2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Stationary Trajectories, Singular Hamiltonian Systems and Ill-posed Interconnection

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***Abstract*— We study Hamiltonian systems, namely, systems comprising of trajectories which are 'stationary' with respect to a quadratic performance index: they play a central role in many optimal control problems. A typical assumption in the literature is that of 'regularity': the resulting first-order dynamical system is a *regular* state space system and not a singular descriptor system. In this paper we show that the first order representation of a Hamiltonian is a singular descriptor system if and only if the interconnection of a related MIMO system $G(s)$ with its dual (i.e. its adjoint) is ill-posed. We address the possibility of existence of inadmissible initial conditions, i.e. initial conditions that give rise to impulsive solutions. We characterize conditions on $G(s)$ under which the corresponding singular Hamiltonian system has inadmissible initial conditions. Under suitable simplifying assumptions, which amount to studying an extreme case of ill-posedness, our main result states that there exist no inadmissible initial conditions if and only if the skew-symmetric part of the first moment about $s = \infty$ of the transfer matrix $G(s)$ is nonsingular; a condition we show that is opposite to that for $G(s)$ to be an all-pass filter. As a corollary, ill-posed interconnection of a square MIMO system with odd number of inputs (in particular, SISO systems) with its adjoint always contains in inadmissible initial conditions.**

***Keywords:* Hamiltonian matrix, zeros at infinity, dissipativity, all-pass filter, inadmissible initial conditions, well-posedness, adjoint system, dual system**

I. INTRODUCTION

Dissipativity of dynamical systems plays an important role in the analysis and design of control problems. In the theory of dissipativity, the set of trajectories that are lossless with respect to a cost functional plays a very important role. For example, in LQR optimal control problems, through the Pontryagin maximum principle, these lossless trajectories turn out to be candidates for optimal solutions (see [17]). This set of lossless trajectories is the central object of study in this paper. In fact, we show that the set of lossless trajectories can be viewed as a system itself obtained by interconnecting the plant with its 'orthogonal complement' (defined below in Section III). We also show that the benefit of viewing the lossless trajectories as an interconnection is that certain irregularities in the theory of dissipativity, for example, losslessness at infinity, can be related with well-known system theoretic issues of ill-posed interconnections. Further, it is well-known that in the state-space formulation of various optimal control and dissipativity issues, the algebraic Riccati equation/inequality (ARE/ARI) plays a central role. Importantly, the existence of the ARE/ARI too requires a regularity assumption on the feed-through term D: for example, invertibility of $(I - DD^T)$ or of $D + D^T$, depending on the notion of 'power/supply-rate' (defined below in Section IV). We show in this paper, that relaxation of this assumption can be handled by analyzing the above mentioned irregular interconnected system.

It has been shown in [11] that this system of stationary trajectories is a Hamiltonian behavior. Following this, we call the interconnection of a behavior with its orthogonal complement (or Σ-orthogonal complement) as the interconnected Hamiltonian behavior in this paper. In the context of ill-posed interconnection, we assume $(I - DD^T)$ is singular; this assumption is central to this paper. For the notion of power we assume in this paper, namely $u^T u - y^T y$, when a system is dissipative with respect to this supply rate, singularity of $(I - DD^T)$ is equivalent to lack of strictness of dissipativity asymptotically as frequency ω tends to ∞. After characterizing ill-posedness/singularity of the Hamiltonian system, we formulate necessary and sufficient conditions for existence of inadmissible initial conditions, i.e. initial conditions for which the solutions are impulsive. For this the concept of zeros at infinity plays a key role.

A brief overview of the main results in this paper and the paper organization are as follows. The following section contains definitions pertaining to the behavioral approach, quadratic differential forms (QDFs), well-posedness of interconnection and the notion of zeros at infinity of a polynomial matrix and its relation to inadmissible initial conditions, i.e. those initial conditions that cause impulsive solutions. Section III defines the notion of a Hamiltonian system and its relation to the orthogonal complement of a system. This section also relates (in Theorem 3.3) the well-posedness of the interconnection and its adjoint with the drop in the degree of the determinant of a certain para-hermitian matrix later defined as $\partial\Phi(\xi)$: the regularity of the resulting Hamiltonian system and its autonomy are also characterized here.

In Section IV, a state space representation of the behavior and its complement have been considered and a state space representation of the interconnection has been worked out. Using the property relating the degree of determinant of $\partial\Phi(\xi)$ proved in Theorem 3.3, the state space representation of the interconnection is further simplified under mild assumptions on the feed-through matrix D. In Section IV, where we deal with ill-posed interconnections, we obtain (in Theorem 4.4) a necessary and sufficient condition for the interconnection to be autonomous and relate this with

The research was supported in part by the Bharti Centre for Communication at IIT Bombay and SERB, DST.

