Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

New results in dissipativity of uncontrollable systems and Lyapunov functions

Debasattam Pal Madhu N. Belur

Abstract— **Dissipative systems have played an important role in the analysis and synthesis of dynamical systems. The commonly used definition of dissipativity often requires an assumption on the controllability of the system. However, it is very natural to think of Lyapunov functions as storage functions for autonomous systems with power supplied to the system equal to zero. We use a definition of dissipativity that is slightly different (and less often used in the literature) to study a linear, time-invariant, possibly uncontrollable dynamical system. This paper contains various results in the context of uncontrollable dissipative systems that smoothly bridge the gap between storage functions for controllable dissipative systems and Lyapunov functions for autonomous systems. We provide a necessary and sufficient condition for an uncontrollable system to be strictly dissipative with respect to a supply rate under the assumption that the uncontrollable poles are not "mixed"; i.e., no pair of uncontrollable poles is symmetric about the imaginary axis: this condition is known to be related to the solvability of a Lyapunov equation. We show that for an uncontrollable system the set of storage functions is unbounded, and that the unboundedness arises precisely due to the set of Lyapunov functions for an autonomous linear system being unbounded. Further, we show that stabilizability of a system results in this unbounded set becoming bounded from below. Positivity of storage functions is known to be very important for stability considerations because the maximum stored energy that can be drawn out is bounded when the storage function is positive. In this paper we establish the link between stabilizability of an uncontrollable system and existence of positive definite storage functions. In the context of autonomous systems, we prove that the Lyapunov operator is onto if and only if its image has observable symmetric rank one matrices.**

Index Terms— **dissipativity, uncontrollability, storage functions, behaviors, Lyapunov equation**

I. Introduction

Dissipativity of dynamical systems helps in the analysis and design of control systems. An important assumption in some of these developments is that of controllability of the dynamical system. In this paper we study dissipativity of general linear time-invariant systems, possibly uncontrollable.

Dissipativity of a system is about the absence of any source of energy within the system, and hence all interactions with the environment have to satisfy the condition that the "net energy" is directed inwards. This is made precise below in Definition 3.1. For example, a passive electrical network made out of passive circuit elements must continue to be dissipative even if it loses controllability. In this paper we consider a general linear time-invariant system and work on a theory of dissipativity free from any controllability assumption. Our work is based on the signature characteristic of a dissipative system to store energy, i.e., existence of a storage function. An important issue that immediately arises is whether to include unobservable variables to describe this storage of energy (see [10]). Our main result sorts out this issue: for the case of strict dissipativity, we show that a storage function depending only on the manifest variables suffices, and no unobservable variables are necessary.

The present theory of dissipative systems is well-developed primarily for controllable systems because it is possible there to define dissipativity without taking recourse to the existence of a storage function. This is done using an integral inequality involving only the compactly supported trajectories allowed by the system. This definition turns out to be inadequate for a general, possibly uncontrollable, linear behavior. In order to overcome this inadequacy, there has been prior work of taking existence of storage functions satisfying a dissipation inequality as a definition of dissipativity; see [7], for example. In this paper we further develop using this definition. The principal finding is that a certain condition on the uncontrollable poles, which we call the *unmixing* condition, plays a key role. If no pair of the uncontrollable poles of the system is symmetric with respect to the imaginary axis, then the noncontrollability poses no hindrance to strict dissipativity, i.e., the strict dissipativities of the behavior and its controllable part are equivalent (Theorem 3.4). This result is utilized to show useful identities about positive storage functions and unboundedness of the set of storage functions for the case of uncontrollability.

The paper is structured as follows. The rest of this section has a few words about the notation we follow. Section II contains some preliminaries we require regarding behavioral theory. The next section (section III) has some definitions that we need in order to state the main result of this paper. Here we also present the main result: a necessary and sufficient condition for a general linear time-invariant system to be strictly dissipative with respect to a supply rate that depends on the manifest variables, under the assumption that the set of uncontrollable poles satisfies the unmixing condition. Interestingly, this unmixing condition on the uncontrollable poles is reminiscent of the solvability condition of Lyapunov equations: this is elaborated in sections III and VII. In section VI we present some insight on the nature, namely, unboundedness of the set of all storage functions of an uncontrollable dissipative behavior. (The set of storage functions is known to be *bounded* in the case of controllability.) Section VII explores into the extent of necessity of the unmixing property that we have assumed throughout this paper. In this section we show an interesting result about rank one symmetric matrices and solvability of the Lyapunov

Department of Electrical Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India. debpal@ee.iitb.ac.in, belur@ee.iitb.ac.in.

The research was supported in parts by IITB-ISRO-STC and SERC division, DST India.

equation.

