New results and techniques for computation of stored energy in lossless/all-pass systems

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***Abstract*—Lossless and all-pass systems are energy-conservative in the sense that the energy that is extracted from the system always equals that supplied to the system. This stored energy turns out to be independent of the system realization/description. For example, there are different LC realizations of a lossless transfer function, but the energy stored can be uniquely expressed in terms of the port variables and their derivatives. This paper proposes new results and algorithms to compute the stored energy in lossless systems. We present four different techniques to compute the stored energy of lossless/all-pass systems. The first method is LC realization based (Foster, Cauer and their combinations) and the second is based on the Bezoutian of two polynomials. The notion of "balancing controllability/observability Gramians" is used for the third, while the last method is based on adjoint networks. A comparative study among the four methods shows that the first three methods are comparable with respect to computation time, while for numerical accuracy, the Bezoutian method is the best. Three different methods to compute the Bezoutian is also reported here: Euclidean long division, Pseudo-inverse method and the two dimensional discrete Fourier transform.**

Keywords: **Foster-Cauer realizations, Bezoutian, Adjoint network, Gramian balancing.**

1. Introduction

This paper deals with lossless systems (and more generally conservative systems), i.e. systems for which the energy extracted from the system equals the energy supplied to the system. Traditionally, LC realizations of lossless transfer functions are non-unique; Foster 1 & 2, Cauer 1 & 2 and their combinations, for example. The values of the capacitances and inductances would be highly varied across these realizations, due to which, for a given amount of stored energy, the capacitor-voltages and inductor-currents would be different across the realizations. Further, for a given lossless transfer function there are many state-space realizations that need not correspond to an LC realization, this also adds to the non-uniqueness in the values of states for a given stored energy. In spite of this non-uniqueness, it is known (and elaborated in Section 2-D below) that the energy stored, when expressed in terms of the external variables (port-variables) and their derivatives, is exactly the same function (henceforth called the storage function) and is independent of both the LC realization and the state-space realization. This property can be exploited in the sense that the LC realization or state-space realization can be chosen in a form so that new methods (possibly with better numerical/flop-count properties) to compute the stored energy are revealed by the chosen realization. This paper proposes four different approaches to characterize the stored energy; each approach unfolds new results and algorithms to compute the storage function. Note that when the system is not lossless but strictly passive, then the computation of the stored energy is linked to solutions of the so-called Algebraic Riccati Inequality/Equality (ARI/ARE). However, such an inequality/equation cannot be formulated for conservative systems, since certain "regularity conditions" are not satisfied by such systems. Hence methods described in this paper not only provide new ways to compute the storage function but also to solve control problems where the ARE does not exist due to failure of the regularity condition on D. In this paper we develop new results which yield procedures to compute storage function of conservative systems. We propose following four approaches:

1) LC realizations: Foster/Cauer and their combinations,
2) Bezoutian of two polynomials
   a) Euclidean long division,
   b) Pseudo-inverse/Left-inverse,
   c) Two dimensional discrete Fourier transform,
3) balancing of controllability/observability Gramians and
4) dual/adjoint network.

The rest of the paper is organized as follows: Section 2 summarizes the notation and preliminaries required in the paper. In Section 3 we present results based on LC realizations. Section 4 contains results based on the Bezoutian of two polynomials. Three methods for Bezoutian are proposed in the section. It also contains a comparison of these three methods with respect to time and accuracy. Section 5 has results to compute the storage function of all-pass systems using the concept of balanced states. Methods to compute storage function of lossless systems based on adjoint networks is reported in Section 6. In Section 7, we report algorithms for storage function computation of lossless systems. Section 8 has a comparison of the algorithms based on their computational time and numerical accuracy. Concluding remarks are presented in Section 9.

2. Notation and Preliminaries

A. Notation

We use standard notation: $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers respectively. $\mathbb{R}[\xi]$ denotes the ring of polynomials in one indeterminate ξ with real coefficients. The set $\mathbb{R}^{\mathtt{w}\times\mathtt{p}}[\xi]$ denotes all $\mathtt{w}\times\mathtt{p}$ matrices with entries from $\mathbb{R}[\xi]$. We use $\bullet$ when a dimension need not be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows. $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$ denotes the set of polynomial matrices in two indeterminates: ζ and η, having $\mathtt{n}$ rows and $\mathtt{m}$ columns. $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the set of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$

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denotes the subspace of all compactly supported trajectories in $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. A block diagonal matrix A is represented as $\mathrm{diag}(A_1,A_2,\ldots,A_{\mathtt{m}})$ where $A_1,A_2,\ldots,A_{\mathtt{m}}$ are square matrices of possibly different sizes. While vectors are usually column vectors in this paper, sometimes, depending on the context e_i either denotes the i-th row or the i-th column of the identity matrix. $\mathbf{1}_{\mathtt{m}} \in \mathbb{R}^{\mathtt{m}}$ denotes a column vector of all ones. $A=[a_{ij}]$ represents a matrix A with elements a_{ij} where i and j represent the row and column indices of the matrix. The symbol $A \bigotimes B$ represents Kronecker product of matrices A and B. A matrix of the form $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ is represented as $\mathrm{col}(B_1,B_2)$.

B. Behavior

This section contains some essential preliminaries of the behavioral approach to systems theory: an elaborate exposition can be found in [22, Section 2]. A linear differential behavior, denoted by $\mathfrak{B}$, is defined as the set of all infinitely often differentiable trajectories that satisfy a system of ordinary linear differential equations with constant coefficients, i.e.,

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}, \text{where } R(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi].$$

We denote the set of all linear differential behaviors with $\mathtt{w}$ number of variables by $\mathfrak{L}^{\mathtt{w}}$. The behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ can be represented as $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$ called the kernel representation of $\mathfrak{B}$. Without loss of generality, we assume $R(\xi)$ is of full row rank (see [18, Theorem 2.5.23]). Since $R(\xi)$ is of full row rank, there exists a possibly non-unique partition of $R(\xi) = [Q(\xi) \;\; P(\xi)]$ (after permutation of columns of $R(\xi)$ if necessary) where $P(\xi)$ is square and nonsingular. Conforming to this partition of $R(\xi)$, the variable w is partitioned as (u,y) with u as the input and y as the output. The transfer matrix from input u to output y is $-P^{-1}Q$. When this matrix of rational functions is proper, then there exists an input/*state*/output (i/s/o) representation of the form $\dot{x} = Ax + Bu$ and $y = Cx + Du$ such that $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and with $\mathtt{n}$ defined as $\mathtt{n} :=$ deg det P. For the purpose of this paper, we call a behavior $\mathfrak{B}$ controllable, when an i/s/o representation of the system is both controllable and observable; see [18, Chapter 5] for other equivalent definitions. We represent the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. The minimum number of states required for an i/s/o representation of a controllable behavior is called the *McMillan degree* of the system. It is known that the state-space representation of a controllable and observable system is *minimal* and the number of states of such a system is equal to the McMillan degree. One of the various ways of representing a controllable linear differential behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is the *image representation*: there exists $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}}) \text{ such that } w = M\left(\frac{d}{dt}\right)\ell \right\}.$$

In fact, there exists an M such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$; it is called an *observable* image representation (see [22, Section 2]). For the rest of the paper we consider controllable behaviors $\mathfrak{B}$ only and also use the term system and behavior interchangeably.

C. Quadratic Differential Forms and Dissipativity

In this subsection, we provide basic details about quadratic differential forms (QDF): for a detailed study see [22]. Consider a two-variable polynomial matrix

$$\phi(\zeta,\eta) := \sum_{j,k} \phi_{jk}\zeta^j\eta^k \;\in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta], \text{ where } \phi_{jk} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}.$$

$\phi(\zeta,\eta)$ is called *symmetric* if $\phi(\zeta,\eta) = \phi(\eta,\zeta)^T$. For the rest of the paper, we deal with symmetric two-variable polynomial matrices only. The QDF Q_ϕ induced by $\phi(\zeta,\eta)$ is a map $Q_\phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined as

$$Q_\phi(w) := \sum_{j,k} \left(\frac{d^j w}{dt^j}\right)^T \phi_{jk} \left(\frac{d^k w}{dt^k}\right).$$

A quadratic form induced by a real symmetric *constant* matrix is a special case and is often needed in this paper: we denote it by $Q_\Sigma(w) = w^T\Sigma w$, where $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$.

We call a controllable behavior $\mathfrak{B}$ Σ-*dissipative* if

$$\int_{\mathbb{R}} w^T\Sigma w\, dt \geqslant 0 \text{ for every } w \in \mathfrak{B}\cap\mathfrak{D}. \tag{1}$$

Q_Σ is called the *supply rate* or power. The variable w, in terms of which the power is specified, is called the *manifest variable*. For this paper, the manifest variable w contains the port variables like voltage and current. The supplied power and the energy stored may also be expressed in terms of other variables, like the latent variable ℓ and the state x: this is dealt below when dealing with the stored energy.

For a Σ-dissipative controllable behavior $\mathfrak{B}$, the two variable polynomial matrix $\psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is said to induce a *storage function* Q_ψ with respect to the supply rate Q_Σ if

$$\frac{d}{dt}Q_\psi(w) \leqslant w^T\Sigma w \text{ for all } w \in \mathfrak{B}. \tag{2}$$

The notion of storage function captures the intuition that the rate of increase of stored energy in a dissipative system is at most the power supplied. The storage function with respect to a supply rate is not unique in general for a given system. However, lossless systems are a special case for which storage function is unique (see [22, Remark 5.13]). Such a system also satisfies the inequality (2) as an equality: these systems are the focus of this paper and is elaborated in the following subsection.

D. Conservative systems and their storage functions

Conservative systems are a special class of dissipative systems. A controllable behavior $\mathfrak{B}$ is said to be *conservative* with respect to $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ if the inequality in equation (1) is satisfied with *equality*, i.e.

$$\int_{\mathbb{R}} w^T\Sigma w\, dt = 0 \text{ for all } w \in \mathfrak{B}\cap\mathfrak{D}.$$

Thus conservative systems satisfy equation (2) with equality:

$$\frac{d}{dt}Q_\psi(w) = w^T\Sigma w \text{ for all } w \in \mathfrak{B}. \tag{3}$$

Noting that

$$\frac{d}{dt}Q_\Psi(\ell) = Q_\Phi(\ell) \tag{4}$$

whenever $(\zeta+\eta)\Psi(\zeta,\eta)=\Phi(\zeta,\eta)$ and using the image representation of the controllable behavior $w=M\left(\frac{d}{dt}\right)\ell$, one can obtain the two-variable polynomial matrix $\Psi\in\mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$ associated with the storage functions (expressed in variable ℓ) using

$$\Psi(\zeta,\eta)=\frac{\Phi(\zeta,\eta)}{\zeta+\eta}=\frac{M(\zeta)^T\Sigma M(\eta)}{\zeta+\eta}=\frac{\sum_{i,j}\tilde{\Phi}_{ij}\zeta^i\eta^j}{\zeta+\eta} \quad (5)$$

with $\tilde{\Phi}_{i,j}\in\mathbb{R}^{\mathtt{m}\times\mathtt{m}}$. Given an image representation, equation (5) gives us the unique storage function of the conservative system in the latent variables: refer [22, Section 5] for a discussion on conservative systems. Further, since the image representation $w=M\left(\frac{d}{dt}\right)\ell$ can be assumed to be observable without loss of generality, the stored energy can be expressed in terms of the manifest variable w using a polynomial left-inverse of $M(\xi)$ as follows. Let $M^\dagger(\xi)\in\mathbb{R}^{\mathtt{m}\times\mathtt{w}}[\xi]$ be such that $M^\dagger(\xi)M(\xi)=I_{\mathtt{m}}$. Then, the stored energy

$$Q_\Psi(\ell)=Q_{\widehat{\Psi}}(w) \quad \text{with } \widehat{\Psi}(\zeta,\eta):=M^\dagger(\zeta)^T\Psi(\zeta,\eta)M^\dagger(\eta). \quad (6)$$

The special case when the stored energy is to be expressed in terms of the *states* is central to this paper. It is known that for controllable systems, the energy stored can be expressed in the form x^TKx, where $K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is symmetric: see [22, Section 5]. Think of x as the capacitor voltages and inductor currents in an LC realization, but this paper considers other minimal state-space realizations also. Note that ability to express energy in terms of w, instead of ℓ (or x), requires that the variable ℓ (or x) is *observable* from w: this is true for the controllable case. See [12] for significance of observability in storage functions. Since we deal with storage functions expressed in the observable state variable, i.e. x^TKx, for the rest of this paper we use equation (6) to express x^TKx in terms of w and its derivatives. Hence hereafter we consider specific state-space realizations and then focus only on the computation of the matrix K.

