New Results in Optimal Quadratic Supply Rates

Debasattam Pal, Subhrajit Sinha, Madhu N. Belur and Harish K. Pillai

***Abstract*— This paper concerns studying dissipativity of a system with supply rates that depend on one or more parameters. We show that suitable choice of supply rate turns out to make dissipativity equivalent to traditional gain/phase margin conditions for stability. Further, the well-known circle criterion corresponds to a different supply rate, and here optimizing the supply rate is nothing but finding the largest circle such that circle criterion implies absolute stability for time-varying nonlinearities. $\mathcal{L}_\infty$-control is another example of dissipativity with respect to a relevant supply rate, and here we show that, in fact, improper $\mathcal{L}_\infty$-controllers are easily dealt with using our approach (unlike the standard state space methods). We formulate and prove necessary and sufficient conditions for $\mathcal{L}_\infty$-control, and then conclude that optimal controllers always exist (under suboptimal solvability conditions).**

Keywords: Dissipative systems, behaviors, optimal control, circle criterion, $\mathcal{L}_\infty$-control.

I. Introduction and notation

Analysis and design of control systems based on dissipativity has been an active field of research. The theory of dissipativity formalizes the concept of dissipation of energy and gives a firm footing for analysis of systems from energy absorption view-point. Among many others, dissipativity with respect to a quadratic supply function plays a key role because a number of important control theoretic concepts can be generalized into dissipativity with quadratic supply functions, for example, the LQR/LQG control, $\mathcal{H}_\infty$ problem, circle and Popov criteria, and synthesis of passive systems (see [14], [16], [6]). In this paper we relate dissipativity to some system theoretic properties and consider extremizing supply rates to get the optimal such rate. In section III, we bring out a strong relation between dissipativity and the well-known system theoretic ideas of gain and phase margins. In the context of circle criterion, we show how dissipativity concepts can be used in order to find the 'largest' circle satisfying this criterion thus assuring absolute stability for sector-bound time varying nonlinearities (see IV). Next we address the issue of improper controllers for the $\mathcal{L}_\infty$ suboptimal control problem in section V. We present necessary and sufficient conditions for the solvability of the suboptimal problem without making the restrictive assumptions typically assumed in state space control theory (see [10]). A remarkable aspect of our main result for this section is that it can be related to well-known system theoretic concepts of invariant zeros, though *without* an apriori input/output partition of the control variables. Finally we show how the result in this section can be used in the following section to infer the solvability of the $\mathcal{L}_\infty$ optimal control problem (section VI). Here we show that due to nonrequirement of properness of the controller, the optimal controller *always exists* (assuming suboptimal solvability is possible). This result is among the main results of the paper.

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Before we begin with some preliminaries in the following section, we devote a few words about the notation used in this paper. The set $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ means the space of infinitely often differentiable maps from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. The subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ with functions having compact support is denoted by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Sometimes we will drop the argument when it is clear from the context, and write just $\mathfrak{D}$, for example. Also, in order to identify the number of components in a vector w, we simply use $\mathtt{w}$, for example, $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Finally, within text, we often require to stack vectors or matrices into a column: $\mathrm{col}(R_1, R_2)$ denotes $[R_1^T \ R_2^T]^T$.

II. Preliminaries

In this paper, by a linear differential behavior $\mathfrak{B}$, we mean a subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ such that elements $w \in \mathfrak{B}$ satisfy a system of ordinary linear differential equations with constant coefficients. This amounts to existence of a polynomial matrix $R(\xi) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. We denote the set of all such linear differential behaviors with $\mathtt{w}$ number of variables by $\mathfrak{L}^{\mathtt{w}}$. This representation is known as a *kernel representation* of $\mathfrak{B}$. Though there are many possible kernel representations (corresponding to elementary operations on the equations describing $\mathfrak{B}$), the number of inputs and outputs of the system does not depend on the particular representation. We denote the number of inputs and the number of outputs in the system by $\mathtt{m}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B})$ respectively. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$

