Novel representation formulae for discrete $\mathbf{2}$D autonomous systems

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***Abstract*— In this paper, we provide explicit solution formulae for higher order discrete 2D autonomous systems. We first consider a special type of 2D autonomous systems, namely, systems whose quotient modules are *finitely generated* as modules over the one variable Laurent polynomial ring $\mathbb{R}[\sigma_1^{\pm 1}]$. We then show that these solutions can be written in terms of various integer powers of a square 1-variable Laurent polynomial matrix $A(\sigma_1)$ acting on suitable 1D trajectories. We call this form of expressing the solutions a *representation formula*. Then, in order to extend this result to general 2D autonomous systems, we obtain an analogue of a classical algebraic result, called *Noether's normalization lemma*, for the Laurent polynomial ring in two variables. Using this result we show that every 2D autonomous system admits a representation formula through a suitable coordinate transformation in the domain $\mathbb{Z}^2$.**

I. Introduction

The search for first order representations of systems of partial differential or difference equations, has been a topic of active research for the past few decades; see for example [1]–[5]. For ordinary differential/difference equations, a first order representation in input/state/output (or simply i/s/o) form is almost always assumed to be the starting point. This is not the case for nD systems with $n \geqslant 2$ (see [1], [2], [6]). For example, Maxwell's equations are first order, but heat equations or wave equations are not. In [7], Willems demonstrated how, for 1D systems, a first order representation can be obtained from a general higher order representation. For discrete 2D systems, a similar construction was provided in [2] using the behavioral description. In [1] i/s/o representations were constructed for 2D systems described in input/output form.

The importance of a first order representation lies in the explicit solution formula that it entails. For 1D systems, an i/s/o representation provides such a representation formula for the solutions in terms of the 'flow' operator acting on the initial conditions plus the 'input' convolved with the flow. Unfortunately, an analogous representation formula is absent for nD systems. The main difficulty in obtaining such a formula stems from the fact that, unlike the 1D case, nD systems do not have an *a priori* fixed direction of evolution. One way of circumventing this difficulty is by giving one independent variable, namely 'time', preference over the others (see [8]–[10]). In another approach, for the case of continuous autonomous systems, a representation formula is given in terms of integrations on the 'characteristic variety' of the system. This representation formula is known as the *Ehrenpreis-Palamodov integral representation formula*, see [11].

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Unfortunately, the short-coming of the first approach is that there are many systems which cannot be brought to a first order form in the special variable [10]. For these cases, it is more advantageous to treat both the variables equally. On the other hand, the drawback of the integral representation formula is that it first requires a complete knowledge of the points in the characteristic variety, and then an integration to be evaluated on this variety with suitable measures; both of these processes may be computationally very challenging.

In this paper, we shall present a representation theory for discrete 2D autonomous systems, which overcomes the above-mentioned drawbacks. We show that every 2D autonomous system admits a representation formula in terms of a flow matrix acting on initial conditions. Interestingly, it turns out that the initial conditions are either finite dimensional vectors or infinite dimensional trajectories depending upon whether the characteristic variety is zero dimensional or one dimensional.

The article [12] is an earlier, longer and detailed version of this paper. Due to page limit constraints, in this paper, we omit the proofs of *all* the main results. These proofs and various auxiliary results required for the proofs can be found in [12].

A. Notation

We use $\mathbb{R}$ and $\mathbb{C}$ to denote the fields of real and complex numbers, respectively. Consequently, $\mathbb{R}^n$, $\mathbb{C}^n$ denote the n-dimensional vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of integers is denoted by $\mathbb{Z}$, and $\mathbb{Z}^2$ denotes the set of two tuples of elements in $\mathbb{Z}$. In this paper, our main object of study is a particular class of doubly-indexed sequences of elements in $\mathbb{R}^{\mathtt{w}}$, for some positive integer $\mathtt{w}$. We denote the set of doubly-indexed sequences in $\mathbb{R}^{\mathtt{w}}$ by $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, *i.e.*, $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} := \{\mathbb{Z}^2 \to \mathbb{R}^{\mathtt{w}}\}$. The Laurent polynomial ring in two indeterminates σ_1, σ_2, usually written as $\mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]$, will be denoted by $\mathcal{A}$, and the same in one indeterminate σ_1, written as $\mathbb{R}[\sigma_1^{\pm 1}]$, will be denoted by $\mathcal{A}_1$. We use $\mathcal{A}^{\mathtt{w}}$ to denote the free module of rank $\mathtt{w}$ over $\mathcal{A}$, where the elements of $\mathcal{A}^{\mathtt{w}}$ are written as $\mathtt{w}$-tuple of rows. For a set S, we use $S^{m\times n}$ to denote the set of $(m\times n)$ matrices with entries from the set S. The single letter σ is often used to denote the tuple (σ_1, σ_2). Further, for an integer tuple $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$, the symbol σ^ν denotes the monomial $\sigma_1^{\nu_1}\sigma_2^{\nu_2}$. In this paper, we follow the bar notation to denote equivalence classes: for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and a submodule $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$, we use $\overline{r(\sigma)}$ to denote the equivalence class of $r(\sigma)$ in the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$.