In this paper, we use power := 2 input $\times$ output as the supply rate and call it the *passivity supply rate* i.e. for $w=(u,y)$

$$Q_\Sigma(w)=\begin{bmatrix}u\\y\end{bmatrix}^T\Sigma\begin{bmatrix}u\\y\end{bmatrix} \text{ induced by } \Sigma=\begin{bmatrix}0&1\\1&0\end{bmatrix} \quad (7)$$

where u, y are the input/output of the system. A system dissipative with respect to the passivity supply rate is called a *passive system*. In this paper, as in the literature, conservative systems with respect to the passivity supply rate are called *lossless systems*[1]. From the above discussion, it is clear that equation (4) written in terms of state variables x and adapted to passivity supply rate takes the form

$$\frac{d}{dt}x^TKx=2u^Ty\ . \quad (8)$$

Electrical circuits consisting of ideal inductors and/or capacitors have a lossless behavior. For example, consider the lossless system with transfer function $G(s)=\frac{2s}{s^2+1}$. This corresponds to an LC tank circuit (or a mass-spring system) with $C=\frac{1}{2}F$ and $L=2H$. Note that $G(s)+G(-s)=0$. Let v_C and i_L be the capacitor voltage and inductor current respectively. The kernel and image representation of the system (as discussed in Section 2-B) is $\begin{bmatrix}\frac{d^2}{dt^2}+1 & -2\frac{d}{dt}\end{bmatrix}\begin{bmatrix}v_C\\i_L\end{bmatrix}=0$ and $\begin{bmatrix}v_C\\i_L\end{bmatrix}=\begin{bmatrix}2\frac{d}{dt}\\\frac{d^2}{dt^2}+1\end{bmatrix}\ell$ respectively. The stored energy is given by $\frac{1}{2}v_C^2+2i_L^2$.

In order to simplify the exposition in this paper, we shall be using the passivity supply rate and deal with lossless systems only. However, all the methods reported in this paper can be applied to systems conservative with respect to other supply rates too.

E. Adjoint systems and duality

In this section, we give a brief introduction to adjoint systems and duality. We first define the Σ-orthogonal complement behavior $\mathfrak{B}^{\perp\Sigma}$ of a behavior $\mathfrak{B}$.

Definition 2.1. *Consider $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a nonsingular, symmetric $\Sigma\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The Σ-orthogonal complement $\mathfrak{B}^{\perp\Sigma}$ of $\mathfrak{B}$ is defined as*

$$\mathfrak{B}^{\perp\Sigma}:=\{v\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid\int_{-\infty}^{\infty}v^T\Sigma w\,dt=0 \text{ for all } w\in\mathfrak{B}\cap\mathfrak{D}\}.$$

The behavior $\mathfrak{B}^{\perp\Sigma}$ is also known in the literature as the adjoint system of $\mathfrak{B}$: see details in [22, Section 10]. If (A,B,C,D) is a minimal state-space representation of a system $\mathfrak{B}$ then, with respect to passivity supply rate, the system $\mathfrak{B}^{\perp\Sigma}$, with manifest variables (e,f), admits a minimal state-space representation of the form $\dot{z}=-A^Tz+C^Te$, $f=B^Tz-D^Te$. The variable z is called the dual-state and satisfies $\frac{d}{dt}x^Tz=u^Tf+y^Te$ for $\begin{bmatrix}u\\y\end{bmatrix}\in\mathfrak{B}$ and $\begin{bmatrix}f\\e\end{bmatrix}\in\mathfrak{B}^{\perp\Sigma}$.

F. Controller canonical form

Though the controller canonical form is standard, we include it for completeness. Consider a system with a strictly proper transfer function $G(s)=\frac{n(s)}{d(s)}$ where $n(s)=b_{\mathtt{n}-1}s^{\mathtt{n}-1}+b_{\mathtt{n}-2}s^{\mathtt{n}-2}+\cdots+b_0$ and $d(s)=s^{\mathtt{n}}+a_{\mathtt{n}-1}s^{\mathtt{n}-1}+\cdots+a_1s+a_0$. Define the controller canonical form state-space representation of the system

$$\dot{x}=Ax+Bu \qquad \text{and} \qquad y=Cx \quad (9)$$

where $A\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B,C^T\in\mathbb{R}^{\mathtt{n}}$ with A,B,C as

$$A:=\begin{bmatrix}0&1&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&0&1\\-a_0&-a_1&\cdots&-a_{\mathtt{n}-2}&-a_{\mathtt{n}-1}\end{bmatrix},\ B:=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix},\ C:=\begin{bmatrix}b_0\\\vdots\\b_{\mathtt{n}-2}\\b_{\mathtt{n}-1}\end{bmatrix}^T.$$

For $n(s)$ and $d(s)$ defined above, the states of the system corresponding to this controller canonical representation satisfy

$$x:=(\ell,\ \dot{\ell},\ \cdots,\ \ell^{(\mathtt{n}-1)}) \quad \text{and} \quad \begin{bmatrix}u\\y\end{bmatrix}=\begin{bmatrix}d(\frac{d}{dt})\\n(\frac{d}{dt})\end{bmatrix}\ell. \quad (10)$$

[1]Lossless systems, with u input and y output, are conservative with respect to the "passivity supply rate" u^Ty and have $D+D^T=0$. Similarly, all-pass systems are conservative with respect to the "bounded real supply rate" u^Tu-y^Ty. For all-pass systems the feedthrough term D satisfies $I-D^TD=0$. Hence, it can be shown that all arguments made for lossless systems are applicable to all-pass systems as well.

G. LMI and Algebraic Riccati equation

As explained in Subsection 2-D, consider a passive system with minimal i/s/o representation (A,B,C,D) and McMillan degree $\mathtt{n}$. Suppose x^TKx is the stored energy. Using $Q_{\widetilde{\Psi}}(w) = x^TKx$, the i/s/o representation of the passive system and the dissipation inequality (2), we get an LMI of the form

$$\begin{bmatrix} A^TK+KA & KB-C^T \\ B^TK-C & -(D+D^T) \end{bmatrix} \leqslant 0. \tag{11}$$

Contrary to the condition $D+D^T=0$, which happens when a system is lossless, for strictly passive case we have $D+D^T > 0$. For such systems, the Schur complement with respect to $D+D^T$ in LMI (11) gives the well known algebraic Riccati inequality. The algebraic Riccati equation (ARE)[2] associated with such a system with respect to the passivity supply rate is

$$A^TK+KA+(KB-C^T)(D+D^T)^{-1}(B^TK-C)=0. \tag{12}$$

Equation (12) indicates that existence of the ARE depends on the nonsingularity of $D+D^T$. For lossless systems, $D+D^T$ is identically zero and the ARE does not exist. However, for lossless systems the LMI (11) holds with equality[3], i.e.

$$A^TK+KA=0 \qquad \text{and} \qquad B^TK-C=0. \tag{13}$$

H. Minimal Polynomial Basis (MPB)

The *degree* of a polynomial vector $r(s)\in\mathbb{R}^{\mathtt{n}}[s]$ is defined as the maximum degree amongst the $\mathtt{n}$ components of the vector. Degree of the zero polynomial and the zero vector $\mathbb{R}^{\mathtt{n}}[s]$ is defined as $-\infty$.

Consider the polynomial matrix $R(s)\in\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[s]$ of rank $\mathtt{n}$. Suppose $P(s)\in\mathbb{R}^{\mathtt{m}\times(\mathtt{n}-\mathtt{m})}[s]$ is of rank $\mathtt{n}-\mathtt{m}$ and satisfies $R(s)P(s)=0$. Then the columns of $P(s)$ are said to form a basis of the nullspace of $R(s)$. Suppose the columns of $P(s)$ are $\{p_1(s),p_2(s),\ldots,p_{\mathtt{m}-\mathtt{n}}(s)\}$ ordered with degrees $d_1\leqslant d_2\leqslant\ldots\leqslant d_{\mathtt{m}-\mathtt{n}}$. The set $\{p_1(s),p_2(s),\ldots,p_{\mathtt{m}-\mathtt{n}}(s)\}$ is said to be a *minimal polynomial basis* of $R(s)$ if every other nullspace basis $\{q_1(s),q_2(s),\ldots,q_{\mathtt{m}-\mathtt{n}}(s)\}$ with degree $c_1\leqslant c_2\leqslant\ldots\leqslant c_{\mathtt{m}-\mathtt{n}}$ is such that $d_i\leqslant c_i$, for $i=1,2,\ldots,\mathtt{m}-\mathtt{n}$. The degrees of the vectors of minimal polynomial basis of $R(s)$ are called the *Forney invariant minimal indices* or *Kronecker indices* (more details in [11, Section 6.5.4]).

[2]There are different forms of ARE depending on the notion of power being used. For example, consider a system with input/state/output (i/s/o) representation $\dot{x}=Ax+Bu$ and $y=Cx+Du$, then for passivity studies (power = $2u^Ty$) we consider ARE of the form $A^TK+KA+(KB-C^T)(D+D^T)^{-1}(B^TK-C)=0$. On the other hand, for small gain system (power = u^Tu-y^Ty), the ARE takes the form $A^TK+KA+C^TC+(KB+C^TD)(I-D^TD)^{-1}(B^TK+D^TC)=0$. Note that the existence of ARE crucially depends on the nonsingularity of $D+D^T$ and $I-D^TD$ respectively. We call such conditions on the feedthrough term D as "regularity condition". For dissipative systems, ARE exists if and only if this regularity conditions on feedthrough term D is satisfied. When the ARE exists, it finds many applications: for example, see [1], [8], [20], amongst many others.

[3]One way of computing solutions of an ARE is to solve for the LME linked with it. Conservative systems do not admit an ARE but they do admit an LME. However, solving such LME corresponding to a conservative system is not possible with standard procedures like interior point methods due to absence of interior points to work with. Further conservative systems also admit linear matrix equations of the Sylvester form $PX+X^TQ=R$ where P,Q,R,X are matrices of suitable dimensions. Methods to solve the Sylvester equation are known in the literature. However, most of these methods require matrices P and Q to be square, while the matrix equations encountered in the conservative case have P and Q nonsquare.

3. LC REALIZATION BASED METHOD

The rest of this paper contains results for computation of stored energy and algorithms based on these results (see [14] also). This section uses the partial fraction/continued fraction expansion based method. The capacitor voltages and inductor currents in the electrical network are taken as the states while computing the storage function in this section.

A. Foster/Cauer based methods: SISO case

This method is based on viewing the lossless transfer function $G(s)$ as the driving point impedance/admittance of an *LC* network. Since the system is lossless, the poles and zeros of the system are all on the imaginary axis. Expansion of the proper transfer function $G(s)$ into its partial fractions using the Foster form gives

$$G(s)=\frac{r_0}{s}+\sum_{q=1}^{\mathtt{m}}\frac{r_qs}{s^2+\omega_q^2} \tag{14}$$

where[4] $r_0\geqslant 0, r_1,r_2,\ldots,r_{\mathtt{m}}>0$ and each $\omega_q>0$. For a system with proper transfer function $G(s)$ as in equation (14), a minimal i/s/o representation

$$\dot{x}_f=A_fx_f+B_fu_f \qquad \text{and} \qquad y_f=C_fx_f \tag{15}$$

is given by

$$A_f:=\text{diag }(A_0,A_1,\cdots,A_{\mathtt{m}}) \text{ where } A_q:=\begin{bmatrix} 0 & -r_q \\ \dfrac{\omega_q^2}{r_q} & 0 \end{bmatrix},\ A_0:=0,$$

$$B_f:=\begin{bmatrix} r_0 & r_1 & 0 & r_2 & 0 & \cdots & r_{\mathtt{m}} & 0 \end{bmatrix}^T\in\mathbb{R}^{2\mathtt{m}+1}$$

$$C_f:=\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix}\in\mathbb{R}^{2\mathtt{m}+1}, q=1,2,\cdots,\mathtt{m}.$$

