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Finite gain and phase margins as dissipativity conditions

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***Abstract*— It is well-known that the two situations: gain margin being infinite and phase margin being infinite are individually special cases of dissipativities with respect to suitable supply rates; the small gain theorem and passivity result being the respective theorems. However, in practice, the two margins being finite also allows concluding closed loop stability using the Nyquist stability criteria. We show in this paper that the finite and positive gain and phase margin condition is equivalent to dissipativity with respect to a convex combination of the supply rates arising from infinite gain margin and infinite phase margin supply rates. We formulate this situation into a more general framework of Nyquist Plot Compatible (NPC) supply rates and transformations that keep such supply rates invariant.**
Keywords: dissipativity, convex combination, small gain theorem, passivity result, Nyquist plot

I. Introduction and notation

Dissipativity theory has provided a unifying approach to address a large class of control theoretic problems, namely LQR/LQG control, $\mathcal{H}_\infty$ control (see [2], [8]). Among various kinds of dissipativity, quadratic ones play an important role. While the general theory of dissipativity with respect to quadratic supply rates does not put emphasis on the supply rates, there are a few special supply rates that allow an interesting kind of manipulations. In this paper, we formalize this idea of special supply rates: we call them Nyquist Plot Compatible (NPC) supply rates. We show how certain transformations on these supply rates bring out frequency domain properties of systems. We then study the classical gain and phase margin conditions for stability using the Nyquist criteria and relate it to dissipativity. It is well-known that infinite phase margin condition results in closed loop stability (in the negative unity feedback configuration shown in Figure 1, assuming $H(s) = 1$) through the small gain theorem; the relation between small gain theorem and dissipativity is classical too (see [2]). A similar situation is true for the passivity result, namely, feedback interconnection of two positive real transfer functions, at least one being strictly positive real, results in closed loop stability: the relation to dissipativity is again classical. The following question naturally arises as a consequence of the above relations: can the assurance of closed loop stability by finite and positive gain and phase margins be explained using possibly a combination of 'small-gain-like' and 'passivity-like' dissipativities? This paper makes this question precise and resolves it (Theorem 6): a polynomially-convex combination of the two dissipativities indeed results in a sufficient condition to rule out encirclement of the critical point '-1' by the Nyquist plot of the loop gain transfer function. This main result makes use of a novel supply rate that captures non-intersection of the negative real axis as a dissipativity property.

The outline of the paper is as follows. Section II contains preliminaries regarding the problem formulation and the behavioral approach. Section III formalizes certain relations between Nyquist plot and supply rates: this section contains one of the main results about calculus of transformations on the set of 'Nyquist Plot Compatible' supply rates. The next section (Section IV) consists of another main result (Theorem 6) about the finite gain/phase margin condition being nothing but dissipativity with respect to a convex combination of two extreme supply rates. The proof of Theorem 6 follows in Section V. We elaborate on this theorem using examples in Section VI, following which we conclude the paper in Section VII. The rest of this section is devoted to the notation used in this paper. The set $\mathbb{R}$ stands for the real numbers, while $\mathbb{C}$ stands for complex numbers. In the context of stability of the negative unity feedback configuration, the point '-1' on the complex plane plays a key role: we call it the critical point. In a similar context, the unit circle in $\mathbb{C}$ refers to the circle of radius one centered at the origin. The set $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ means the space of infinitely often differentiable maps from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. The subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ with functions having compact support is denoted by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Sometimes, when it is clear from the context, we write just $\mathfrak{D}$.

II. Preliminaries

In this paper, by a linear differential behavior $\mathfrak{B}$ we mean a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that elements $w \in \mathfrak{B}$ satisfy a system of ordinary linear differential equations with constant coefficients. This amounts to existence of a polynomial matrix $R(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0\}.$$

This representation is known as a *kernel representation* of $\mathfrak{B}$. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *controllable* if for every $w', w'' \in \mathfrak{B}$, there exist $w \in \mathfrak{B}$ and $\tau > 0$ such that $w(t) = w'(t)$ for all $t \leqslant 0$ and $w(t) = w''(t)$ for all $t \geqslant \tau$. It was shown in [6], [5] that $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ is controllable if and only if $R(\lambda)$ does not lose rank for any $\lambda \in \mathbb{C}$. In this paper, we consider only SISO systems and also assume the system/behavior to be controllable.