II. Background

By 2D systems, in this paper, we mean systems described by a set of 2D linear partial difference equations with constant real coefficients. Such partial difference equations are often described using the 2D shift operators σ_1 and σ_2. These shift operators act on a doubly-indexed real-valued sequence $w \in \mathbb{R}^{\mathbb{Z}^2}$ as follows: for $\nu' := (\nu_1', \nu_2'), \nu := (\nu_1, \nu_2) \in \mathbb{Z}^2$

$$(\sigma^{\nu'} w)(\nu_1, \nu_2) = w(\nu_1 + \nu_1', \nu_2 + \nu_2'). \tag{1}$$

This definition can be extended naturally to define the action of $\mathcal{A}$, the Laurent polynomial ring in the shifts, on $\mathbb{R}^{\mathbb{Z}^2}$. And, likewise, the action of the row module $\mathcal{A}^{\mathtt{w}}$ on columns of sequences (trajectories) $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ can be defined: for a row-vector $r(\sigma) = [r_1(\sigma), r_2(\sigma), \cdots, r_{\mathtt{w}}(\sigma)]$ and a column-vector $w = \mathrm{col}(w_1, w_2, \ldots, w_{\mathtt{w}}) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ we define $r(\sigma)w := \sum_{i=1}^{\mathtt{w}} r_i(\sigma) w_i$.

A. The kernel representation

The collection of trajectories $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ that satisfy a given set of partial difference equations is called the *behavior* of the system, and is denoted by $\mathfrak{B}$. The above description of the action of $\mathcal{A}^{\mathtt{w}}$ on $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ gives the following representation of behaviors of 2D partial difference equations:

$$\mathfrak{B} := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid R(\sigma) w = 0\}, \tag{2}$$

where $R(\sigma) \in \mathcal{A}^{\mathtt{g} \times \mathtt{w}}$; the number $\mathtt{g}$ gives the number of equations present in the mathematical model of a system. The above equation (2) is called a *kernel representation* of $\mathfrak{B}$ and written as $\mathfrak{B} = \ker(R(\sigma))$. Note that many different matrices can have the same kernel. Importantly, all matrices having the same row-span over $\mathcal{A}$ result in the same behavior. This leads to the following equivalent definition of behaviors: let $R(\sigma) \in \mathcal{A}^{\mathtt{g} \times \mathtt{w}}$ and $\mathcal{R} := \mathrm{rowspan}(R(\sigma))$,

$$\mathfrak{B}(\mathcal{R}) := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid r(\sigma) w = 0 \text{ for all } r(\sigma) \in \mathcal{R}\}. \tag{3}$$

The submodule $\mathcal{R}$ generated by the rows of a kernel representation matrix is called *the equation module* of $\mathfrak{B}$. It was shown in [13] that the submodules of $\mathcal{A}^{\mathtt{w}}$ and 2D behaviors with $\mathtt{w}$ number of manifest variables are in one-to-one correspondence.

B. Autonomous systems

In this paper, we provide representation formulae for a special type of 2D systems, namely *autonomous systems*. Among several equivalent definitions of 2D autonomous systems (see [14], [15]), in this paper, we stick to the following Definition 2.1. In Definition 2.1 we need the notion of *characteristic ideal*[1] of a behavior.

Definition 2.1: A 2D system is said to be *autonomous* if the characteristic ideal $\mathcal{I}(\mathfrak{B})$ is nonzero. Further, an autonomous behavior is said to be *strongly autonomous* if the quotient ring $\mathcal{A}/\mathcal{I}(\mathfrak{B})$ is a finite dimensional vector space over $\mathbb{R}$.

Note that this condition of $\mathcal{I}(\mathfrak{B})$ being non-zero is equivalent to saying that $\mathfrak{B}$ admits a kernel representation matrix $R(\sigma)$ that is *full column rank*. For 1D systems, this also means that $\mathtt{g} = \mathtt{w}$, however, this is not the case for 2D systems (see [16]).

[1]Let $\mathfrak{B}$ be given by a kernel representation $\mathfrak{B} = \ker(R(\sigma))$ with $R(\sigma) \in \mathcal{A}^{\mathtt{g} \times \mathtt{w}}$. The *characteristic ideal* of $\mathfrak{B}$, denoted by $\mathcal{I}(\mathfrak{B})$, is defined as the ideal of $\mathcal{A}$ generated by the $(\mathtt{w} \times \mathtt{w})$ minors of $R(\sigma)$. For $\mathtt{g} < \mathtt{w}$, $\mathcal{I}(\mathfrak{B})$ is defined to be the zero ideal.

C. The quotient module

Given a behavior $\mathfrak{B} = \ker(R(\sigma))$, let $\mathcal{R}$ be the submodule of $\mathcal{A}^{\mathtt{w}}$ spanned by the rows of $R(\sigma)$. We define

$$\mathcal{M} := \mathcal{A}^{\mathtt{w}}/\mathcal{R},$$

and call it the *quotient module* of $\mathfrak{B}$. This quotient module $\mathcal{M}$ plays a central role in this paper. We often let elements from $\mathcal{M}$ act on $\mathfrak{B}$. This action is defined as follows: for $m \in \mathcal{M}$, the action of m on $w \in \mathfrak{B}$ is defined to be the action of a lift of m in $\mathcal{A}^{\mathtt{w}}$ on w. For example, let $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ be such that $\overline{r(\sigma)} = m \in \mathcal{M}$, then[2]

$$mw := r(\sigma) w. \tag{4}$$

Now note that it follows from Definition 2.1 above that $\mathfrak{B}$ is autonomous *if and only if* the quotient module $\mathcal{M}$ is a *torsion module*, *i.e.*, for every $\overline{m(\sigma)} \in \mathcal{M}$ there exists a $f(\sigma) \in \mathcal{A}$ such that $\overline{f(\sigma) m(\sigma)} = 0 \in \mathcal{M}$. In that case we get the following ideal called the *annihilator ideal* of $\mathcal{M}$.