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existence of an all-pass subsystem in the system described by $G(s)$ (in Remark 5.4). In Section V, in Theorem 5.3, we derive a necessary and sufficient condition for a singular Hamiltonian behavior to have no inadmissible initial conditions: this main result is obtained as a special case of a more general result (Lemma 5.2) about existence of inadmissible initial conditions during an ill-posedness interconnection of two state-space systems. The rest of this section is devoted to the notation we use.

The sets $\mathbb{R}$ and $\mathbb{C}$ stand for that of all real and complex numbers respectively. The set of polynomials in the indeterminate ξ with coefficients from $\mathbb{R}$ is denoted as $\mathbb{R}[\xi]$, while matrices with entries from $\mathbb{R}[\xi]$ and having $\mathtt{p}$ rows and $\mathtt{m}$ columns is denoted by $\mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$. The spaces $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}\times\mathtt{w}})$ and $\mathfrak{L}_1^{\text{loc}}$ stand for the spaces of infinitely often differentiable functions and locally integrable functions each from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The set of those elements in $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}\times\mathtt{w}})$ which have compact support is denoted by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}\times\mathtt{w}})$. When the domain and co-domain are clear from the context, then they are skipped and we write just $\mathfrak{C}^\infty$ or $\mathfrak{L}_1^{\text{loc}}$.

II. Preliminaries

This section contains the essential preliminaries. The following subsection reviews required results from the behavioral approach to dynamical systems. Subsection II-B introduces the notion of dissipativity of a behavior with respect to QDF. Subsection II-C explains about interconnection and gives the definition when an interconnection is well-posed. Subsection II-D deals with zeros at infinity of a matrix. Subsection II-E defines inadmissible initial conditions.

A. The behavioral approach

A system behavior is the set of all trajectories that the system allows. More precisely, a linear differential behavior $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of the solutions to a set of ordinary linear differential[1] equations with constant coefficients; i.e.,

$$\mathfrak{B} := \left\{ w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\},$$

where $R(\xi)$ is a polynomial matrix having $\mathtt{w}$ number of columns: $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. This representation is called a *kernel representation* of $\mathfrak{B}$ and w is called the *manifest variable*. We assume a kernel representation matrix $R(\xi)$ to be of full row rank without loss of generality (see [9]) and such a full row rank kernel representation is called a minimal kernel representation. Since R is full row rank, there exists a nonsingular matrix P such that $R = [P \quad Q]$ (after permutation of columns of R, if necessary) and this results in a corresponding partition of variables w into $w = (y, u)$ with u as the input and y as the output: we call such a partition of w into (y, u) as an input/output partition. Though the i/o partitioning is non-unique, the number of components in the input in any i/o partition depends only on the behavior $\mathfrak{B}$: this is called the input cardinality, and is denoted by $\mathtt{m}(\mathfrak{B})$.

[1] The differential equations are required to be satisfied in only a weak sense, i.e. in the distributional sense.

For a given partition as above, the transfer matrix from u to y is $-P^{-1}Q$. In this paper, we interchangeably use the words system and its behavior, and the transfer matrix too, when the i/o partition is clear from the context: for example in Fig. 1.

The familiar steerability definition of Kalman state controllability generalizes to behaviors and similarly the PBH rank test generalizes (see [9]) to the result that for a behavior $\mathfrak{B}$ with minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, the behavior is controllable if and only if $R(\lambda)$ has full row rank for every $\lambda \in \mathbb{C}$. Such an R is called *left-prime*, and the set of controllable behaviors is denoted as $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. It is also known that a behavior $\mathfrak{B}$ is controllable if and only if there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that[2]

$$\begin{aligned}\mathfrak{B} = \{ w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid & \text{ there exists} \\ & \ell \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{m}}), w = M\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\ell \}\end{aligned} \tag{1}$$

This representation of $\mathfrak{B}$ is called an image representation. It turns out (see [9]) that for an image representation, without loss of generality, one can assume $M(\xi)$ is right-prime: M is called right-prime if M^T is left-prime. A *square* left-prime matrix U is also right-prime: U is in fact unimodular, i.e. its determinant is a nonzero constant. A right-prime polynomial matrix M is called *column reduced* if the degrees of its columns (after permutation, if required, of the columns to have the column degrees nondecreasing) are the lowest amongst that of all MU for U unimodular.

B. Quadratic differential forms

In this subsection we review the essential notions of Quadratic Differential Forms (QDFs): see [18] for more details. Consider a two variable polynomial matrix $\Phi(\zeta,\eta) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ with $\Phi(\zeta,\eta) := \sum_{i,k} \Phi_{ik}\zeta^i\eta^k$, where $\Phi_{ik} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. A Quadratic Differential Form (QDF) Q_Φ induced by $\Phi(\zeta,\eta)$ is a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{i,k} \left(\frac{\mathrm{d}^i w}{\mathrm{d}t^i}\right)^T \Phi_{ik} \left(\frac{\mathrm{d}^k w}{\mathrm{d}t^k}\right).$$

Note that only a finite number of derivatives of w need to exist for the QDF to be defined. Secondly, when dealing with quadratic forms in w and its derivatives, we assume without loss of generality that $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^T$; see [18].