The notation we follow is standard. $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers. The ring of polynomials in ξ with real coefficients is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ stands for the set of $\mathtt{p}\times\mathtt{w}$ matrices with entries from $\mathbb{R}[\xi]$. Likewise $\mathbb{R}[\zeta,\eta]$ denotes the set of real polynomials in the indeterminates ζ and η; $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ stands for the set of $\mathtt{w}\times\mathtt{w}$ matrices with entries in $\mathbb{R}[\zeta,\eta]$. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\mathtt{w})$ denotes the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^\mathtt{w}$, and $\mathfrak{D}(\mathbb{R},\mathbb{R}^\mathtt{w})$ denotes its subspace of all compactly supported trajectories. We use $\bullet$ when it is unnecessary to specify a dimension. For example, $R\in\mathbb{R}^{\bullet\times\mathtt{w}}$ means R is a real matrix with $\mathtt{w}$ columns. When dealing with many variables, in order to keep track of the dimensions, we use the same letter as a generic variable w, but in typewriter font $\mathtt{w}$, to denote the number of components; for example, $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\mathtt{w})$. In the context of stability, we require certain regions of the complex plane $\mathbb{C}$. The open left and right half complex planes are denoted by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. To improve readability within text, we use $\operatorname{col}(\cdot,\cdot)$ to stack its arguments into a column, i.e., $\operatorname{col}(w_1,w_2)=[w_1^T \quad w_2^T]^T$.

For page limit constraints, we have omitted the proofs of the main results; the proofs can be found in [2].

II. Behaviors, QDFs, and state representations

A linear differential behavior $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\mathtt{w})$ consisting of solutions to a set of ordinary linear differential equations with constant coefficients; i.e.,

$$\mathfrak{B}:=\left\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\mathtt{w})\mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w=0\right\},$$

where $R(\xi)$ is a polynomial matrix having $\mathtt{w}$ number of columns: $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. We shall denote the set of linear differential behaviors with $\mathtt{w}$ number of variables by $\mathfrak{L}^\mathtt{w}$. The linear differential behavior $\mathfrak{B}\in\mathfrak{L}^\mathtt{w}$ can also be written as $\mathfrak{B}=\ker\, R(\frac{\mathrm{d}}{\mathrm{d}t})$. That is why this representation is called a *kernel representation* of $\mathfrak{B}$. We call w the *manifest variable*; these are the variables of interest. In this paper, w is the variable through which the system exchanges energy with the environment. It turns out that we can assume, without loss of generality, that $R(\xi)$ is of full row rank (see [4]); in this paper, a kernel representation matrix $R(\xi)$ is assumed to be of full row rank. For a behavior $\mathfrak{B}=\ker\, R(\frac{\mathrm{d}}{\mathrm{d}t})$, the row rank of $R(\xi)$ gives the output cardinality (number of outputs in the system). Though the variables w can often be partitioned into inputs and outputs in more than one way, the output cardinality remains the same: $\operatorname{rank} R$. Further, the cardinality does not depend on the R used to define it, but depends only on $\mathfrak{B}$. In this sense, the output cardinality is an integer invariant of $\mathfrak{B}$ and we denote it by $\mathtt{p}(\mathfrak{B})$. The number of inputs to the system, the input cardinality, is another integer invariant of $\mathfrak{B}$. This integer is denoted by $\mathtt{m}(\mathfrak{B})$ and is calculated using $\mathtt{m}(\mathfrak{B})=\mathtt{w}-\mathtt{p}(\mathfrak{B})$, where $\mathtt{w}$ is the number of components in the manifest variable w.

A concept of central importance for this paper is that of controllability. A behavior $\mathfrak{B}\in\mathfrak{L}^\mathtt{w}$ is said to be *controllable* if for every $w',w''\in\mathfrak{B}$, there exists a $w\in\mathfrak{B}$ and a $\tau>0$ such that

$$\begin{aligned} w(t) &= w'(t) \quad \text{for all } t\leqslant 0,\\ &= w''(t) \quad \text{for all } t\geqslant\tau. \end{aligned}$$

We denote the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^\mathtt{w}_{\mathrm{cont}}$. A behavior $\mathfrak{B}=\ker\, R(\frac{\mathrm{d}}{\mathrm{d}t})$ is controllable if and only if $R(\lambda)$ does not lose rank for any $\lambda\in\mathbb{C}$.

In the context of uncontrollable systems, we use the key notion of *uncontrollable poles* and *uncontrollable characteristic polynomial*. Suppose $\mathfrak{B}=\ker\, R(\frac{\mathrm{d}}{\mathrm{d}t})$ and suppose $\mathfrak{B}$ is *not* controllable. Then there exist one or more complex numbers λ such that $R(\lambda)$ loses rank. These complex numbers, together with multiplicities, are defined as uncontrollable poles in the definition below. Uncontrollable poles are the roots of a monic polynomial called the uncontrollable characteristic polynomial. Definition 2.1 below is for easy reference of this definition.

