\left[\, 0_{\mathtt{n},\mathtt{g}} \ V_{2\mathtt{e}} \,\right] = V_{2\Lambda}$ (see Remark 2.11). From Lemma 4.2, we get that $V_{2\Lambda}^T W = \left[\begin{smallmatrix} 0 \\ V_{2\mathtt{e}}^T \end{smallmatrix}\right] W = 0 \Leftrightarrow V_{2\mathtt{e}}^T W = 0$. Further, by Statement 2 of Lemma 2.12, we have $V_{2\mathtt{e}}^T V_{\mathtt{g}} = 0$. Consequently, from equation (15) we get that $V_{2\mathtt{e}}^T V_{1\mathtt{e}} z_2 = 0$. But, from Statement 3 of Lemma 2.12, we know that $V_{2\mathtt{e}}^T V_{1\mathtt{e}}$ is non-singular. Thus, $z_2 = 0$. So, equation (15) reduces to

$$V_{\mathtt{g}}z_1 + Wz_3 = 0. \tag{16}$$

Recall from Remark 2.8 that the primal Σ is a left-invertible system. So, by [2, Theorem 3.26] it follows that $\mathtt{img}V_{\mathtt{g}} \cap \mathtt{img}W = \{0\}$, because the columns of $V_{\mathtt{g}}$ and the columns of W form bases for the good slow space $\mathcal{V}_{\mathtt{g}}$ and the fast space $\mathcal{R}_s$ of the primal Σ, respectively (see Proposition 3.2). Thus, we conclude from equation (16) that $z_1 = 0, z_3 = 0$. This is a contradiction, because we have assumed that $\mathtt{col}\,(z_1, z_2, z_3) \neq 0$. Hence, $X_{1\Lambda}$ is invertible.
2. We need to show that $X_{2\Lambda}{X_{1\Lambda}}^{-1} = (X_{2\Lambda}{X_{1\Lambda}}^{-1})^T \Leftrightarrow X_{2\Lambda}^T X_{1\Lambda} - X_{1\Lambda}^T X_{2\Lambda} = 0$. It follows from the definitions of $X_{1\Lambda}$ and $X_{2\Lambda}$ that

$$X_{2\Lambda}^T X_{1\Lambda} - X_{1\Lambda}^T X_{2\Lambda} = \begin{bmatrix} V_{2\Lambda}^T V_{1\Lambda} - V_{1\Lambda}^T V_{2\Lambda} & V_{2\Lambda}^T W \\ -W^T V_{2\Lambda} & 0 \end{bmatrix}. \tag{17}$$

By Lemma 2.12 and Lemma 4.2 we get $V_{2\Lambda}^T V_{1\Lambda} = V_{1\Lambda}^T V_{2\Lambda}$ and $V_{2\Lambda}^T W = 0$, respectively. Thus, from equation (17), we get $X_{2\Lambda}^T X_{1\Lambda} - X_{1\Lambda}^T X_{2\Lambda} = 0$. Hence, $K_{\mathtt{max}}$ is symmetric.
3. By taking Schur complement with respect to $\widehat{R}$, we get that K is a solution of the LMI (5) if and only if K satisfies

$$(A^TK + KA + Q - (KB_2 + S_2)\widehat{R}^{-1}(B_2^TK + S_2^T)) \geqslant 0, \text{ and } KB_1 = 0,$$
$$\Leftrightarrow \mathcal{L}_{\mathtt{r}}(K) := \widetilde{A}^TK + K\widetilde{A} + \widetilde{Q} - KA_zK \geqslant 0, \text{ and } K\widetilde{B} = 0, \tag{18}$$

where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{Q}$, and A_z are as defined in equation (10). Note that $K_{\mathtt{max}}X_{1\Lambda} = X_{2\Lambda}X_{1\Lambda}^{-1}X_{1\Lambda} = X_{2\Lambda}$. Therefore,

