Dissipativity analysis of SISO systems using Nyquist-Plot-Compatible (NPC) supply rates

N. Santosh Kumar Debasattam Pal

***Abstract*— In this paper we deal with a special class of quadratic supply rates, dissipativity with respect to which can be directly read off from a system's Nyquist plot. These supply rates are called Nyquist-plot-compatible (NPC) supply rates [8]. The characterizing property of these supply rates is that to each of them a specific region in the complex plane can be associated: dissipativity w.r.t. these supply rates is equivalent to systems' Nyquist plots being contained in the region. The classical results of small gain and passivity theorems are special cases of dissipativity w.r.t. NPC supply rates. We show in this paper, that apart from the aforementioned two special cases, there are many more such NPC supply rates. In particular, we construct supply rates for regions in the complex plane given by the right-half (or, left-half) of a vertical line, interiors (or, exteriors) of circles of various radii with centers on the real-axis. We then show that a system's Nyquist plot being contained in the union of two regions is equivalent to dissipativity w.r.t. a frequency weighted combination of the corresponding two NPC supply rates. We finally give an algorithm for finding out these weighting polynomial functions.**

I. Introduction

Dissipativity theory has rightfully become one of the cornerstones in modern control theory. The evidence of this fact lies in its successful application to myriad control problems, e.g., LQR/LQG optimal control [21], $\mathcal{H}_\infty$ optimal/suboptimal control [18], [22], absolute stability of interconnection (the Luré problem) [5]–[7], [9], [14], [15], etc. This paper deals with a particular issue concerning the latter-most topic: problem of stability of interconnection using dissipativity.

The key idea behind the solution to the absolute stability problem is: given that the plant is dissipative w.r.t. a quadratic supply rate, if a controller too is made dissipative w.r.t. a supply rate determined by the plant's supply rate and the interconnection topology, then the interconnected system is guaranteed to be stable [6], [14]. Note, however, that a design procedure using this idea necessarily presumes that the plant is *a priori* known to be dissipative w.r.t. a certain quadratic supply rate. This raises the issue that given a plant, how do we know whether it is dissipative w.r.t a quadratic supply rate? And, if so, how do we find out a suitable supply rate so that the plant is dissipative w.r.t that? In this paper we provide a partial answer to this issue. Our inspiration comes from the classical results of small-gain and passivity.

In small-gain and passivity theorems (when applied to SISO LTI systems), the system's Nyquist plot turns out to have a direct relation with dissipativity w.r.t. two special quadratic supply rates. Indeed, for a SISO system with u, y as input and output, respectively, dissipativity w.r.t. $r^2u^2 - y^2$ is equivalent to the system's Nyquist plot being contained in the disk of radius r around the origin. Likewise, for passivity, i.e., dissipativity w.r.t. uy, it is necessary and sufficient that the system's Nyquist plot be contained in the right half of the complex plane.

In this paper, we show that there is a sizeable class of quadratic supply rates, each of which has a specified region in the complex plane such that a system is dissipative w.r.t. a supply rate in this class if and only if its Nyquist plot is contained in the designated region of the complex plane. Naturally, the small-gain and passivity supply rates turn out to be special cases of these supply rates. We call these supply rates Nyquist-Plot-Compatible (NPC) supply rates.[1]

NPC supply rates help us in coming up with a supply rate given a SISO system and its Nyquist plot. Typical regions, as shown in this paper, for NPC supply rates include interiors (or, exteriors) of circles with centers on the real axis, right half or left half of a vertical line, etc. However, quite often, given a Nyquist plot it may not be apparent whether there is some NPC supply rate w.r.t. which the system is dissipative[2]. Sometimes the Nyquist plot may turn out to lie in the *union* of the dissipativity regions of two well-known NPC supply rates. We show in this paper, that this is equivalent to dissipativity w.r.t. a frequency-weighted combination of the two corresponding NPC supply rates. We conclude the paper with an algorithm to find these frequency dependent weighting functions. A noteworthy point here is that such mixing of two supply rates is not new; it has been done in [2], [3] for small-gain and passivity, and in [10] for small-gain and negative-imaginary. Our result here differs from those in the references mentioned on at least two counts. First, our result is not specific to special supply rates, it holds for *any* two NPC supply rates. Secondly, the weighting functions in our case are polynomials (unlike the ones in the aforementioned references). The crucial benefit of polynomials is that they have a functional meaning as

[1]The fact that Nyquist plots are used in order to define the supply rates is not surprising, for dissipativity can be defined using a frequency domain inequality (see Proposition 2.1). However, what singles out NPC supply rates is that each of them has a designated region in the complex plane where a system's Nyquist plot must lie in order to be dissipative. See section III for details.