QDFs play a central role in the notion of dissipativity. Consider $\Phi(\zeta,\eta) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. A controllable behavior $\mathfrak{B}$ is said to be Φ-dissipative if

$$\int_{-\infty}^{\infty} Q_\Phi(w)\,\mathrm{d}t \geqslant 0 \text{ for all } w \in \mathfrak{B}\cap\mathfrak{D}.$$

Suppose $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an image representation of the behavior. Then $\mathfrak{B}$ is dissipative if and only if $M(-j\omega)^T\Phi(-j\omega,j\omega)M(j\omega)$ is non-negative for each $\omega \in \mathbb{R}$. In this context, for a given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, we often require the one variable polynomial matrix $\partial\Phi(\xi)$ defined as $\Phi(-\xi,\xi)$. As we shall see in the sequel, the notion of a Hamiltonian system is very closely related to $\partial\Phi$.

[2] Due to differential equations being satisfied in just the weak sense, equality of the sets in Equation (1) requires mild conditions on smoothness of the trajectories concerned; see Footnote 1 and [8].

C. Well-posed interconnection

This subsection contains definitions about interconnection aspects in the behavioral approach. One of the salient features of this approach is that control is viewed as *restriction* of the plant behavior $\mathfrak{B}_1$ to a desired sub-behavior. This restriction is achieved by designing new laws that the system variables have to satisfy in addition to the existing equations. These additional laws themselves constitute a dynamical system, whose behavior is say $\mathfrak{B}_2$. Hence, upon *interconnection* of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, the trajectories allowed in the controlled system are those that satisfy the laws of both $\mathfrak{B}_1$ and $\mathfrak{B}_2$, and thus they lie in the intersection $\mathfrak{B}_1 \cap \mathfrak{B}_2$. While the intersection is indeed of significance in control, there are key properties of the *interconnection* $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ of systems $\mathfrak{B}_1$ and $\mathfrak{B}_2$ that we study in this paper, for example well-posedness. For this purpose, we require the following definition of the slow McMillan degree of a polynomial matrix.

Definition 2.1: Consider a full rank polynomial matrix $T(\xi) \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$ and let $\mathtt{r} = \min(\mathtt{n},\mathtt{m})$ be its rank. The *slow McMillan degree* of $T(\xi)$ denoted by $\mathtt{n}(T)$ is defined as the maximum amongst the determinantal degrees of all $\mathtt{r}\times\mathtt{r}$ minors of $T(\xi)$.

The slow McMillan degree plays a key role in the dimension of a minimal state space realization of a proper transfer matrix as follows. Suppose $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ is left-prime and $\mathtt{n}$ is its slow McMillan degree. Then for any partition of $R = [P \quad Q]$, degree of $\det P$ equals $\mathtt{n}$ if and only if $P^{-1}Q$ is proper. Further, the dimension of the minimal state space realization of $P^{-1}Q$ is precisely $\mathtt{n}$. See [12] for an elaborate development. We often need an input/state/output representation of the behavior $\mathfrak{B}$; this is the familiar form $\dot{x} = Ax + Bu$ and $y = Cx + Du$. An i/o partition of w into (u, y) such that there exists such an i/s/o representation can also be obtained using properness/slow McMillan degree: this is also treated in [12].

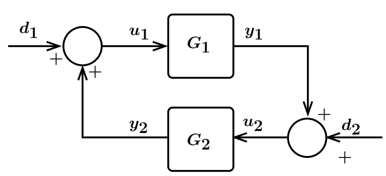

Fig. 1. Well-posedness of an interconnection

Definition 2.2: Consider the interconnection of the behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as shown in the Fig. 1. The interconnection of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is said to be *well-posed* if for any d_1, d_2 $\in \mathfrak{L}_1^{\text{loc}}$, there exist unique $u_1, y_1, u_2, y_2 \in \mathfrak{L}_1^{\text{loc}}$ such that the laws[3] in Fig. 1 are satisfied.

The following result from [3, Theorem 2.1] relates well-posedness and slow McMillan degrees. (See also [15, Theorem 7.1].)

[3]Strictly speaking, an input/output partition as assumed in Fig. 1 is not required in the definition of well-posedness. However, in this paper, due to the dissipativity assumption, we always will have an i/o partition such that the corresponding transfer matrix is proper: hence we do not pursue a definition of well-posedness that is independent of the i/o partition. This is clarified in Proposition 2.3 to some extent and further elaborated in [3, Theorem 2.1] and [15, Theorem 7.1].