$$\begin{aligned} &X_{1\Lambda}^T(\widetilde{A}^TK_{\mathtt{max}} + K_{\mathtt{max}}\widetilde{A} + \widetilde{Q} - K_{\mathtt{max}}A_zK_{\mathtt{max}})X_{1\Lambda} \\ &= X_{1\Lambda}^T\widetilde{A}^TX_{2\Lambda} + X_{2\Lambda}^T\widetilde{A}X_{1\Lambda} + X_{1\Lambda}^T\widetilde{Q}X_{1\Lambda} - X_{2\Lambda}^TA_zX_{2\Lambda} \\ &= \left[\, X_{2\Lambda}^T \ -X_{1\Lambda}^T \,\right] \begin{bmatrix} \widetilde{A} & -A_z \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} X_{1\Lambda} \\ X_{2\Lambda} \end{bmatrix} = \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \\ 0 & -W^T \end{bmatrix} \begin{bmatrix} \widetilde{A} & -A_z \\ -\widetilde{Q} & -\widetilde{A}^T \end{bmatrix} \begin{bmatrix} V_{1\Lambda} & W \\ V_{2\Lambda} & 0 \end{bmatrix} \\ &= \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \\ 0 & -W^T \end{bmatrix} \begin{bmatrix} \widetilde{A}V_{1\Lambda} - A_zV_{2\Lambda} & \widetilde{A}W \\ -\widetilde{Q}V_{1\Lambda} - \widetilde{A}^TV_{2\Lambda} & -\widetilde{Q}W \end{bmatrix}. \end{aligned}$$

Further, using equation (11) in the above equation, we have

$$\begin{aligned} &X_{1\Lambda}^T(\widetilde{A}^TK_{\mathtt{max}} + K_{\mathtt{max}}\widetilde{A} + \widetilde{Q} - K_{\mathtt{max}}A_zK_{\mathtt{max}})X_{1\Lambda} \\ &= \begin{bmatrix} V_{2\Lambda}^T & -V_{1\Lambda}^T \\ 0 & -W^T \end{bmatrix} \begin{bmatrix} V_{1\Lambda}\Gamma - \widetilde{B}V_{3\Lambda} & \widetilde{A}W \\ V_{2\Lambda}\Gamma & -\widetilde{Q}W \end{bmatrix} \\ &= \begin{bmatrix} (V_{2\Lambda}^TV_{1\Lambda} - V_{1\Lambda}^TV_{2\Lambda})\Gamma & -\Gamma^TV_{2\Lambda}^TW \\ -W^TV_{2\Lambda}\Gamma & W^T\widetilde{Q}W \end{bmatrix}. \end{aligned} \tag{19}$$

Application of Lemma 4.2 and Lemma 2.12 further yields

$$X_{1\Lambda}^T(\widetilde{A}^TK_{\mathtt{max}} + K_{\mathtt{max}}\widetilde{A} + \widetilde{Q} - K_{\mathtt{max}}A_zK_{\mathtt{max}})X_{1\Lambda} = \begin{bmatrix} 0 & 0 \\ 0 & W^T\widetilde{Q}W \end{bmatrix}. \tag{20}$$

Since $\widetilde{Q} \geqslant 0$ and $X_{1\Lambda}$ is non-singular, we conclude that $(\widetilde{A}^TK_{\mathtt{max}} + K_{\mathtt{max}}\widetilde{A} + \widetilde{Q} - K_{\mathtt{max}}A_zK_{\mathtt{max}}) \geqslant 0$.

Next, by equation (11), we have $\widetilde{B}^TV_{2\Lambda} = 0$. Therefore,

$$\widetilde{B}^TK_{\mathtt{max}} = \widetilde{B}^TX_{2\Lambda}{X_{1\Lambda}}^{-1} = \widetilde{B}^T\left[\, V_{2\Lambda} \ 0 \,\right]{X_{1\Lambda}}^{-1} = 0.$$

Hence, $K_{\mathtt{max}}$ is a solution of the singular LQR LMI (5). $\square$

The following theorem shows that $K_{\mathtt{max}}$ is a rank-minimizing solution of the singular LQR LMI.

Theorem 4.4: $K_{\mathtt{max}}$ is a rank-minimizing solution of the singular LQR LMI (5); that is, $\mathtt{rank}\,\mathcal{L}_{\mathtt{t}}(K_{\mathtt{max}}) \leqslant \mathtt{rank}\,\mathcal{L}_{\mathtt{t}}(K)$ for any K that satisfies $\mathcal{L}_{\mathtt{t}}(K) \geqslant 0$.