[2]Note that, in many cases, the Nyquist plot may be contained in the right-half (or left-half) of a vertical line chosen sufficiently away on the left (or right) of the imaginary axis. And hence, an NPC supply rate can be obtained for such a system easily. However, this will result in supply rates with very large coefficients, which might cause problem for controller design.

The authors are with the Department of EEE, IIT Guwahati, India - 781039. email: ns.kumar@iitg.ernet.in, debasattam@iitg.ernet.in.

differential operators. This fact is advantageous when it comes to designing a controller using the mixed supply rate.

II. Notation and Preliminaries

A. Notation

The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The ring of real polynomials in ξ is denoted by $\mathbb{R}[\xi]$. The set of matrices with $\mathtt{m}$ rows and $\mathtt{p}$ columns having real polynomials for their entries is denoted by $\mathbb{R}^{\mathtt{m}\times\mathtt{p}}[\xi]$. $\mathbb{R}^{\mathtt{w}\times\bullet}$, $\mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ denotes the set of (real constant or real polynomial) matrices having $\mathtt{w}$ rows, where the no. of columns is unspecified. Likewise, $\mathbb{R}^{\bullet\times\mathtt{p}}$ and $\mathbb{R}^{\bullet\times\mathtt{p}}[\xi]$ denote the set (real constant or real polynomial) of matrices with $\mathtt{p}$ columns. $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\bullet})$ denotes the space of smooth (infinite times differentiable) functions from $\mathbb{R}$ to $\mathbb{R}^{\bullet}$. The subset of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ with functions having compact supports is denoted by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. The set of all $\mathtt{w}\times\mathtt{w}$ bivariate polynomial matrices in ζ,η with real coefficients will be denoted by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

B. Behavior

In this paper, we follow the definition of dissipativity[3] given in [19]. For this purpose, we need to define what is known as the *behavior* of an LTI differential system [16]. For a SISO system with u as input and y as output the behavior is defined as

$$\mathfrak{B}=\left\{\begin{pmatrix} u\\ y\end{pmatrix}\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^2)\ \middle|\ \text{such that } u,y \text{ satisfy the system's differential equations}\right\}.$$

As shown in [16], if $\mathfrak{B}$ is controllable then

$$\mathfrak{B}=\left\{\begin{bmatrix} D(\frac{d}{dt})\\ N(\frac{d}{dt})\end{bmatrix}\ell\ \middle|\ \ell\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})\right\}, \tag{1}$$

where $N(\xi),D(\xi)\in\mathbb{R}[\xi]$ are coprime. Equation (1) is called an *image representation* of $\mathfrak{B}$. In this paper, only SISO systems are taken into account and we also assume that the system is controllable.

C. Quadratic differential forms

Dissipativity analysis of linear differential behaviors is done using *quadratic differential forms (QDF)* as shown in [19]. In this subsection, we provide the rudimentary details of QDFs and their use in dissipativity analysis that will be important for this paper. Details can be found in [19].

Quadratic forms are special polynomial functions on real or complex Euclidean vector spaces. Quadratic forms that involve system variables as well as their derivatives can be defined using QDFs. A QDF Q_Φ is induced by a two-variable polynomial matrix with real constant coefficients, $\Phi(\zeta,\eta)$. This is done as follows: let $\Phi(\zeta,\eta)$ be given by

$$\Phi(\zeta,\eta):=\sum_{i,k}\Phi_{ik}\zeta^i\eta^k\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta],$$

[3]It may be noted that many other definitions of dissipativity are prevalent in the literature; see for example [4], [19]. However, perhaps the one in [19] suits the purpose of this paper the best.

where $\Phi_{ik}\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Then Q_Φ is a map $Q_\Phi:\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})\to\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})$ defined as

$$Q_\Phi(w)=\sum_{i,k}\left(\frac{d^iw}{dt^i}\right)^T\Phi_{ik}\left(\frac{d^kw}{dt^k}\right). \tag{2}$$

While dealing with quadratic forms in w and its derivatives, we can assume without loss of generality, that $\Phi(\zeta,\eta)=\Phi^T(\eta,\zeta)$ where $(\bullet)^T$ denotes the usual matrix transposition. Such a $\Phi(\zeta,\eta)$ is called symmetric.