Proposition 2.3: Let R_1, R_2 be minimal kernel representation matrices of behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Let $\mathtt{n}_1$ and $\mathtt{n}_2$ be the slow McMillan degrees of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Let $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ be square and nonsingular and let $\mathtt{n}_3$ be the slow McMillan degree of $\mathfrak{B}_1 \cap \mathfrak{B}_2$. Then $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ is well-posed if and only if $\mathtt{n}_1 + \mathtt{n}_2 = \mathtt{n}_3$.

Well-posedness and ill-posedness of interconnections play a central role in this paper and the zeros at infinity of a polynomial matrix play a key role when studying the ill-posed case; this is reviewed next.

D. Zeros at infinity

We review the notion of zeros of polynomial/rational matrices at infinity. An elaborate treatment can be found in [2], for example. For any $\lambda \in \mathbb{C}$ and $P \in \mathbb{R}^{q\times q}(s)$ there exist square, nonsingular, rational matrices U and $V \in \mathbb{R}^{q\times q}(s)$ such that $U(\lambda)$ and $V(\lambda)$ exist, are nonsingular and $UPV = \text{diag}\left((s-\lambda)^{n_i(\lambda)}\right)$ with the integers $n_i(\lambda)$ nondecreasing in i for $i = 1, ..., q$. It turns out that the integers $n_i(\lambda)$ depend only on P and not on the U and V matrices. If $n_q > 0$ we say that P has (one or more) zeros at λ, and the positive $n_i(\lambda)$'s are called the structural zero indices at λ. The zeros and their structural indices of $P \in \mathbb{R}^{q\times q}(s)$ at infinity are defined as those of $Q(s) \in \mathbb{R}^{q\times q}(s)$ at $s = 0$ with $Q(s) := P(\lambda)$ and $\lambda := 1/s$. A more direct count of the zeros at infinity can be obtained by counting the so-called valuations at infinity for a rational matrix as elaborated in [2].

E. Inadmissible initial conditions

We define an inadmissible initial condition vector for an autonomous system $P(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$, with $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ nonsingular; see Verghese et al [14], Dai [1] and Vardulakis [13] for a similar treatment. Let $\mathtt{z}$ be equal to the degree of the highest degree entry in $P(\xi)$. Let $w(0)$, $w^{(1)}(0)$,...,$w^{(\mathtt{z}-1)}(0)$ be the values of w, $\frac{\mathrm{d}}{\mathrm{d}t}w$, ..., $\frac{\mathrm{d}^{\mathtt{z}-1}}{\mathrm{d}t^{\mathtt{z}-1}}w$ at time $t = 0^-$. Define the vector $\overline{w}(0) = (w(0), w^{(1)}(0), ..., w^{(\mathtt{z}-1)}(0))$. We call $\overline{w}(0) \in \mathbb{R}^{\mathtt{zw}}$ an initial condition vector. A vector $\overline{w}(0) \in \mathbb{R}^{\mathtt{zw}}$ is said to be an *inadmissible* initial condition vector if the corresponding solution $w(t)$ contains the Dirac impulse $\delta(t)$ and/or its distributional derivatives.

The following result from Vardulakis [13, Theorem 4.32], which gives conditions for existence of inadmissible initial conditions for autonomous systems.

Proposition 2.4: Consider the autonomous system defined by $P(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$ where $P \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is nonsingular, and suppose $\mathtt{z}$ is the degree of the highest degree entry in $P(\xi)$. Then, there exist no inadmissible initial conditions for $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ if and only if P has no zeros at infinity.

III. Interconnection and Stationarity: the Well-posed Case

Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a real symmetric nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Following [11], we call a trajectory $w \in \mathfrak{B}$ *stationary with respect to* Σ (or simply,

Σ-*stationary*) if for all $v \in \mathfrak{B} \cap \mathfrak{D}$ we have

$$\int_{-\infty}^{\infty} w^T \Sigma v \,\mathrm{d}t = 0. \tag{2}$$

Throughout this paper, we assume Σ is a real, symmetric and a nonsingular matrix: we will skip specifying this explicitly. In fact, without loss of generality, throughout this paper we assume $\Sigma := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}$: this amounts to a coordinate transformation S in the variables of the system, with the S obtained from a congruence transformation of a general symmetric and nonsingular matrix Σ.

The Σ-stationary trajectories in $\mathfrak{B}$ are closely related to the Σ-orthogonal complement of $\mathfrak{B}$. This is defined as follows.