is said to be *controllable* if for every $w', w'' \in \mathfrak{B}$, there exists a $w \in \mathfrak{B}$ and a $\tau > 0$ such that $w(t) = w'(t)$ for all $t \leqslant 0$ and $w(t) = w''(t)$ for all $t \geqslant \tau$. We denote the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^{\mathtt{w}}_{\rm cont}$. It was shown in [13], [9] that $\mathfrak{B} = \ker R(\frac{\rm d}{{\rm d}t})$ is controllable if and only if $R(\lambda)$ does not lose rank for any $\lambda \in \mathbb{C}$. Controllable behaviors are precisely the behaviors that admit an *image representation*: there exists an $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that $\mathfrak{B} := \{w \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ such that $w = M(\frac{\rm d}{{\rm d}t})\ell\}$. For the purpose of this paper, we need the image representation to have the property that ℓ can be deduced from $w \in \mathfrak{B}$; this is called observability. The image representation above is said to be observable if $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. It turns out that image representations can be assumed to be observable without loss of generality.

Sometimes, a *full behavior* $\mathfrak{B}_{\rm full}$ is associated with $\mathfrak{B}$ that has a latent variable representation by taking the latent variables as manifest variables, that is $\mathfrak{B}_{\rm full} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{m}})$ is given by: $\mathfrak{B}_{\rm full} := \{(w, \ell) \mid$ such that $R(\frac{\rm d}{{\rm d}t})w = M(\frac{\rm d}{{\rm d}t})\ell\}$.

Another important concept required for this paper is the notion of a quadratic differential form (QDF). (See [15] for a detailed exposition.) A QDF Q_Φ induced by a two-variable polynomial matrix with real constant coefficients, $\Phi(\zeta, \eta) := \sum_{i,k} \Phi_{ik}\zeta^i\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, where $\Phi_{ik} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, is a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \rightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined as $Q_\Phi(w) := \sum_{i,k}(\frac{{\rm d}^i w}{{\rm d}t^i})^T \Phi_{ik}(\frac{{\rm d}^k w}{{\rm d}t^k})$. When dealing with quadratic forms in w and its derivatives, we can assume, without loss of generality, that $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$; such a $\Phi(\zeta, \eta)$ is called *symmetric* and the set of all such symmetric two-variable polynomial matrices is denoted by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\rm s}[\zeta, \eta]$. We often require the one-variable polynomial matrix $\Phi(-\xi, \xi)$ obtained from $\Phi(\zeta, \eta)$: define $\partial\Phi(\xi) := \Phi(-\xi, \xi)$.

This paper concerns optimizing the dissipativity property of a controllable behavior $\mathfrak{B}$. We call a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\rm cont}$ dissipative on $\mathbb{R}$ with respect to a symmetric two-variable polynomial matrix $\Phi(\zeta, \eta)$ if

$$\int_{\mathbb{R}} Q_\Phi(w){\rm d}t \geqslant 0 \qquad (1)$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}$. We will make use of the following result from [15], which relates the dissipativity of a behavior to the non-negativity of a certain para-Hermitian matrix on the imaginary axis.

Proposition 1: Consider $\mathfrak{B} = {\rm im} M(\frac{\rm d}{{\rm d}t})$ and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\rm s}[\zeta, \eta]$. Then $\mathfrak{B}$ is dissipative with respect to $\Phi(\zeta, \eta)$ on $\mathbb{R}$ if and only if

$$M^T(-i\omega)\partial\Phi(i\omega)M(i\omega) \geqslant 0 \text{ for all } \omega \in \mathbb{R}. \qquad (2)$$

For the purpose of this paper[1] $\mathfrak{B}$ is said to be *strictly* dissipative if the integral in inequality (1) satisfies a strict inequality for all *nonzero* $w \in \mathfrak{B} \cap \mathfrak{D}$. One can check that strict dissipativity is equivalent to $\partial\Phi(i\omega) > 0$ for almost all ω.

[1]There are other more stringent definitions of strict dissipativity; see [15].