Another important concept required for this paper is that of a quadratic differential form (QDF). (See [7] for a detailed exposition.) A QDF Q_Φ induced by a two-variable polynomial matrix with real constant coefficients, $\Phi(\zeta, \eta) :=$

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$\sum_{i,k} \Phi_{ik}\zeta^i\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, where $\Phi_{ik} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, is a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined as

$$Q_\Phi(w) := \sum_{i,k} (\frac{\mathrm{d}^i w}{\mathrm{d}t^i})^T \Phi_{ik} (\frac{\mathrm{d}^k w}{\mathrm{d}t^k}).$$

When dealing with quadratic forms in w and its derivatives, we can assume, without loss of generality, that $\Phi(\zeta,\eta) = \Phi^T(\eta,\zeta)$; such a $\Phi(\zeta,\eta)$ is called *symmetric*. We often require the one-variable polynomial matrix $\Phi(-\xi,\xi)$ obtained from $\Phi(\zeta,\eta)$: define $\partial\Phi(\xi) := \Phi(-\xi,\xi)$. In this context, the notion of *para-Hermitian* plays an important role. A square polynomial matrix $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is called para-Hermitian if $P(-\xi) = P^T(\xi)$. The significance of P being para-Hermitian is that $P(j\omega)$ is Hermitian for all $\omega \in \mathbb{R}$.

We call a controllable behavior $\mathfrak{B}$ dissipative on $\mathbb{R}$ with respect to a symmetric two-variable polynomial matrix $\Phi(\zeta,\eta)$ if

$$\int_{\mathbb{R}} Q_\Phi(w)\mathrm{d}t \geqslant 0 \qquad \text{for all } w \in \mathfrak{B}\cap\mathfrak{D}. \tag{1}$$

For the purpose of this paper[1] $\mathfrak{B}$ is said to be *strictly* dissipative if the integral in inequality (1) satisfies a strict inequality for all nonzero $w \in \mathfrak{B}\cap\mathfrak{D}$.

III. Transformations on supply rates

Dissipativity is closely related to frequency domain characterization of the system (see [7]). For example, dissipativity of a system with respect to the supply rate $u^2 - y^2$ is equivalent to the Bode magnitude plot being below the 0 dB line, i.e., the Nyquist plot being within the unit circle. Supply rates that allow such statements in terms of the Nyquist plot of a transfer function allow an interesting kind of manipulations and also play a common role in various dissipativity studies. This is the focus of this section.

In order to formalize these notions, we present the following definition of a collection Ω of 'Nyquist Plot Compatible' supply rates. In this paper we restrict ourselves to SISO systems; a useful description of controllable SISO systems is obtained by looking into their kernel representations. Suppose $n(s)$ and $d(s)$ are coprime polynomials such that $G(s) = \frac{n(s)}{d(s)}$, we then consider the behavior $\mathfrak{B}_G$ described as

$$\mathfrak{B}_G := \{(u,y) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^2) \mid d(\tfrac{\mathrm{d}}{\mathrm{d}t})y - n(\tfrac{\mathrm{d}}{\mathrm{d}t})u = 0\}. \tag{2}$$

Definition 1: A supply rate $\Phi(\zeta,\eta) \in \mathbb{R}^{2\times2}[\zeta,\eta]$ is said to induce a trichotomy of the complex plane $\mathbb{C}$ if corresponding to $\Phi(\zeta,\eta)$ there exists a 3-tuple of disjoint sets $\{\mathcal{A}_\Phi^+, \mathcal{A}_\Phi^0, \mathcal{A}_\Phi^-\}$ whose union is $\mathbb{C}$ such that for every $\mathfrak{B}_G$, dissipativity (lossless-ness) is equivalent to the Nyquist plot of G being contained in $\mathcal{A}_\Phi^+$ (contained in $\mathcal{A}_\Phi^0$). The set of all such supply rates that induce trichotomies of $\mathbb{C}$, called the Nyquist-Plot-Compatible (NPC) supply rates $\Omega \subset \mathbb{R}^{2\times2}[\zeta,\eta]$ is defined as

$$\Omega := \{\Phi(\zeta,\eta) \in \mathbb{R}^{2\times2}[\zeta,\eta] \mid \Phi \text{ induces a trichotomy of } \mathbb{C} = \mathcal{A}_\Phi^+ \cup \mathcal{A}_\Phi^0 \cup \mathcal{A}_\Phi^-\}.$$

[1] There are other more stringent definitions of strict dissipativity; see [7].

An example of Φ that is within this set is $\Phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (arising out of $Q_\Phi(u,y) = u^2 - y^2$). This Φ gives $\mathcal{A}_\Phi^0$ equal to the unit circle, and $\mathcal{A}_\Phi^+$ as the interior of the unit circle. We describe some more important and familiar elements of Ω after the following theorem.

We show below that Ω is closed under certain important transformations. In what follows, we use this idea to relate interesting frequency domain properties such as gain/phase margins with dissipativities of supply rates obtained by combining two key supply rates (see Section IV). The following theorem is one of the main results of this paper; it shows closure of Ω under congruence transformations and finite combinations.