Proof Due to the notion of Schur complement, there exists a non-singular matrix $\widetilde{U} \in \mathbb{R}^{(\mathtt{n}+\mathtt{m})\times(\mathtt{n}+\mathtt{m})}$ such that $\widetilde{U}^T\mathcal{L}_{\mathtt{t}}(K)\widetilde{U} = \left[\begin{smallmatrix} \mathcal{L}_{\mathtt{r}}(K) & K\widetilde{B} & 0 \\ \widetilde{B}^T K & 0 & 0 \\ 0 & 0 & \widehat{R}\end{smallmatrix}\right]$, where $\mathcal{L}_{\mathtt{r}}(K)$ is as defined in equation (18). Further, for any K satisfying $\mathcal{L}_{\mathtt{t}}(K) \geqslant 0$, we must have $K\widetilde{B} = 0$. So, for an arbitrary solution K of LQR LMI (5), we have $\mathtt{rank}\mathcal{L}_{\mathtt{t}}(K) = \mathtt{rank}\mathcal{L}_{\mathtt{r}}(K) + \mathtt{rank}\widehat{R}$. So, it suffices to show that $\mathtt{rank}\mathcal{L}_{\mathtt{r}}(K_{\mathtt{max}}) \leqslant \mathtt{rank}\mathcal{L}_{\mathtt{r}}(K)$. Now,

$$\begin{aligned} & \mathtt{rank}\,\mathcal{L}_{\mathtt{r}}(K) \geqslant \mathtt{rank} W^T\mathcal{L}_{\mathtt{r}}(K)W \\ = \;& \mathtt{rank}\{W^T(\widetilde{A}^TK + K\widetilde{A} + \widetilde{Q} - KA_zK)W\}. \end{aligned} \tag{21}$$

By equation (20) and non-singularity of $X_{1\Lambda}$, we infer that

$$\mathtt{rank}\mathcal{L}_{\mathtt{r}}(K_{\mathtt{max}}) = \mathtt{rank}X_{1\Lambda}^T\mathcal{L}_{\mathtt{r}}(K_{\mathtt{max}})X_{1\Lambda} = \mathtt{rank}W^T\widetilde{Q}W.$$

Thus, using Lemma 4.1 in equation (21), we conclude that

$$\mathtt{rank}\mathcal{L}_{\mathtt{r}}(K) \geqslant \mathtt{rank}W^T\widetilde{Q}W = \mathtt{rank}\mathcal{L}_{\mathtt{r}}(K_{\mathtt{max}}).$$

This completes the proof. $\square$

Following theorem shows that $K_{\mathtt{max}}$ is the maximal solution of the singular LQR LMI.

Theorem 4.5: Assume that K is an arbitrary solution of LQR LMI (5). Then, $K \leqslant K_{\mathtt{max}}$.

Proof We first claim that $\Delta := V_{1\Lambda}^T(K - K_{\mathtt{max}})V_{1\Lambda} \leqslant 0$. To prove this claim, we evaluate the quantity $\frac{d}{dt}(x^TKx) + \left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right]^T \left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right]$ for all trajectories $\mathtt{col}\,(x, u_1, u_2)$ that belongs to the primal Σ (see equation (6)) to get:

$$\begin{aligned} &\tfrac{d}{dt}(x^TKx) + \left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right]^T \left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right]^T \underbrace{\left[\begin{smallmatrix} A^TK+KA+Q & 0 & KB_2+S_2 \\ 0 & 0 & 0 \\ B_2^TK+S_2^T & 0 & \widehat{R}\end{smallmatrix}\right]}_{\mathcal{L}_{\mathtt{t}}(K)} \left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right] \text{ for all } t \geqslant 0. \end{aligned}$$

Since $\mathcal{L}_{\mathtt{t}}(K) \geqslant 0$, from the above equation it is clear that

$$\frac{d}{dt}(x^TKx) + \left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right]^T \left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ u_1 \\ u_2\end{smallmatrix}\right] \geqslant 0 \text{ for all } t \geqslant 0. \tag{22}$$