On the other hand, expansion of a proper transfer function $G(s)=g_q(s)$ in continued fraction using Cauer-II methods involves the following iteration:

$$g_q(s)=\frac{h_q}{s}+\frac{1}{g_{q+1}(s)}, \quad g_{\mathtt{n}}(s):=\frac{h_{\mathtt{n}}}{s} \tag{16}$$

where $q=1,2,\cdots,\mathtt{n}$ and $\mathtt{n}$ is the McMillan degree of the system. For the sake of simplicity, we assume that the McMillan degree $\mathtt{n}$ of the system is odd. Consider the vectors $V:=\begin{bmatrix} v_1 & v_2 & \cdots & v_{\mathtt{m}} \end{bmatrix}^T$, $I:=\begin{bmatrix} i_1 & i_2 & \cdots & i_{\mathtt{m}-1} \end{bmatrix}^T$ and $B_2:=\begin{bmatrix} h_1 & h_3 & \cdots & h_{\mathtt{n}} \end{bmatrix}^T\in\mathbb{R}^{\mathtt{m}}$ where $\mathtt{m}:=\frac{\mathtt{n}+1}{2}$. For $p=1,2,\cdots,\mathtt{m}-1$, define $H^u,H^l\in\mathbb{R}^{(\mathtt{m}-1)\times(\mathtt{m}-1)}$ such that

$$H^u_{pj}:=\begin{cases} 0 & \text{for } p>j \\ h_{2p} & \text{for } p\leqslant j \end{cases} \quad \& \quad H^\ell_{pj}:=\begin{cases} h_{2p+1} & \text{for } p\geqslant j \\ 0 & \text{for } p<j \end{cases}$$

A minimal representation of the system $G(s)$,

$$\dot{x}_c=A_cx_c+B_cu_c \qquad \text{and} \qquad y_c=C_cx_c \tag{17}$$

is given by the following matrices:

$$A_c:=\begin{bmatrix} & H^u & \\ & 0 & \\ -H^\ell & & \end{bmatrix},\ B:=\begin{bmatrix} 0 \\ B_2 \end{bmatrix} \text{ and } C:=\begin{bmatrix} 0 \\ \mathbf{1}_{\mathtt{m}} \end{bmatrix}^T \text{ where } x_c:=\begin{bmatrix} I \\ V \end{bmatrix}.$$

[4]The residues in this expansion are assumed non-negative primarily to make contact with LC realization studies, where the residues affect the inductance and capacitance parameters. This is closely linked to positive definiteness of the obtained storage function and plays no further role in the algorithm. We do not dwell further on this: see [22, Section 6].

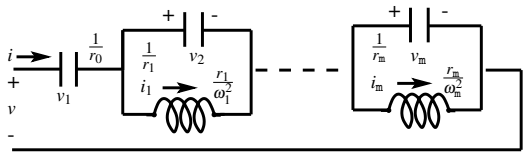

Fig. 1. LC realization based on partial fractions: Foster-I form

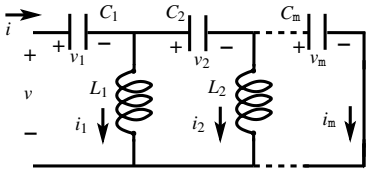

Fig. 2. LC realization based on continued fractions: Cauer-II form

The physical realization of transfer function in equation (14) in an LC network can be done using the Foster method (as shown in Figure 1) and the Cauer method (as in Figure 2).

Theorem 3.1. *Consider a controllable, lossless system with a strictly proper transfer function $G(s)$ of the form given in equations* (14) *and* (16)*. Assume the McMillan degree of $G(s)$ is odd. For each of the two cases (a) & (b) below corresponding to Foster and Cauer realizations, consider the state-space realizations in which the states are the capacitor voltages and inductor currents. Then, the stored energy*

$$x^T Kx = \sum_{L_j} L_j i_j^2 + \sum_{C_q} C_q v_q^2.$$

More precisely,

(a) For systems with proper impedance function as in equation (14) *and a minimal i/s/o representation as in* (15)*, the unique storage function is $x_f^T K_f x_f$, with the diagonal matrix $K_f \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ defined by*

$$K_f := \text{diag}\left(\frac{1}{r_0}, K_1, K_2, \ldots, K_{\mathtt{m}}\right) \text{ where } K_q := \begin{bmatrix} r_q^{-1} & 0 \\ 0 & r_i\omega_i^{-2} \end{bmatrix}$$

for $q = 1, 2, \cdots, \mathtt{m}$.

(b) For systems with proper admittance function as in equation (16) *and a minimal i/s/o representation as in* (17)*, the unique storage function is $x_c^T K_c x_c$, with the diagonal matrix $K_c \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ defined by*

$$K_c := \text{diag}\left(\frac{1}{h_2}, \frac{1}{h_4}, \cdots, \frac{1}{h_{\mathtt{n}-1}}, \frac{1}{h_1}, \frac{1}{h_3}, \cdots, \frac{1}{h_{\mathtt{n}}}\right).$$

Proof. **a)** Note that $\omega_q \neq \omega_j$ for $q \neq j$ guarantees controllability and observability[5] of the system. We prove the theorem for $\mathtt{n} = 5$, the general case follows from it. The transfer function in partial fraction form is $G(s) = \frac{r_0}{s} + \frac{r_1 s}{s^2+\omega_1^2} + \frac{r_2 s}{s^2+\omega_2^2}$. Hence $A = \text{diag}\left\{0, \begin{bmatrix} 0 & -r_1 \\ \frac{r_1}{\omega_1^2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -r_2 \\ \frac{r_2}{\omega_2^2} & 0 \end{bmatrix}\right\}$, $B = \begin{bmatrix} r_0 & r_1 & 0 & r_2 & 0 \end{bmatrix}^T$, $C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Consider $K_f = K_f^T = [k_{xy}] \in \mathbb{R}^{5\times 5}$ where $x, y = 0, 1, \cdots, 4$. Since $\omega_1 \neq \omega_2$ and K_f satisfies $A_f^T K_f + K_f A_f = 0$, we have $K_f = \text{diag}\,\{k_{00}, k_{11}, \cdots, k_{44}\}$, $\frac{k_{22}}{k_{11}} = \frac{r_1}{r_2}$, and $\frac{k_{44}}{k_{33}} = \frac{r_3}{r_4}$. Further use of the equation $B_f^T K_f - C_f = 0$, we get a unique K_f of the form $K_f = \text{diag}\left\{\frac{1}{r_0}, \frac{1}{r_1}, \frac{r_1}{\omega_1^2}, \frac{1}{r_2}, \frac{r_2}{\omega_2^2}\right\}$. This completes the proof Statement a) of Theorem 3.1.

b) We give a brief outline of the proof here due to paucity of space. We show it for a system with McMillan degree $\mathtt{n} = 5$. The proof for the general case follows from it. Using equation (17), we have

$$A_c = \begin{bmatrix} & & & h_2 & h_2 \\ & & & 0 & h_4 \\ & & 0 & & \\ -h_3 & 0 & & & \\ -h_5 & -h_5 & & & \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ h_1 \\ h_3 \\ h_5 \end{bmatrix}, C_c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T$$

Solving the matrix equations $A_c^T K_c + K_c A_c = 0$ and $B_c^T K_c - C_c = 0$, we get the unique diagonal matrix $K_c := \text{diag}\left(\frac{1}{h_2}, \frac{1}{h_4}, \frac{1}{h_1}, \frac{1}{h_3}, \frac{1}{h_5}\right)$. Hence K_c induces the storage function $x_c^T K_c x_c$ of the system. This completes the proof Statement b) of Theorem 3.1. □

Based on the above result, we report an algorithm referred to as the 'LC realization based' algorithm in Section 7.1 to compute storage function of lossless systems. Results in Theorem 3.1 is easily extendable to lossless systems with even McMillan degree. We present an example next to demonstrate Theorem 3.1.

Example 3.2. *Consider a lossless behavior $\mathfrak{B}$ with transfer function $G(s) = \frac{8s^2+1}{6s^3+s}$.*

LC realization based method:

$$G(s) = \frac{1}{s} + \frac{s/3}{s^2 + 1/6} = \frac{1}{s} + \frac{1}{\frac{1}{2s} + \frac{1}{1/3s}}$$

Clearly, $r_0 = 1, r_1 = \frac{1}{3}, \omega_1^2 = \frac{1}{6}$ and $h_1 = 1, h_2 = \frac{1}{2}, h_3 = \frac{1}{3}$.

Foster realization (Theorem 3.1(a)): $x = (v_{C1}, v_{C2}, i_L)$

$$A_f = \begin{bmatrix} 0 & & \\ & 0 & -\frac{1}{3} \\ & \frac{1}{2} & 0 \end{bmatrix}, B_f = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{3} \end{bmatrix}, C_f = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T, K_f = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 2 \end{bmatrix}$$

Cauer realization (Theorem 3.1(b)):$x = (i_L, v_{C1}, v_{C2})$

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}, C_c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^T, K_c = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 3 \end{bmatrix}$$

B. Partial fraction based method – MIMO case

In this section we generalize the SISO result based on Foster LC realization to MIMO lossless systems. Gilbert's realization is a well known method to find the i/s/o representation of MIMO systems [11, Section 6.1]. However, such a method does not guarantee an i/s/o representation with the inductor currents and capacitor voltages as the states in an LC realization. We need such a form of A since the proposed method uses energy stored in inductor and capacitor as the storage function. In this section, we present a method to find the i/s/o representation of a lossless system such that the inductor

[5]It can be verified that the controllability matrix $[B_f\ A_f B_f\ \cdots\ A_f^{n-1} B_f]$ and observability matrix $[C_f^T\ A_f^T C_f^T\ \cdots\ A_f^{n-1^T} C_f^T]$ has full rank if and only if $\omega_q \neq \omega_j$ for $q \neq j$.

current and capacitor voltage are the states of the system. We then proceed to compute the storage function matrix K with respect to these states.

1) Gilbert's realization: Before we present the main results of the section, we revisit Gilbert's theorem as given in [6, Theorem 7]. This proposition gives a method to compute the McMillan degree of a MIMO system.

Proposition 3.3. *Consider a proper rational transfer-function matrix $G(s)$ whose elements have semi-simple poles at $s = \lambda_i$, $i = 1,\cdots,\mathtt{q}$ in the s-plane. Let the partial fraction expansion of $G(s)$ be $\sum_{i=1}^{\mathtt{q}} \frac{R_i}{s-\lambda_i} + D$ where $R_i = \lim_{s\to\lambda_i}(s-\lambda_i)G(s)$ and $D = \lim_{s\to\infty} G(s)$. Corresponding to each λ_i, let the rank of the R_i matrix be $\mathtt{r}_i$. Then McMillan degree corresponding to the system is given by $\mathtt{n} = \sum_{\mathtt{i}=1}^{\mathtt{q}} \mathtt{r}_\mathtt{i}$.*

For systems with imaginary axis poles, Proposition 3.3 is adapted and presented as Lemma 3.5 in the next subsection.

2) Adapted Gilbert's MIMO realization for lossless case: We first review necessary and sufficient condition for a real rational matrix to be positive real[6] and/or lossless (see [20]).

Proposition 3.4. *A transfer matrix $G(s) \in \mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$ is positive real if and only if*

1) *Each element of $G(s)$ is analytic in the open right half s-plane.*
2) *$G(j\omega) + G(-j\omega)^T \geqslant 0$ for all $\omega \in \mathbb{R}$ such that $j\omega$ is not a pole of any element of $G(s)$.*

Further, a positive real rational transfer matrix $G(s)$ is lossless if and only if $G(s) + G(-s)^T = 0$.

Let $G(s) = [g_{ij}]$ and poles of g_{ij} be represented as $\mathscr{P}(g_{ij})$. Using Proposition 3.4, we conclude that a necessary condition for a positive real transfer matrix to be lossless is $\mathscr{P}(g_{ij})_{i\neq j} \subseteq \mathscr{P}(g_{ij})_{i=j}$ i.e. the poles of the off-diagonal entries of $G(s)$ are also poles of the diagonal entries. This means that the residue matrix corresponding to the poles of the diagonal entries of $G(s)$ that are not poles of the off-diagonal entries of $G(s)$ will be diagonal. Therefore a more generalized case would be when the poles of the off-diagonal and diagonal entries of $G(s)$ are the same: thus ensuring non-diagonal residue matrix. We deal with such systems. Suppose the poles of $G(s)$ are $\mathscr{P}(G(s)) = \{\pm j\omega_1, \pm j\omega_2, \cdots, \pm j\omega_\mathtt{q}\}$. Using partial fraction method, we write $G(s) = \Sigma_{\ell=1}^{\mathtt{q}} G_\ell(s)$ where $G_\ell(s) = \frac{R_\ell(s)}{s^2+\omega_\ell^2}$, $R_\ell(s) \in \mathbb{R}[s]^{\mathtt{p}\times\mathtt{p}}$.

Next we adapt Proposition 3.3 for the case of imaginary axis poles and present it as a lemma. We will use the lemma to construct the minimal i/s/o representation of a MIMO lossless system.