$$\mathrm{ann}(\mathcal{M}) := \{f(\sigma) \in \mathcal{A} \mid f(\sigma) m = 0 \ \forall m \in \mathcal{M}\}.$$

D. Change of coordinates

Change of coordinates in $\mathbb{Z}^2$ plays a crucial role throughout this paper. By a coordinate change we mean a $\mathbb{Z}$-linear map from $\mathbb{Z}^2$ to itself of the form

$$\begin{array}{rcl} T : \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ \mathrm{col}(\nu_1, \nu_2) =: \nu & \mapsto & T\nu, \end{array}$$

where $T \in \mathbb{Z}^{2\times 2}$ is a *unimodular* matrix (*i.e.*, $\det(T) = \pm 1$). Note that because of unimodularity, the columns of T span the whole of $\mathbb{Z}^2$ as a $\mathbb{Z}$-module. Such a coordinate transformation T induces the following two maps.

$$\begin{array}{rcl} \varphi_T : \mathcal{A} & \to & \mathcal{A} \\ \sigma^{\nu} & \mapsto & \sigma^{T\nu} \end{array} \quad \Bigg| \quad \begin{array}{rcl} \Phi_T : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} & \to & (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \\ w(\nu) & \mapsto & w(T\nu), \end{array} \tag{5}$$

for all $\nu \in \mathbb{Z}^2$. Unimodularity of T makes both these maps *bijective*. In fact, Φ_T is an automorphism of the $\mathbb{R}$-vector space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, while φ_T is an automorphism of the $\mathbb{R}$-algebra $\mathcal{A}$. As a consequence, an *ideal* $\mathfrak{a} \subseteq \mathcal{A}$ is mapped to another *ideal* $\varphi_T(\mathfrak{a})$. The map φ_T can be extended to a map from $\mathcal{A}^{\mathtt{w}}$ to itself by applying φ pointwise. That is, define $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$ by mapping $[f_1(\sigma), f_2(\sigma), \cdots, f_{\mathtt{w}}(\sigma)]$ to $[\varphi_T(f_1(\sigma)), \varphi_T(f_2(\sigma)), \cdots, \varphi_T(f_{\mathtt{w}}(\sigma))]$. The map $\widehat{\varphi}_T$ is an $\mathcal{A}$-module morphism via the automorphism φ_T, *i.e.*, for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and $f(\sigma) \in \mathcal{A}$,

$$\widehat{\varphi}_T(f(\sigma) r(\sigma)) = \varphi_T(f(\sigma)) \widehat{\varphi}_T(r(\sigma)).$$

The bijective property of φ_T extends to the module case: as a result, $\widehat{\varphi}_T(\mathcal{R})$, the image of a *submodule* $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ under $\widehat{\varphi}_T$, is also a *submodule*.

Theorem 2.2 brings out precisely how the two maps Φ_T and $\widehat{\varphi}_T$ are related with each other. Given a behavior $\mathfrak{B}$, we define

$$\Phi_T(\mathfrak{B}) := \{v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid v = \Phi_T(w) \text{ for some } w \in \mathfrak{B}\}. \tag{6}$$

[2]Note that m may have several distinct lifts in $\mathcal{A}^{\mathtt{w}}$, but all of them have the same action on $w \in \mathfrak{B}$, because two lifts differ by an element in the equation module.

Theorem 2.2: Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule with behavior $\mathfrak{B}(\mathcal{R})$, and let $T \in \mathbb{Z}^{2\times 2}$ be unimodular. Then we have

$$\mathfrak{B}(\mathcal{R}) = \Phi_T(\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))). \tag{7}$$

III. Representation formula for a special type of autonomous systems

It is well-known that strongly autonomous systems admit first order representations with a pair of system matrices in the following manner (see [2], [14]).

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \middle| \begin{array}{l} \exists x \in (\mathbb{R}^n)^{\mathbb{Z}^2} \text{ such that} \\ \sigma_1 x = A_1 x,\ \sigma_2 x = A_2 x,\ w = Cx \end{array} \right\},$$

where n is a positive integer, $A_1, A_2 \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{\mathtt{w}\times n}$, with A_1, A_2 nonsingular and satisfying $A_1A_2 = A_2A_1$. Consequently, trajectories in a strongly autonomous behavior admit the following representation formula: for all $(\nu_1, \nu_2) \in \mathbb{Z}^2$,

$$w(\nu_1, \nu_2) = CA_1^{\nu_1}A_2^{\nu_2}x(\mathbf{0}), \tag{8}$$

where $x(\mathbf{0}) \in \mathbb{R}^n$ is an arbitrary initial condition. For this reason, we consider in this paper only those systems which are *not* strongly autonomous. We aim for a representation formula, analogous to equation (8) above, for general autonomous behaviors which are not strongly autonomous

It may be recalled that the crucial fact that leads to the representation formula (equation (8)) for strongly autonomous systems is that for these systems the quotient module $\mathcal{M}$ happens to be a finite dimensional vector space over $\mathbb{R}$. While this is clearly not true for general autonomous systems, there are, however, a large class of autonomous systems, whose quotient modules have the structure of a *finitely generated module* over the one variable Laurent polynomial ring $\mathbb{R}[\sigma_1^{\pm 1}]$ (denoted by $\mathcal{A}_1$ in the sequel). In this section, we concentrate on these special autonomous systems. We call them *strongly σ_2-relevant*. The name is inspired by the notion of *time/space-relevant* autonomous systems introduced in [10]. Due to page limit constraints, we do not go into details of the relation between strong σ_2-relevance and time/space-relevance of [10]. The interested reader may refer to [12] for the same.