Following [19], we call a controllable behavior $\mathfrak{B}$ *dissipative with respect to a symmetric two-variable polynomial matrix* $\Phi(\zeta,\eta)$, or simply Φ-*dissipative* if

$$\int_{\mathbb{R}}Q_\Phi(w)dt\geqslant 0\quad\forall w\in\mathfrak{B}\cap\mathfrak{D}. \tag{3}$$

The behavior $\mathfrak{B}$ is said to be *strictly* Φ-dissipative if the above-mentioned inequality (3) is satisfied with a strict inequality.

The QDF Q_Φ, corresponding to which dissipativity of a behavior is sought, is often called a *supply rate*, for it generalizes the idea of a power supply (see [19]). With a slight abuse of terminology we often call $\Phi(\zeta,\eta)$ also a supply rate since there is no risk of ambiguity. The following result, Proposition 2.1, from [19] is our main tool for dissipativity analysis throughout this paper.

Proposition 2.1: Consider the system $G(s)=\frac{N(s)}{D(s)}$ and $\Phi\in\mathbb{R}^{2\times 2}[\zeta,\eta]$. Then, the behavior corresponding to G is dissipative with respect to $\Phi(\zeta,\eta)$ if and only if

$$M^T(-j\omega)\partial\Phi(j\omega)M(j\omega)\geqslant 0\quad\forall\omega\in\mathbb{R} \tag{4}$$

where $M(j\omega)=[D(j\omega)\quad N(j\omega)]^T$. Further, the system is strictly dissipative if and only if the above inequality is strict for almost all $\omega\in\mathbb{R}$.

D. Factorization of Para-Hermitian matrices

We frequently need the one-variable polynomial matrix $\Phi(-\xi,\xi)$ obtained from $\Phi(\zeta,\eta)$; for notational convenience, we denote this matrix by $\partial\Phi(\xi)$. The matrix $\partial\Phi(\xi)$ has a special property: it is *para-Hermitian*. A polynomial matrix $P(\xi)\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is called para-Hermitian if $P(\xi)=P^T(-\xi)$. The significance of P being para-Hermitian is that $P(j\omega)$ is Hermitian for all $\omega\in\mathbb{R}$. We now define two important parameters associated with para-Hermitian matrices: the "inertia" and the "worst inertia" of a para-Hermitian matrix.

Definition 2.2: [11], [12] Suppose $P(\xi)\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is para-Hermitian and assume $P(\xi)$ is nonsingular as a polynomial matrix, i.e., $\det(P(\xi))\not\equiv 0$. Let $\omega_0\in\mathbb{R}$ be such that $j\omega_0$ is not a zero of $P(\xi)$, i.e., $\det(P(j\omega_0))\neq 0$. Then the inertia of $P(j\omega_0)$ is defined as the 2-tuple: $(\sigma_-(P(j\omega_0)),\sigma_+(P(j\omega_0)))$ where $\sigma_-(P(j\omega_0))$ and $\sigma_+(P(j\omega_0))$ are the numbers of negative and positive eigenvalues of $P(j\omega_0)$, respectively. If $P(j\omega_0)$ is singular, then the inertia is undefined at that point.

Let ν_{max} be the maximum number of negative eigenvalues of $P(j\omega)$ as ω varies over $\mathbb{R}$, i.e., $\nu_{max}:=\max_{\omega\in\mathbb{R}}\{\sigma_-(P(j\omega))\}$. The *worst inertia* of $P(\xi)$ is defined as $(\nu_{max},\mathtt{w}-\nu_{max})$, and correspondingly, the *worst inertia matrix* (see [11]) is defined as $J_{worst} := \begin{bmatrix} I_{\mathtt{w}-\nu_{max}} & 0 \\ 0 & -I_{\nu_{max}} \end{bmatrix}$.

The following result from [11, Theorem 3.6.5] (see also [12]) concerns factorization of para-Hermitian polynomial matrices that might not have constant inertia almost everywhere on the imaginary axis. This result will be crucial importance for us in the sequel.

Proposition 2.3: Let $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be para-Hermitian and nonsingular and let $J_{worst} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be its worst inertia matrix. Then there exist polynomial matrices $K \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $L \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, with K square and nonsingular, such that

$$P(\xi) = K^T(-\xi)J_{worst}K(\xi) + L^T(-\xi)L(\xi). \tag{5}$$

III. NPC SUPPLY RATES AND SOME STANDARD EXAMPLES

A. Nyquist-plot-compatible (NPC) supply rates

The main object of study in this paper is a special type of supply rates, dissipativity with respect to which can be read off from the systems' Nyquist plots. We call such supply rates Nyquist-plot-compatible (NPC) supply rates. As mentioned earlier, in various situations, given a plant we are required to know whether that system is dissipative with respect to some supply rate. The NPC supply rates come very handy in these situations. For example, dissipativity with respect to $\sum_{br} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is equivalent to the system's Nyquist plot being contained in the unit disk. This property is the key behind the well-known small-gain theorem. Another equally important result, the passivity theorem, is likewise related to dissipativity with respect to $\sum_{pr} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Both these supply rates and many other come under the class of NPC supply rates. We formally define NPC supply rates now.