Definition 2.1: Let $R\in\mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ have full row rank and suppose $R(\frac{\mathrm{d}}{\mathrm{d}t})w=0$ is a kernel representation for $\mathfrak{B}$. Consider a factorization of R into $R(\xi)=F(\xi)R_{\mathrm{cont}}(\xi)$ such that $R_{\mathrm{cont}}\in\mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$, $R_{\mathrm{cont}}(\lambda)$ has full row rank for every complex number λ, and $\det F$ is a monic polynomial. The *uncontrollable characteristic polynomial* of $\mathfrak{B}$, denoted by $\chi_{\mathrm{un}}(\mathfrak{B})$, is defined as $\det F$. The set of *uncontrollable poles* is defined as $\operatorname{roots}(\chi_{\mathrm{un}})$, and is denoted by $\Lambda_{\mathrm{un}}(\mathfrak{B})$.

If the behavior $\mathfrak{B}$ is clear from the context, we write just χ_{un} and Λ_{un}. Notice that if $\mathfrak{B}$ is controllable, then $\chi_{\mathrm{un}}=1$. When a behavior is not controllable, we often require the *controllable part* of $\mathfrak{B}$. This is the largest controllable behavior contained in $\mathfrak{B}$; the controllable part of $\mathfrak{B}$ is denoted by $\mathfrak{B}_{\mathrm{cont}}$. Consider the above definition in which R has been factorized as described to obtain R_{cont}. A kernel representation for $\mathfrak{B}_{\mathrm{cont}}$ is induced by R_{cont}. For a detailed exposition on behaviors, controllability, and uncontrollable characteristic polynomial, we refer the reader to [4].

This paper deals with dissipativity and in this context we deal with quadratic forms in the system variables and a finite number of their derivatives. It turns out to be very natural to associate two variable polynomial matrices to such quadratic forms. Consider a two variable polynomial matrix $\Phi(\zeta,\eta):=\sum_{i,k}\Phi_{ik}\zeta^i\eta^k\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, where $\Phi_{ik}\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. A Quadratic Differential Form (QDF) Q_Φ induced by $\Phi(\zeta,\eta)$ is a map $Q_\Phi:\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\mathtt{w})\to\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined by

$$Q_\Phi(w):=\sum_{i,k}\left(\frac{\mathrm{d}^i w}{\mathrm{d}t^i}\right)^T\Phi_{ik}\left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right).$$

A quadratic form induced by a real symmetric constant matrix $S\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is a special QDF and we shall often need this in this paper. Throughout this paper, ample use is made of the well-developed theory of QDFs; only the essential results of which are reviewed here. See [7] for a thorough and complete treatment on QDFs.

III. Dissipative systems: Definition and main result

Dissipative systems are those that have no source of energy within, and hence any energy stored within the system has to have been supplied from its environment. This intuitive physical concept was made concrete in [9], [7] using the *dissipation inequality*: at any time instant, the rate of increase of stored energy is at most the power supplied to the system. In this paper, the power supplied and the stored energy are both QDFs in the manifest variables w of the system. In this paper we use the following definition of dissipativity; its relation to other definitions is discussed below.

Definition 3.1: A linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *dissipative* with respect to supply rate $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ if there exists a quadratic differential form $Q_\Psi(w)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_S(w) \text{ for all } w \in \mathfrak{B}. \quad (1)$$

The quadratic differential form Q_Ψ is called a *storage function* for $\mathfrak{B}$ with respect to the supply rate S.

The inequality (1) above is called the dissipation inequality. In some control problems like in LQR and the suboptimal $\mathcal{H}_\infty$ control, a stricter notion of dissipativity plays a key role. In this paper we shall deal primarily with strict dissipativity, although many of our results are valid for just dissipativity also. We define strict dissipativity as follows.

Definition 3.2: A linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *strictly dissipative* with respect to $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ if there exists an $\epsilon > 0$ and a storage function $Q_\Psi(w)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_S(w) - \epsilon|w|^2 \quad \text{for all } w \in \mathfrak{B}.$$

Because the above definitions require the existence of a hitherto unknown storage function, it has been common to use an equivalent statement for the definition of (strict) dissipativity when dealing with *controllable* systems. The following result from [7] shows the equivalence.

Proposition 3.3: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Then the following statements are equivalent.