Due to equation (11), it is straightforward to verify that corresponding to the initial condition $x_{0s} = V_{1\Lambda}\alpha$, where $\alpha \in \mathbb{R}^{\mathtt{n_s}}$ is arbitrary, $x_s := V_{1\Lambda}e^{\Gamma t}\alpha$, $u_{s_1} := V_{3\Lambda}e^{\Gamma t}\alpha$, and $u_{s_2} := -\widehat{R}^{-1}(S_2^TV_{1\Lambda} + B_2^TV_{2\Lambda})e^{\Gamma t}\alpha$ satisfy $\dot{x}_s = Ax_s + B_1u_{s_1} + B_2u_{s_2}$. Thus, the trajectory $\mathtt{col}\,(x_s, u_{s_1}, u_{s_2})$ belongs to Σ. So, by equation (22), we have

$$\frac{d}{dt}(x_s^TKx_s) \geqslant -\left[\begin{smallmatrix} x_s \\ u_{s_1} \\ u_{s_2}\end{smallmatrix}\right]^T \left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_s \\ u_{s_1} \\ u_{s_2}\end{smallmatrix}\right] \text{ for all } t \geqslant 0. \tag{23}$$

Next, by simple algebraic manipulations using the definitions of x_s, u_{s_1}, and u_{s_2} along with equation (11) and the identities $K_{\mathtt{max}}V_{1\Lambda} = V_{2\Lambda}$, $K_{\mathtt{max}}\widetilde{B} = 0$, and $V_{1\Lambda}^TV_{2\Lambda} = V_{2\Lambda}^TV_{1\Lambda}$ we get that

$$\frac{d}{dt}(x_s^TK_{\mathtt{max}}x_s) + \left[\begin{smallmatrix} x_s \\ u_{s_1} \\ u_{s_2}\end{smallmatrix}\right]^T \left[\begin{smallmatrix} Q & 0 & S_2 \\ 0 & 0 & 0 \\ S_2^T & 0 & \widehat{R}\end{smallmatrix}\right]\left[\begin{smallmatrix} x_s \\ u_{s_1} \\ u_{s_2}\end{smallmatrix}\right] = 0 \text{ for all } t \geqslant 0. \tag{24}$$

Subtracting equation (24) from equation (23), it follows that

$$\frac{d}{dt}\{x_s^T(K - K_{\mathtt{max}})x_s\} \geqslant 0 \text{ for all } t \geqslant 0. \tag{25}$$

We substitute $x_s = V_{1\Lambda}e^{\Gamma t}\alpha$ and $\dot{x}_s = V_{1\Lambda}\Gamma e^{\Gamma t}\alpha$ in equation (25) and then evaluate the expression at $t = 0$ to yield

$$\alpha^T(\Gamma^T\Delta + \Delta\Gamma)\alpha \geqslant 0, \text{ where } \Delta = V_{1\Lambda}^T(K - K_{\mathtt{max}})V_{1\Lambda}.$$

Since $\alpha \in \mathbb{R}^{\mathtt{n_s}}$ is arbitrary, we must have $\Gamma^T\Delta + \Delta\Gamma \geqslant 0$. In view of the fact that $\sigma(\Gamma) \subsetneq \mathbb{C}_-$, we conclude that $\Delta = V_{1\Lambda}^T(K - K_{\mathtt{max}})V_{1\Lambda} \leqslant 0$. This proves the claim.

Now, we prove that $K - K_{\mathtt{max}} \leqslant 0$ for all K satisfying $\mathcal{L}_{\mathtt{t}}(K) \geqslant 0$. Due to the non-singularity of $X_{1\Lambda}$, it suffices to show that $X_{1\Lambda}^T(K - K_{\mathtt{max}})X_{1\Lambda} \leqslant 0$. Now,

$$X_{1\Lambda}^T(K - K_{\mathtt{max}})X_{1\Lambda} = \left[\begin{smallmatrix} V_{1\Lambda}^T(K-K_{\mathtt{max}})V_{1\Lambda} & V_{1\Lambda}^T(K-K_{\mathtt{max}})W \\ W^T(K-K_{\mathtt{max}})V_{1\Lambda} & W^T(K-K_{\mathtt{max}})W\end{smallmatrix}\right]. \tag{26}$$