Let $G(s) = \frac{N(s)}{D(s)}$ be the transfer function of a SISO system. As mentioned earlier, the behavior $\mathfrak{B}_G$ of this system is described as

$$\mathfrak{B}_G := \left\{ \begin{bmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{bmatrix} \ell \;\middle|\; \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \right\}. \tag{6}$$

We denote by $M(\xi)$ the column vector $[D(\xi), N(\xi)]^T$ in the sequel.

Definition 3.1: [8] A supply rate $\Phi(\zeta,\eta) \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ is said to induce a trichotomy of the complex plane $\mathbb{C}$ if corresponding to $\Phi(\zeta,\eta)$ there exists a 3-tuple of disjoint sets $\{\mathcal{A}_\Phi^+, \mathcal{A}_\Phi^0, \mathcal{A}_\Phi^-\}$, whose union is $\mathbb{C}$, such that for every $\mathfrak{B}_G$, we have the following:

1) The Nyquist plot of G at a frequency $\omega \geqslant 0$ is contained in $\mathcal{A}_\Phi^+ \iff M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) > 0$.
2) The Nyquist plot of G at a frequency $\omega \geqslant 0$ is contained in $\mathcal{A}_\Phi^0 \iff M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) = 0$.
3) The Nyquist plot of G at a frequency $\omega \geqslant 0$ is contained in $\mathcal{A}_\Phi^- \iff M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) < 0$.

If a supply rate satisfies all these properties, then it is called a *Nyquist-plot-compatible (NPC)* supply rate.

Note that under Definition 3.1 strict dissipativity is equivalent to the Nyquist plot of G being contained in $\mathcal{A}_\Phi^+$ for almost all positive frequencies. From now on we refer to $\mathcal{A}_\Phi^+$ as NPC-region, and $\mathcal{A}_\Phi^0$ as NPC-boundary associated with the NPC supply rate Φ. We denote by Ω the collection of all NPC supply rates. There are many supply rates present in the set of Ω. Some standard and basic NPC supply rates are discussed below.

The small-gain supply rate: $\sum_{br} = \left[\begin{smallmatrix} r^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is the small-gain supply rate (arising out of $Q_\Phi(u,y) = r^2u^2 - y^2$). This supply rate arises in Bounded Real Lemma. $\frac{0.6s-0.6}{s^2+s+1}$ and $\frac{5s+5}{s^3+s^2-s+2}$ are two transfer functions which are strictly dissipative w.r.t. $\sum_{br}$ ($r = 3.6$) according to Proposition 2.1. The NPC-region is shown in Figure 1.

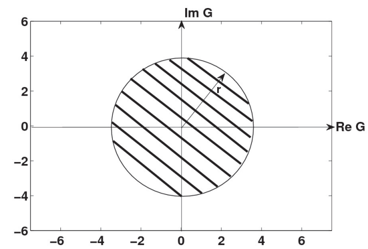

Fig. 1. Associated region of small-gain supply rate.

The passivity supply rate: Another example of Φ that is within the set of Ω is $\sum_{pr} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ (arising out of $Q_\Phi(u,y) = 2uy$). $\sum_{pr}$ is related to Positive Real Lemma. Based on Proposition 2.1, $\frac{2s-2}{s^2-s-1}$ and $\frac{s-3}{s^4-4s^3+2s^2-3s-3}$ are two transfer functions which are strictly dissipative w.r.t. $\sum_{pr}$. Figure 2 depicts the corresponding NPC region.

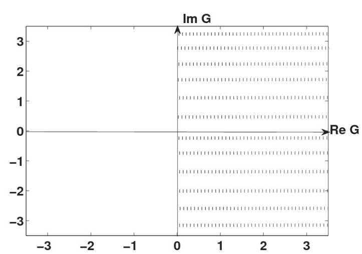

Fig. 2. Associated region of passivity supply rate

The negative imaginary supply rate [15]: One more example of Φ that is in Ω is $\sum_{ni} = \begin{bmatrix} 0 & \eta \\ \zeta & 0 \end{bmatrix}$ (arising out of $Q_\Phi(u,y) = 2u\frac{dy}{dt}$). In accordance with Proposition 2.1 $\frac{-1}{s^2-2s+0.5}$ and $\frac{5s-5}{s^3-s^2-2}$ are two transfer functions which are strictly dissipative w.r.t. $\sum_{ni}$. The NPC-region is displayed in Figure 3.