1. There exists a storage function $Q_\Psi(w)$ such that $\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant Q_S(w) - \epsilon|w|^2$ for all $w \in \mathfrak{B}$.
2. The integral inequality $\int_{\mathbb{R}} Q_S(w)\mathrm{d}t \geqslant \epsilon \int_{\mathbb{R}} |w|^2 \mathrm{d}t$ is satisfied for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

The above proposition shows that the existence of a storage function satisfying the dissipation inequality is equivalent to saying that the total energy transferred into the system is strictly positive whenever we start the system from rest and bring the system back to rest. Statement 2 was used as the definition of strict dissipativity in [7]. With $\epsilon = 0$, we get the definition of nonstrict dissipativity given in [7], [8]. It is important to note here that the second statement above holds over only compactly supported trajectories in $\mathfrak{B}$, while the first holds for all $w \in \mathfrak{B}$: controllability of $\mathfrak{B}$ is crucial here. However, for an uncontrollable behavior, Statement 2 of Proposition 3.3 puts no restrictions on the trajectories in the behavior which are outside the controllable part (see [3]), and hence this cannot be used as a definition of dissipativity for uncontrollable systems.

We define signature of a real symmetric nonsingular matrix S, denoted by $\sigma(S)$ as the pair of integers $\sigma(S) = (\sigma_-(S), \sigma_+(S))$, where $\sigma_-(S)$ and $\sigma_+(S)$ are the number of negative and positive eigenvalues of S, respectively. In this paper we shall deal only with the case when the positive signature $\sigma_+(S)$ equals the input cardinality $\mathtt{m}(\mathfrak{B})$ of the behavior $\mathfrak{B}$.

We are now ready to state one of the main results of this paper. The following theorem tells that if a certain unmixing condition is satisfied for the uncontrollable poles, then the controllable part of a behavior being strictly dissipative is equivalent to the existence of a storage function for the whole behavior's strict dissipativity. Recall from Definition 2.1 that the uncontrollable characteristic polynomial χ_{un} of $\mathfrak{B}$ is the monic polynomial whose roots (with suitable multiplicities) are those complex numbers where $R(\xi)$ (of a minimal kernel representation) loses rank. Theorem 3.4 below states that if the uncontrollable poles are such that no pair of the uncontrollable poles is symmetric with respect to the imaginary axis, then noncontrollability of $\mathfrak{B}$ poses no hindrance to strict dissipativity of $\mathfrak{B}$; i.e., strict dissipativities of $\mathfrak{B}$ and $\mathfrak{B}_{\mathrm{cont}}$ are equivalent.

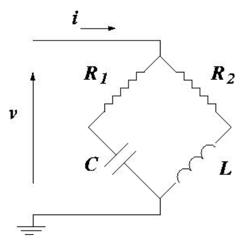

Fig. 1. An LCR circuit.

Theorem 3.4: Consider a linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and a nonsingular $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ with the input cardinality of $\mathfrak{B}$ equal to the positive signature of S: $\mathtt{m}(\mathfrak{B}) = \sigma_+(S)$. Assume that the uncontrollable characteristic polynomial of $\mathfrak{B}$, χ_{un}, is such that $\chi_{\mathrm{un}}(\xi)$ and $\chi_{\mathrm{un}}(-\xi)$ are coprime. Then, $\mathfrak{B}$ is strictly S-dissipative if and only if its controllable part $\mathfrak{B}_{\mathrm{cont}}$ is strictly S-dissipative.

We call the condition of coprimeness of $\chi_{\mathrm{un}}(\xi)$ and $\chi_{\mathrm{un}}(-\xi)$ the *unmixing condition*. In the context of autonomous systems, it is well known that the unmixing condition is a necessary and sufficient condition for the existence of a unique solution to the Lyapunov equation.

Throughout this paper, we shall assume S has the following form:

$$\Sigma := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}. \quad (2)$$

This is without loss of generality using Sylvester's law of inertia. The following sections relate the above main result to properties of storage functions and Lyapunov functions.

IV. Dissipativity of uncontrollable behaviors

We first consider an example of a simple electrical circuit as shown in Figure 1. Under the condition $R_1 C \neq L/R_2$, the port variables (manifest variables) (v, i) satisfy the following differential equation:

$$R(\frac{\mathrm{d}}{\mathrm{d}t}) \begin{bmatrix} v \\ i \end{bmatrix} = 0,$$

where $R(\xi) = [p(\xi) \;\; q(\xi)]$ with $p(\xi) = LC\xi^2 + (R_1 + R_2)C\xi + 1$ and $q(\xi) = -\big(R_1 LC\xi^2 + (R_1 R_2 C + L)\xi + R_2\big)$.