Definition 3.1: (See [18]) Given a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, *the Σ-orthogonal complement of* $\mathfrak{B}$, denoted by $\mathfrak{B}^{\perp_\Sigma}$ is the set of all the trajectories $v \in \mathfrak{L}^{\text{loc}}_1(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that $\int_{-\infty}^{\infty} v^T \Sigma w \,\mathrm{d}t = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

It then easily follows that the set of Σ-stationary trajectories is equal to $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$. Suppose $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal kernel representation and $\mathfrak{B} = \text{im } M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is an observable image representation of $\mathfrak{B}$. It is known that $\mathfrak{B}^{\perp_\Sigma}$ then has a minimal kernel representation $\mathfrak{B}^{\perp_\Sigma} = \ker M^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma$. Therefore, $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ must satisfy

$$\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \left\{ w \in \mathfrak{L}^{\text{loc}}_1(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \middle| \begin{bmatrix} R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ M^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma \end{bmatrix} w = 0 \right\}. \tag{3}$$

Next we view this intersection of $\mathfrak{B}$ and its Σ-orthogonal complement as an *interconnection* of two behaviors. The benefit of this approach is that it allows addressing various irregularity issues by looking at whether the said interconnection is well-posed. Our first main result Theorem 3.3 below relates the question of well-posedness with the para-Hermitian polynomial matrix $\partial\Phi(\xi) := M^T(-\xi)\Sigma M(\xi)$. Lemma 3.2 is preliminary for Theorem 3.3.

Lemma 3.2: Consider a controllable behavior $\mathfrak{B} = \text{im } M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ with a minimal i/s/o representation (A, B, C, D). Suppose $\partial\Phi(\xi) = M^T(-\xi)\Sigma M(\xi)$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma) = \mathtt{m}$. Let $\mathtt{n}$ be the dimension of the state-space. Then $(I_{\mathtt{m}} - D^T D)$ is singular if and only if $\deg\det \partial\Phi(\xi) < 2\mathtt{n}$.

Theorem 3.3: Let the behavior be given by $\mathfrak{B} = \text{im } M(\frac{\mathrm{d}}{\mathrm{d}t})$, where $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ is right prime. Let $\mathtt{m}(\mathfrak{B}) = \mathtt{m}$. Then the following statements hold:

1) $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = M(\frac{\mathrm{d}}{\mathrm{d}t})(\ker \partial\Phi(\frac{\mathrm{d}}{\mathrm{d}t}))$.
2) The interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is autonomous if and only if $\det(\partial\Phi(\xi)) \neq 0$.
3) $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is well-posed if and only if $\deg\det \partial\Phi(\xi) = 2n$.

In [11] the behavior $\ker \partial\Phi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ has been called a *Hamiltonian behavior*. Statement 1) of Theorem 3.3 above shows that $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ and $\ker \partial\Phi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ are related by an observable $M(\frac{\mathrm{d}}{\mathrm{d}t})$. This hints toward the fact that $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ too is a Hamiltonian behavior. Indeed, when $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is well-posed, the interconnected system admits a state representation with a system matrix which is Hamiltonian[4]. This is proved later below. We provide a definition of a Hamiltonian system using the various results in the literature that link a Hamiltonian system, stationarity w.r.t. a quadratic form, and $\ker \partial\Phi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$: see [17, Equations (10), (12) and (18)] and [11, Proposition 4.1 and Theorem 3.4] for the literature perhaps closest to this paper.

Definition 3.4: A behavior $\mathfrak{B}_{\text{Ham}} \in \mathfrak{L}^{\mathtt{w}}$ is said to be a Hamiltonian behavior if there exist a $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that $\mathfrak{B}_{\text{Ham}} = \mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$.

In this paper we deal with three situations: when $\mathfrak{B}_{\text{Ham}}$ is non-autonomous, and when $\mathfrak{B}_{\text{Ham}}$ is autonomous, but the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is either well-posed or ill-posed. For the ill-posed case, we characterize conditions for existence of initial conditions resulting in impulsive conditions. First, for the well-posed case, we show that the resulting state-transition matrix turns out to be a Hamiltonian matrix.

Suppose $\mathfrak{B}$ has the following minimal i/s/o representation

$$\begin{aligned} \dot{x} &= Ax + Bw_1 \\ w_2 &= Cx + Dw_1 \end{aligned} \tag{4}$$

where x is the state vector, w_1 is the input vector, w_2 is the output vector. A state space representation of $\mathfrak{B}^{\perp_\Sigma}$ is given by

$$\begin{aligned} \dot{z} &= -A^T z - C^T v_1 \\ v_2 &= B^T z + D^T v_1 \end{aligned} \tag{5}$$

Under the interconnection $w_2 = v_1$ and $w_1 = v_2$, a first order representation of the interconnected system evaluates to

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ 0 \end{bmatrix} = \begin{bmatrix} A & BB^T & BD^T \\ 0 & -A^T & -C^T \\ -C & -DB^T & I_{\mathtt{p}} - DD^T \end{bmatrix} \begin{bmatrix} x \\ z \\ v_1 \end{bmatrix} \tag{6}$$