B. Congruence transformations on supply rates

The following result from [8] shows closure of Ω under congruence transformations.

Proposition 3.2: Consider the set $\Omega \subset \mathbb{R}^{2\times 2}[\zeta,\eta]$ of Nyquist-Plot-Compatible supply rates, and let $\Phi \in \Omega$. Then for any nonsingular $T \in \mathbb{R}^{2\times 2}$, the supply rate $T^T\Phi T$ also belongs to Ω.

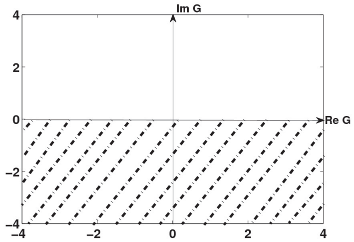

Fig. 3. Associated region of negative imaginary supply rate

The above result is based on Möbius transformations. We show now that by suitable congruence transformations on Σ_{br} one can get the NPC supply rates with NPC-regions as interiors or exteriors of circles with various radii and centers on the real axis, and, also, RHS or LHS of lines parallel to the imaginary axis.

Lemma 3.3: Let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$ be a non-singular matrix and $\sum_{br} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ be the small-gain supply rate. Then the new supply rate $\Phi = T^T \sum_{br} T$ is an NPC supply rate, and its corresponding $\mathcal{A}_\Phi^+$ is one of the following:

1) If $b = d$, then the the boundary, $\mathcal{A}_\Phi^0$, is a line parallel to the imaginary axis. Further, if $ab - cd > 0$ (or, if $ab - cd < 0$) then $\mathcal{A}_\Phi^+$ is the RHS (LHS) of the line $\mathcal{A}_\Phi^0$.
2) If $b \neq d$ then the boundary, $\mathcal{A}_\Phi^0$, is a circle with center on the real axis. Further, the corresponding $\mathcal{A}_\Phi^+$ is the interior (or the exterior) of the circle if $b^2 - d^2 < 0$ ($b^2 - d^2 > 0$).

Proof. The proof is omitted due to page limit constraints; it can be found in the appendix to this paper kept at `www.ee.iitb.ac.in/~debasattam`. □

Remark 3.4: Note that, $\sum_{br}$ in Lemma 3.3 can be replaced by $\sum_{pr}$. Indeed, since $\sum_{br}$ and $\sum_{pr}$ are themselves related with each other by a congruence transformation, any Φ obtained from $\sum_{br}$ by a congruence transformation can also be obtained from $\sum_{pr}$, albeit, by a different congruence transformation.

IV. Mixing two NPC supply rates

As mentioned previously, in many situations, we are faced with this converse question: given a plant, is there a suitable supply rate with respect to which the plant is dissipative? We have seen that NPC supply rates help in answering this question for a number of transfer functions. However, there are still many transfer functions, which may not be dissipative with respect to any obvious NPC supply rates. In this section, we deal with a special case of this situation: we assume that the plant's Nyquist plot is contained in the *union* of two known NPC regions of the complex plane (that is, $\mathcal{A}_{\Phi_1}^+ \cup \mathcal{A}_{\Phi_2}^+$, where Φ_1 and Φ_2 are known). We show that in this situation, the plant turns out to be dissipative with respect to a frequency-weighted combination of the two NPC supply rates. This issue of 'mixing' two supply rates in the frequency domain is not new. In [2], [3], [8], this issue has been looked at considering the small gain and the passivity supply rates, while, in [10], the same has been done for small-gain and negative imaginary (that is, Σ_{ni} in the notation of this paper) supply rates. We do not restrict ourselves to special NPC supply rates. Theorem 4.1 below shows that any two NPC supply rates can be mixed provided the plant's Nyquist plot is contained in the union.

Theorem 4.1: Consider a SISO LTI system given by the transfer function $G(s)$ and let $\mathfrak{B}_G = \mathrm{im} M(\frac{d}{dt})$ be its image representation. Let Φ_1 and Φ_2 be NPC supply rates. Then the following two statements are equivalent:

1) G has Nyquist plot contained in $\mathcal{A}_{\Phi_1}^+ \cup \mathcal{A}_{\Phi_2}^+$ for almost all $\omega \geqslant 0$.
2) There exist $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta,\eta) := p(\zeta)\Phi_1(\zeta,\eta)p(\eta) + q(\zeta)\Phi_2(\zeta,\eta)q(\eta) \quad (7)$$