The notion of orthogonal complement of a behavior is related to dissipativity, and we require it in this paper. Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\rm cont}$ having kernel representation $R(\frac{\rm d}{{\rm d}t})w = 0$. The orthogonal complement $\mathfrak{B}^\perp$ of the behavior $\mathfrak{B}$ is defined as $\mathfrak{B}^\perp := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{\mathbb{R}} w^T v {\rm d}t = 0$ for all $v \in \mathfrak{B} \cap \mathfrak{D}\}$. It turns out that $\mathfrak{B}^\perp \in \mathfrak{L}^{\mathtt{w}}_{\rm cont}$ and, in fact, has image representation $w = R^T(-\frac{\rm d}{{\rm d}t})\ell$.

III. Gain/phase margins and dissipativity

Concepts of gain margin and phase margin (abbreviated here as GM and PM respectively) have been important in frequency domain methods of design of control systems. They nicely capture the essence of certain properties like relative stability and robustness which are desirable for a control system. In this section we relate these concepts with dissipativity. It is well-known that GM and PM are related to small gain and passivity respectively, which again are only special cases of dissipativity. Our main result in this section shows that, the GM and PM conditions are equivalent to dissipativity with respect to a supply rate $\Phi(\zeta, \eta)$ obtained as a combination of the small gain matrix $\Sigma_{\rm sg} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the passivity matrix $\Sigma_{\rm pa} := \begin{bmatrix} 0 & \zeta \\ \eta & 0 \end{bmatrix}$.

The small gain matrix $\Sigma_{\rm sg}$ and the passivity matrix $\Sigma_{\rm pa}$ are such that they are supply rates with respect to which the unity feedback path is 'least dissipative': in our case 'least dissipative' means that inequality (1) is satisfied with an equality. (See figure 1 for the unity feedback interconnection; the ϕ in the figure is equal to 1.) The passivity matrix $\Sigma_{\rm pa}$ is adapted from the more commonly used matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ due to the same reason: to ensure that the unity feedback path of our case is least dissipative.

If some system is dissipative with respect to just $\Sigma_{\rm sg}$, this would mean that the unity feedback results in asymptotic stability, using the small gain theorem. While the interconnection of two passive systems (at least one being strictly passive) results in closed loop asymptotic stability: this is captured by the $\Sigma_{\rm pa}$ matrix.

However, gain and phase margin conditions (assuming open loop system is stable) being positive also assure us of closed loop stability. The following theorem, our main result of this section, says that the gain and phase margin conditions are closely related to a suitable polynomial combination of

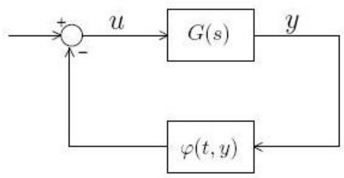

Fig. 1. Feedback Interconnection

these two special supply rates $\Sigma_{\rm sg}$ and $\Sigma_{\rm pa}$. We skip the proof for page limt constraints; see [7] for the same.

Theorem 2: Consider a SISO LTI system given by the transfer function $G(\xi) = \frac{Y(\xi)}{U(\xi)}$ (or equivalently, $(u, y) \in \mathfrak{B} = \text{im}\begin{bmatrix} U(\frac{\rm d}{{\rm d}t}) \\ Y(\frac{\rm d}{{\rm d}t}) \end{bmatrix}$). Then the following two statements are equivalent:

1) there exist $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}$ is strictly dissipative with respect to $\Phi(\zeta, \eta) := p(\zeta)\Sigma_{\rm sg}p(\eta)+q(\zeta)\Sigma_{\rm pa}q(\eta)$,
2) for almost all $\omega \in \mathbb{R}$, either $\| \; G(i\omega) \; \| < 1$, or $\omega \text{Im}(G(i\omega)) < 0$, or both.

Remark 3: Condition 2) in above theorem formalizes the well-known condition that, assuming transfer function G has positive DC gain, if the open loop system is stable then the gain and phase margins being positive guarantees closed loop stability. The above theorem relates this to strict dissipativity with respect to a suitable supply rate.