Theorem 2: Let $T_1, T_2 \in \mathbb{R}^{2\times2}$ be nonsingular. Then for all $\Phi_1, \Phi_2 \in \Omega$ the following holds:

$$T_1^{\mathrm{T}}\Phi_1 T_1 + T_2^{\mathrm{T}}\Phi_2 T_2 \in \Omega. \tag{3}$$

Proof : First we prove that Ω is closed under congruence transformations. Let $T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \in \mathbb{R}^{2\times2}$ be a nonsingular matrix, define the corresponding bilinear transformation (also known as Möbius transformation) $\mathcal{B}_T : \mathbb{C} \to \mathbb{C}$ as $\mathcal{B}_T(z) = \frac{t_1 z + t_2}{t_3 z + t_4}$. Suppose $\Phi(\zeta,\eta) \in \Omega$, and $\widetilde{\Phi} := T^{\mathrm{T}}\Phi T$, then for a transfer function G, the corresponding behavior $\mathfrak{B}_G$ is $\widetilde{\Phi}$-dissipative if and only if the behavior corresponding to $\widetilde{G} := \mathcal{B}_T(G)$ is Φ-dissipative. Since $\Phi \in \Omega$, $\mathfrak{B}_{\widetilde{G}}$ is Φ-dissipative if and only if the Nyquist plot of $\widetilde{G}$ is contained in $\mathcal{A}_\Phi^+$, which is equivalent to Nyquist plot of G being contained in $\mathcal{B}_T(\mathcal{A}_\Phi^+)$. This proves $T^T\Phi T \in \Omega$.

It remains to prove that Ω is closed under addition. We first note the following observation: if the complex plane is identified with $\mathbb{R}^2$, and $z = (x+iy)$ is represented by $[x \;\; y]^{\mathrm{T}}$, then the set $\mathcal{A}_\Phi^+$ induced by a $\Phi(\zeta,\eta) \in \Omega$ is given by an algebraic inequality $f(x,y) > 0$, where $f(x,y) \in \mathbb{R}[x,y]$. Let $\Phi_1, \Phi_2 \in \Omega$ be such that $\mathcal{A}_{\Phi_1}^+$ and $\mathcal{A}_{\Phi_2}^+$ are given by $f(x,y) > 0$ and $g(x,y) > 0$. It then follows that $\mathcal{A}_{\Phi_1+\Phi_2}^+$ is given by $f(x,y) + g(x,y) > 0$, For any G such that the Nyquist plot of G is in $\mathcal{A}_{\Phi_1}^+$ and $\mathcal{A}_{\Phi_2}^+$, it follows that $f(x,y) > 0$ and $g(x,y) > 0$, and hence $f(x,y)+g(x,y)$ also is positive for all points on the Nyquist plot, and conversely. This proves $\Phi_1 + \Phi_2 \in \Omega$. □

The set Ω has some familiar elements; we already saw $\Sigma_{\mathrm{br}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which arises in the Bounded Real Lemma, and the associated trichotomy, namely the unit circle and its interior, etc.. We now elaborate on some more examples.

1: The $\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ results in $\mathcal{A}_\Phi^0$ equal to the imaginary axis and $\mathcal{A}_\Phi^+$ as the open right half complex plane. Of course, dissipativity with respect to this supply rate is precisely by systems having a positive real transfer function, and this is equivalent to Nyquist plot being in the right half complex plane. (We have ignored the stability aspect in the definition of positive real; this matters only over half-line dissipativity and not in this paper.)

2: Consider $\Phi = \begin{bmatrix} 2\alpha\beta & \alpha+\beta \\ \alpha+\beta & 2 \end{bmatrix}$ which plays a key role in sector nonlinearities and is related to the circle criterion. See [2] for the link between Integral Quadratic Constraints (IQC), dissipativity and the circle criterion. The trichotomy in this case is such that $\mathcal{A}_\Phi^0$ is the circle centered at $(\beta+\alpha)/(2\beta\alpha)$ and radius $(\beta-\alpha)/(2\beta\alpha)$. Assuming $\alpha, \beta > 0$, $\mathcal{A}_\Phi^+$ is the exterior of this circle. It turns out that this matrix can be obtained by applying a congruence transformation on $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. However, such a transformation is not unique; one of them is given by $T = \begin{bmatrix} \alpha\sqrt{\beta} & \sqrt{\beta} \\ \sqrt{\beta} & 1/\sqrt{\beta} \end{bmatrix}$.

3: Consider $\Phi = \begin{bmatrix} 0 & \eta \\ \zeta & 0 \end{bmatrix}$. It turns out that $\mathcal{A}_\Phi^0$ for this case is the real axis and $\mathcal{A}_\Phi^+$ is the open lower half complex plane (see [3]).

In the context of Example 3 above, notice that a supply rate such that $\mathcal{A}_\Phi^0$ is just the negative real axis is useful, for one can then characterize non-intersections of the negative real axis (i.e., infinite gain margin) as a dissipativity property. This is the subject of the following section, in particular, Corollary 5, and is one of the main results of this paper.