Definition 3.1: Let $\mathfrak{B}$ be an autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$. Then $\mathfrak{B}$ is said to be *strongly σ_2-relevant* if the quotient module $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$.

Note that strongly autonomous systems are trivially strongly σ_2-relevant. Indeed, for strongly autonomous systems $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$, which is trivially a finitely generated module over $\mathcal{A}_1$. However, there are other strongly σ_2-relevant systems which are not strongly autonomous. The following is a scalar (*i.e.*, $\mathtt{w} = 1$) example of one such system.

Example 3.2: Consider the behavior

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2^2 - 2\sigma_2 + 1 \\ \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1 \end{bmatrix}.$$

Since $\mathfrak{B}$ above is having only one manifest variable, here the equation module $\mathcal{R}$ is the ideal $\mathfrak{a} := \langle \sigma_2^2 - 2\sigma_2 + 1, \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1\rangle$. Consequently, the quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$. The presence of the polynomial $\sigma_2^2 - 2\sigma_2 + 1$ in the equation ideal $\mathfrak{a}$ implies that this $\mathcal{M}$ is a finitely generated module over $\mathcal{A}_1$. Indeed, every element in $\mathcal{M}$ can be written as a linear combination of $\{\overline{1}, \overline{\sigma_2}\}$ with coefficients coming from $\mathcal{A}_1$. This can be seen in the following manner. First note that $\sigma_2^2 - 2\sigma_2 + 1 \in \mathfrak{a}$ implies that $\sigma_2 - 2 + \sigma_2^{-1} \in \mathfrak{a}$, which means $(\overline{\sigma_2})^{-1} = -\overline{\sigma_2} + 2 \in \mathcal{M}$. By taking higher positive powers we get that for all $i \in \mathbb{Z}$, $i > 0$, we have $(\overline{\sigma_2})^{-i}$ is equal to a polynomial in non-negative powers of $\overline{\sigma_2}$ with coefficients in $\mathbb{R}$. As a consequence, every Laurent polynomial in $\overline{\sigma_1}$ and $\overline{\sigma_2}$ is equal to a polynomial in *non-negative* powers of $\overline{\sigma_2}$ with coefficients from $\mathbb{R}[\overline{\sigma_1}^{\pm 1}] = \mathcal{A}_1/(\mathfrak{a} \cap \mathcal{A}_1)$. However, note that $\sigma_2^2 - 2\sigma_2 + 1$ is monic. Therefore, given any polynomial, say $f(\sigma) \in \mathcal{A}$, having only non-negative powers of σ_2 with coefficients in $\mathcal{A}_1$, we can carry out Euclidean division algorithm by $\sigma_2^2 - 2\sigma_2 + 1$ to obtain $a_1(\sigma_1), a_0(\sigma_1) \in \mathcal{A}_1$ and $q(\sigma) \in \mathcal{A}$ such that

$$f(\sigma) = q(\sigma)(\sigma_2^2 - 2\sigma_2 + 1) + a_1(\sigma_1)\sigma_2 + a_0(\sigma_1).$$

In other words, $\overline{f(\sigma)} = \overline{a_1(\sigma_1)}\overline{\sigma_2} + \overline{a_0(\sigma_1)}$. This, together with the fact that every element in $\mathcal{M}$ can be reduced to a polynomial with only non-negative powers of $\overline{\sigma_2}$, proves that every element in $\mathcal{M}$ can be written as a linear combination of $\overline{1}$ and $\overline{\sigma_2}$ with coefficients from $\mathcal{A}_1$. That is, $\mathcal{M}$ is finitely generated as a module over $\mathcal{A}_1$. Thus, $\mathfrak{B}$ above is strongly σ_2-relevant.

□

The defining property of strongly σ_2-relevant systems, that is, the quotient module $\mathcal{M}$ is finitely generated over $\mathcal{A}_1$, leads to a representation formula, akin to equation (8). We present this representation formula in Theorem 3.4 below. Before we get to this theorem we need the following construction of various 1-variable Laurent polynomial matrices, which are guaranteed to exist once $\mathcal{M}$ is assumed to be a finitely generated module over $\mathcal{A}_1$.