From Lemma 4.1, we have $KW = K_{\mathtt{max}}W = 0$ and thus from equation (26) we get $X_{1\Lambda}^T(K - K_{\mathtt{max}})X_{1\Lambda} = \left[\begin{smallmatrix} \Delta & 0 \\ 0 & 0\end{smallmatrix}\right]$. But, since $\Delta \leqslant 0$, it is evident that $X_{1\Lambda}^T(K - K_{\mathtt{max}})X_{1\Lambda} \leqslant 0$. Hence, $K - K_{\mathtt{max}} \leqslant 0$ for any arbitrary solution K of the singular LQR LMI (5). This completes the proof. $\square$

Corollary 4.6: $K_{\mathtt{max}}$ satisfies the property $K_{\mathtt{max}} \geqslant 0$.

Proof As $0 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ solves the LQR LMI (5), by Theorem 4.5 it evident that $K_{\mathtt{max}} \geqslant 0$. $\square$

Remark 4.7: From the definition of $X_{1\Lambda}$ and Remark 2.11 we know that $X_{1\Lambda} = \left[\begin{smallmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W\end{smallmatrix}\right]$. By Theorem 4.3, $X_{1\Lambda}$ is non-singular. Consequently, the state-space $\mathbb{R}^{\mathtt{n}}$ admits the direct-sum decomposition $\mathbb{R}^{\mathtt{n}} = \mathtt{img}\,V_{\mathtt{g}} \oplus \mathtt{img}\,V_{1\mathtt{e}} \oplus \mathtt{img}\,W$. Now, from [1, Theorem 2] we know that the optimal cost for the singular LQR Problem 1.2 must be $J_0 = x_0^TK_{\mathtt{max}}x_0$ for a given initial condition x_0. Thus, if x_0 is from the good slow space of the primal, that is, $x_0 = V_{\mathtt{g}}\alpha$ for an arbitrary $\alpha \in \mathbb{R}^{\mathtt{g}}$, then the cost incurred by the optimal input is $J_0 = \alpha^TV_{\mathtt{g}}^TK_{\mathtt{max}}V_{\mathtt{g}}\alpha$. But $K_{\mathtt{max}}V_{\mathtt{g}} = \left[\begin{smallmatrix} 0 & V_{2\mathtt{e}} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W\end{smallmatrix}\right]^{-1}V_{\mathtt{g}} = 0$. Therefore, the optimal input incurs zero cost. Similarly, if x_0 is from the fast space of the primal, that is, $x_0 = W\gamma$ for an arbitrary $\gamma \in \mathbb{R}^{\mathtt{n_f}}$, then the optimal cost $J_0 = \gamma^TW^TK_{\mathtt{max}}W\gamma = 0$. So, if $x_0 = \left[\begin{smallmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W\end{smallmatrix}\right]\left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma\end{smallmatrix}\right]$ with $\alpha \in \mathbb{R}^{\mathtt{g}}, \beta \in \mathbb{R}^{(\mathtt{n_s}-\mathtt{g})}$, and $\gamma \in \mathbb{R}^{\mathtt{n_f}}$ is an arbitrary initial condition, then the optimal cost $J_0 = \beta^TV_{1\mathtt{e}}^TK_{\mathtt{max}}V_{1\mathtt{e}}\beta$. But, $K_{\mathtt{max}}V_{1\mathtt{e}} = \left[\begin{smallmatrix} 0 & V_{2\mathtt{e}} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix} V_{\mathtt{g}} & V_{1\mathtt{e}} & W\end{smallmatrix}\right]^{-1}V_{1\mathtt{e}} = V_{2\mathtt{e}}$. Hence, the optimal cost for an arbitrary initial condition is given by $J_0 = \beta^TV_{1\mathtt{e}}^TV_{2\mathtt{e}}\beta$. $\square$