For the case that $R_1 C = L/R_2$, and $R_1 = R_2$, the system becomes uncontrollable. The corresponding kernel representation is

$$\left[\left(R_2 C\frac{\mathrm{d}}{\mathrm{d}t} + 1\right) - \left(L\frac{\mathrm{d}}{\mathrm{d}t} + R_2\right)\right] \begin{bmatrix} v \\ i \end{bmatrix} = 0.$$

If the voltage across the capacitor v_C and current through the inductor i_L are considered as internal system variables, then we can write the following dissipation inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Cv_C^2 + Li_L^2\right) \leqslant \begin{bmatrix} v & i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}.$$

However, it turns out that the latent variables (v_C, i_L) are not observable from (v, i), and so the storage function in the left-hand side of above inequality cannot be written in terms of a QDF in just the manifest variables. We ask the question: is it possible to find a storage function in terms of the manifest variables, or do we have to have, for some cases, storage functions in terms of "hidden" variables only (variables that are unobservable from the manifest variables are also said to be hidden)? Our main result Theorem 3.4 addresses this issue under the unmixing assumption, and gives a necessary and sufficient condition for the existence of a storage function in terms of manifest variables. Thus Theorem 3.4 rules out the necessity of hidden variables to construct storage functions.

For the case of the above example, as derived in [10], $q(v - R_1 i)^2$ with any $q > 0$ is a storage function, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} q(v - R_1 i)^2 \leqslant vi,$$

which is a dissipation inequality in just the manifest variables. This storage function has no apparent interpretation as physical energy (see [2, Remark 5.1]). Further, $q > 0$ makes the set of storage functions unbounded for this case: in section VI we shall prove the unboundedness for general uncontrollable behaviors.

Remark 4.1: Dissipativity is closely related with solvability of a class of LMIs. and to algebraic Riccati inequalities/equations (abbreviated as ARI/ARE); see [5], [2]. The question of solvability of the positive-real LMI without imposing system theoretic assumptions like controllability or observability has been dealt with in [1]. However, a very restrictive assumption made there is that the whole set of eigenvalues of the system matrix A satisfies the unmixing property, i.e., $\mathrm{spec}(A) \cap (\mathrm{spec}(-A)) = \phi$. According to our main result (Theorem 3.4) this assumption is not necessary. It is sufficient that only the uncontrollable poles satisfy the unmixing property. We shall see later in section VII the extent of necessity of this unmixing property. The following example shows how the positive-real LMI is solvable when some elements of $\mathrm{spec}(A)$ have symmetry with respect to the imaginary axis and the system is uncontrollable.

Example 4.2: Consider an i/s/o system with the following A, B, C, D matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 1.$$

Observe that $\mathrm{spec}(A) = \{1, -1\}$, which is symmetric with respect to the imaginary axis. Here $\Lambda_{\mathrm{un}} = \{-1\}$, and the other eigenvalue ($= 1$) is controllable. An equivalent kernel representation of the manifest behavior is given by

$$\begin{bmatrix} \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\mathrm{d}}{\mathrm{d}t} - 2\right) & -\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 1\right) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.$$

We ask the question: is this i/s/o system dissipative with respect to $S := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or equivalently, is there a real symmetric solution $K = K^T \in \mathbb{R}^{2\times 2}$ for the following LMI (the positive-real LMI)

$$\begin{bmatrix} -A^T K - KA & C^T - KB \\ C - B^T K & D + D^T \end{bmatrix} \geqslant 0?$$

Obviously, $\Lambda_{\mathrm{un}} = \{-1\}$ satisfies the unmixing property, and one can check that the controllable part $\mathfrak{B}_{\mathrm{cont}} = \ker \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} - 2 & \frac{\mathrm{d}}{\mathrm{d}t} + 1 \end{bmatrix}$ is strictly S-dissipative, which from Theorem 3.4 implies that $\mathfrak{B}$ is strictly S-dissipative. This can be verified by checking that the following real symmetric matrix induces a storage function that satisfies the dissipation inequality

$$K = \begin{bmatrix} -0.957 & -1.457 \\ -1.457 & -1.957 \end{bmatrix},$$

and therefore solves the LMI.

V. Positive storage functions and stabilizability

In this section we establish an important link between stabilizability of systems and positive definiteness of storage functions of strictly dissipative systems. The importance of this link lies in the fact that the energy stored in physical systems is a nonnegative quantity and dissipative physical systems satisfy an additional property that, if the system was initially discharged, then the net energy supplied into the system *upto any time instant* is nonnegative; this is called *half-line dissipativity*. We review these concepts (from [7]) below and prove similar results for uncontrollable systems in this section.

For this paper, we need half-line dissipativity for only the negative half of the real line: $\mathbb{R}_-$. A controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$ is said to be Σ-dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^{0} Q_\Sigma(w)\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Half-line dissipativity is related to (semi-)definiteness of the storage function. A storage function Q_Ψ is called nonnegative if $Q_\Psi(w)(t) \geqslant 0$ for all $t \in \mathbb{R}$ and $w \in \mathfrak{B}$. For controllable behaviors, it was shown in [7] that existence of a nonnegative storage function is equivalent to dissipativity of $\mathfrak{B}$ on $\mathbb{R}_-$. The importance of nonnegative storage functions is due to such functions being bounded from below (namely, by zero), because of which we expect that when the supply of energy is stopped, then the trajectories cannot become unbounded. This link to stability was made precise and proved in [8, Proposition 1, Part I].