IV. Circle criterion: computation of the largest sector

In this section we develop a systematic approach to calculate the largest sector that a sector-nonlinearity can belong to, keeping an inter-connection stable. Consider the interconnection of the linear system (with transfer function G) and the sector nonlinearity φ in the feedback path as shown in figure 1. Let φ belong to the sector $[\alpha, \beta]$ with $\alpha < \beta$ and $\beta > 0$. The sector condition on ϕ can be expressed as $\alpha y^2 \leqslant y \cdot \varphi(t, y) \leqslant \beta y^2$, which in matrix form

$$\begin{bmatrix} y \\ \varphi(t,y) \end{bmatrix}^T \begin{bmatrix} -\alpha\beta & \alpha \\ \beta & -1 \end{bmatrix} \begin{bmatrix} y \\ \varphi(t,y) \end{bmatrix} \geqslant 0 \qquad (3)$$

for all t and for all $y \in \mathfrak{L}^2(\mathbb{R}, \mathbb{R})$ (where $\mathfrak{L}^2$ denotes the space of functions that are square integrable over its domain).

Let $G(\xi) = \frac{Y(\xi)}{U(\xi)}$ where $U(\xi), Y(\xi) \in \mathbb{R}[\xi]$. The well-known circle criteria can be rewritten using the Integral Quadratic Constraints (IQC) formulation of [6] as follows. If G satisfies the following inequality strictly[2]

$$\int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} 2\alpha\beta & (\alpha+\beta) \\ (\alpha+\beta) & 2 \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} {\rm d}t > 0 \qquad (4)$$

$\forall \; u \in \mathfrak{L}^2[0, \infty)$, then we have absolute stability for all nonlinearities, possibly time-varying, φ in the sector $[\alpha, \beta]$. Suppose β happens to be a gain that stabilizes the transfer function G, *i.e.*, $\varphi(t, y) := \beta y$ results in closed loop stability, then the question arises as to the minimum value of α that results in absolute stability. This brings us to the problem addressed in this section: the question of optimizing the supply rate $\begin{bmatrix} 2\alpha\beta & (\alpha+\beta) \\ (\alpha+\beta) & 2 \end{bmatrix}$.

Problem Statement: Consider an LTI system having transfer function $G(\xi)$ in the forward path with a nonlinearity φ in the feedback path as shown in figure 1. Suppose β is such that $1+\beta G(\xi)$ has all its zeros in the left half complex plane. Find the smallest α such that we have absolute stability due to circle criterion for φ in the sector $[\alpha, \beta]$.

The following lemma answers the above problem and is easily convertible into an algorithm. Its proof is straightforward and is hence skipped.

Lemma 4: Let $G(\xi) \in \mathbb{R}(\xi)$ and let $\beta > 0$ be such that $1 + \beta G(\xi)$ has all its zeros in the left half complex plane. The minimum α such that we have absolute stability due to circle criterion for all φ in the sector $[\alpha, \beta]$ is the least α that satisfies the inequality:

$$\begin{bmatrix} G(i\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 2\alpha\beta & (\alpha+\beta) \\ (\alpha+\beta) & 2 \end{bmatrix} \begin{bmatrix} G(i\omega) \\ 1 \end{bmatrix} \geqslant 0 \quad (5)$$

is satisfied for all $\omega \in \mathbb{R}$.

Using coprime polynomials $U(\xi)$ and $Y(\xi)$ such that $G(\xi) = Y(\xi)/U(\xi)$ we can rewrite the above *rational* inequality as a *polynomial* inequality by replacing $[G(i\omega) \;\; 1]$ by $[Y(i\omega) \;\; U(i\omega)]$. In fact, for a fixed β, the L.H.S. of inequality (5) is a polynomial in ω and α. Considered as a polynomial in ω with coefficients from $\mathbb{R}[\alpha]$, this polynomial is even in ω:

$$p_\alpha(\omega)a_0(\alpha) + a_2(\alpha)\omega^2 + ... + a_{2n}(\alpha)\omega^{2n} \geqslant 0 \qquad (6)$$

with $a_i \in \mathbb{R}[\alpha]$. We now elaborate on how one can find the minimum α such that inequality (5) is satisfied for all $\omega \in \mathbb{R}$.