Such transformations as above can be related to the idea of multipliers used in nonlinear systems analysis. In this context, dynamic multipliers give rise to transformations that are polynomial matrices. An analogue of the above result in this situation requires further investigation. In the sequel, we further explore the set Ω by considering polynomial scalar combination of two key supply rates in Ω and we relate this to gain/phase margin properties.

IV. Gain/phase margins and dissipativity

In this section we relate the traditional gain and phase margin conditions to dissipativity. It is well-known that $G(j\omega) \leqslant 1$ for all $\omega \in \mathbb{R}$ if and only if the system (more precisely, the behavior $\mathfrak{B}_G$ defined in Equation (2)) is dissipative with respect to the supply rate $u^2 - y^2$, where u is the input and y is the output. Similarly, $|\angle G(j\omega)| \leqslant 90^o$ is equivalent to dissipativity with respect to the supply rate uy. Our main result Theorem 6 shows that the situation when both gain and phase margins are finite and positive is equivalent to dissipativity of the system with respect to a convex combination of supply rates corresponding to these conditions individually. Precise statement requires the following development.

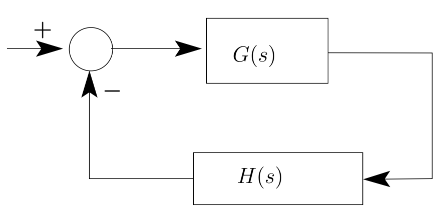

Fig. 1. Standard unity feedback configuration

Consider the negative unity feedback configuration shown in Figure 1. The Nyquist plot of GH does not encircle the point '-1' on the complex plane if the transfer function GH satisfies $|GH(j\omega)| < 1$ for all real ω, or if $|\angle GH(j\omega)| < 180^o$ for all real ω. Consequently, we obtain the following classical sufficient condition for closed loop stability. In the context of the feedback configuration shown in 1, we use well-posedness: the interconnection is said to be well-posed if $1 + G(\infty)H(\infty) \neq 0$.

Proposition 3: Let G and H be two stable proper real rational transfer functions, and suppose their interconnection is well-posed. The closed loop is asymptotically stable if any one of the following conditions hold:

1) For each $\omega \in \mathbb{R}$, $G(s)H(s)$ satisfies $|GH(j\omega)| < 1$.
2) For each $\omega \in \mathbb{R}$, G and H satisfy: $|\angle G(j\omega)| < 90^o$ and $|\angle H(j\omega)| < 90^o$.

Of course, the above conditions are not essential for ruling out encirclements of the critical point by the Nyquist plot. Amongst the various ways in which the above result can be strengthened, we focus on the traditional gain/phase margin conditions for closed loop stability: this is a *combination* of the conditions 1 and 2 above. We state this as a proposition for easy reference.

Proposition 4: Let G and H be two stable proper real rational transfer functions. Suppose the transfer function GH satisfies

$$\angle GH(j\omega) = 180^o \Rightarrow |GH(i\omega)| < 1. \quad (4)$$

Then, the feedback interconnection is asymptotically stable.

While the condition stated in Equation (4) is not necessary[2], it captures gain/phase margin conditions taught in a first level control course. Further, the condition relaxes both the conditions stated in Proposition 3. This section deals with the question: can condition (4) be obtained as a 'convex combination' of conditions 1 and 2 in Proposition 3? We now formulate this question using dissipativity with respect to suitable supply rates, and show that a convex combination indeed gives condition (4).

Since it is only the loop gain GH that affects closed loop stability conditions, we assume without loss of generality that $H = 1$. Though this does not affect stability, this crucially affects the supply rate corresponding to passivity, and our modification of the supply rate to handle non-intersection of the negative real axis is one of the novel aspects of this paper. This is addressed further below; we now continue with just $G(s)$. The condition $G(j\omega) \leqslant 1$ for all ω in $\mathbb{R}$ is known to be equivalent to dissipativity of $\mathfrak{B}$ with respect to the supply rate $Q_{sg}(u, y) := u^2 - y^2$. The following result is the analogue in the context of infinite gain margin: we write this as a corollary, since it follows from Theorem 6 below. The supply rate $\Sigma_{\rm pa}$ formalizes in a novel way the fact that non-intersection of the negative real axis is, in fact, dissipativity with respect to a quadratic supply rate.