It is known (see [6] or [2, Corollary 5.6]) that every storage function is a static function of the states of the system, i.e. for a dissipative behavior $\mathfrak{B}$ with a minimal state map $X \in \mathbb{R}^{\mathrm{n}\times\mathrm{w}}[\xi]$, a storage function Q_Ψ is associated to a symmetric matrix $K \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ such that $Q_\Psi(w) = (X(\frac{\mathrm{d}}{\mathrm{d}t})w)^T K X(\frac{\mathrm{d}}{\mathrm{d}t})w$. Hence Q_Ψ is nonnegative if and only if $K \geqslant 0$ (see [7]). In the context of strict dissipativity, we define a *positive definite* storage function: a storage function Q_Ψ is called positive definite if $K > 0$.

The following result is one of the main results of this paper. It relates existence of positive definite storage functions to stability of the autonomous part of the uncontrollable dissipative behavior. A behavior with a stable autonomous part is nothing but a *stabilizable* behavior. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ is called stabilizable if for every $w \in \mathfrak{B}$, there exists a $w' \in \mathfrak{B}$ such that $w(t) = w'(t)$ for $t \leqslant 0$ and $w'(t) \to 0$ as

$t \to \infty$. A behavior is stabilizable if and only if $\Lambda_{\rm un} \subset \mathbb{C}^-$ (see [4]).

Theorem 5.1: Let a linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ be strictly Σ-dissipative with $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then there exists a positive definite storage function if and only if the following are satisfied:

1. there exists $\epsilon > 0$ such that $\int_{\mathbb{R}_-} Q_\Sigma(w)\mathrm{d}t \geqslant \int_{\mathbb{R}_-} \epsilon|w|^2\mathrm{d}t$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$ and
2. $\Lambda_{\rm un} \subset \mathbb{C}^-$.

The first condition is clearly a necessary condition for existence of a positive definite storage function; namely, the controllable part has to be strictly dissipative on $\mathbb{R}_-$. The second condition is also necessary because of the notion that the storage function behaves like a Lyapunov function for an autonomous system, and as is well known, a positive Lyapunov function exists if and only if the autonomous system is asymptotically stable. The fact that these two conditions are together *sufficient* for the existence of a positive definite storage function for the whole behavior is one of the main contributions of this paper. Also notice that $\Lambda_{\rm un} \subset \mathbb{C}^-$ is a very special case of the unmixing condition. Thus the uncontrollability of the stabilizable behavior poses no hindrance to existence of a storage function for strict dissipativity as long as the controllable/autonomous parts allow storage/Lyapunov functions individually. As noted above, this is the principal finding of this paper.

It is shown in [2] that, in fact, *every* storage function for the behavior is positive definite. This has been shown for the controllable case in [7, Theorem 6.4]. Intuitively, storage functions being positive is closer to their interpretation as energy-like functions. Also, the meaning of dissipativity that there is no source of energy in the system appeals to both positive definite storage functions and the stabilizability of the system. In the following section we explore other properties of the set of storage functions, like (un)boundedness of this set.

VI. Set of all storage functions for an uncontrollable system

An important topic of interest is the set of all storage functions of a dissipative behavior. For LQR/LQG theory and $\mathcal{H}_\infty$ control, certain extremum storage functions give stabilizing controllers. In this section we show that the set of storage functions is unbounded for uncontrollable dissipative systems and that for stabilizable systems, this set is bounded from below.

Theorem 6.1: Let $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ be uncontrollable, and suppose the set of its uncontrollable poles $\Lambda_{\rm un}$ satisfies the unmixing property, i.e., $\Lambda_{\rm un} \cap (-\Lambda_{\rm un}) = \phi$. Further, let $\mathfrak{B}$ be strictly Σ-dissipative. Then the set of all storage functions is an unbounded convex set.

The example below shows how the Lyapunov equation due to the behavior's uncontrollability plays a key role in making the set of storage functions unbounded.