Lemma 3.5. *Consider a rational transfer matrix $G(s)$ whose elements have simple poles at $s = 0$ and/or $s = \pm j\omega_i$, $i = 1,2,\cdots\mathtt{q}$. Let the partial expansion of $G(s)$ be*

$$\frac{R_0}{s} + \sum_{l=1}^{\mathtt{q}} \frac{R_\ell(s)}{s^2+\omega_\ell^2} + D = \frac{R_0}{s} + \sum_{l=1}^{\mathtt{q}} \left(\frac{Z_\ell}{s+j\omega_\ell} + \frac{Z_\ell^*}{s-j\omega_\ell} \right) + D$$

where $R_0 = \lim_{s\to 0} sG(s) \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}, D = \lim_{s\to\infty} G(s) \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$ are the residual matrices and $R_\ell(s)$ is the residue matrix corresponding to the conjugate pair of poles on the imaginary axis. Let $\mathtt{r}_0 =$ rank of R_0 and $\mathtt{r}_\ell =$ rank of Z_ℓ. Then the minimal order of the i/s/o representation of the system is $\mathtt{n} = \mathtt{r}_0 + \sum_{\ell=1}^{\mathtt{q}} 2\times\mathtt{r}_\ell$.

Proof. Note that rank(Z_ℓ) = rank(Z_ℓ^*) = $\mathtt{r}_\ell$. Hence using Proposition 3.3, minimum number of states for the system is $\mathtt{n} = \mathtt{r}_0 + \sum_{\ell=1}^{\mathtt{q}} \text{rank}(Z_\ell) + \sum_{\ell=1}^{\mathtt{q}} \text{rank}(Z_\ell^*) = \mathtt{r}_0 + \sum_{\ell=1}^{\mathtt{q}} 2\times\mathtt{r}_\ell$. This proves the lemma. □

From Proposition 3.4, we know that for a lossless system $G(s) = -G(-s)^T$. Hence the partial fractions corresponding to each of the poles ω_ℓ have a skew symmetric structure. Consider $G_\ell = [g_{ij}^\ell]$. The general structure[7] of $G_\ell(s)$ is given by

$$g_{ij}^\ell(s) = \frac{\alpha_{ij}^\ell s - \beta_{ij}^\ell}{s^2+\omega_i^2} \text{ where } \beta_{ij}^\ell = -\beta_{ji}^\ell. \tag{18}$$

We state and prove a theorem next which gives a procedure for construction of the (A,B,C) matrices for lossless systems. For simplicity, we consider that the transfer matrix has only one pair of conjugate poles on the imaginary axis i.e. we consider $\mathtt{q} = 1$ in $G(s) = \Sigma_{\ell=1}^{\mathtt{q}} G_\ell(s)$. (For the general case i.e. $\mathtt{q} > 1$ we just have to apply the method described in Theorem 3.6 (for $\mathtt{q} = 1$) on each partial fraction.)

Theorem 3.6. *Consider a lossless transfer matrix $G(s) = \frac{R(s)}{s^2+\omega^2} = \frac{Z}{s+j\omega} + \frac{Z^*}{s-j\omega}$ where $R(s) = sX - Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[s]$ and $Z,X,Y \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$. Elements of $G(s)$ are of the form given in equation* (18)*. Let the rank of Z be $\mathtt{r}$. The i-th rows of $R(s)$, X and Y are represented as z_i, x_i and y_i respectively. Then the following state-space realization is minimal.*

1) *$A := \text{diag } (J_\omega, J_\omega, \ldots, J_\omega) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ where $\mathtt{n} := 2\times\mathtt{r}$ and $J_\omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$.*
2) *Construct $B_j := \begin{bmatrix} x_j \\ \frac{y_j}{\omega} \end{bmatrix}$ for each $j = 1,2,\ldots,\mathtt{p}$ such that $B_j \neq 0$ and define*

$$B := \text{col } (B_1, B_2, \cdots, B_\mathtt{r}) \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}.$$

3) *There exist row vectors $c_1, c_2, \cdots, c_j \in \mathbb{R}^{\mathtt{n}}$ such that $C := \text{col } (e_1, e_3, \cdots, e_{2\mathtt{r}-1}, c_1, c_2, \cdots c_j) \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$ where $c_j \in$ span of $\{e_1, e_2, \cdots, e_{2\mathtt{r}-1}\}$ and e_i is the i-th row of the identity matrix.*

Proof. The proof for the general case follows from the proofs of the following two special cases:

1) $G(s)$ is nonsingular and $\mathtt{p} = 2$.
2) $G(s)$ is singular and $\mathtt{p} = 3$.

Case 1: $G(s)$ is nonsingular: For $\mathtt{p} = 2$. Using equation (18),

$$R(s) = s\begin{bmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{12} & \alpha_2 \end{bmatrix} - \begin{bmatrix} 0 & \beta_{12} \\ -\beta_{12} & 0 \end{bmatrix} = sX - Y.$$

Since $G(s)$ is nonsingular, $\mathtt{r} = 2$ and $\mathtt{n} = 4$ (By Lemma 3.5). Here $A = \text{diag } \left\{ \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \right\} \in \mathbb{R}^{4\times 4}$.

[6] We focus on lossless systems in this paper. Proposition 3.4 can hence be taken as definition of positive real.

[7] In general the elements of the transfer matrix $G(s)$ may not be proper. However, there always exists an input-output partition such that the transfer matrix is proper [22, Section 2]. Hence without loss of generality, we assume the transfer matrix to be proper.

$$B^T=\begin{bmatrix}\alpha_1 & 0 & \alpha_{12} & -\frac{\beta_{12}}{\omega}\\ \alpha_{12} & \frac{\beta_{12}}{\omega} & \alpha_2 & 0\end{bmatrix},\quad C=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}.$$

Using (A,B,C), it is easy to verify that $C(sI-A)^{-1}B=G(s)$. Applying the same construction procedure for the matrices A,B,C, the theorem is proved for any nonsingular $G(s)$ of arbitrary order $\mathtt{n}$.

Case 2: $G(s)$ is singular: For $\mathtt{p}=3$. Using equation (18),

$$R(s)=s\begin{bmatrix}\alpha_1 & \alpha_{12} & \alpha_{13}\\ \alpha_{12} & \alpha_2 & \alpha_{23}\\ \alpha_{13} & \alpha_{23} & \alpha_3\end{bmatrix}-\begin{bmatrix}0 & \beta_{12} & \beta_{13}\\ -\beta_{12} & 0 & \beta_{23}\\ -\beta_{13} & -\beta_{23} & 0\end{bmatrix}=sX-Y.$$

Since $G(s)$ is singular, consider the case when the rows of $R(s)$ are related by the following relation $a\times z_1+b\times z_2=:z_3$. Here $\mathtt{r}=2$ and hence $\mathtt{n}=4$ (By Lemma 3.5). Therefore consider $A=\operatorname{diag}\left\{\begin{bmatrix}0&-\omega\\ \omega&0\end{bmatrix},\begin{bmatrix}0&-\omega\\ \omega&0\end{bmatrix}\right\}\in\mathbb{R}^{4\times4}$.

$$B^T=\begin{bmatrix}\alpha_1 & 0 & \alpha_{12} & -\frac{\beta_{12}}{\omega}\\ \alpha_{12} & \frac{\beta_{12}}{\omega} & \alpha_2 & 0\\ \alpha_{13} & \frac{\beta_{13}}{\omega} & \alpha_{23} & \frac{\beta_{23}}{\omega}\end{bmatrix}\text{ and }C=\begin{bmatrix}1&0&0&0\\0&0&1&0\\a&0&b&0\end{bmatrix}.$$

With the constructed triplet (A,B,C), we have $C(sI-A)^{-1}B=G(s)$. This completes the proof. □

Given the above minimal state-space realization of $G(s)$, we compute the storage function associated with the system. The storage function associated to a lossless transfer function $G(s)$ must satisfy the matrix equations (13) where the storage function is induced by the symmetric matrix K. Let $K=[k_{ij}]$ and $K=K^T$. Since K satisfies the first matrix equation in equation (13), K has to have the form

$$K=\begin{bmatrix}k_{11} & 0 & k_{13} & k_{14} & \cdots & k_{1(\mathtt{n}-1)} & k_{1\mathtt{n}}\\ 0 & k_{11} & -k_{14} & k_{13} & \cdots & -k_{1\mathtt{n}} & k_{1(\mathtt{n}-1)}\\ k_{13} & -k_{14} & k_{33} & 0 & \cdots & k_{3(\mathtt{n}-1)} & k_{3\mathtt{n}}\\ k_{14} & k_{13} & 0 & k_{33} & \cdots & -k_{3\mathtt{n}} & k_{3(\mathtt{n}-1)}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ k_{1(\mathtt{n}-1)} & -k_{1\mathtt{n}} & k_{3(\mathtt{n}-1)} & -k_{3\mathtt{n}} & \cdots & k_{\mathtt{nn}} & 0\\ k_{1\mathtt{n}} & k_{1(\mathtt{n}-1)} & k_{3\mathtt{n}} & k_{3(\mathtt{n}-1)} & \cdots & 0 & k_{\mathtt{nn}}\end{bmatrix}.$$

Writing the $\frac{\mathtt{n}^2}{4}$ unknown elements in K as a vector y_k, we define

$$y_k^T:=\begin{bmatrix}k_{11} & k_{13} & \cdots & k_{1\mathtt{n}} & k_{33} & k_{35} & \cdots & k_{3\mathtt{n}} & \cdots & k_{\mathtt{nn}}\end{bmatrix}\in\mathbb{R}^{\frac{\mathtt{n}^2}{4}}. \tag{19}$$

The matrix K has to further satisfy the second matrix equation in (13). Hence we have $\mathtt{p}\times\mathtt{n}$ linear equations of the form $Py_k=q$ where $q\in\mathbb{R}^{\mathtt{pn}}$. Solution to these set of linear equations[8] gives us the desired storage function K.

Note that Theorem 3.6 gives a minimal realization of $G(s)$. With the same A obtained as in Statement 1 of Theorem 3.6, we can have different (B,C) pairs. Depending on the specific (B,C), we get different sets of linear equations. We compute the storage function of the system using the triplet (A,B,C) obtained in Theorem 3.6. The unknown elements of K are hence given by $y_k=P^\dagger q$ where $P^\dagger$ is the pseudo-inverse of P.

[8]Note that for a *controllable* conservative system there exists a unique symmetric matrix K that induces the storage function x^TKx. Hence for such a system the vector y_k defined in equation (19) is unique as well. Further, the facts that every conservative system admits a storage function and the unique K satisfies matrix equations (25) together guarantee that $q\in\operatorname{img}P$ in equation $Py_k=q$.

The special structure of the triplet (A,B,C) is used to create P and q in the set of linear equations $Py_k=q$. For simplicity, we show the construction of P and q for a lossless nonsingular $G(s)\in\mathbb{R}^{2\times2}(s)$. Since $G(s)$ is nonsingular, the minimum number of states of $G(s)$ is 4.

Construction of $P\in\mathbb{R}^{8\times4}$ and $q\in\mathbb{R}^8$:

Let $B^T:=\begin{bmatrix}b_1 & b_2 & b_3 & b_4\end{bmatrix}\in\mathbb{R}^{2\times4}$ and $C:=\begin{bmatrix}c_1 & c_2 & c_3 & c_4\end{bmatrix}\in\mathbb{R}^{2\times4}$.

1) Construction of matrix P:
   Partition $P^T=\begin{bmatrix}P_1^T & P_2^T & \cdots & P_{\frac{\mathtt{n}}{2}}^T\end{bmatrix}$ where $P_i\in\mathbb{R}^{2\mathtt{p}\times\mathtt{p}^2}$. Further, each P_i is partitioned as $\begin{bmatrix}P_{i1} & P_{i2} & \cdots & P_{i\mathtt{p}}\end{bmatrix}$ where $P_{ij}\in\mathbb{R}^{2\mathtt{p}\times(2\mathtt{p}-2j+1)}$. Therefore

$$P:=\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}=\left[\begin{array}{ccc|c}b_1 & b_3 & b_4 & 0\\ b_2 & b_4 & -b_3 & 0\\ \hline 0 & b_1 & -b_2 & b_3\\ 0 & b_2 & b_1 & b_4\end{array}\right].$$

2) Construction of vector q:

$$q=\operatorname{col}\begin{pmatrix}c_1, & c_2, & c_3, & c_4\end{pmatrix}.$$

This construction is to be used for any lossless system with nonsingular $G(s)$. For a singular $G(s)$, a slightly modified construction procedure is to be used after appropriate zero padding in B and C: this is skipped since the procedure is straightforward.