A. The state matrix $A(\sigma_1)$

Since $\mathcal{M}$ is finitely generated over $\mathcal{A}_1$, we can find a finite generating set $\mathcal{G} := \{g_1, g_2 \ldots, g_n\} \subseteq \mathcal{M}$ for $\mathcal{M}$ as a module over $\mathcal{A}_1$. This enables us to set up the following $\mathcal{A}_1$-module homomorphism:

$$\begin{array}{rccl} \psi: & \mathcal{A}_1^n & \to & \mathcal{M} \\ & e_i & \mapsto & g_i \text{ for all } 1 \leqslant i \leqslant n \end{array} \tag{9}$$

where e_i is the standard basis row-vector in $\mathcal{A}_1^n$, *i.e.*,

$$e_i := \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}}^{n \text{ entries}} \in \mathcal{A}_1^n.$$
$$\qquad\qquad\uparrow$$
$$\qquad\qquad i^{\text{th}} \text{ position}$$

Now, let $\mu : \mathcal{M} \to \mathcal{M}$ denote the map 'multiplication by σ_2' in $\mathcal{M}$. Clearly, μ is $\mathcal{A}_1$-linear, which means, under the chosen generating set $\mathcal{G}$, μ can be represented by an $n \times n$ matrix with entries from $\mathcal{A}_1$. More precisely, if for $1 \leqslant i \leqslant n$,

$$\mu(g_i) = \sigma_2 g_i = a_{i,1}(\sigma_1)g_1 + a_{i,2}(\sigma_1)g_2 + \cdots + a_{i,n}(\sigma_1)g_n$$

then we get the matrix

$$A(\sigma_1) := \begin{bmatrix} a_{1,1}(\sigma_1) & a_{1,2}(\sigma_1) & \cdots & a_{1,n}(\sigma_1) \\ a_{2,1}(\sigma_1) & a_{2,2}(\sigma_1) & \cdots & a_{2,n}(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(\sigma_1) & a_{n,1}(\sigma_1) & \cdots & a_{n,n}(\sigma_1) \end{bmatrix}. \quad (10)$$

Note that the map μ is invertible on $\mathcal{M}$, and its inverse is the map given by multiplication by σ_2^{-1}, although, the matrix $A(\sigma_1)$ representing μ may not be automatically invertible (unimodular) in $\mathcal{A}_1^{n\times n}$. However, one can always choose a suitable generating set for $\mathcal{M}$ as a module over $\mathcal{A}_1$ such that the corresponding $A(\sigma_1)$ is indeed invertible.

Lemma 3.3: Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule such that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Then there exists a finite generating set $\{g_1, g_2, \dots, g_n\}$ of $\mathcal{M}$ as a module over $\mathcal{A}_1$ such that the corresponding matrix $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$, as defined in equation (10), is invertible (unimodular) in $\mathcal{A}_1^{n\times n}$.

Keeping Lemma 3.3 in mind, in the sequel, we always assume that $A(\sigma_1)$ is invertible in $\mathcal{A}_1^{n\times n}$. Inverse of $A(\sigma_1)$ clearly represents the map defined by multiplication by σ_2^{-1}. The matrix $A(\sigma_1)$ thus defined leads to the following commutative diagram of $\mathcal{A}_1$-module maps: for any $i \in \mathbb{Z}$, define $\mu^i : \mathcal{M} \ni \overline{m} \mapsto \overline{\sigma_2^i m} \in \mathcal{M}$, then we have

$$\begin{array}{ccc} \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{M} \\ A(\sigma_1)^i \downarrow & & \downarrow \mu^i \\ \mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \mathcal{M} \end{array} \quad (11)$$

where $A(\sigma_1)^i : \mathcal{A}_1^n \ni r(\sigma_1) \mapsto r(\sigma_1)A(\sigma_1)^i \in \mathcal{A}_1^n$.

B. The matrix of relations $R_1(\sigma_1)$

The finitely generated $\mathcal{A}_1$-module $\mathcal{M}$ may *not* be *free*, that is, the generators may satisfy nontrivial relations among themselves over $\mathcal{A}_1$. In that case, recalling the map $\psi : \mathcal{A}_1^n \to \mathcal{M}$ defined by equation (9), we must have $\ker(\psi)$ to be a nontrivial submodule of $\mathcal{A}_1^n$. Since $\mathcal{A}_1^n$ is a Noetherian module, this submodule $\ker(\psi)$ must be finitely generated. Let $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$ be a matrix whose rows generate $\ker(\psi)$, *i.e.*

$$\operatorname{rowspan}(R_1(\sigma_1)) = \ker(\psi). \quad (12)$$

We call this matrix $R_1(\sigma_1)$ a *matrix of relations* of $\mathcal{G}$.

C. The output matrix $C(\sigma_1)$

Next, let e_i be the standard i^{th} basis row-vector in $\mathcal{A}^{\mathtt{w}}$. Suppose $\overline{e_i} \in \mathcal{M}$, the image of e_i under the surjection $\mathcal{A}^{\mathtt{w}} \twoheadrightarrow \mathcal{A}^{\mathtt{w}}/\mathcal{R} = \mathcal{M}$, is given by a linear combination of $\{g_1, g_2, \dots, g_n\}$ over $\mathcal{A}_1$ as

$$\overline{e_i} = c_{i,1}(\sigma_1)g_1 + c_{i,2}(\sigma_1)g_2 + \cdots + c_{i,n}(\sigma_1)g_n.$$

Define

$$C(\sigma_1) := \begin{bmatrix} c_{1,1}(\sigma_1) & c_{1,2}(\sigma_1) & \dots & c_{1,n}(\sigma_1) \\ c_{2,1}(\sigma_1) & c_{2,2}(\sigma_1) & \dots & c_{2,n}(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{\mathtt{w},1}(\sigma_1) & c_{\mathtt{w},2}(\sigma_1) & \dots & c_{\mathtt{w},n}(\sigma_1) \end{bmatrix}. \quad (13)$$