Example 6.2: Consider behavior $\mathfrak{B}$ having an i/s/o representation with $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and supply rate $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This system is close to being uncontrollable; it loses controllability when $\epsilon = 0$. Inspection of this almost uncontrollable system helps in obtaining a useful appreciation of the fact that uncontrollability forces the set of all storage functions to become unbounded. With $\epsilon = 0.001$, there are four real symmetric solutions for $K$to the corresponding algebraic Riccati equation are: $K_1 = \begin{bmatrix} 0.268 & 0.366 \\ 0.366 & 0.567 \end{bmatrix}$, $K_2 = \begin{bmatrix} 3.735 & -1.370 \\ -1.370 & 1.437 \end{bmatrix}$, $K_3 = \begin{bmatrix} 0.000 & -0.002 \\ -0.002 & 1.492 \end{bmatrix} \times 10^7$, $K_4 = \begin{bmatrix} 0.00 & 0.04 \\ 0.04 & 107.04 \end{bmatrix} \times 10^4$. The last two solutions are clearly much larger (elementwise) than the first two. The reason for this can be linked to the corresponding 'lambda-sets' being close to inadmissible for the last two cases (see [2] for details). The example brings out the tendency of the set of ARI solutions, and therefore the set of storage functions, to become unbounded as the system loses controllability. At uncontrollability, i.e. when $\epsilon = 0$, the set becomes unbounded and the direction in which the set goes off to infinity is given by $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

A very interesting fact about the set of all storage functions comes up for the case when the behavior is uncontrollable but stabilizable; i.e., the set of uncontrollable poles $\Lambda_{\rm un}$ is contained in the open left half of the complex plane (see the previous section for the definition and related results about stabilizability). We show below that for stabilizability, the set of storage functions, though an unbounded set, is bounded from below. In other words, there exists a storage function $Q_{\Psi\_}$ such that every storage function Q_Ψ satisfies $Q_\Psi(w) - Q_{\Psi\_}(w) \geqslant 0$ for all $w \in \mathfrak{B}$. We state this result as a theorem below.

Theorem 6.3: Let $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ be an uncontrollable, strictly Σ-dissipative behavior. Also assume that the set of uncontrollable poles satisfies $\Lambda_{\rm un} \subset \mathbb{C}^-$. Then the set of all storage functions is bounded from below; i.e., there exists a storage function $Q_{\Psi\_}(w)$ for $\mathfrak{B}$ such that for each storage function $Q_\Psi(w)$ for $\mathfrak{B}$, $Q_{\Psi\_}(w) \leqslant Q_\Psi(w)$ for all $w \in \mathfrak{B}$.

Note the analogy of this result with that for controllable behaviors, where the set of storage functions is *bounded* and has a maximum and a minimum element (see [7, Theorem 5.7]). While we have shown unboundedness for the case of uncontrollability, stabilizability ensures the existence of a minimum element in this unbounded set.

We saw in the previous section that, for a strictly dissipative and stabilizable behavior $\mathfrak{B}$, dissipativity on $\mathbb{R}_-$ of $\mathfrak{B}_{\rm cont}$ assures the existence of positive definite storage functions. Combining this result with the one above, we infer that the lower bound of the set of storage functions is, in fact, positive (see discussions following Theorem 5.1). This formalizes the intuition that such a system is devoid of any energy sources within it, and hence the maximum extractable energy[1] from any given state is bounded.

Using a very similar argument as in the above proof, one can show that if the behavior is antistabilizable, meaning all the uncontrollable poles are unstable, i.e., $\Lambda_{\rm un} \subset \mathbb{C}^+$, then the set of storage functions is bounded *from above*.

VII. Autonomous systems: the Lyapunov equation

As seen in Theorem 3.4, the unmixing property of the uncontrollable poles makes strict dissipativity of the con-

[1]This has been called *available storage* in [7].

trollable part equivalent to that of the whole behavior. As mentioned in the introduction, the unmixing property serves as a sufficient condition for solvability of a Lyapunov equation and the corresponding Lyapunov operator becomes singular when this condition is not satisfied. We shall see in this section that the Lyapunov operator is onto if and only if there exists an observable rank one symmetric matrix in its image.

It is possible to show that, when the eigenvalues of A are mixed, the Lyapunov equation solution, if one exists, need not be symmetric. However, the existence of a nonsymmetric solution guarantees existence of a symmetric solution: if K is a solution to the Lyapunov equation, then so are K^T and $(K + K^T)/2$. With this simple observation we now give a necessary and sufficient condition for the existence of solution to a Lyapunov equation for the special case that the constant term is of rank one. Interestingly, for this case when the constant term is rank 1, the unmixing condition becomes *necessary*.

Theorem 7.1: Let $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and $C \in \mathbb{R}^{\mathrm{p}\times\mathrm{n}}$. Consider the Lyapunov equation $A^TK+KA+C^TC = 0$ with (C, A) pair observable. Assume $\mathrm{rank}(C^TC) = 1$. Then there exists a solution K to the Lyapunov equation if and only if $\mathrm{spec}(A)\cap \mathrm{spec}(-A) = \phi$.