V. CONCLUSIONS

The method to compute the maximal rank-minimizing solution $K_{\mathtt{max}}$ of the singular LQR LMI provided in this paper shows that $K_{\mathtt{max}}$ is intimately related to the slow and the fast space of the Hamiltonian system. For single-input systems it has been shown in [4] that $K_{\mathtt{max}}$ plays a pivotal role in order to design a P-D feedback law for the optimal input. The main advantage of using this Hamiltonian system based approach is that it enables us to solve the singular LQR problem for arbitrary initial conditions. It is well-known in the literature that for arbitrary initial conditions the optimal trajectories are, in general, impulsive in nature. This Hamiltonian system based method is able to handle these impulsive optimal trajectories as well. In [15] a method to design a P-D feedback controller has been provided under certain assumptions by using behavioral theoretic ideas. In our forthcoming work we wish to provide a P-D feedback solution for the multi-input case of the singular LQR problem using the results that have been developed here. We expect that such a solution will be devoid of such assumptions.

APPENDIX

A. Proof of Lemma 3.1

Proof We denote the fast space of the system Ω by $\widetilde{\mathcal{R}}_s$.
$\mathcal{R}_s \subseteq \mathcal{R}_s$: By Proposition 2.6, there exists $G\in\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ such that

$$(A+GC)\mathcal{R}_s \subseteq \mathcal{R}_s \text{ and } \mathtt{img}\,[\,B_1 \;\; B_2+GD_2\,] \subseteq \mathcal{R}_s. \tag{27}$$

Next, notice that $(\widetilde{A}+G\widetilde{C}) = (A+GC) - (B_2+GD_2)\widehat{R}^{-1}S_2^T$. From equation (27), we know that $\mathtt{img}(B_2+GD_2) \subseteq \mathcal{R}_s$. Therefore, $(\widetilde{A}+G\widetilde{C})\mathcal{R}_s \subseteq \mathcal{R}_s$. Again, since $\widetilde{B} = B_1$, from equation (27), we further get that $\mathtt{img}\widetilde{B} \subseteq \mathcal{R}_s$. Hence, by Proposition 2.6 we conclude that $\widetilde{\mathcal{R}}_s \subseteq \mathcal{R}_s$.
$\mathcal{R}_s \subseteq \widetilde{\mathcal{R}}_s$: By Proposition 2.6, there exists $\widetilde{G}\in\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ such that

$$(\widetilde{A}+\widetilde{G}\widetilde{C})\widetilde{\mathcal{R}}_s \subseteq \widetilde{\mathcal{R}}_s \text{ and } \mathtt{img}\,\widetilde{B} \subseteq \widetilde{\mathcal{R}}_s. \tag{28}$$

Define $G_1 := (\widetilde{G} - \widetilde{G}D_2\widehat{R}^{-1}D_2^T - B_2\widehat{R}^{-1}D_2^T)$. Then, $(\widetilde{A}+\widetilde{G}\widetilde{C}) = (A+G_1C)$ and $(B_2+G_1D_2)$=0. So, by equation (28), we get that $(A+G_1C)\widetilde{\mathcal{R}}_s \subseteq \widetilde{\mathcal{R}}_s$ and $\mathtt{img}\,[\,B_1 \;\; B_2+G_1D_2\,] \subseteq \widetilde{\mathcal{R}}_s$. Thus, we infer that $\mathcal{R}_s \subseteq \widetilde{\mathcal{R}}_s$. Hence, $\mathcal{R}_s = \widetilde{\mathcal{R}}_s$. $\square$

B. Proof of Lemma 4.1

Proof Recall from equation (18) that K is a solution of LMI (5) if and only if it satisfies:

$$\mathcal{L}_{\mathtt{r}}(K) = \widetilde{A}^TK + K\widetilde{A} + \widetilde{Q} - KA_zK \geqslant 0 \text{ and } K\widetilde{B} = 0. \tag{29}$$