Corollary 5: Consider the feedback interconnection shown in Figure 1. Suppose $H(s) = 1$, $G(s) = \frac{n(s)}{d(s)}$,

[2]This is due to the situation that $G(0) < 0$ or due to multiple intersections of the negative real axis/unit circle: there could possibly be zero encirclements though the condition in Proposition 4 (Equation (4)) is not satisfied. See Section VI for an example.

and let the behavior $\mathfrak{B}_G$ be as defined in (2). Define $\Sigma_{\rm pa} \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ as

$$\Sigma_{\rm pa}(\zeta,\eta) = \begin{bmatrix} n(\zeta)n(\eta) & -d(\zeta)n(\eta)+\epsilon \\ -n(\zeta)d(\eta)+\epsilon & d(\zeta)d(\eta) \end{bmatrix}. \quad (5)$$

Then, the following are equivalent.

- The Nyquist plot of G does not intersect the negative real axis.
- $|\angle G(j\omega)| < 180^o$ for all $\omega \in \mathbb{R}$.
- $\mathfrak{B}_G$ is $\Sigma_{\rm pa}(\zeta,\eta)$ dissipative.

The proof of this corollary follows from our main result (Theorem 6 below). The more general statement in Proposition 4 that gain and phase margins each being positive assures closed loop stability, notwithstanding that there are both negative real axis intersections and unit circle intersections by the Nyquist plot, is addressed in the theorem below: a polynomially convex combination suffices for this purpose.

Theorem 6: Consider a SISO LTI system given by the transfer function $G(\xi) = \frac{n(\xi)}{d(\xi)}$ and let $\mathfrak{B}_G$ be the set

$$\{(u,y) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^2) \mid d(\tfrac{\rm d}{{\rm d}t})y - n(\tfrac{\rm d}{{\rm d}t})u = 0\}.$$

Define $\Sigma_{\rm pa}(\zeta,\eta) \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ as in Equation (5) above and let $\Sigma_{\rm br} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then the following two statements are equivalent:

1) there exist $p, q \in \mathbb{R}[\xi]$ and $\epsilon > 0$ such that $\mathfrak{B}$ is strictly dissipative with respect to
$$\Phi_\epsilon(\zeta,\eta) := p(\zeta)\Sigma_{\rm br}p(\eta) + q(\zeta)\Sigma_{\rm pa}q(\eta),$$
2) for each $\omega \in \mathbb{R}$, either $|G(j\omega)| < 1$, or $|\angle(G(j\omega))| <$ 180 degrees, or both.

Notice that the second condition in the above theorem rules out encirclements of the critical point -1 by the Nyquist plot of G. The above theorem shows the equivalence between this and dissipativity of the system with respect to a supply rate that depends on the two extreme supply rates: small gain and passivity supply rates. The first extreme case is classical, the second extreme situation has been elaborated as Corollary 5 above. In other words, when a transfer function $G(s)$ has infinite phase margin (i.e., its Nyquist plot lies within the unit circle), or when the Nyquist plot does not intersect the negative real axis (i.e., infinite gain margin), then either q or p can be taken to be zero respectively, and Theorem 6 reduces to the small gain theorem and Corollary 5.

V. Proof of Theorem 6

The proof requires some more preliminaries. The first concerns controllable behaviors: see Section II for the definition. Controllable behaviors happen to be precisely the behaviors that admit an *image representation*: there exists an $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that $\mathfrak{B} := \{w \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})$ such that $w = M(\frac{\rm d}{{\rm d}t})\ell\}$. For the purpose of this paper, we need the image representation to have the property that ℓ can be deduced from $w \in \mathfrak{B}$; this is called observability. The image representation above is said to be observable if $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. It turns out that image representations can be assumed to be observable without loss of generality.

We make use of the following result from [7], which relates the dissipativity of a behavior to the non-negativity of a certain matrix on the imaginary axis.

Proposition 7: Consider $\mathfrak{B} = {\rm im} M(\frac{\rm d}{{\rm d}t})$ and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Then $\mathfrak{B}$ is dissipative with respect to $\Phi(\zeta,\eta)$ on $\mathbb{R}$ if and only if

$$M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) \geqslant 0 \text{ for all } \omega \in \mathbb{R}. \quad (6)$$

One can check that strict dissipativity is equivalent to $M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) > 0$ for almost all ω, i.e. $M^T(-j\omega)\partial\Phi(j\omega)M(j\omega)$ is nonsingular as a polynomial matrix, and satisfies non-negativity for all $\omega \in \mathbb{R}$.

We also need the notion of the inertia of a polynomial matrix. The inertia of a Hermitian constant matrix $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is the triple $(\sigma_-(S), \sigma_0(S), \sigma_+(S))$, the number of negative, zero and positive eigenvalues of S respectively. When dealing with a nonsingular Hermitian matrix S, we abuse this notation of inertia by skipping the middle integer and just write the 2-tuple $(\sigma_-(S), \sigma_+(S))$. In our case, the concerned Hermitian matrix depends on ω, therefore the inertia of such a matrix, also depends on ω. Moreover, when the polynomial matrix $S(\omega)$ is nonsingular, i.e. $\det(S(\omega)) \not\equiv 0$, we require to use the 3-tuple notation for inertia only at finitely many points. Hence we stick to 2-tuple notation, which is sufficient *almost everywhere*. We make this precise after considering a simple example.