4. Bezoutian based method

This section contains results to compute storage function of lossless system using the idea of Bezoutian. We deal with the SISO case first and then move on to the MIMO case.

A. Bezoutian based method – SISO case

Consider a lossless SISO system $G(s)=\frac{n(s)}{d(s)}$ with an observable image representation $w=M(\frac{d}{dt})\ell$ where $w=\begin{bmatrix}u\\y\end{bmatrix}$, $M(\xi):=\begin{bmatrix}d(\xi)\\n(\xi)\end{bmatrix}$. Here $\Sigma=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $d(s)$ is a monic polynomial. From equation (5),

$$\Psi(\zeta,\eta)=\frac{d(\zeta)n(\eta)+d(\eta)n(\zeta)}{\zeta+\eta}=\frac{\sum_{i,j}\tilde{\Phi}_{i,j}\zeta^i\eta^j}{\zeta+\eta}. \tag{20}$$

Here $\tilde{\Phi}_{i,j}\in\mathbb{R}$. The storage function can be calculated by what may be called, "polynomial long division technique" which is based on Euclidean long division of polynomials. We state this as a result below.

Theorem 4.1. *Consider a Σ-lossless behavior $\mathfrak{B}$ with transfer function $G(s)=\frac{n(s)}{d(s)}$ where $d(s)$ is a monic polynomial and the controller canonical state-space realization:*

$$\dot{x}=Ax+Bu\qquad\text{and}\qquad y=Cx. \tag{21}$$

Construct the two variable polynomial $z_b(\zeta,\eta)$, induced by the "Bezoutian" of the polynomials $n(s)$ and $d(s)$ by

$$z_b(\zeta,\eta):=\frac{n(\zeta)d(\eta)+n(\eta)d(\zeta)}{\zeta+\eta}=\begin{bmatrix}1\\ \zeta\\ \vdots\\ \zeta^{\mathtt{n}-1}\end{bmatrix}^T Z_b\begin{bmatrix}1\\ \eta\\ \vdots\\ \eta^{\mathtt{n}-1}\end{bmatrix}, \tag{22}$$

where $Z_b \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is the corresponding symmetric matrix. Then, $x^T Z_b x$ is the unique storage function for the Σ-lossless system with state-space description (21)*, i.e. $\frac{d}{dt}x^T Z_b x = 2uy$.*

Proof. Consider the image representation of $\mathfrak{B}$: $w = M(\frac{d}{dt})\ell = \begin{bmatrix} d(s) \\ n(s) \end{bmatrix}\ell$. Using equations (5) and (7), we have $\Psi(\zeta,\eta) =$

$$\frac{\begin{bmatrix} d(\zeta) \\ n(\zeta) \end{bmatrix}^T \Sigma \begin{bmatrix} d(\eta) \\ n(\eta) \end{bmatrix}}{\zeta+\eta} = \frac{n(\zeta)d(\eta)+n(\eta)d(\zeta)}{\zeta+\eta} = z_b(\zeta,\eta).$$

Further, $\Psi(\zeta,\eta)$ is a symmetric 2-variable polynomial in ζ and η. Hence

$$\Psi(\zeta,\eta) = \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{\mathtt{n}-1} \end{bmatrix}^T \tilde{\Psi} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \\ \vdots \\ \eta^{\mathtt{n}-1} \end{bmatrix} \quad \text{(Note that } \tilde{\Psi} = Z_b\text{)}.$$

The storage function is $Q_\Psi(\frac{d}{dt})\ell = x^T\tilde{\Psi}x$ where $x = (\ell, \dot{\ell}, \cdots, \ell^{(\mathtt{n}-1)})$ as in equation (10), and x corresponds to the state of the behavior $\mathfrak{B}$. Hence $x^T\tilde{\Psi}x = x^T Z_b x$ is the storage function of $\mathfrak{B}$. □

Remark 4.2. The conventional Bezoutian of two polynomials $p(x)$ and $q(x)$ is defined as $B_z(x,y) := \frac{p(x)q(y)-p(y)q(x)}{x-y}$. Notice the change in sign between this conventional Bezoutian definition and the one defined in equation (22): inspite of the sign change, we call z_b the Bezoutian due to the following reasons. In any lossless transfer function $\frac{n(s)}{d(s)}$, when the order of the system is even then $n(s)$ is an odd polynomial i.e. $n(-s) = -n(s)$ and $d(s)$ is even polynomial i.e. $d(-s) = d(s)$. Thus our definition is same as the conventional one if we substitute $x = -\zeta$; $y = \eta$ when the order of the system is even and $x = \zeta$; $y = -\eta$ when the order of the system is odd. Hence for lossless case $B_z(x,y) = z_b(\zeta,\eta)$ where $x = -\zeta$, $y = \eta$ for even order systems and $x = \zeta$, $y = -\eta$ for odd order systems. In fact Z_b of eq. (22) is nonsingular if and only if $n(s)$ and $d(s)$ are coprime. This justifies the use of 'Bezoutian' for $z_b(\zeta,\eta)$ defined in equation (22).

Methods to compute the Bezoutian: There are various methods of finding the Bezoutian z_b of two polynomials. In this paper we propose three different methods to compute z_b:

(a) Euclidean long division method,
(b) Pseudo-inverse method, and
(c) 2 dimensional discrete Fourier transform method.

Euclidean long division method: Though Theorem 4.1 involves *bivariate* polynomial manipulation, Algorithm 7.3 below uses only *univariate* polynomial operations. The algorithm is similar to Euclidean long division. As in [3], write $\Phi(\zeta,\eta) = \phi_0(\eta) + \zeta\phi_1(\eta) + \cdots + \zeta^{\mathtt{n}}\phi_{\mathtt{n}}(\eta)$. Then the storage function $\Psi(\zeta,\eta) = \psi_0(\eta) + \zeta\psi_1(\eta) + \cdots + \zeta^{\mathtt{n}-1}\psi_{\mathtt{n}-1}(\eta)$ can be computed by the following recursion (see [3, Section 6.5]) with $k = 1, \ldots, \mathtt{n}-1$:

$$\psi_0(\xi) := \frac{\phi_0(\xi)}{\xi}, \qquad \psi_k(\xi) := \frac{\phi_k(\xi) - \psi_{k-1}(\xi)}{\xi}. \tag{23}$$

Pseudo-inverse method: Consider

$$\Psi(\zeta,\eta) = \sum_{i,j} \tilde{\Psi}_{ij}\zeta^i\eta^j \text{ where } [\tilde{\Psi}_{ij}] =: \tilde{\Psi} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}.$$
$$\Phi(\zeta,\eta) = \sum_{i,j} \tilde{\Phi}_{ij}\zeta^i\eta^j \text{ where } [\tilde{\Phi}_{ij}] =: \tilde{\Phi} \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}.$$

From equation (5), we have

$$(\zeta+\eta)\Psi(\zeta,\eta) = M(\zeta)^T\Sigma M(\eta) = \Phi(\zeta,\eta). \tag{24}$$

Using equation (24), we have

$$\sigma_R(\tilde{\Psi}) + \sigma_D(\tilde{\Psi}) = \tilde{\Phi} \tag{25}$$

where $\sigma_R(\tilde{\Psi}) := \begin{bmatrix} \mathbf{0}^T & \tilde{\Psi} \\ 0 & \mathbf{0} \end{bmatrix}$, $\sigma_D(\tilde{\Psi}) := \begin{bmatrix} \mathbf{0} & 0 \\ \tilde{\Psi} & \mathbf{0}^T \end{bmatrix}$ and $\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$: see also [21]. Due to the symmetry of $\Psi(\zeta,\eta)$ and $\Phi(\zeta,\eta)$ the number of unknowns in $\tilde{\Psi}$ is $\frac{\mathtt{n}(\mathtt{n}+1)}{2}$ and the number of distinct elements in $\tilde{\Phi}$ is $\frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2}$. Hence the matrix equation (25) can be rewritten as linear equations of the form

$$P_b y = q_b \quad \text{where } P_b \in \mathbb{R}^{\frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2}\times\frac{\mathtt{n}(\mathtt{n}+1)}{2}}, q_b \in \mathbb{R}^{\frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2}} \tag{26}$$
$$\text{and } y := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \cdots & \Psi_{1n} & \Psi_{22} & \cdots & \Psi_{2n} & \cdots & \Psi_{nn} \end{bmatrix}^T.$$

For a dissipative system, a storage function exists and this guarantees $q_b \in \operatorname{im} P_b$. We compute the pseudo-inverse[9] of the matrix P_b i.e. $P_b^\dagger$ and obtain $y := P_b^\dagger q_b$. This gives the storage function. The above discussion is a proof of the following corollary.

Corollary 4.3. *Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with transfer matrix $G(s) = \frac{n(s)}{d(s)}$ where $d(s)$ is a monic polynomial and construct $\tilde{\Phi} = [\tilde{\Phi}_{i,j}]$ as in equation* (20)*. Let the matrix equation* (25) *be written in the linear equation form $P_b y = q_b$. Then the following are equivalent*

1) *$\mathfrak{B}$ is lossless.*
2) *There exists a unique symmetric K such that $x^T Kx = 2u^T y$.*
3) *P_b is full column rank and $q_b \in$ img P_b.*

2 dimensional discrete Fourier transform method (2D-DFT): Another technique we propose to compute the Bezoutian of equation (22) is the 2D-DFT method. Consider the matrix $\tilde{I} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that for a system with McMillan degree $\mathtt{n}$, the term $\zeta+\eta$ in equation (24) can be written as

$$\zeta+\eta = \sum_{i,j} \tilde{Q}_{ij}\zeta^i\eta^j \text{ where } [\tilde{Q}_{ij}] = \tilde{Q} = \begin{bmatrix} \tilde{I} & 0 \\ 0 & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}.$$

Here $\mathbf{0}$ is the zero matrix of dimension $(\mathtt{n}-1)\times(\mathtt{n}-1)$. Appending a row and column in Z_b (from Theorem 4.1), we rewrite the Bezoutian as

$$z_b(\zeta,\eta) = \sum_{i,j} \tilde{Z}_{b_{ij}}\zeta^i\eta^j \text{ where } [\tilde{Z}_{b_{ij}}] = \tilde{Z}_b = \begin{bmatrix} Z_b & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}.$$

We compute Z_b using the formula

$$Z_b = \tilde{Z}_b(1:\mathtt{n}, 1:\mathtt{n}) \text{ where } \tilde{Z}_b = \mathscr{F}^{-1}[\mathscr{F}(\tilde{\Phi})./\mathscr{F}(\tilde{Q})]$$

where 2D-DFT and its inverse operation are represented as $\mathscr{F}$ and $\mathscr{F}^{-1}$ respectively and the symbol ./ represents *element-wise* division. Note that for the element-wise division to be

[9]The solution of $P_b y = q_b$ can also be found using Scilab's so-called 'backslash' implementation (i.e. $P_b \backslash q_b$). Technically, using the backslash method is faster. However, in the case of Scilab and for the orders we considered, both the backslash and the pseudo-inverse methods require the same computational time. Hence we proceed with pseudo-inverse method only.

possible, $\mathscr{F}(\tilde{Q})$ must have every element nonzero. Using the definition/formula of two dimensional DFT, it can be shown that $\mathscr{F}(\tilde{Q})$ has all entries nonzero if $\mathtt{n}$ is even. Hence the procedure described here is directly applicable for even order systems. With a straightforward and meticulous modification, one deals with the odd order case also: we hence skip this. We summarize the above 3 methods in the following theorem for easy reference. The description of the methods contains the proof.

Theorem 4.4. *Consider the problem of finding* $K = [K_{ij}] \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *such that* $\sum_{i,j} K_{ij}\zeta^i\eta^j = \frac{\Phi(\zeta,\eta)}{\zeta+\eta}$. *Then each of the following algorithms gives* K

1) *Euclidean long division using equation (equation* (23)*),*
2) *Pseudo-inverse of* P *in the linear equation* $Py = q$ *(using Corollary 4.3),*
3) *Two dimensional Fourier transform followed by element wise division and inverse Fourier transform.*

After having described these procedures to compute the Bezoutian, we next compare them for accuracy and time.

B. Experimental setup and procedure

Experimental setup: The experiments were carried out on an Intel Core i3 computer at 3.30 GHz with 4 GB RAM using Ubuntu 14.04 LTS operating system. The relative machine precision is $\varepsilon \approx 2.22 \times 10^{-16}$. Numerical computational package Scilab 5.5 (which, like Matlab and Python-SciPy, NumPy, relies on LAPACK for core numerical computations) has been used to implement the algorithms.