We now present the representation formula for strongly σ_2-relevant autonomous systems. It is important to note at this point that elements from $\mathcal{A}_1^n$ act on n-tuples of 1D trajectories: for $r(\sigma_1) = \begin{bmatrix} r_1(\sigma_1) & r_2(\sigma_1) & \cdots & r_n(\sigma_1) \end{bmatrix} \in \mathcal{A}_1^n$ and $x = \operatorname{col}(x_1, x_2, \dots, x_n) \in (\mathbb{R}^n)^{\mathbb{Z}}$ the action of $r(\sigma_1)$ on x is defined as

$$r(\sigma_1)x = r_1(\sigma_1)x_1 + r_2(\sigma_1)x_2 + \cdots + r_n(\sigma_1)x_n \in \mathbb{R}^{\mathbb{Z}}. \quad (14)$$

Theorem 3.4: Let $\mathfrak{B}$ be an autonomous behavior with equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$. Suppose $\mathfrak{B}$ is strongly σ_2-relevant, that is, $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Let $\{g_1, g_2, \dots, g_n\} \subseteq \mathcal{M}$ be a set of generators of $\mathcal{M}$ as an $\mathcal{A}_1$-module and consider the $\mathcal{A}_1$-module map $\psi : \mathcal{A}_1^n \to \mathcal{M}$ as in equation (9). Further, let $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$, $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$ and $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$ be as defined in equations (12), (13) and (10), respectively, with $A(\sigma_1)$ invertible in $\mathcal{A}_1^{n\times n}$. Then $w \in \mathfrak{B}$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ satisfying

$$R_1(\sigma_1)x = 0 \quad (15)$$

such that for all $\nu = \operatorname{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$

$$w(\nu) = \left(C(\sigma_1)A(\sigma_1)^{\nu_2}x\right)(\nu_1). \quad (16)$$

The key step in obtaining Theorem 3.4 is defining the 'state' variable x by

$$x(\bullet) = \left(\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} w \right)(\bullet, 0).$$

We illustrate the result of Theorem 3.4 in the following example.

Example 3.5: Consider the scalar ($\mathtt{w} = 1$) strongly σ_2-relevant behavior of Example 3.2,

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2^2 - 2\sigma_2 + 1 \\ \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1 \end{bmatrix}.$$

Here the equation module is the ideal $\mathfrak{a} = \langle \sigma_2^2 - 2\sigma_2 + 1, \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1 \rangle$, and consequently, the quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$. As we have already seen, here $\mathcal{M}$ is a finitely generated module over $\mathcal{A}_1$; $\{\overline{1}, \overline{\sigma_2}\}$ generate $\mathcal{M}$ as an $\mathcal{A}_1$-module. In this case, we have $n = 2$ and the 1-variable Laurent polynomial matrices $R_1(\sigma_1)$, $C(\sigma_1)$ and $A(\sigma_1)$ are given by

1) $R_1(\sigma_1) = \begin{bmatrix} (\sigma_1 - 1) & -(\sigma_1 - 1) \end{bmatrix}$,
2) $C(\sigma_1) = \begin{bmatrix} 1 & 0 \end{bmatrix}$,
3) $A(\sigma_1) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

So every solution in $\mathfrak{B}$ is of the form

$$w(\nu_1, \nu_2) = \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}^{\nu_2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)(\nu_1),$$

where $\operatorname{col}(x_1, x_2) \in (\mathbb{R}^2)^{\mathbb{Z}}$ satisfies

$$\begin{bmatrix} (\sigma_1 - 1) & -(\sigma_1 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

□

The assumption of $\mathfrak{B}$ being strongly σ_2-relevant in Theorem 3.4 is quite restrictive. There are many systems which do not satisfy this requirement. For example, consider a scalar behavior given by a single equation $\mathfrak{B} = \ker(f(\sigma))$, where $f(\sigma) \in \mathcal{A}$ is of the form:

$$f(\sigma) = \sigma_2^n + \alpha_{n-1}(\sigma_1)\sigma_2^{n-1} + \cdots + \alpha_1(\sigma_1)\sigma_2 + \alpha_0(\sigma_1),$$

where n is a positive integer, with $\alpha_i(\sigma_1) \in \mathcal{A}_1$ for $0 \leqslant i \leqslant n-1$. Suppose that $\alpha_0(\sigma_1)$ is *not* a unit in $\mathcal{A}_1$. It can be shown, in that case, that the quotient ring $\mathcal{M} = \mathcal{A}/\langle f(\sigma)\rangle$ will not be finitely generated as a module over $\mathcal{A}_1$. Consequently, $\mathfrak{B}$ *cannot* be strongly σ_2-relevant. A concrete example of such an $f(\sigma)$ is: $f(\sigma) = \sigma_2 - \sigma_1 - 1$. Another example of a scalar behavior that is not strongly σ_2-relevant is $\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)$. In the coming Sections IV and V we overcome this drawback of Theorem 3.4 and present a representation formula for general autonomous systems in Theorem 5.3. The main idea behind this is that every autonomous system can be converted to a strongly σ_2-relevant system by a suitable change of coordinates on the indexing set $\mathbb{Z}^2$. This is achieved by the following algebraic procedure: given a quotient module $\mathcal{M}$, we construct a suitable change of variables (via a coordinate transformation on $\mathbb{Z}^2$) so that the image of $\mathcal{M}$ under this transformation becomes a finitely generated module over $\mathcal{A}_1$. In the next section, we show how to achieve this transformation for ideals; we call this result the *discrete version of Noether's normalization lemma*. Then in Section V we first extend the normalization process to submodules and then use this result to give the general representation formula (Theorem 5.3).