Consider $P(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ defined as the diagonal matrix with $(1+\xi^2)$ and $(9+\xi^2)$ on the diagonal. This polynomial matrix is para-Hermitian, i.e. P satisfies $P(-\xi) = P(\xi)^T$. As a result, for each $\omega \in \mathbb{R}$, $P(j\omega)$ is Hermitian. Since $P(j\omega)$ is nonsingular, we define its inertia at $\omega_0 \in \mathbb{R}$ to be the 2-tuple $(\sigma_-(P(j\omega_0)), \sigma_+(P(j\omega_0)))$: assuming, of course, $P(j\omega_0)$ is nonsingular. This allows us to define the inertia for a nonsingular para-Hermitian polynomial matrix $P(\xi)$ for almost all points on the imaginary axis. The inertia of $P(j\omega_0)$ is left undefined if $j\omega_0$ is a zero of $P(\xi)$. Thus inertia of $P(2i) = (1,1)$, and that of $P(0) = (0,2)$, while inertias of $P(3i)$ and $P(i)$ are undefined. The definition below makes this precise for general para-Hermitian polynomial matrices.

Definition 8: Suppose $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is para-Hermitian and assume $P(\xi)$ is nonsingular. Let $\omega_0 \in \mathbb{R}$ be such that $j\omega_0$ is not a zero of $P(\xi)$, i.e. $\det(P(j\omega_0)) \neq 0$. Then, the inertia of $P(j\omega_0)$ is defined as the 2-tuple: $(\sigma_-(P(j\omega_0)), \sigma_+(P(j\omega_0)))$. If $P(j\omega_0)$ is singular, then the inertia is undefined.

The above definition allows inertia of a nonsingular para-Hermitian polynomial matrix to be defined for almost all values on the imaginary axis. Notice in the example above that the inertia can change about zeros of $P(\xi)$ on the imaginary axis. Due to dissipativity condition of a behavior requiring a certain minimum number of positive eigenvalues at every point on the imaginary axis (see [1]), we need the inertia to be such that the number of negative eigenvalues are not too many at any $\omega \in \mathbb{R}$. For this purpose we define a total ordering over the set of two tuples $(\sigma_-(S), \sigma_+(S))$ that satisfy $\sigma_-(S) + \sigma_+(S) = \mathtt{w}$ (the integer $\mathtt{w}$ playing the role of the size of the supply rate matrix, i.e. the number of variables).

Definition 9: Consider two 2-tuples (ν_1, π_1) and (ν_2, π_2) that satisfy $\nu_1+\pi_1 = \nu_2+\pi_2 = \mathtt{w}$. We say $(\nu_1, \pi_1) < (\nu_2, \pi_2)$ if $\nu_1 > \nu_2$. In this case, (ν_1, π_1) is said to be worse than (ν_2, π_2).

Let $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be para-Hermitian and nonsingular. Define $\nu_{\max}$ to be the maximum number of negative eigenvalues of $P(j\omega)$ as ω varies over $\mathbb{R}$, i.e. $\nu_{\max} := \max_{\omega\in\mathbb{R}}\{\sigma_-(P(j\omega))\}$. The *worst inertia* of $P(\xi)$ is defined as $(\nu_{\max}, \mathtt{w}-\nu_{\max})$, and correspondingly, the worst inertia matrix (see [4]) is defined as

$$J_{\text{worst}} := \begin{bmatrix} I_{\mathtt{w}-\nu_{\max}} & 0 \\ 0 & -I_{\nu_{\max}} \end{bmatrix}. \qquad (7)$$

Notice that in our case for any $\omega \in \mathbb{R}$ we have $(\sigma_-(\partial\Phi(j\omega)) + \sigma_0(\partial\Phi(j\omega)) + \sigma_+(\partial\Phi(j\omega))) = 2$. The following result from [4, Theorem 3.6.5] concerns factorization of para-Hermitian polynomial matrices that might not have constant inertia almost everywhere on the imaginary axis.