Experimental procedure: Randomly generated transfer functions of lossless systems are used to test the algorithms. Computation time and error for each transfer function order has been averaged over three different randomly generated transfer functions. To nullify the effect of CPU delays the computation time to calculate K for each transfer function is further averaged over hundred iterations.

C. Experimental results

1) Computation Time: Figure 3 shows a comparison of the time taken by each of the Bezoutian based methods viz. Euclidean long division, Pseudo-inverse and 2D-DFT. From the plot it is clear that the Euclidean long division based method and pseudo-inverse methods are comparable to each other and are much faster than 2D-DFT.

2) Computation error: As discussed in Section 2-G, lossless systems satisfy LMI (11) with *equality*. In view of this, define the error associated with the computation of K as

$$\mathrm{Err}(K) := \left\| \begin{bmatrix} A^TK + KA & KB - C^T \\ B^TK - C & 0 \end{bmatrix} \right\|_2. \qquad (27)$$

The matrix K obtained from the above procedures must ideally yield $\mathrm{Err}(K) = 0$. Figure 4 shows the error in computation of storage function using the three Bezoutian based methods discussed above. All the three methods have comparable errors. From the comparison it is clear that Euclidean long division performs better that the other two methods. Hence we present an example next to show how the algorithm for

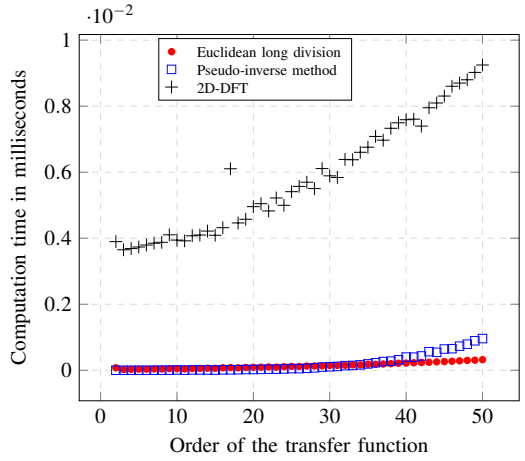

Fig. 3. Plot of computation time versus system's order.

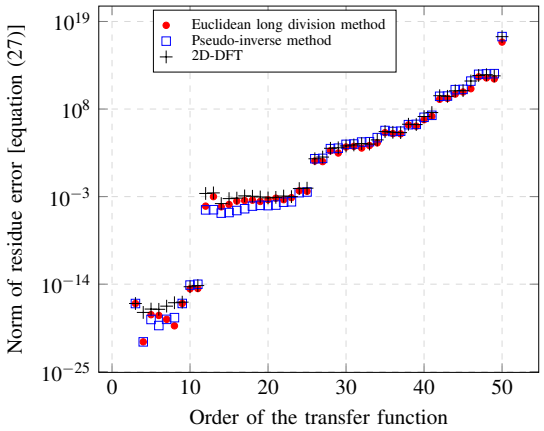

Fig. 4. Plot of error (see equation (27)) versus system's order.

computation of the Bezoutian based on the Euclidean long division method is constructed.

Example 4.5. Bezoutian based method *: Consider the system* $G(s) = \frac{8s^2+1}{6s^3+s} = \frac{\frac{8}{6}s^2+\frac{1}{6}}{s^3+\frac{1}{6}s}$. *Here* $N = [0 \ \frac{8}{6} \ 0 \ \frac{1}{6}]$, $D = [1 \ 0 \ \frac{1}{6} \ 0]$ *and* $\mathtt{n} = 3$.
The i/s/o representation of the system is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{6} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \frac{1}{6}\begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}^T$$

Hence $\Phi(\zeta,\eta) = n(\zeta)d(\eta) + n(\eta)d(\zeta) =$

$$\frac{1}{36}\Big\{\underbrace{(\eta+6\eta^3)}_{\phi_0(\eta)} + \underbrace{(1+8\eta^2)}_{\phi_1(\eta)}\zeta + \underbrace{(8\eta+48\eta^3)}_{\phi_2(\eta)}\zeta^2 + \underbrace{(6+48\eta^2)}_{\phi_3(\eta)}\zeta^3\Big\}$$

$$= \frac{1}{36}\begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \zeta^3 \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 1 & 0 & 6 \\ 1 & 0 & 8 & 0 \\ 0 & 8 & 0 & 48 \\ 6 & 0 & 48 & 0 \end{bmatrix}}_{N^TD + D^TN = K_b} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \\ \eta^3 \end{bmatrix}$$

This corresponds to step (3) of Algorithm 7.3. Using the

equations (23)*, we have*

$$\psi_0(\xi) = \frac{\phi_0(\xi)}{\xi} = \frac{1+6\xi^2}{36},\ \psi_1(\xi) = \frac{\phi_1(\xi)-\psi_0(\xi)}{\xi} = \frac{2\xi}{36}$$
$$\psi_2(\xi) = \frac{\phi_2(\xi)-\psi_1(\xi)}{\xi} = \frac{6+48\xi^2}{36}$$

Note that the polynomial subtraction and division shown in these steps can also be done using corresponding vector shift and subtraction operations. This is implemented with Step (4) to Step (10) of Algorithm 7.3. Hence the storage function is

$$\Psi(\zeta,\eta) = \frac{1}{36}\left\{(1+6\eta^2)+2\eta\zeta+(6+48\eta^2)\zeta^2\right\}$$
$$= \frac{1}{36}\begin{bmatrix}1\\ \zeta\\ \zeta^2\end{bmatrix}\underbrace{\begin{bmatrix}1 & 0 & 6\\ 0 & 2 & 0\\ 6 & 0 & 48\end{bmatrix}}_{K}\begin{bmatrix}1\\ \eta\\ \eta^2\end{bmatrix}$$

D. Bezoutian based method – MIMO case

In this section we propose an extension of the Bezoutian based method for the SISO case to MIMO case when each of the elements in $G(s)$ are considered to be lossless i.e. we consider each element of $G(s)$ to have poles on the imaginary axis. In general the elements of $G(s)$ have the form given in equation (18). However, since we consider each element of $G(s)$ to be lossless therefore $\beta_{ij} = 0$. For such systems, we consider each of the elements in $G(s)$ as lossless systems and use the procedure described in Section 4-A to compute the storage function of the system. We present a theorem next to compute the storage function of MIMO systems with the Bezoutian method.

Theorem 4.6. *Consider a lossless transfer matrix $G(s)$ with the (i,j)-th element represented as g_{ij}. Recall again that g_{ij} has the form of equation* (18) *with $\beta_{ij} = 0$. The controller canonical form of each element g_{ij} is represented by the triplet (A_{ij}, b_{ij}, c_{ij}). Construct matrices $B_{ij} \in \mathbb{R}^{2\times \mathtt{p}}$ such that j-th column of B_{ij} := b_{ij} and rest entries zero. Let $C_{ij} \in \mathbb{R}^{\mathtt{p}\times 2}$ be matrices with i-th row of C_{ij} := c_{ij} and rest entries zero. Suppose K_{ij} represents the storage function corresponding to each g_{ij} given by Theorem 4.1. Then (possibly nonminimal) state-space representation of the system $G(s)$ is given by the following (A,B,C) matrices.*

1) $A = \operatorname{diag}\left(A_{11}, A_{12}, \ldots, A_{1\mathtt{p}}, A_{21}, \ldots, A_{\mathtt{pp}}\right) \in \mathbb{R}^{2\mathtt{p}^2\times 2\mathtt{p}^2}$.
2) $B = \operatorname{col}\left(B_{11}, B_{12}, \ldots, B_{1\mathtt{p}}, B_{21}, \ldots, B_{\mathtt{pp}}\right) \in \mathbb{R}^{2\mathtt{p}^2\times \mathtt{p}}$.
3) $C = \begin{bmatrix}C_{11} & C_{12} & \ldots & C_{1\mathtt{p}} & C_{21} & \ldots & C_{\mathtt{pp}}\end{bmatrix} \in \mathbb{R}^{\mathtt{p}\times 2\mathtt{p}^2}$.

The K matrix that induces the storage function of the lossless system $G(s)$ with respect to the triplet (A,B,C) is given by $K = \operatorname{diag}\left(K_{11}, K_{12}, \cdots, K_{1\mathtt{p}}, K_{21}, \cdots, K_{\mathtt{pp}}\right) \in \mathbb{R}^{2\mathtt{p}^2\times 2\mathtt{p}^2}$.

Proof. We present the proof for a specific case. The general proof is essentially a book-keeping version of this simplified case. Consider the transfer matrix of the form $G(s) = \begin{bmatrix}g_{11}(s) & g_{12}(s)\\ g_{21}(s) & g_{22}(s)\end{bmatrix} := \begin{bmatrix}\frac{\alpha_{11}s}{s^2+\omega^2} & \frac{\alpha_{12}s}{s^2+\omega^2}\\ \frac{\alpha_{21}s}{s^2+\omega^2} & \frac{\alpha_{22}s}{s^2+\omega^2}\end{bmatrix}$. Consider $c_{ij}(sI - A_{ij})^{-1}b_{ij} = g_{ij}$ where $i,j = 1,2$. Construct $A = \operatorname{diag}\ \{A_{11}, A_{12}, A_{21}, A_{22}\}$, $B = \begin{bmatrix}b_{11} & 0\\ 0 & b_{12}\\ b_{21} & 0\\ 0 & b_{22}\end{bmatrix}$ and $C = \begin{bmatrix}c_{11} & c_{12} & 0 & 0\\ 0 & 0 & c_{21} & c_{22}\end{bmatrix}$. It can be verified that $C(sI-A)^{-1}B = G(s)$.
The storage function corresponding to each g_{ij} is K_{ij}. Hence $A_{ij}^T K_{ij} + K_{ij}A_{ij} = 0$ is satisfied. Further, $b_{ij}^T K_{ij} - c_{ij} = 0$. Construct $K = \operatorname{diag}\ \{K_{11}, K_{12}, K_{21}, K_{22}\}$. From the construction of A and K it follows that $A^T K + KA = 0$. Further it can also be verified that $B^T K - C = 0$. This proves that K gives the required storage function. □

Note that the K matrix obtained by the method described in Theorem 4.6 is not minimal in general. This is due the non-minimal state-space representation obtained for the lossless transfer matrix $G(s)$. It is not clear whether a minimal state-space realization is always possible for a lossless MIMO system with the states of the form given in Section 2-F: for more on nonminimality of RLC circuits in general see [5], [9]. This requires further investigation and is not dealt here.

5. Gramian balancing method

In this subsection we present a method to compute storage function of *all-pass* systems. Note that only for the Gramian balancing method, we shift to all-pass instead of lossless systems. Both lossless and all-pass systems are conservative with respect to a suitable supply rate. It is based on the notion of balancing of controllability and observability Gramian. We proceed to state the result in the form of a theorem next.

Theorem 5.1. *Consider an $\mathtt{n}^{th}$ order all-pass and stable system $G(s) \in \mathbb{R}(s)^{\mathtt{p}\times\mathtt{p}}$ with a minimal balanced state-space realization $\dot{x} = Ax + Bu,\ y = Cx + Du$ i.e. in this basis, $A + A^T + BB^T = 0$ and $A^T + A + C^T C = 0$. Then the storage function associated is $x^T x$.*

Note there always exists a balanced state-space representation for any stable system. The proof is skipped here since this paper focuses on algorithms. The algorithm for Gramian balancing method consists of two main steps. Firstly, for the given system compute a transformation that yields a balanced state-space realization. The procedure to find such a balanced state-space representation involves use of the Cholesky decomposition and SVD: for the detailed algorithm refer [1, Section 7.3]. Let S be the transformation matrix. Secondly, note that in the balanced state-space realization the storage function is induced by $I \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Hence the storage function in the original state space basis is $K := S^T I S = S^T S$.

Example 5.2. *For the transfer function $G(s)$ in Example 4.5 above, consider the all-pass system $\frac{1-G(s)}{1+G(s)} = \frac{6s^3-8s^2+s-1}{6s^3+8s^2+s+1}$. The storage function is $x^T x$ for*

$$A = \begin{bmatrix}-0.04 & -0.36 & -0.04\\ 0.36 & 0 & 0.01\\ -0.42 & -0.01 & -1.29\end{bmatrix}, B = \begin{bmatrix}-0.29\\ 0\\ -1.61\end{bmatrix},\ C = -B^T,\ D = 1$$

Remark 5.3. Obtaining a minimal state-space realization of system $G(s)$ such that the controllability and observability Gramians (W_r & W_o) are equal is achieved using the so called *simultaneous diagonalization* method: see [1, Section 7.1]. The basis in which we obtain $W_r = W_o$ is called a *balanced basis*. Hence we call the method of obtaining storage function of all-pass systems based on

Theorem 5.1 as the "Gramian balancing method" when comparing the time and accuracy in Figures 5 and 6. See also [19, Theorem 3] for related work about balancing, though the development is solely for the strict dissipative case.