Assume that the columns of $\mathtt{col}(N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}}) := N$ form a basis for $\mathtt{ker}\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}$, where $N_0, N_1, \ldots, N_{\mathtt{n_f}-\mathtt{d}} \in \mathbb{R}^{\mathtt{d}\times\mathtt{n_f}}$. Define $W_i := [\,\widetilde{B}\;\widetilde{A}\widetilde{B}\;\ldots\;\widetilde{A}^i\widetilde{B}\,]\,\mathtt{col}(N_{\mathtt{n_f}-\mathtt{d}-\mathtt{i}}, \ldots, N_{\mathtt{n_f}-\mathtt{d}}), i \in \{0, 1, \ldots, (\mathtt{n_f}-\mathtt{d})\}$. Then, we claim that $KW_i = 0$. We prove this claim by induction.
Base case ($i=0$): $KW_0 = K\widetilde{B}N_{\mathtt{n_f}-\mathtt{d}}$. But, since K satisfies equation (29), we have $K\widetilde{B} = 0$. Hence $KW_0 = 0$.
Inductive step: Assume that $KW_{i-1} = 0$ for some $1 \leqslant i \leqslant \mathtt{n_f}-\mathtt{d}$. We need to show that $KW_i = 0$. Using $KW_{i-1}=0$, we have that $W_{i-1}^T\mathcal{L}_{\mathtt{r}}(K)W_{i-1} = W_{i-1}^T\widetilde{Q}W_{i-1}$. Now,

$$\begin{aligned}\widetilde{C}W_{i-1} &= \widetilde{C}\,[\,\widetilde{B}\;\widetilde{A}\widetilde{B}\;\ldots\;\widetilde{A}^{i-1}\widetilde{B}\,]\,\mathtt{col}(N_{\mathtt{n_f}-\mathtt{d}-\mathtt{i}}, \ldots, N_{\mathtt{n_f}-\mathtt{d}})\\ &= [\,0_{\mathtt{p},\mathtt{d}(\mathtt{n_f}-\mathtt{d}-\mathtt{i})}\;\widetilde{C}\widetilde{B}\;\widetilde{C}\widetilde{A}\widetilde{B}\;\ldots\;\widetilde{C}\widetilde{A}^{i-1}\widetilde{B}\,]\,\mathtt{col}(N_0, \ldots, N_{\mathtt{n_f}-\mathtt{d}}).\end{aligned}$$

Notice that, for $1 \leqslant i \leqslant \mathtt{n_f}-\mathtt{d}$, $[\,0_{\mathtt{p},\mathtt{d}(\mathtt{n_f}-\mathtt{d}-\mathtt{i})}\;\widetilde{C}\widetilde{B}\;\widetilde{C}\widetilde{A}\widetilde{B}\;\ldots\;\widetilde{C}\widetilde{A}^{i-1}\widetilde{B}\,]$ is the $(i+1)^{st}$ block row of $\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}$. Thus, using $\mathtt{img}N = \mathtt{ker}\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}$ in the above equation, we get that $\widetilde{C}W_{i-1} = 0$

$$\Rightarrow W_{i-1}^T\widetilde{C}^T\widetilde{C}W_{i-1} = W_{i-1}^T\widetilde{Q}W_{i-1} = W_{i-1}^T\mathcal{L}_{\mathtt{r}}(K)W_{i-1} = 0.$$

Since $\mathcal{L}_{\mathtt{r}}(K) \geqslant 0$, this further implies that $\mathcal{L}_{\mathtt{r}}(K)W_{i-1} = 0$. Using the inductive hypothesis and $\widetilde{Q}W_{i-1} = 0$, we have

$$\begin{aligned}&\mathcal{L}_{\mathtt{r}}(K)W_{i-1} = K\widetilde{A}W_{i-1} = 0\\ \Leftrightarrow\quad & K\,[\,\widetilde{B}N_{\mathtt{n_f}-\mathtt{d}-\mathtt{i}}\;\;\widetilde{A}W_{i-1}\,] = KW_i = 0 \quad (\because K\widetilde{B}=0).\end{aligned}$$

This proves the claim. Next, since $W_{\mathtt{n_f}-\mathtt{d}} = W$, it is clear that $KW = 0$. This completes the proof. $\square$