The minimum α for which $p_\alpha(\omega)$ is non-negative results in $p_\alpha(\omega)$ and $q_\alpha(\omega) := \frac{\partial p_\alpha(\omega)}{\partial \omega}$ to lose coprimeness. Using a Sylvester resultant condition on these two polynomials (with coefficients that are polynomials in α), one can get all real candidates α's ($< \beta$) that cause loss of coprimeness

[2] In this section, we do not dwell on the precise form of the strictness of the inequalities since the focus is on computing the largest sector.

of p_α and q_α, and for these finitely many candidates checking non-negativity of inequality of $p_\alpha(\omega)$ gives us the required minimum value of α.

Remark 5: Notice that the above method works even when $G(\xi)$ is not stable, unlike the IQC result. The analogue of the above method for optimizing a slightly different supply rate gives rise to another method to compute the $\mathcal{H}_\infty$ norm of a transfer matrix. This has been dealt in [2]. Another important point to note is that we have addressed the optimization of the sector *with respect to* the circle criterion. We do not address the issue that for certain cases, the circle criterion can itself be unnecessarily conservative; for example, if non-quadratic storage is allowed (see [3], [4]). We thank Prof. Jan C. Willems for his useful inputs.

The following example shows how the minimum α computed by the above method gives rise to the circle that touches the Nyquist plot of the unstable transfer function $G(s)$, and is encircled twice anti-clockwise by the Nyquist plot (due to two unstable poles of $G(s)$).

Example 6: Consider $G(s) = \frac{(-s^2+19s+6)}{2(s-1)(s-2)}$ whose Nyquist plot is shown in figure 2. Let $\beta = 5/3$, which happens to result in closed loop stability (for the gain β). In this case we have $Y(s) = (-s^2 + 19s + 6)$ and $U(s) = (2s^2 - 6s + 4)$. Inequality (5) gives $(4 - 2\alpha)\omega^4 + (2998\alpha - 1100)\omega^2 + (504\alpha + 336) \geqslant 0$ for all $\omega \in \mathbb{R}$. In this case, we get $\alpha = 0.34$ as the required minimum α corresponding to $\beta = 5/3$; the circle corresponding to this pair (α, β) is shown in Figure 2. (The other value of α obtained by the above method corresponds to a non-real frequency ω, and hence is ignored.)

V. Improper controllers for $\mathcal{L}_\infty$ control

In this section we will address the solvability of the $\mathcal{L}_\infty$ (or $\mathcal{H}_\infty$) control problem when the restrictive regularity assumptions on the "feed-through" terms of the plant are relaxed. The regularity assumptions are required in order to have the controller in the conventional observer-state-feedback structure, which is equivalent to the properness of the controller transfer function. These assumptions are restrictive in the sense that even when they are violated, which can make the $\mathcal{L}_\infty$ optimal control problem unsolvable with a proper controller, an improper controller might still exist that succeeds in making the controlled system achieve the optimal $\mathcal{L}_\infty$ norm. Our main result provides necessary and sufficient conditions for the solvability of the $\mathcal{L}_\infty$ control problem without making any assumptions implying properness of the controller transfer function. However, before we state our main result we give a simple motivating example to show that situations where the controller has improper transfer function comes quite naturally in linear systems.

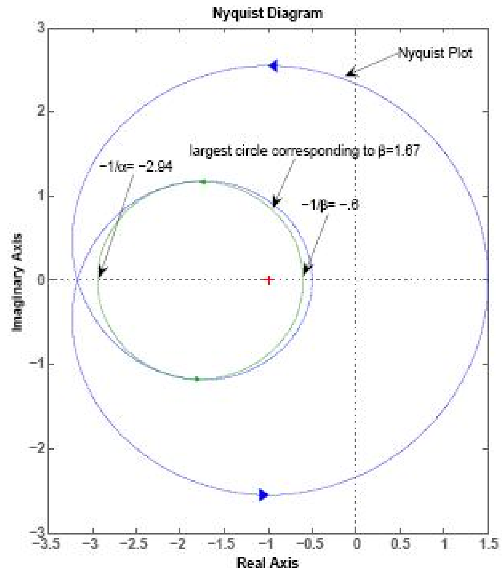

Fig. 2. Nyquist Plot

Example 7: Consider the state-space description of the plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d,$$