6. Adjoint network method

In this section we report new properties of the storage function of lossless systems based on adjoint network. Using these new properties we propose an Algorithm in Section 7. As discussed in Section 2-E, the system $\mathfrak{B}^{\perp_\Sigma}$ is interpreted as the adjoint behavior of $\mathfrak{B}$: refer [10], [17]. In network theory terminology $\mathfrak{B}^{\perp_\Sigma}$ represents the adjoint network corresponding to the given network behavior $\mathfrak{B}$. Note that an $\mathtt{n}$-dimensional minimal i/s/o of $\mathfrak{B}$ being $\dot{x} = Ax + Bu$, $y = Cx + Du$, the adjoint system $\mathfrak{B}^{\perp_\Sigma}$ admits a corresponding $\mathtt{n}$-dimensional i/s/o representation with respect to the passivity supply rate:

$$\dot{z} = -A^T z + C^T u \quad \text{and} \quad y = B^T z - D^T u$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ and $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}$.

An interconnection of the adjoint system (dual system) with its primal system gives a new behavior $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$. Call this behavior $\mathfrak{B}_{\text{Ham}}$. Though it was shown in the context of strict passivity, it can be shown along the lines like in [20] that $\mathfrak{B}_{\text{Ham}}$ admits a first order representation of the form

$$R\left(\frac{d}{dt}\right)\begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0 \quad \text{with} \quad R(\xi) = \xi E - H \tag{28}$$

where $E := \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $H := \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D+D^T \end{bmatrix}$.
For reasons elaborated in [10], call $R(\xi)$ a "Hamiltonian pencil". For a lossless behavior $\mathfrak{B}$, a first order representation of $\mathfrak{B}_{\text{Ham}}$ is

$$\begin{bmatrix} \xi I_n - A & 0 & -B \\ 0 & \xi I_n + A^T & -C^T \\ -C & B^T & 0 \end{bmatrix}\begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0. \tag{29}$$

It turns out that when a behavior $\mathfrak{B}$ is lossless with respect to $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and input cardinality of $\mathfrak{B}$ is equal to the positive signature of Σ, then $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \mathfrak{B}$: see [2, Lemma 11]. Hence the McMillan degree of $\mathfrak{B}_{\text{Ham}}$ and $\mathfrak{B}$ is the same i.e. $\mathtt{n}$. However, the representation of the behavior $\mathfrak{B}_{\text{Ham}}$ in equation (29) has $2\mathtt{n}$ states and hence x and z satisfy static relations amongst each other. This is made precise in Proposition 6.1 below, whose proof can be found in [4]. We use this to prove the main result of this section: Theorem 6.2. The theorem helps extract the static relations of the first order representation (29) of behavior $\mathfrak{B}_{\text{Ham}}$ and in the process yields the storage function for the lossless behavior $\mathfrak{B}$.

Proposition 6.1. *Consider a lossless behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal i/s/o representation $\dot{x} = Ax + Bu$, $y = Cx + Du$ and define the Hamiltonian pencil $R(\xi)$ as in equation* (28)*. Then, there exists a symmetric $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that*

$$\frac{d}{dt}x^T K x = 2u^T y \qquad \textit{for all } \begin{bmatrix} u \\ y \end{bmatrix} \in \mathfrak{B}. \tag{30}$$

if and only if

$$\textit{rank}\begin{bmatrix} sI-A & 0 & -B \\ 0 & sI+A^T & -C^T \\ -C & B^T & 0 \\ -K & I & 0 \end{bmatrix} = \textit{rank}\begin{bmatrix} sI-A & 0 & -B \\ 0 & sI+A^T & -C^T \\ -C & B^T & 0 \end{bmatrix}. \tag{31}$$

Next we report one of the main results of this paper. Algorithm 7.4 to compute the storage function of lossless systems is based on this result. The definition of minimal polynomial basis can be found in Section 2-H.

Theorem 6.2. *Consider $R(\xi) := \xi E - H \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}$ as defined in equation* (28) *constructed for the lossless behavior $\mathfrak{B} \in \mathfrak{L}^{2\mathtt{p}}_{\text{cont}}$. Let $M(\xi) \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times\mathtt{p}}$ be any minimal polynomial nullspace basis (MPB) for $R(\xi)$. Partition $M = \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix}$ with $M_1 \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$. Let $N(\xi)$ be any MPB for $M_1(\xi)^T$. Then, the following statements are true.*

1) *Each of the first $\mathtt{n}$ Forney invariant minimal indices of $N(\xi)$ are 0, i.e. first n columns of $N(\xi)$ are constant vectors.*
2) *Partition N into $\begin{bmatrix} N_1 & N_2(\xi) \end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$ and further partition $N_1 = \begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix}$ with $N_{12} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then N_{12} is invertible and $K := -N_{11}N_{12}^{-1}$ is the storage function for $\mathfrak{B}$, i.e. $\frac{d}{dt}x^T K x = 2u^T y$ for all system trajectories.*

Proof. **1**: We first prove that the first $\mathtt{n}$ minimal indices of the Hamiltonian pencil $R(\xi)$ are 0. For lossless systems $\det R(\xi) = 0$. Since rank $R(\xi) = 2\mathtt{n}$ where $\mathtt{n}$ is the McMillan degree of behavior $\mathfrak{B}$ and $R(\xi) \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}[\xi]$, find $M(\xi) \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times\mathtt{p}}[\xi]$ with rank $M(\xi) = \mathtt{p}$ such that $R(\xi)M(\xi) = 0$.
From Proposition 6.1, we have $\begin{bmatrix} -K & I & 0 \end{bmatrix}$ is in the row span of $R(\xi)$. Therefore,

$$\begin{bmatrix} -K & I & 0 \end{bmatrix} M(\xi) = 0 \ \text{ i.e. } \ \begin{bmatrix} -K & I & 0 \end{bmatrix}\begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix} = 0$$

where $M_1 \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$. This implies that

$$\begin{bmatrix} -K & I \end{bmatrix} M_1(\xi) = 0 \qquad \text{i.e.} \qquad M_1(\xi)^T \begin{bmatrix} -K \\ I \end{bmatrix} = 0.$$

The nullspace of $M_1(\xi)^T$ must have $\mathtt{n}$ constant polynomial vectors. Hence the first $\mathtt{n}$ (Forney invariant) minimal indices are 0. This proves Statement 1 of Theorem 6.2.
2: A minimal polynomial nullspace basis of $M_1(\xi)^T$ is the set of columns of $N(\xi) \in \mathbb{R}[\xi]^{2\mathtt{n}\times(2\mathtt{n}-\mathtt{p})}$. Partition N into $\begin{bmatrix} N_1 & N_2(\xi) \end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$ and further partition $N_1 = \begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix}$ with $N_{12} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Further

$$\text{span}\begin{bmatrix} N_{11} \\ N_{12} \end{bmatrix} = \text{span}\begin{bmatrix} -K \\ I \end{bmatrix}.$$

Therefore $K = -N_{11}N_{12}^{-1}$. The construction of $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ in the proof is done such a way that $\begin{bmatrix} -K & I & 0 \end{bmatrix}$ is in the row span of $R(\xi)$. From Proposition 6.1, the matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ satisfies equation (31) and hence $\frac{d}{dt}x^T K x = 2u^T y$ for all $(u,y) \in \mathfrak{B}$.

Note that lossless systems satisfies equation (8) for all $(u,y) \in \mathfrak{B}$. Hence K induces the storage function for $\mathfrak{B}$. Thus Statement 2 of the theorem follows. This completes the proof of the theorem. □

Example 6.3. *Consider the system:* $G(s)=\frac{8s^2+1}{6s^3+s}$. *One of the state space representation of the system is*

$$\frac{d}{dt}x=\begin{bmatrix}0&1&0\\0&0&1\\0&-\frac{1}{6}&0\end{bmatrix}x+\begin{bmatrix}0\\0\\1\end{bmatrix}u \qquad y=\frac{1}{6}\begin{bmatrix}1\\0\\8\end{bmatrix}^T u$$

The pencil corresponding to $\mathfrak{B}\cap\mathfrak{B}^{\perp_\Sigma}$ *is*

$$R(\xi)=\frac{1}{6}\begin{bmatrix}6\xi&-6&0&&&&0\\0&6\xi&-6&&&&0\\0&1&6\xi&&&&-6\\&&&6\xi&0&0&-1\\&&&6&6\xi&-1&0\\&&&0&6&6\xi&-8\\-1&0&-8&0&0&6&0\end{bmatrix}$$

The minimal polynomial basis of $R(\xi)$ *is*

$$[\underbrace{36 \quad 36\xi \quad 36\xi^2 \quad 1+6\xi^2 \quad 2\xi \quad 6+48\xi^2}_{M_1(\xi)^T} \quad 6\xi+36\xi^3]^T$$

This is step 2 of Algorithm 7.4. By Theorem 6.2 the first $\mathtt{n}=3$ *columns of the minimal polynomial basis of the* $M_1(\xi)^T$ *have Forney indices 0. The first* 3 *columns of the minimal polynomial basis of* $M_1(\xi)^T$ *are*

$$\left[\begin{array}{rrr}-0.0189&0.0025&-0.0987\\-0.0002&-0.0554&-0.0013\\-0.0960&0.0195&-0.7921\\\hline 0.9938&-0.0017&-0.0470\\0.0028&0.9981&0.0243\\-0.0522&-0.0144&0.6000\end{array}\right]=:\begin{bmatrix}N_{11}\\N_{12}\end{bmatrix}$$

This step corresponds to step (3) and step (4) of Algorithm 7.4. Therefore $K=-N_{11}N_{12}^{-1}=\frac{1}{36}\begin{bmatrix}1&0&6\\0&2&0\\6&0&48\end{bmatrix}$.

7. Algorithms based on the proposed methods

In this section, we present four algorithms based on the results developed in previous sections. Algorithm 7.1 and Algorithm 7.2 are based on partial fraction expansion (Foster method for LC realization) described in Section 3-A and Section 3-B respectively. The Cauer realization is analogous. The transfer function of the lossless system is an input to each of the algorithms and output of the algorithm is a unique symmetric matrix K that induces the storage function. Algorithm for lossless systems without pole at the origin is almost the same and hence is not presented here.

Algorithm 7.1 LC realization based algorithm - SISO.

Input: Strictly proper transfer function of the lossless system $G(s)$.

Output: $K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Calculate the partial fraction expansion: $G(s)=\frac{r_0}{s}+\sum_{i=1}^{\mathtt{m}}G_i(s)$ (say) where $G_i(s)=\frac{r_i s}{s^2+\omega_i^2}$, $i=1,\cdots,\mathtt{m}$ and $\omega_i>0$.
2: For each $G_i(s)$, obtain (A_i,B_i,C_i) triple, where $A_i\in\mathbb{R}^{2\times2}$, $B_i\in R^{2\times1}$ and $C_i\in R^{1\times2}$ using Equation (15).
3: Obtain K_i from each triple (A_i,B_i,C_i) using Theorem 3.1.
4: Define $K:=\text{diag}\left(\frac{1}{r_0},K_1,K_2,\ldots,K_{\mathtt{m}}\right)\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Algorithm 7.2 Partial fraction expansion algorithm - MIMO.

Input: Strictly proper transfer function matrix of the lossless system $G(s)$.