IV. Discrete version of Noether's normalization

Recall that given a unimodular $T \in \mathbb{Z}^{2\times 2}$, it defines an automorphism of $\mathcal{A}$ as

$$\begin{array}{rccc} \varphi_T : & \mathcal{A} & \to & \mathcal{A} \\ & \sigma^\nu & \mapsto & \sigma^{T\nu} \end{array}. \tag{17}$$

In this section, we show that, given a nonzero ideal $\mathfrak{a} \subseteq \mathcal{A}$, either $\mathcal{A}/\mathfrak{a}$ is a finite dimensional vector space over $\mathbb{R}$, or there exists a unimodular $T \in \mathbb{Z}^{2\times 2}$ such that under the corresponding φ_T the quotient ring $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated *faithful* module over $\mathcal{A}_1$. This observation constitutes the main Theorem 4.2 of this section. Theorem 4.2 can be thought as an analogue, applicable for Laurent polynomial rings, of the well-known Noether's normalization lemma, which applies to polynomial rings. Hence we call the result the discrete version of Noether's normalization lemma.

Before we get to Theorem 4.2, we first state the following Lemma 4.1, which is a precursor to Theorem 4.2. The lemma shows that given a 2D Laurent polynomial, there exists a unimodular T such that under φ_T the given Laurent polynomial is mapped to a Laurent polynomial with a special structure: when written as a Laurent polynomial in σ_2 with coefficients from $\mathcal{A}_1$, these coefficients are all *units* in $\mathcal{A}_1$. A similar result can be found in [17], where the result has been used in a different context, namely design of inverse 2D filters.

Lemma 4.1: Let $0 \neq f(\sigma) \in \mathcal{A}$ be given by

$$f(\sigma) = \sum_{\nu\in\mathbb{Z}^2} \alpha_\nu \sigma^\nu, \quad \alpha_\nu \in \mathbb{R},$$

with only finitely many $\alpha_\nu \neq 0$. Then there exists a unimodular $T \in \mathbb{Z}^{2\times 2}$ such that under the corresponding automorphism φ_T given by equation (17), we have

$$\varphi_T(f(\sigma)) = \left(\sum_{k=0}^{\delta} u_k(\sigma_1)\sigma_2^k\right) u(\sigma_2), \tag{18}$$

where $\{u_0(\sigma_1), u_1(\sigma_1), \ldots, u_\delta(\sigma_1)\} \subseteq \mathcal{A}_1$ and $u(\sigma_2) \in \mathbb{R}[\sigma_2^{\pm 1}]$ are all units in $\mathcal{A}$ and δ is some finite positive integer.

We now state the discrete version of Noether's normalization lemma.

Theorem 4.2: Suppose $\{0\} \neq \mathfrak{a} \subseteq \mathcal{A}$ is an ideal. Then exactly one of the following statements is true:

1) $\mathcal{A}/\mathfrak{a}$ is a finite dimensional vector space over $\mathbb{R}$.
2) There exists $T \in \mathbb{Z}^{2\times 2}$ unimodular, such that under the corresponding ring automorphism $\varphi_T : \mathcal{A} \to \mathcal{A}$, the quotient ring $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated faithful module over $\mathcal{A}_1$.

When statement (1) of Theorem 4.2 above does not hold, the process of obtaining a unimodular $T \in \mathbb{Z}^{2\times 2}$ to get the automorphism $\varphi_T : \mathcal{A} \to \mathcal{A}$ so that statement (2) holds will be referred to in the sequel as *Noether's normalization*.

V. Representation formula for general autonomous systems

In this section, we utilize the discrete version of Noether's normalization lemma to obtain a representation formula for a general 2D autonomous system. This is stated as Theorem 5.3 below. In order to make use of Noether's normalization, we need first to extend Theorem 4.2 to the module case. This extension, namely Theorem 5.2 is done using the following technical lemma, which relates annihilators of two quotient modules after a coordinate change, as in the Noether's normalization process, is done. Recall from Subsection II-D how a unimodular $T \in \mathbb{Z}^{2\times 2}$ induces a map $\widehat{\varphi}_T : \mathcal{A}^{\mathrm{w}} \to \mathcal{A}^{\mathrm{w}}$.

Lemma 5.1: Let $T \in \mathbb{Z}^{2\times 2}$ be unimodular and let $\widehat{\varphi}_T : \mathcal{A}^{\mathrm{w}} \to \mathcal{A}^{\mathrm{w}}$ be the corresponding map of $\mathcal{A}$-modules via the ring map $\varphi_T : \mathcal{A} \to \mathcal{A}$. Then

$$\varphi_T(\mathrm{ann}(\mathcal{M})) = \mathrm{ann}(\mathcal{A}^{\mathrm{w}}/\widehat{\varphi}_T(\mathcal{R})).$$

Lemma 5.1 above, together with Theorem 4.2, leads to Theorem 5.2 below.