with to-be-regulated variables z_1 and z_2 satisfying $z_1 = x_1$ and $z_2 = u$, and the measurement y satisfying $y = x_1$. It can be checked that a state-space controller cannot restrict this plant to a controlled behavior whose $\mathcal{H}_\infty$ norm is at most one. However, a controller of the form $u = \dot{y}$, which is improper, solves the problem.

It is important to note that the closed loop system and the open loop system are both of dynamic order two. This was possible only because the transfer function from disturbance d to the measurements y was strictly proper. Of course, from Theorem 10 of [16, part II], it is expected that this transfer function's strict properness is necessary for controller's improperness.

Through this observation we notice that more general conditions are expected to be necessary and sufficient for solvability.

It is well-known that, in state space $\mathcal{L}_\infty$ optimal control, invariant zeros of the system plays an important role in determining the solvability of the problem. It almost always remains as a standing assumption that the system has no invariant zeros on the imaginary axis (see [12]). Interestingly our main result is very much reminiscent of the invariant zeros condition. It turns out that these conditions are related quite expectedly to certain stabilizability and detectability conditions (see remark 11). However, note that no input/output partitions are assumed on the control variables, and as shown by the above example, improper controllers are pretty easily accommodated by $\mathcal{L}_\infty$ (and hence eventually $\mathcal{H}_\infty$) controllers.

Our description of the plants is similar to that in [16]. The system variables are partitioned into exogenous disturbance d, to be regulated output z and control variable c. The full-behavior of the plant is denoted here by $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{d}+\mathtt{z}+\mathtt{c}}$. The associated plant behavior $\mathcal{P}$ is obtained by eliminating c from $\mathcal{P}_{\text{full}}$, which is defined as: $\mathcal{P} := \{(d,z) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{z}}) \mid \exists c \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{c}}) \text{ such that } (d,z,c) \in \mathcal{P}_{\text{full}}\}$. The control objective is to restrict this plant behavior to a sub-behavior $\mathcal{K}$ to meet the control specifications. In such a formulation of the control problem the controller is allowed to put in restrictions on the control variable c only. In $\mathcal{H}_\infty$ control, the specification is given in terms of the dissipativity on $\mathbb{R}_-$ of the controlled behavior with respect to a real constant matrix

$$\Sigma_\gamma := \begin{bmatrix} \gamma^2 I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{z}} \end{bmatrix} \tag{7}$$

and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma_\gamma)$ (see [16] for a detailed formulation of the problem). It is shown in [16] that a controlled behavior, $\mathcal{K}$ with the controller putting restrictions only on the control variables exists if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$, where $\mathcal{N}$, called the 'hidden behavior' is given by,

$$\mathcal{N} := \{(d,z) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{z}}) \mid (d,z,0) \in \mathcal{P}_{\text{full}}\}. \tag{8}$$

If we relax the condition of dissipativity on $\mathbb{R}_-$ to that on $\mathbb{R}$, we get the corresponding $\mathcal{L}_\infty$ control problem. In this section we restrict ourselves to the case of $\mathcal{L}_\infty$ control only. The $\mathcal{L}_\infty$ problem is said to be solvable for a plant if there exists a controlled behavior $\mathcal{K}$ as above and some positive real γ such that $\mathcal{K}$ is Σ_γ dissipative on $\mathbb{R}$ and $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma_\gamma)$. We now state a portion of Theorem 7.2.1 from [1], which we will utilize to prove our claims. We state this as a proposition below.

Proposition 8: The $\mathcal{L}_\infty$ control problem is solvable if and only if

- $\mathcal{N}$ is Σ_γ dissipative on $\mathbb{R}$ and
- $\mathcal{P}^\perp$ is $-\Sigma_\gamma^{-1}$ dissipative on $\mathbb{R}$.