Proof. We give only a sketch of the proof due to page limit constraints; the complete proof can be found in the appendix to this paper kept at `www.ee.iitb.ac.in/~debasattam`. To get an idea of the proof, note that $\mathfrak{B}_G$ is strictly dissipative with respect to the $\Phi(\zeta,\eta)$ defined in equation (7) if and only if $p(\xi), q(\xi)$ satisfy

$$\begin{bmatrix} p(-j\omega) \\ q(-j\omega) \end{bmatrix} \begin{bmatrix} \Gamma(-j\omega,j\omega) & 0 \\ 0 & \Pi(-j\omega,j\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} > 0 \quad (8)$$

for almost all $\omega \in \mathbb{R}$, where Γ and Π are defined as

$$\left.\begin{array}{rcl} \Gamma(-j\omega,j\omega) & := & M^T(-j\omega)\partial\Phi_1(j\omega)M(j\omega) \\ \Pi(-j\omega,j\omega) & := & M^T(-j\omega)\partial\Phi_2(j\omega)M(j\omega) \end{array}\right\}. \quad (9)$$

Now note that equation (9) is true if and only if the auxiliary behavior, $\mathfrak{B}_{aux} := \mathrm{im}\begin{bmatrix} p(\frac{d}{dt}) \\ q(\frac{d}{dt}) \end{bmatrix}$ is strictly dissipative with respect to the auxiliary supply rate, $\Phi_{aux}(\zeta,\eta) = \begin{bmatrix} \Gamma(\zeta,\eta) & 0 \\ 0 & \Pi(\zeta,\eta) \end{bmatrix}$. It has been shown in [11], [12] that it is possible to find a $\mathfrak{B}_{aux}$ if and only if the worst inertia of Φ_{aux} is *not* $(2,0)$. This fact, again, is equivalent to Statement 1) of the theorem. □

Example 4.2: (*Mixing of small-gain and passivity*) The transfer function $G = \frac{3}{s^2+3s+2}$ has Nyquist plot (for positive frequencies) contained in the union of the unit circle region ($r = 1$ in $\sum_{br}$) and the right half plane region as shown in the Figure 6. So according to the theorem there exists $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta,\eta) = p(\zeta)\sum_{br} p(\eta) + q(\zeta)\sum_{pr} q(\eta) \quad (10)$$

So the required p, q found using the steps presented in the algorithm in Section V are

$$\begin{array}{rcl} p(\xi) & = & 2.449\xi^3 + 2.449\xi^2 + 0.3709\xi + 2.0781 \\ q(\xi) & = & 1.3163\xi^3 - 2.65256\xi^2 - 0.36314\xi - 2.236. \end{array}$$

Example 4.3: (*Mixing of passivity and negative imaginary*) The transfer function $G = \frac{2s-1}{s^3+2s^2+2s}$ has Nyquist plot (only positive frequencies shown) contained in the union of the right half plane region and the lower half plane region as

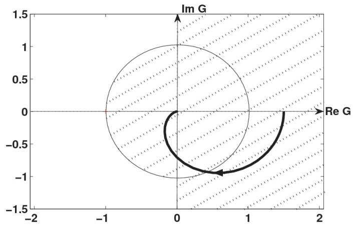

Fig. 4. Mixing of small-gain and passivity

shown in the following Figure 7. According to the theorem there exists $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta,\eta) = p(\zeta)\sum_{pr} p(\eta) + q(\zeta)\sum_{ni} q(\eta) \tag{11}$$

The required p, q found using the steps presented in algorithm of Section V are

$$\begin{aligned} p(\xi) &= -2.69282\xi^3 - 1.30718\xi^2 - 2.0\xi \\ q(\xi) &= -2.0\xi^4 - 2.0\xi^3 + 0.0784\xi^2 - 2.0784\xi. \end{aligned}$$

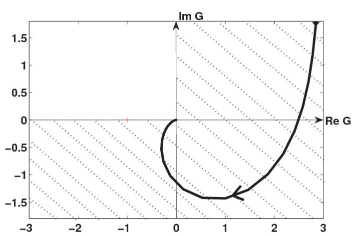

Fig. 5. Mixing of passivity and negative imaginary

V. Algorithm for finding $p(\xi), q(\xi)$ polynomials

It is possible to give an explicit algorithm for the computation of the polynomial weighting functions p, q of Theorem 4.1 . The crucial step in this computation is that of factorizing a para-Hermitian matrix following Proposition 2.3. There are many methods available for such factorizations of para-Hermitian polynomial matrices (see [1], [17]). In this section, we adapt the existing methods to provide a simplified one that suits our requirement of factorizing the matrix

$$S(\xi) := \begin{bmatrix} \Gamma(-\xi,\xi) & 0 \\ 0 & \Pi(-\xi,\xi) \end{bmatrix}. \tag{12}$$