As seen before, when $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is well-posed, $(I_{\mathtt{m}} - D^T D)$ is nonsingular, which is equivalent to $(I_{\mathtt{p}} - DD^T)$ being nonsingular. Therefore, the last line in the matrix-vector equation (6) can be rewritten as $v_1 = (I_{\mathtt{p}} - DD^T)^{-1}Cx + (I_{\mathtt{p}} - DD^T)^{-1}DB^T z$. Substituting this in equation (6) to eliminate v_1 we get

$$I_{2\mathtt{n}} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix} \qquad \text{with } H \text{ defined as:}$$

$$\begin{bmatrix} A + BD^T(I_{\mathtt{p}} - DD^T)^{-1}C & BB^T + BD^T(I_{\mathtt{p}} - DD^T)^{-1}DB^T \\ -C^T(I_{\mathtt{p}} - DD^T)^{-1}C & -(A^T + C^T(I_{\mathtt{p}} - DD^T)^{-1}DB^T) \end{bmatrix}. \tag{7}$$

The $(2\mathtt{n} \times 2\mathtt{n})$ matrix H above is a Hamiltonian matrix. Thus a well-posed $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is indeed a Hamiltonian system. Keeping this result in mind, in Definition 3.4, we have called $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ a *Hamiltonian system*. Note, however, that the above derivation fails when $(I_{\mathtt{m}} - D^T D)$ is singular. The nonsingularity of $(I_{\mathtt{m}} - D^T D)$ remains a standing assumption in various applications. Equation (7) is familiar since this matrix arises in many optimal control problems. For example, the Algebraic Riccati equations that arise in the linear quadratic control problem, $\mathcal{H}_\infty$-control and the $\mathcal{H}_2$-control problem are all related to corresponding Hamiltonian matrices of the form (7): see [4], [10], for example.

We address this issue of $(I_{\mathtt{m}} - D^T D)$ being singular by analysing the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ when it is not well-posed.

[4]Rather than using the (slightly) more general notion of a 'symplectic form', for the purpose of this paper, a matrix $H \in \mathbb{R}^{2\mathtt{n}\times 2\mathtt{n}}$ is said to be a *Hamiltonian matrix* if H satisfies $HK = -KH^T$ with $K = \begin{bmatrix} 0 & I_{\mathtt{n}} \\ -I_{\mathtt{n}} & 0 \end{bmatrix}$.

IV. Ill-posed interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$

In this section, we obtain a state space representation of $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ for the case that the interconnection is not well-posed, i.e. $(I_{\mathtt{p}} - DD^T)$ is singular. In this context we obtain a singular descriptor state space system. Consider the following assumption on the behavior and on the feed-through term D.

Assumption 4.1: Consider $\Sigma = \text{diag}\ (I_{\mathtt{m}}, -I_{\mathtt{m}})$ and let $\mathfrak{B} \in \mathfrak{L}^{2\mathtt{m}}_{\text{cont}}$ have $\mathtt{m}$ outputs. Consider the partition of w corresponding to partition of Σ as $w = (u, y)$ and assume the transfer matrix G from u to y has a minimal state space realization $(A, B, C, I_{\mathtt{m}})$. Also assume B is full column rank.

Remark 4.2: The reasons behind the assumption that $D = I$ are as follows. Clearly, for ill-posedness, $I - DD^T$ is singular, i.e. one or more of the singular values of D are equal to one. Further, when the system is Σ-dissipative, i.e. when the transfer matrix has $\mathcal{L}_\infty$-norm at most one, the remaining singular values are strictly less than one. The singular values of D that are strictly less than one do not cause ill-posedness of the interconnection and hence a state space similarity transformation combined with a coordinate transformation in u and y variables (see [7, Eqn (A.3)]) result in a modified $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ such that $\tilde{D}$ is diagonal with all diagonal entries being either zero or one. The diagonal entries being zero are as good as the corresponding transfer matrix being *strictly proper*. Since the strictly proper case and the situation when $I - DD^T > 0$, both result in the well-understood regular case, this particular aspect in the more general singular $(I - DD^T)$ case can be handled by a corresponding regular part in the final singular descriptor state space system. In order to analyze the situation due to singularity, we thus focus on the extreme case of ill-posedness, namely, when D is the identity matrix $I_{\mathtt{m}}$. As a special case, for a SISO system, assuming $\Sigma = \text{diag}\ (1, -1)$, ill-posedness of the interconnection is *equivalent* to $d = 1$.