Theorem 5.2: Let $\mathcal{R} \subseteq \mathcal{A}^{\mathrm{w}}$ be a submodule such that $\mathcal{M} = \mathcal{A}^{\mathrm{w}}/\mathcal{R}$ is a torsion module. Then exactly one of the following statements is true:

1) $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$.
2) There exists $T \in \mathbb{Z}^{2\times 2}$ such that under the corresponding module map $\widehat{\varphi}_T : \mathcal{A}^{\mathrm{w}} \to \mathcal{A}^{\mathrm{w}}$, the quotient module $\mathcal{A}^{\mathrm{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated faithful module over $\mathcal{A}_1$.

It is well-known that statement (1) of Theorem 5.2 above corresponds to $\mathfrak{B}(\mathcal{R})$ being strongly autonomous. Since such behaviors are already known to have a representation formula given by equation (8), in the sequel we shall concentrate only on autonomous systems which are not strongly autonomous. Recall that strongly autonomous systems are always strongly σ_2-relevant. As a consequence of Theorem 5.2 above and Theorem 2.2 it follows that for every 2D autonomous system $\mathfrak{B}$ there exists a coordinate transformation T such that $\mathfrak{B}$

is related with a strongly σ_2-relevant behavior, say $\mathfrak{B}'$, by $\mathfrak{B} = \Phi_T(\mathfrak{B}')$. This is the key idea behind the general representation formula stated in Theorem 5.3 below.

Theorem 5.3: Suppose $\mathfrak{B}$ is an autonomous behavior whose equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ is such that the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is not a finite dimensional vector space over $\mathbb{R}$. Then there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular, two positive integers n, n', and the following 1-variable Laurent polynomial matrices

- $R_1(\sigma_1) \in \mathcal{A}_1^{n'\times n}$,
- $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w}\times n}$,
- $A(\sigma_1) \in \mathcal{A}_1^{n\times n}$,

with $A(\sigma_1)$ invertible in $\mathcal{A}_1^{n\times n}$, such that $w \in \mathfrak{B}$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ which satisfies

$$R_1(\sigma_1)x = 0$$

and for all $\nu = \mathrm{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$

$$w(\nu) = \left(C(\sigma_1)A(\sigma_1)^{(T\nu)_2}x\right)((T\nu)_1),$$

where $T\nu = \mathrm{col}((T\nu)_1, (T\nu)_2)$.

Example 5.4: Consider the scalar behavior

$$\mathfrak{B} = \ker(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1).$$

The equation module is the principal ideal $\mathfrak{a} = \langle \sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1\rangle$. The quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$ is clearly not a finitely generated module over $\mathcal{A}_1$. Therefore, $\mathfrak{B}$ is not strongly σ_2-relevant. However, under the coordinate transformation $T = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ the transformed ideal $\varphi_T(\mathfrak{a})$ turns out to be

$$\varphi_T(\mathfrak{a}) = \langle \varphi_T(\sigma_1\sigma_2 - \sigma_1 - \sigma_2 + 1)\rangle = \langle \sigma_2^3 - \sigma_2^2 - \sigma_1^{-1}\sigma_2 + \sigma_1^{-1}\rangle.$$

Clearly, $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1$. Generators can be chosen to be $\{\overline{1}, \overline{\sigma_2}, \overline{\sigma_2}^2\}$. In fact, these generators freely generate $\mathcal{A}/\varphi_T(\mathfrak{a})$ as an $\mathcal{A}_1$-module. Here, $n = 3$ and

- $R_1(\sigma_1) = 0$,
- $A(\sigma_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma_1^{-1} & \sigma_1^{-1} & 1 \end{bmatrix}$
- $C(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Hence, solutions in $\mathfrak{B}$ are given by

$$w(\nu_1, \nu_2) = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma_1^{-1} & \sigma_1^{-1} & 1 \end{bmatrix}^{2\nu_1+\nu_2} x\right)(\nu_1),$$

where $x \in (\mathbb{R}^3)^{\mathbb{Z}}$ is arbitrary.

□

VI. Concluding Remarks

In this paper, we looked into novel representation formulae for discrete 2D autonomous systems. These representation formulae generalize the solution formula for 1D autonomous systems given by a flow acting on initial conditions. The crucial difference in the 2D case is that here the initial conditions are given by 1D trajectories as opposed to real vectors in the 1D case. Moreover, instead of a constant matrix, here in the 2D case the flow operator is a 1-variable Laurent polynomial matrix. We first looked at systems whose corresponding quotient modules are finitely generated as modules over $\mathbb{R}[\sigma_1^{\pm 1}]$. We showed that these systems admit representation formulae of the above-mentioned type. Then we used a discrete version of Noether's normalization to obtain representation formulae for general 2D autonomous systems. A crucial step in the normalization process is finding a suitable coordinate transformation in $\mathbb{Z}^n$.

There are a number of issues related with the results presented in this paper that have not been addressed here. For example, the question of how to get minimal size of the 1-variable Laurent polynomial matrix $A(\sigma_1)$, or algorithms for computing the matrix. The extension of the formulae to non-autonomous systems is also another important unresolved question.

Acknowledgement

The authors would like to thank an anonymous reviewer for various important and interesting suggestions. The first author would like to thank Ms Uddipana Dowerah for reading the manuscript and suggesting corrections.

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