Proposition 10: Let $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be para-Hermitian and nonsingular and let $J_{\text{worst}} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be its worst inertia matrix. Then there exist polynomial matrices[3] K and $L \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, with K square and nonsingular, such that

$$P(\xi) = K^T(-\xi)J_{\text{worst}}K(\xi) + L^T(-\xi)L(\xi). \qquad (8)$$

Proof of Theorem 6: 1) $\Rightarrow$ 2): Assume that there exist polynomials $p, q \in \mathbb{R}[\xi]$ and $\epsilon > 0$ such that $\mathfrak{B}$ is strictly Φ-dissipative. From proposition 7, this means for all $\omega \in \mathbb{R}$ the following holds:

$$\begin{bmatrix} d(j\omega) \\ n(j\omega) \end{bmatrix}^* \partial\Phi_\epsilon(j\omega) \begin{bmatrix} d(j\omega) \\ n(j\omega) \end{bmatrix} > 0.$$

The LHS simplifies to

$$\begin{array}{l} |p(j\omega)|^2 \left(|d(j\omega)|^2 - |n(j\omega)|^2\right) \\ +|q(j\omega)|^2 \left(2\epsilon\text{Re}(n(-j\omega)d(j\omega)) + 4(\text{Im}(n(-j\omega)d(j\omega)))^2\right). \end{array}$$

This implies

$$\begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix}^* \begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi(\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} > 0, \qquad (9)$$

where

$$\begin{array}{ll} \Gamma(\omega) & := |d(j\omega)|^2 - |n(j\omega)|^2 \qquad \text{and} \\ \Pi(\omega) & := 2\epsilon\text{Re}(n(-j\omega)d(j\omega)) + 4(\text{Im}(n(-j\omega)d(j\omega)))^2. \end{array}$$

Since both $p(-j\omega)p(j\omega)$ and $q(-j\omega)q(j\omega)$ are non-negative for each $\omega \in \mathbb{R}$, the above inequality rules out existence of any ω_0 such that $\Gamma(\omega_0) < 0$ and $\Pi(\omega_0) < 0$. Therefore, for almost all $\omega \in \mathbb{R}$, either $|d(j\omega)|^2 - |n(j\omega)|^2 > 0$ or $2\epsilon\text{Re}(n(-j\omega)d(j\omega)) + 4(\text{Im}n(-j\omega)d(j\omega))^2 > 0$ or both. Thus statement 2) follows.

2) $\Rightarrow$ 1): Condition 2) implies that $\begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi(\omega) \end{bmatrix}$ has worst inertia $(1,1)$ or $(0,2)$. (Any one of the two inequalities within Condition 2) implies worst inertia of $(1,1)$ and when both inequalities are satisfied, the worst inertia is $(0,2)$.) The latter case requires no proof since any pair of polynomials (p, q) satisfying coprimeness on the imaginary axis also satisfies inequality (9), thus proving condition 1). For the former case, from proposition 10 above, $\begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Pi(\omega) \end{bmatrix}$ can be written as

$$K^T(-j\omega)J_{\text{worst}}K(j\omega) + L^T(-j\omega)L(j\omega), \qquad (10)$$

where $J_{\text{worst}} = \text{diag}(1, -1)$ and matrices $L(\xi)$ and $K(\xi) \in \mathbb{R}^{\bullet\times 2}[\xi]$, with K square and nonsingular.

As done in Theorem 3.4.7 in [4], equation (10) is used to find p, q that meet the requirements in statement 1 as follows:

1) Choose a transfer function $h(\xi) \in \mathbb{R}(\xi)$ with $\mathcal{L}_\infty$ norm strictly less than one. Its coprime numerator and denominator polynomials, say p' and q' respectively, satisfy $p'(-j\omega)p'(j\omega) - q'(-j\omega)q'(j\omega) > 0$ for all ω.
2) Next, construct the adjugate $\text{adj}\ K(\xi)$ of $K(\xi)$. The required p, q are given by $\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} := (\text{adj}\ K(\xi)) \begin{bmatrix} p'(\xi) \\ q'(\xi) \end{bmatrix}$.

Using this p and q (after cancelling common factors, if any), one can reverse the chain of arguments before inequality (9) in order to conclude strict dissipativity. This concludes the proof of 2) $\Rightarrow$ 1). ☐

VI. Examples

In this section we study two examples; the first one having a Nyquist plot with finite gain and phase margins and hence is dissipative with respect to a suitable polynomially convex combination of the extreme supply rates, while the second one has a Nyquist plot without encirclements of the critical point, but is not dissipative with any such convex combination.

Consider the transfer function $G(s) = \frac{30}{(s+1)(s+2)(s+3)}$, whose Nyquist plot is as shown in Figure 2. Since the point of intersection of the negative real axis and the Nyquist plot lies inside the unit circle, Condition 2) of Theorem 6 is satisfied and hence the system is dissipative.

We next consider the transfer function $G(s) = \frac{50(s^2+2s+100)}{s^3+6s^2+11s+6}$: see Figure 3 for its Nyquist plot. The negative real axis intersections are at -61 and at -4/3. While the critical point -1 is not encircled, the negative real axis is intersected twice, and both outside the unit circle. This example shows how non-encirclements of the point -1 does not imply dissipativity with respect to some convex combination of the two supply rates. Of course, this is an example of how the close loop is stable for $k = 1$ (the gain k is multiplied to the loop gain for this analysis), but not for arbitrary *decrease* of k: the closed loop is unstable for $\frac{1}{61} < k < \frac{3}{4}$. It is important to note that traditionally gain and phase 'margins' applied to situations where decrease in the gain did not cause instability.