Our adapted procedure is much simplified owing to the fact that $S(\xi)$ is diagonal. However, the problematic part is $S(\xi)$ has non-constant inertia on the imaginary axis. In order to make the description of our procedure to factorize $S(\xi)$ easy to follow, we make two simplifying assumptions:
1) The number of *crossover frequencies*, i.e., frequencies at which the system's Nyquist plot crosses from one NPC-region to another, is only two. 2) The roots of the polynomials $\Gamma(-j\omega, j\omega)$ and $\Pi(-j\omega, j\omega)$ are known precisely[4].

[4]Note that these polynomials are polynomials in ω with real coefficients.

Under these two assumptions we describe below an algorithm for finding $p, q \in \mathbb{R}[\xi]$ to meet the requirements of Theorem 4.1.
Step 1: Input: the transfer function $G(s)$ and the two NPC supply rate matrices $\Phi_1(\zeta,\eta)$ and $\Phi_2(\zeta,\eta)$.
Step 2: Find the $S(\xi)$ matrix, as defined in equation (12), using equation (9).
Step 3: Compute the worst inertia of the matrix $S(\xi)$.
Step 4: If the worst inertia is $(2,0)$ then statement 1) of Theorem 4.1 does not hold. So, by Theorem 4.1, p, q cannot be found.
Step 5: If the worst inertia is $(0,2)$ then, as shown in the proof of Theorem 4.1, any pair of polynomials p, q satisfying coprimeness will ensure that $\mathfrak{B}_G$ is dissipative with respect to $\Phi(\zeta,\eta) = p(\zeta)\Phi_1(\zeta,\eta)p(\eta) + q(\zeta)\Phi_2(\zeta,\eta)q(\eta)$.
Step 6: If the worst inertia is $(1,1)$ then, by Proposition 2.3 $S(j\omega)$ admits a factorization as

$$K^T(-j\omega)J_{worst}K(j\omega) + L^T(-j\omega)L(j\omega)$$

where $J_{worst} = \operatorname{diag}(1,-1)$ and matrices $K(\xi) \in \mathbb{R}^{2\times 2}[\xi]$ and $L(\xi) \in \mathbb{R}^{\bullet\times 2}[\xi]$, with K non-singular.
Step 7: To achieve the above factorization we must first decompose the para-Hermitian matrix $S(\xi)$ as

$$K^T(-\xi)J_{worst}K(\xi) + L^T(-\xi)L(\xi)$$

Step 8: First factorize the matrix $S(\xi)$ as $S(\xi) = Y^T(-\xi)S_2(\xi)Y(\xi)$, where $Y(\xi)$ is called the symmetric factor of $S(\xi)$. This factor consists of the following:

1) Zeros of $S(\xi)$ having nonzero real parts (which always come in reflection symmetry about the imaginary axis because $S(\xi)$ is para-Hermitian; see [17]).
2) Purely imaginary zeros of $S(\xi)$ that have *even* multiplicities.

Assumption 2), that is, precise knowledge of the roots of $\Gamma(\omega)$ and $\Pi(\omega)$ makes it possible to extract the symmetric factor $Y(\xi)$ easily. Note that, $S(\xi)$ and $S_2(\xi)$ both have the same worst inertia of $(1,1)$.
Step 9: Now $S_2(\xi)$ will be in the form as shown below because of assumption 1), i.e., there are only two crossover frequencies

$$S_2(\xi) = \begin{bmatrix} \xi^2+\alpha^2 & 0 \\ 0 & -(\xi^2+\beta^2) \end{bmatrix},$$

where we may assume that $\alpha \geqslant \beta \geqslant 0$ and $\alpha \neq 0$ without loss of generality[5].
Step 10: Extracting the symmetric factor out reduces the problem of factorizing $S(\xi)$ to factorizing $S_2(\xi)$ in the form of

$$K^T(-\xi)J_{worst}K(\xi) + L^T(-\xi)L(\xi)$$

The $S_2(\xi)$ matrix is a real regular 2×2 matrix polynomial satisfying $S_2(\xi) = [S_2(-\xi)]^T$ having variable inertia on the