Output: $K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Find the minimal state-space realization of $G(s)$ using Theorem 3.6. Result: Triplet (A,B,C).
2: Define a matrix P such that it is partitioned into row blocks $P_i\in\mathbb{R}^{2\mathtt{p}\times\mathtt{p}^2}$.
3: Partition each P_i in column blocks $P_{ij}\in\mathbb{R}^{2\mathtt{p}\times(2\mathtt{p}-2\mathtt{j}+1)}$. P_{ij} is the i-th row block and j-th column block of P.
4: **if** $i=j$ **then**
5: $\quad\hat{P}_{ii}=\begin{bmatrix}b_{2i-1}&b_{2i}&b_{2i+2}&b_{2i+3}&\cdots&b_{2\mathtt{p}}\\b_{2i}&-b_{2i-1}&b_{2i+3}&-b_{2i+2}&\cdots&-b_{2\mathtt{p}-1}\end{bmatrix}$
6: $\quad$Delete second column of $\hat{P}_{ii}$. Result: $P_{ii}\in\mathbb{R}^{2\mathtt{p}\times(2\mathtt{p}-2\mathtt{i}+1)}$.
7: **else**
8: $\quad$**if** $i<j$ **then**
9: $\quad\quad P_{ij}=0\in\mathbb{R}^{2\mathtt{p}\times(2\mathtt{p}-2i+1)}$
10: $\quad$**else** (i.e. if $i>j$)
11: $\quad\quad$Construct
12: $\quad\quad L_j:=\begin{bmatrix}b_{2j-1}&-b_{2j}\\b_{2j}&b_{2j-1}\end{bmatrix}$, $\hat{L}_j=\begin{bmatrix}0&L_j\otimes I_{\mathtt{p}-j}\end{bmatrix}$
13: $\quad\quad$col $(P_{(j+1)j},P_{(j+2)j},\cdots,P_{pj}):=\hat{L}_j$
14: $\quad\quad\quad$where $j=1,2,\cdots,\mathtt{p}-1$.
15: $\quad$**end if**
16: **end if**
17: $q=$ col $(c_1,c_2,\cdots,c_{2\mathtt{p}})$
18: Compute $y_k=P^\dagger q$ where y is as defined in equation (19).

Algorithm 7.3 is based on the Bezoutian of polynomials described in Section 4-A. Many methods like long division, 2D-DFT, Linear matrix equation can be used to find the Bezoutian matrix Z_b defined in Theorem 4.1. The algorithm here is based on long division method.

Algorithm 7.3 Bezoutian based algorithm - SISO.

Input: Transfer function of a lossless system $G(s)=\frac{n(s)}{d(s)}$ of order $\mathtt{n}$ where $d(s)$ is monic and $G(s)$ proper.

Output: $K\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Extract coefficients of the polynomials $n(s)$ and $d(s)$ into arrays $N\in\mathbb{R}^{1\times\mathtt{n}}$ and $D\in\mathbb{R}^{1\times(\mathtt{n}+1)}$ with constant term coefficient first.
2: Equate length of N and D by $N(\mathtt{n}+1):=0$.
3: Compute Bezoutian coefficient matrix using equation (22)

$$K_b:=N^TD+D^TN\ \in\mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}.$$

4: Implement the division in first equation of (23) by constructing a row vector from the first row of K_b

$$F_{\text{old}}:=\begin{bmatrix}K_b(1,2:\mathtt{n}+1)&0\end{bmatrix}\in\mathbb{R}^{1\times(\mathtt{n}+1)}.$$

5: Set $F_{\text{new}}:=F_{\text{old}}$.
6: Append new rows to get $F_{\text{new}}\in\mathbb{R}^{\mathtt{n}\times(\mathtt{n}+1)}$ by implementing the division in (23) by the following iteration:
7: **for** i=2,..., n **do**
8: $\quad r:=K_b(i,:)-F_{\text{new}}(i-1,:)$
9: $\quad F_{\text{new}}:=\begin{bmatrix}F_{\text{old}}\\r(2:\mathtt{n}+1)\qquad0\end{bmatrix}$
10: $\quad F_{\text{old}}:=F_{\text{new}}$
11: **end for**
12: Define $K:=F_{\text{new}}(1:\mathtt{n},1:\mathtt{n})$.

Algorithm 7.4 is based on extraction of static relations in first order representation of the behavior $\mathfrak{B}_{\text{Ham}}$ described in Section 6. The algorithm takes as input the Hamiltonian pencil $R(\xi)$ and gives a unique symmetric matrix K that induces storage function of the lossless behavior.

Algorithm 7.4 Adjoint network algorithm.

Input: Recall $R(\xi) := \xi E - H \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}$, a rank $2\mathtt{n}$ polynomial matrix.
Output: $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Compute a minimal polynomial nullspace basis of $R(\xi)$. Result: A full column rank polynomial matrix $M(\xi) \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times\mathtt{p}}$.
2: Partition $M(\xi)$ as $\begin{bmatrix} M_1(\xi) \\ M_2(s) \end{bmatrix}$ where $M_1(\xi) \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$.
3: Compute a minimal polynomial nullspace basis of $M_1(\xi)^T$. Result: A full column rank polynomial matrix $N(\xi) \in \mathbb{R}[\xi]^{2\mathtt{n}\times(2\mathtt{n}-\mathtt{p})}$.
4: Partition $N(\xi) = \begin{bmatrix} N_{11} & N_{12}(\xi) \\ N_{21} & N_{22}(\xi) \end{bmatrix}$ with $N_{11}, N_{21} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. (See Theorem 6.2)
5: Define $K := -N_{11}N_{21}^{-1} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Algorithm 7.4 is based on computation of nullspace basis of polynomial matrices. Efficient and stable computation of nullspace basis of a polynomial matrix can be done by block Toeplitz matrix algorithm: more details can be found in [13]. However, instead of dealing with polynomial computations, we use the Zassenhaus subspace intersection algorithm with a QR adaptation proposed in [15] for Figure 5. This results in an improvement of about 8 times in the computation time.

Remark 7.1. Note that different state representations used in the above algorithms are related to each other by a similarity transform and the storage function matrix K obtained by these techniques are correspondingly related by a congruence transform. For example, if K is the storage function matrix obtained in Method 1, and if $S \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ maps the state space representation used in Method 1 to that used in Method 2, then $\hat{K} = S^TKS$ induces the storage function in Method 2. This ability to relate the K's across realizations allows choice of the computationally best algorithm for a specific instance: this is a key contribution of this paper.

Remark 7.2. The choice of the storage function computation method depends on the system description: for example transfer function or state space. Loosely speaking, a few of the key factors that help in the choice of the algorithm are

(a) Ability to diagonalize the system matrix A using a well-conditioned matrix (i.e., the so-called 'departure from normality')[10].
(b) Extents of uncontrollability/unobservability.
(c) McMillan degrees of the elements in the transfer matrix.

A detailed and thorough investigation would be a matter of further research: we indicate our preliminary observations next.

Partial fraction expansion algorithm is about 'summing' over terms, this algorithm is favourable for a system whose transfer function is obtained as a 'sum'-of-parallel blocks (see [11, Section 2.1.3]). Further, for a system whose system matrix A is normal the similarity transform matrix S that diagonalizes A is well-conditioned (see [16] and also [7, P7.2.3]). Hence for such systems, use of partial fraction based method is suitable. ***Bezoutian based algorithm*** is best suited for systems whose matrix A is non-diagonalizable: this is due to non-diagonalizability being linked to a *chain-of-'integrators'* type of interpretation. Hence systems with controller canonical forms and with A matrices not diagonalizable are candidates for this algorithm. ***Gramian balancing method*** uses inversion of a diagonal matrix for simultaneous diagonalizability. These diagonal elements represent, in a loose sense, the "distance from simultaneous uncontrollability and unobservability": see [1, Section 7.3]. Hence this method is not favourable for systems which are "nearly uncontrollable" or "nearly unobservable" as this will result in inversion of an ill-conditioned matrix. The ***adjoint network algorithm***[11] is favourable for systems where the McMillan degree of the system is much higher than the degrees of the denominators of the transfer matrix of the system (this is especially relevant, in general, for MIMO systems). For such systems, the nullspace of the matrix $R(\xi)$ in Algorithm 7.4 will have a smaller degree and this will result in less computational effort and less error in the computation of K. Hence given a MIMO system realization which has neither the sum-of-parallel blocks form nor the controller canonical form, the adjoint network method is favourable.

[10] A matrix $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is called normal if $AA^T = A^TA$.

8. Comparison of the Methods for Computational Time and Numerical Error

Using the experimental setup and procedure described in Section 4-B, we compare the three methods described in Section 3, 4, 5 and 6 in this section.

Computation time: The plot in Figure 5 shows the time taken by each algorithm to compute the matrix K for lossless systems of different orders. The Bezoutian long division method, the LC realization based method and the Gramian balancing method take relatively less computation time compared to adjoint network method inspite of the Zassenhaus algorithm based modification proposed in [15].

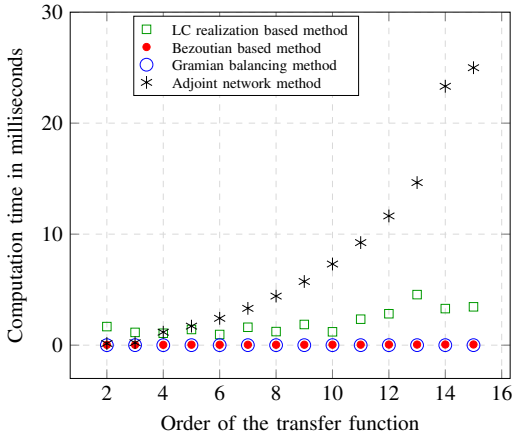

Fig. 5. Plot of computation time versus system's order.

Computation error: Error in K is computed using equation (27) and is plotted for comparison. We calculate $\text{Err}(K)$ for test cases used above for computation time. Figure 6 shows a comparison of the error associated in the computation of K using the four methods presented in this paper. The error has been plotted in the logarithmic scale for better comparison of data. From the plot we infer that Bezoutian based method is marginally better than LC realization based method, Gramian

[11] The adjoint network method is based on finding minimal polynomial basis of the polynomial matrix $R(\xi)$. The algorithm of finding the minimal polynomial basis, as reported in [13], is an iterative algorithm and is based on writing the matrix $R(\xi)$ as $\sum_{i=0}^{\mathtt{d}} R_i\xi^i$ and then using the co-efficient matrices R_i to form Toeplitz matrices at each iteration. Consider the matrices R_i have size $\mathtt{N}\times\mathtt{N}$, rank col$(R_0, R_1, \ldots, R_{\mathtt{d}}) =: \mathtt{r}_0$ and the iteration step is t then the Toeplitz matrix will have a size $(\mathtt{d}+1+\mathtt{t})\mathtt{r}_0 \times (\mathtt{r}_0 + \mathtt{r}_{\mathtt{t}-1})$. At each iterations, SVD of such augmented matrices needs to be computed to find the minimal polynomial basis of $R(\xi)$. Hence, the algorithm being iterative and the large size of the augmented matrix results in more error and computation time. Further the operation of finding minimal basis is done twice in Algorithm 7.4 and this also adds to the error and computation time.

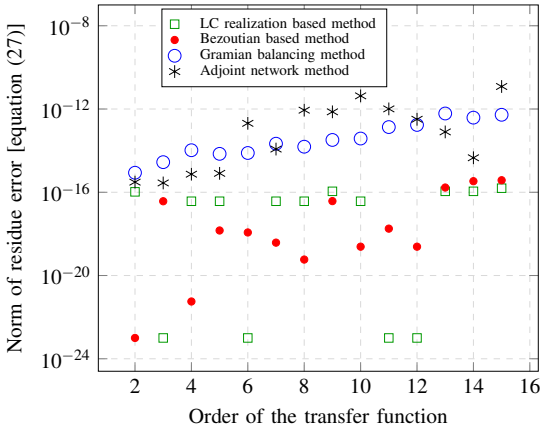

Fig. 6. Plot of error residue versus system's order.

balancing and adjoint network method.

9. Concluding Remarks

This paper dealt with the computation of the stored energy in lossless/all-pass systems. We presented four different conceptual methods to compute the unique storage function for lossless/all-pass systems.

1) **LC realization based method**: This uses Foster/Cauer method (Theorem 3.1 and Theorem 3.6) and capacitor voltages & inductor currents as states.
2) **Bezoutian based method**: (Theorem 4.1) States corresponding to controller canonical form are used in this method. Three different techniques are presented to compute the Bezoutian of such systems.
   a) Euclidean long division,
   b) 2D discrete Fourier transform,
   c) Pseudo-inverse.
3) **Controllable/Observable Gramians balancing method**: (Theorem 5.1). The method uses states in a balanced basis: 'balanced' loosely means that the extent of controllability and observability is equal for each state.
4) **Adjoint network method**: Unlike the three methods mentioned above, this method takes an arbitrary minimal state-space realization as the starting point.

All the methods mentioned in the paper can also be used to solve optimal control problems where ARE does not exist due to failure of regularity conditions on D. Based on the results of the paper, algorithms were formulated and compared for computation time and numerical accuracy: see Figures 5 and 6. All computations were done in Scilab: the time comparison would be thorough if the implementation were in C or Fortran language for example. This paper focussed only on 'proof-of-concept' algorithms emanating out of the main results.

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