Remark 4.3: Consider the assumption of B being full column rank. Under the situation that D is the identity matrix, it can be proved that if B is not full column rank, then the inputs corresponding to the null-space of B result in a non-autonomous all-pass subsystem in the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$. We outline this proof within this remark. Suppose v is a constant nonzero vector such that $Bv = 0$. Then $u = v\ell$ for any nonzero compactly supported function ℓ has the corresponding output $y = v\ell$, assuming initial condition is zero. Clearly, (u, y) is an element of both $\mathfrak{B}$ and of $\mathfrak{B}^{\perp_\Sigma}$ and is a nonzero compactly supported function. This proves that the intersection is non-autonomous. Thus the assumption that B is full column rank is a necessary condition for the interconnected system to be autonomous.

For the rest of this paper, we assume $D = I$ and B is full column rank. Then the state-space representation for the interconnected system in (6) is changed suitably. Define

$$\widetilde{A} := \begin{bmatrix} A & BB^T \\ 0 & -A^T \end{bmatrix}, \widetilde{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}, \widetilde{C} := \begin{bmatrix} C & B^T \end{bmatrix} \tag{8}$$

Recall that the assumption $D = I$ for the transfer function $G(s)$ is an extreme case of ill-posedness (see Remark 4.2 above). These matrices are used to characterize the situation when the interconnection is autonomous. See Remark 5.4 below for the relation with all-pass behavior of the system.

Theorem 4.4: Consider the interconnection of the behaviors $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$. Then the following statements are equivalent.

1) The interconnected system is autonomous.
2) $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is nonsingular for some t.
3) $\ker\ (\widetilde{C}\widetilde{B}) \cap \ker\ (\widetilde{C}\widetilde{A}\widetilde{B}) \cap \cdots \cap \ker\ (\widetilde{C}\widetilde{A}^{2\mathtt{n}-1}\widetilde{B}) = 0$

V. Zeros at infinity of the interconnected system

This section formulates necessary and sufficient conditions for the Hamiltonian system to have zeros at infinity. The following lemma provides a condition for the absence of inadmissible initial conditions in a singular system.

Lemma 5.1: Let $E\dot{x} = Ax$, $E, A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ describe an autonomous singular state space system. Assume the rank of E to be $\mathtt{r}$. Consider $M \in \mathbb{R}^{(\mathtt{n}-\mathtt{r})\times\mathtt{n}}$, full row rank such that $ME = 0$. Then the following are equivalent:

1) $\dim\ (MA \ker\ E) = \mathtt{n} - \text{rank}\ E$.
2) There are no inadmissible initial conditions for this system.

The above lemma is used to prove the following result about existence of inadmissible initial conditions when two state space systems are interconnected.

Lemma 5.2: Consider interconnection of state space systems S_1 and S_2, both with feed-through matrix I, and output of one connected as input of the other:

$$S_1: \quad \begin{aligned} \dot{x} &= A_1x + B_1u \\ y &= C_1x + u \end{aligned} \qquad \text{and } S_2: \quad \begin{aligned} \dot{z} &= A_2z + B_2y \\ u &= C_2z + y \end{aligned}$$

for $A_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B_1, C_1^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}}$, $A_2 \in \mathbb{R}^{\ell\times\ell}$, B_2 and $C_2^T \in \mathbb{R}^{\ell\times\mathtt{p}}$. Assume the interconnected system is autonomous. Define $\widetilde{A} := \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}$, $\widetilde{B} := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ $\widetilde{C} := \begin{bmatrix} C_1 & C_2 \end{bmatrix}$. Then the following are equivalent:

1) The interconnected system has no zeros at infinity.
2) $\text{rank}\ (C_1B_1 + C_2B_2) = \mathtt{p}$
3) $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is nonsingular for $t = 0$.
4) $\ker\ (\widetilde{C}\widetilde{B}) = 0$

The following result is a special case of the above lemma: we interconnect behaviors $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$. This result is one of the main results of this paper; the relation of the conditions with all-pass characteristics of a MIMO system is elaborated in Remark 5.4. Also compare the corresponding equivalent statements in Theorem 4.4 where we characterized just autonomy.

Theorem 5.3: Consider the state space representation of the systems $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ as in Equations (4) and (5) and their interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$. Assume the resulting Hamiltonian system $E\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H\begin{bmatrix} x \\ z \end{bmatrix}$ for E and $H \in \mathbb{R}^{2\mathtt{n}\times 2\mathtt{n}}$ is autonomous. Define $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ as in (8). Then the following are equivalent:

1) The singular Hamiltonian system has no inadmissible initial conditions.
2) $\det\ (CB - (CB)^T) \neq 0$.
3) $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is nonsingular at $t = 0$.
4) $\ker\ (\widetilde{C}\widetilde{B}) = 0$.

The proof is straightforward since we only need to use Lemma 5.2 with S_2 as the adjoint system of S_1. The following remark relates condition 2 of the above Theorem with an all-pass MIMO transfer matrix.