We are now in a position to state our main result of this section, which gives necessary and sufficient conditions for the $\mathcal{L}_\infty$ control problem to be solvable, purely in terms of the kernel representation of the plant behavior devoid of any explicit dissipativity conditions. An interesting feature of the following result is that it can be related to well-known systems theoretic concepts of stabilizability and detectability, but it assumes no conditions implying controller properness. Let the full plant behavior $\mathcal{P}_{\text{full}}$ be given by the following kernel representation:

$$\mathcal{P}_{\text{full}} := \{(d,z,c) \mid R_{\text{d}}(\tfrac{\text{d}}{\text{d}t})d + R_{\text{z}}(\tfrac{\text{d}}{\text{d}t})z + R_{\text{c}}(\tfrac{\text{d}}{\text{d}t})c = 0\}. \tag{9}$$

The following is a kernel representation of the plant behavior $\mathcal{P} \subseteq \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{z}})$ associated with this full behavior.

$$\mathcal{P} := \{(d,z) \mid R_{\text{de}}(\tfrac{\text{d}}{\text{d}t})d + R_{\text{ze}}(\tfrac{\text{d}}{\text{d}t})z = 0\}. \tag{10}$$

The theorem below is the main result of this section.

Theorem 9: Consider the kernel representation of the full plant behavior as in equation (9). The associated plant behavior $\mathcal{P}$ be given by equation (10). Suppose the hidden behavior $\mathcal{N}$ and the plant behavior $\mathcal{P}$ are controllable. Then the $\mathcal{L}_\infty$ control problem is solvable if and only if the following four conditions below are true.

1) $R_{\text{z}}(\lambda)$ has full column rank for all $\lambda \in i\mathbb{R}$.
2) There exists a partitioning of d into (d_1, d_2) such that d_1 is input and (d_2, z) is output for $\mathcal{N}$ and the corresponding transfer function from d_1 to (d_2, z) is proper.
3) $R_{\text{ze}}(\lambda)$ is full row rank for every $\lambda \in i\mathbb{R}$.
4) There exists a partitioning of z into (z_1, z_2) such that (d, z_1) is input and z_2 is output for $\mathcal{P}$ and the corresponding transfer function from (d, z_1) to z_2 is proper.

See [7] for proof, which we skip due to page limit constraints. However, we state the following lemma about polynomial matrices, which plays a crucial role in our proof of Theorem 9. See [7] for a proof of the lemma. We need the notion of column zeros of a polynomial matrix $R(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$: define $\text{colzeros}(R(\xi)) := \{\lambda \in \mathbb{C} \mid \exists\, 0 \neq v \in \mathbb{C}^{\mathtt{q}}$ such that $R(\lambda)v = 0\}$. In case $R(\xi)$ is not full column rank, $\text{colzeros}(R(\xi))$ turns out naturally to be the whole of $\mathbb{C}$. Otherwise, it is a finite set.

Lemma 10: Consider $R(\xi) := \begin{bmatrix} R_1(\xi) & R_2(\xi) \end{bmatrix}$, with $R_1(\xi) \in \mathbb{R}^{(\mathtt{d}+\mathtt{z}-1)\times\mathtt{d}}[\xi]$ and $R_2(\xi) \in \mathbb{R}^{(\mathtt{d}+\mathtt{z}-1)\times\mathtt{z}}[\xi]$. Let $M(\xi) := \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix}$, with $M_1(\xi) \in \mathbb{R}^{\mathtt{d}\times 1}[\xi]$ and $M_2(\xi) \in \mathbb{R}^{\mathtt{z}\times 1}[\xi]$ be such that $R(\xi)M(\xi) = 0$ and $M(\lambda)$ full column rank for all $\lambda \in \mathbb{C}$. Then,
1) $\text{colzeros}(M_1(\xi)) \subseteq \text{colzeros}(R_2(\xi))$.
2) In particular, if $R_2(\xi)$ is full column rank then so is $M_1(\xi)$.
3) Further, if $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, then $\text{colzeros}(R_2(\xi)) = \text{colzeros}(M_1(\xi))$.