As mentioned in Footnote 2, another situation is the transfer function $G(s) = \frac{-(2s+15)}{s+5}$. This system is also not

[3]The number of columns of K and L equals w, but number of rows depends on the particular $\partial\Phi(\xi)$ and hence is left unspecified. We use $\bullet$ to leave the relevant integer unspecified.

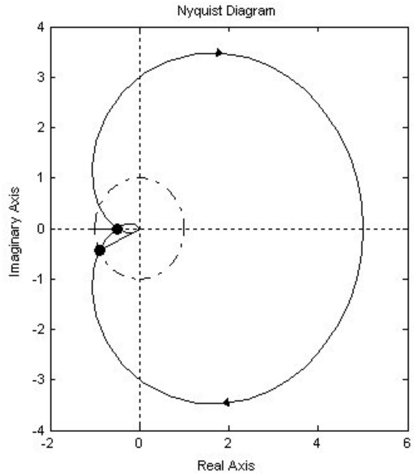

Fig. 2. Nyquist plot of $G(s) = \frac{30}{(s+1)(s+2)(s+3)}$

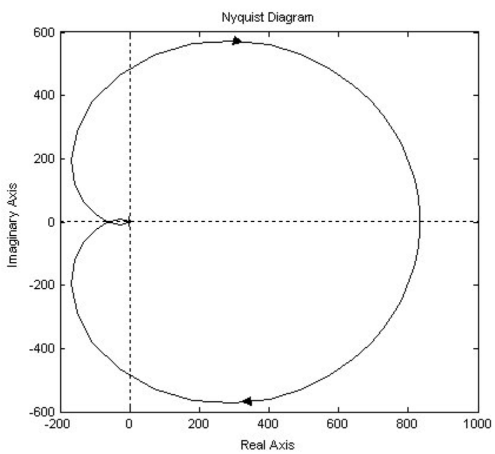

Fig. 3. Nyquist plot of $G(s) = \frac{50(s^2+2s+100)}{s^3+6s^2+11s+6}$

dissipative with respect to any convex combination of the two supply rates, though there are no encirclements of the critical point. Like the previous example, this system has unstable closed loop for k in the range of $(\frac{1}{3}, \frac{1}{2})$.

VII. Concluding remarks

In this section, we summarize the key results in this paper. We formulated properties that certain special supply rates have using the Nyquist plot of system transfer functions. Such supply rates, called Nyquist Plot Compatible (NPC) supply rates, are closed under addition and under the congruence transformation (Theorem 2). Various familiar integral quadratic constraints (IQC's) and frequency domain properties could be explained using this formulation.

The general formulation about NPC supply rates was specialized to the gain and phase margin conditions for stability. While infinite phase margin condition, and hence stability, could be captured as dissipativity with respect to a supply rate, the analogue for infinite gain margin is not as straightforward. The passivity result handles the case of two positive real transfer functions resulting in closed loop stability, but this supply rate does not appear to allow a combination with $\Sigma_{\rm br}$ (of the Bounded Real Lemma) to yield the finite gain/phase margin situations. We proposed a new supply rate such that dissipativity with respect to this is equivalent to non-intersection of the negative real axis, and hence infinite gain margin: Corollary 5. A polynomially convex combination of the two supply rates immediately yields the traditional result that, assuming open loop stability, finite and positive gain and phase margin conditions on the open loop results in closed loop stability (Theorem 6).

The polynomials p and q of Theorem 6 can be thought of as the numerator and denominator of a filter F which when combined with G in series results in dissipativity of $\mathfrak{B}_{TG}$ with respect to one of the two extreme supply rates. The close relation between the bounded real lemma and the positive real lemma probably plays a role in making this precise. This remains to be explored further.

A noteworthy point about the supply rate $\Sigma_{\rm pa}(\zeta,\eta)$ corresponding to non-intersection of the negative real axis by the Nyquist plot of a transfer function G is that $\Sigma_{\rm pa}(\zeta,\eta)$ depends on G, and dissipativity is required to be satisfied for some $\epsilon > 0$, (with ϵ possibly depending on G). The seeming dependence of the supply rate $\Sigma_{\rm pa}(\zeta,\eta)$ on the system is unlike other supply rates such as $\Sigma_{\rm br} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\Sigma_{\rm pr} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which respectively correspond to non-intersection of the unit circle and the imaginary axis. This property is perhaps linked to the fact that there is no essential difference between lines and circles on the complex plane, while half-lines are intrinsically different. Consequently, the small gain theorem (infinite phase margin condition) applies to positive *and negative* phase shifts, while the infinite gain margin condition applies to only positive gain shifts. This matter too requires further investigation.

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