C. Proof of Lemma 4.2

Proof First, we claim that $V_{2\Lambda}^T\widetilde{A}^i\widetilde{B} = \sum_{k=0}^{i-1}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{Q}\widetilde{A}^{(i-1-k)}\widetilde{B}$ for all $i \in \mathbb{N}$. We prove this using mathematical induction.
Base case (i= 1): Recall from equation (11) that

$$-V_{1\Lambda}^T\widetilde{Q} - V_{2\Lambda}^T\widetilde{A} = \Gamma^TV_{2\Lambda}^T \text{ and} \tag{30}$$

$$V_{2\Lambda}^T\widetilde{B} = 0 \tag{31}$$

We post-multiply equation (30) by $\widetilde{B}$ and then use equation (31) to get $V_{2\Lambda}^T\widetilde{A}\widetilde{B} = -V_{1\Lambda}^T\widetilde{Q}\widetilde{B} - \Gamma^TV_{2\Lambda}^T\widetilde{B} = -V_{1\Lambda}^T\widetilde{Q}\widetilde{B}$. This proves the base case.
Inductive step: Assume that $V_{2\Lambda}^T\widetilde{A}^i\widetilde{B} = \sum_{k=0}^{i-1}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{Q}\widetilde{A}^{(i-1-k)}\widetilde{B}$. We need to show that $V_{2\Lambda}^T\widetilde{A}^{i+1}\widetilde{B} = \sum_{k=0}^{i}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{Q}\widetilde{A}^{(i-k)}\widetilde{B}$. Post-multiplying equation (30) by $\widetilde{A}^i\widetilde{B}$ and then using the inductive hypothesis, we have

$$\begin{aligned}&V_{2\Lambda}^T\widetilde{A}^{i+1}\widetilde{B} = -V_{1\Lambda}^T\widetilde{Q}\widetilde{A}^i\widetilde{B} - \Gamma^TV_{2\Lambda}^T\widetilde{A}^i\widetilde{B}\\ &= -V_{1\Lambda}^T\widetilde{Q}\widetilde{A}^i\widetilde{B} - \Gamma^T\sum_{k=0}^{i-1}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{Q}\widetilde{A}^{(i-1-k)}\widetilde{B}\\ &= \sum_{k=0}^{i}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{Q}\widetilde{A}^{(i-k)}\widetilde{B}.\end{aligned}$$

This proves the claim. Using this claim along with the fact that $\widetilde{C}^T\widetilde{C} = \widetilde{Q}$ we get that

$$\begin{aligned}&V_{2\Lambda}^TW = V_{2\Lambda}^T\,[\,\widetilde{B}\;\widetilde{A}\widetilde{B}\;\cdots\;\widetilde{A}^{\mathtt{n_f}-\mathtt{d}}\widetilde{B}\,]\,N\\ &= \left[\,0\;\; -V_{1\Lambda}^T\widetilde{Q}\widetilde{B}\;\cdots\;\textstyle\sum_{k=0}^{\mathtt{n_f}-\mathtt{d}-1}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{Q}\widetilde{A}^{(\mathtt{n_f}-\mathtt{d}-1-\mathtt{k})}\widetilde{B}\,\right]N\\ &= \left[\,0\;\; -V_{1\Lambda}^T\widetilde{C}^T\widetilde{C}\widetilde{B}\;\cdots\;\textstyle\sum_{k=0}^{\mathtt{n_f}-2}(-1)^{k+1}(\Gamma^T)^kV_{1\Lambda}^T\widetilde{C}^T\widetilde{C}\widetilde{A}^{(\mathtt{n_f}-\mathtt{d}-1-\mathtt{k})}\widetilde{B}\,\right]N\\ &= [\,0\;\;(-1)^{\mathtt{n_f}-\mathtt{d}}(\Gamma^T)^{\mathtt{n_f}-\mathtt{d}-1}V_{1\Lambda}^T\widetilde{C}^T\;\cdots\;\Gamma^TV_{1\Lambda}^T\widetilde{C}^T\;\; -V_{1\Lambda}^T\widetilde{C}^T\,]\,\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}N.\end{aligned}$$

But, since $\mathcal{M}_{\mathtt{n_f}-\mathtt{d}+1}N = 0$, it is evident that $V_{2\Lambda}^TW = 0$. $\square$

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