Remark 5.4: Condition 2 of the above theorem is kind of opposite to the condition required for a transfer function $G(s)$ to be all-pass[5]. More precisely, consider a square MIMO transfer function $G(s) \in \mathbb{R}^{p \times p}(s)$ which is all-pass, i.e. $I - G(-s)^T G(s) = 0$ for every $s \in j\mathbb{R}$. The feed-through term D of such a transfer matrix can be assumed to be I by considering a change of coordinates in either the u or the y variables. With this assumption on D, the all-pass condition on G results in various conditions on matrices A, B and C of its state space realization, which turn out to be

$$CB - B^T C^T = 0,\ CAB + (CAB)^T - B^T C^T CB = 0,\ \cdots \quad (9)$$

Notice that CB is nothing but the first moment of $G(s)$ about $s = \infty$. Thus a necessary condition on the first moment for G to be all-pass is that the skew-symmetric part of CB is zero. On the other hand, Condition 2 of the above theorem requires the skew-symmetric part to be nonsingular. In this sense, the necessary and sufficient condition on $G(s)$ for the singular Hamiltonian system to not have any inadmissible initial conditions is opposite to the requirement that $G(s)$ is all-pass.

It is noteworthy that the relation with all-pass is also visible in Theorems 4.4 and 3.3, where we characterized conditions on $G(s)$ for the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ to be autonomous. The necessary and sufficient conditions described there were nothing but ruling out an all-pass subsystem. This follows since ker $(\partial\Phi)(\frac{\mathrm{d}}{\mathrm{d}t})$ corresponds to latent variables that result in the all-pass subsystem, while Condition 3 of Theorem 4.4 just says that at least one of the conditions for all-pass (see (9) above) are not satisfied.

Of course, relaxing Assumption 4.1 would mean Condition 2 is required only for the subsystem that corresponds to the nullspace of $I - DD^T$ and requires the normalization of that restricted transfer function's feed-through term to identity matrix.

Another consequence of condition 2 above, under Assumption 4.1, is that square MIMO systems with an odd-number of inputs, in particular SISO systems, will always have inadmissible initial conditions. This is seen in the following example.

Example 5.5: Consider $G(s) = \frac{s+1}{s+2}$ with input u and output y and consider its state space realization $(A, B, C, D) = (-2, 1, -1, 1)$. The dual system has transfer function $\frac{s-1}{s-2}$ and $(2, 1, 1, 1)$ is a state space description. The interconnected system (described in the variables: state x and output y of the system G and state z of its dual) turns out to be:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\z\\y\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\-1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\z\\y\end{bmatrix}$$

It can be checked that the matrix pencil corresponding to the above first order differential equation has a zero at infinity. After elimination of the variable y too, the differential equation in just x and z turns out to contain inadmissible initial conditions. Theorem 5.3 can be used to obtain the same inference: since $CB - B^T C^T = 0$, we conclude that there exist inadmissible initial conditions. Of course, as noted in Remark 5.4, for SISO systems, ill-posed interconnection implies existence of inadmissible initial conditions.

VI. Conclusive remarks

We studied the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ since this interconnection contains precisely the set of trajectories stationary with respect the quadratic performance index arising from the supply rate: we hence called these 'Hamiltonian systems'. We formulated necessary and sufficient conditions for the intersection to be autonomous. When the intersection is autonomous, the dimension of this set is at most twice the McMillan degree of the system $\mathfrak{B}$, with equality equivalent to well-posedness of the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$. We called this the regular Hamiltonian system.

It is well-known that when an interconnection is not well-posed, there necessarily are trajectories that result in an impulse upon interconnection. However, there may or may not be inadmissible initial conditions for the interconnected system. When the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ is not well-posed, under suitable regularizing assumptions on the feed-through term D in the state space realization and full column rank condition on B (Assumption 4.1), we formulated necessary and sufficient conditions for existence of inadmissible initial conditions for the 'singular' Hamiltonian system in terms of the first moment about $s = \infty$ of the transfer matrix: our second main result. We noted that the condition on the skew-symmetric part of the first moment was opposite to that for the MIMO transfer matrix to be an all-pass filter. As a corollary to this, we deduced that singular Hamiltonian systems satisfying Assumption 4.1 and arising from a transfer matrix with *odd* number of inputs/outputs, in particular SISO systems, always have inadmissible initial conditions.

[5]This remark is relevant for the case that the supply rate corresponds to $u^T u - y^T y$, for which 'lossless' corresponds to all-pass characteristics. When dealing with the supply rate $u^T y$, relevant in passivity analysis, it is singularity of $(D + D^T)$ matrix that plays a role for the results of this paper; $G(s) + G(-s)^T$ then replaces $I - G(-s)^T G(s)$ for the statements made in this remark.

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