[5]Indeed, if $\alpha < \beta$ then $S_2(\xi)$ (correspondingly $S(\xi)$) will have worst inertia of $(2,0)$. Then as said in Step 4 of the algorithm p, q cannot be found. Further, if $\alpha = 0$ then the worst inertia being $(1,1)$ forces $\beta = 0$. In that case, $S_2(\xi) = \begin{bmatrix} \xi^2 & 0 \\ 0 & -\xi^2 \end{bmatrix}$ can be readily factorized as $S_2(\xi) = \begin{bmatrix} 0 & -\xi \\ -\xi & 0 \end{bmatrix} J_{worst} \begin{bmatrix} 0 & \xi \\ \xi & 0 \end{bmatrix}$.

imaginary axis. Following a crucial step as done in [17] we modify $S_2(\xi)$ into a matrix that has constant inertia for almost all points on the imaginary axis. This can be done by adding the diagonal element $(\xi^2+\alpha^2)(\xi^2+\beta^2)$ to $S_2(\xi)$ obtaining a 3×3 diagonal polynomial matrix. So we define $S_{new}(\xi)$ as

$$S_{new}(\xi)=\begin{bmatrix}\xi^2+\alpha^2 & 0 & 0\\ 0 & -(\xi^2+\beta^2) & 0\\ 0 & 0 & (\xi^2+\alpha^2)(\xi^2+\beta^2)\end{bmatrix}.$$

Note that $S_{new}(\xi)$ has constant inertia $(2,1)$ on the imaginary axis for almost all points $\xi\in i\mathbb{R}$. After doing suitable row, column transformations and manipulations on $S_{new}(\xi)$ we can arrive at a very easy formula for $S_2(\xi)$

$$S_2(\xi)=\begin{bmatrix}\xi^2+\alpha^2 & 0\\ 0 & -(\xi^2+\beta^2)\end{bmatrix}=\begin{bmatrix}\alpha & -\xi\\ -\frac{\beta\xi}{\alpha} & -\beta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha & \frac{\beta\xi}{\alpha}\\ \xi & -\beta\end{bmatrix}+\begin{bmatrix}0\\ \frac{-\xi\sqrt{\gamma}}{\alpha}\end{bmatrix}\begin{bmatrix}0 & \frac{\xi\sqrt{\gamma}}{\alpha}\end{bmatrix},$$

where $\gamma=\alpha^2-\beta^2$, $\alpha\neq 0$ and $\alpha\geqslant\beta$.

Step 11: Now we will get the exact $K(\xi)$ and $L(\xi)$ after right multiplying the symmetric factor matrix to the above formula. So

$$S(\xi)=Y^T(-\xi)\left\{\begin{bmatrix}\alpha & -\xi\\ -\frac{\beta\xi}{\alpha} & -\beta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha & \frac{\beta\xi}{\alpha}\\ \xi & -\beta\end{bmatrix}+\begin{bmatrix}0\\ \frac{-\xi\sqrt{\gamma}}{\alpha}\end{bmatrix}\begin{bmatrix}0 & \frac{\xi\sqrt{\gamma}}{\alpha}\end{bmatrix}\right\}Y(\xi)$$

Step 12: Hence final $K(\xi)$ used to find p,q is as follows:

$$K(\xi)=\begin{bmatrix}\alpha & \frac{\beta\xi}{\alpha}\\ \xi & -\beta\end{bmatrix}Y(\xi)$$

Step 13: As done in the proof of Theorem 4.1, this $K(\xi)$ is used to find $p(\xi),q(\xi)$ polynomials.

VI. CONCLUSION

We summarize here the key results of this paper and future scope of the presented work. We started with a special type of supply rates called NPC supply rates. Using congruence transformations on standard NPC supply rates we created new ones. For example, NPC supply rates with NPC-boundaries like circles of any radius and with centers anywhere on the real axis can be synthesized by this method. Similarly a supply rate with NPC-boundary as any line parallel to imaginary axis can be found using a suitable congruence transformation. The result regarding the union of NPC-regions has been theoretically proved and supported with examples for the case of two distinct NPC supply rates. Perhaps this result can be extended to the case of three or more finite NPC supply rates by an induction argument[6]. If this is possible, then stabilization by interconnection of many complex (interconnected) systems can be established. Till now our results are applicable to linear differential systems only. It is our hope that similar results might be applicable to time-varying, nonlinear cases. These issues have to be examined in more depth. The issue of spectral factorization plays an important role in the main Theorem 4.1 and can be handled using the steps mentioned in the algorithm of Section V. The described algorithm has to be modified accordingly to include multi-crossover (more than two crossovers) frequency type Nyquist plots. The drawback of the algorithm is that the exact roots of the polynomials $\Gamma(\omega)$ and $\Pi(\omega)$ have to be known. More work is required to rectify this drawback.

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