It is well known that the unmixing condition is equivalent to existence and uniqueness of solution to the Lyapunov equation. In other words, the unmixing condition is equivalent to the image of the Lyapunov operator containing *all* symmetric matrices. The above theorem shows that unmixing is necessary and sufficient for the image to contain a symmetric matrix of rank one (satisfying observability conditions). equation is not solvable.

That the unmixing is not necessary in general for more than one output is quite expected. The following example gives one such simple instance.

Example 7.2: Consider the autonomous system with i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax$, $y = Cx$, where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Observe that A has "mixed" eigenvalues, i.e., $\Lambda_{\mathrm{un}} \cap (-\Lambda_{\mathrm{un}}) \neq \phi$. Σ-dissipativity of such an autonomous system together with $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$ is equivalent to existence of a real symmetric solution to the following Lyapunov inequality: $A^TK+KA+C^TC \leqslant 0$. Notice that $K = \begin{bmatrix} 2+b & a \\ a & -2-c \end{bmatrix}$ with $a, b, c \in \mathbb{R}$ and $b, c \geqslant 0$ gives a solution to the above Lyapunov inequality. This example shows that the unmixing condition of uncontrollable poles is not necessary for the system to be dissipative.

VIII. Concluding remarks

In this paper we studied dissipativity for a general, possibly uncontrollable, LTI system. Our starting point was a more appropriate, though less often used, definition of dissipativity in terms of a differential inequality called the dissipation inequality. With this definition we brought out an equivalence between the dissipativities of a behavior and its controllable part, under the important unmixing condition (Theorem 3.4). For the case of strict dissipativity, Theorem 3.4 also settles the issue of whether to allow unobservable variables in the defining dissipation inequality: the theorem rules out the requirement of unobservable variables. The important intuitive idea that storage of energy should take place through the state variables comes as a natural consequence of Theorem 3.4.

Next we looked into the set of all storage functions for a strictly Σ-dissipative system. It is well known that this set is a bounded convex polyhedron for a controllable system. We showed that for an uncontrollable system the set loses its boundedness property. Further, this set becomes bounded from below if the system is stabilizable. If in addition the controllable part is strictly Σ-dissipative on $\mathbb{R}_-$, then we showed that this lower bound on the set is positive. We used this result to formalize the physical notion of stored energy being finite in a dissipative system that has no source of energy within: it is *not* possible to extract an indefinite amount of energy from a stabilizable system whose controllable part is strictly dissipative on $\mathbb{R}_-$.

The unmixing condition plays a crucial role in most of the main results of this paper. We showed that unmixing is not necessary in general for existence of a Lyapunov function and therefore for dissipativity. However, an interesting situation arises when the system has only one output. In Theorem 7.1 we showed that under suitable observability conditions a singular Lyapunov operator cannot have a rank one symmetric matrix in its image. The extent of necessity of the unmixing condition for a more general situation remains to be investigated.

In this paper we have dealt only with the maximum input cardinality case, i.e., the case when the number of inputs is equal to the positive signature of the supply rate function Σ. A study of the general case can also be utilized for dissipativity synthesis problems for uncontrollable systems.

References

[1] A. Ferrante and L. Pandolfi, *On the solvability of the positive real lemma equations*, Sys. Control Lett., 47 (2002), pp. 211–219.

[2] D. Pal and M.N. Belur, *Dissipativity of uncontrollable systems, storage functions and Lyapunov functions*, SIAM J. Control Optim., 47 (2008), pp. 2930–2966.

[3] H. K. Pillai and S. Shankar, *A behavioural approach to control of distributed systems*, SIAM J. Control Optim., 37 (1998), pp. 388–408.

[4] J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Springer-Verlag, New York, 1998.

[5] C. W. Scherer, P. Gahinet, and M. Chilali, *Multi-objective output-feedback control via lmi optimization*, IEEE Trans. Automat. Control, 42 (1997), pp. 896–911.

[6] H. L. Trentelman and J. C. Willems, *Every storage function is a state function*, Systems Control Lett., 32 (1997), pp. 249–259.

[7] J. C. Willems and H. L. Trentelman, *On quadratic differential forms*, SIAM J. Control Optim., 36 (1998), pp. 1703–1749.

[8] J. C. Willems and H. L. Trentelman, *Synthesis of dissipative systems using quadratic differential forms: Parts* I *and* II, IEEE Trans. Automat. Control, 47 (2002), pp. 53–86.

[9] J. C. Willems, *Dissipative dynamical systems - Part* I: *General theory, Part* II: *Linear systems with quadratic supply rates*, Archive for Rational Mechanics and Analysis, 45 (1972), pp. 321–351, 352–393.

[10] J. C. Willems, *Dissipative dynamical systems*, Eur. J. Control, 13 (2007), pp. 134–151.