Remark 11: Condition 1) of the above theorem says that $\mathcal{L}_\infty$ control is solvable only if R_{z} is full column rank on the imaginary axis. This is equivalent to the to-be-regulated

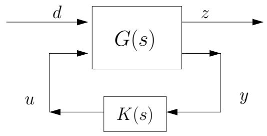

Fig. 3. Standard control problem

variable z being detectable from the rest of the system variables (see [9] for detectability) on the imaginary axis. On the other hand condition 3) states that for the $\mathcal{L}_\infty$ control problem to be solvable, it is also necessary that z be stabilizable on the imaginary axis through the control the variable c. This is nothing but the stabilizability on the imaginary axis of just the z variables through the control variable c, of the unforced plant defined as: $\mathcal{P}_{\text{full,unforced}} := \{(z,c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{z}+\mathtt{c}}) \mid (0,z,c) \in \mathcal{P}_{\text{full}}\}$.

VI. $\mathcal{L}_\infty$ OPTIMAL CONTROL

In this section we address the problem of solving an $\mathcal{L}_\infty$ optimal control problem, *i.e.*, find a controller that minimizes the $\mathcal{L}_\infty$ norm of the closed loop system in the following configuration. The results of the previous section allow us to conclude, as shown below, that if the (suboptimal) $\mathcal{L}_\infty$ control problem is solvable for some γ, then, in fact, the optimal control problem too is solvable.

Recall the definition of Σ_γ from equation (7). Like the supply rate for sector nonlinearities (in section IV) where there was a parameter α to be extremized, we have here the parameter γ to be minimized. The key issue here is that if a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is dissipative with respect to Σ_γ for some $\gamma > 0$, then one can find the minimum γ such that this dissipativity holds. In order to compute this γ, one uses the method described in [2]. Define $\gamma_{\mathcal{N}} := \min_{\gamma \in \mathbb{R}_+} \{\mathcal{N} \text{ is } \Sigma_\gamma \text{ dissipative}\}$, as the minimum γ for which $\mathcal{N}$ is Σ_γ dissipative. Similarly, define $\gamma_{\mathcal{P}}$ as the minimum γ such that $\mathcal{P}^\perp$ is $-(\Sigma_\gamma)^{-1}$ dissipative. (The minima exist and are finite if the corresponding conditions in Theorem 9 are satisfied.) Using these values, one can in fact solve the *optimal* $\mathcal{L}_\infty$ control problem, as stated in the following theorem. The proof is skipped since it follows from the main theorem of the previous section, suitably combined with the algorithm to compute the minimum γ values; see [2] for the algorithm.

Theorem 12: Consider $\mathcal{N}$ and $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, the hidden and the plant behaviors of a system. Suppose the $\mathcal{L}_\infty$ control problem is solvable for some $\gamma > 0$, equivalently, the necessary and sufficient conditions listed in Theorem 9 are satisfied. Then the $\mathcal{L}_\infty$-optimal control problem is also solvable. The optimal γ value is $\gamma_{\text{opt}} = \max(\gamma_{\mathcal{N}}, \gamma_{\mathcal{P}})$

One of the important consequences of the above theorem is that the optimal controller always exists. This is possible essentially because, for a state space way of addressing such optimal control problems, the issue of nonproperness at optimality is not addressable.

VII. CONCLUDING REMARKS

We first proved how gain and phase margin conditions for stability are nothing but dissipativity with respect to a convex-like combination of two important supply rates that arise from small-gain theorem and the passivity theorem. We then showed that the sector for circle criterion can be improved to its largest using our proposed systematic method. We stated necessary and sufficient conditions for $\mathcal{L}_\infty$-control, which turn out to relax properness conditions on the controller (had an apriori input/output partition been fixed for the control variables). As an important consequence of this, we concluded that optimal $\mathcal{L}_\infty$-control problem admits a solution whenever the suboptimal case admits one.

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