Structural properties of the state space of discrete 2D autonomous systems

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***Abstract*— In this paper, we present a state space like construction for discrete 2D autonomous systems. We first construct the state space and then explore some structural properties of it. We show that the state space can be viewed as a 1D system. We then relate some 1D systems theoretic properties – like autonomy, controllability – to properties of the original 2D system.**

I. Introduction

State space approach has been a centerpiece in 1D systems theory. The reasons behind the state space approach's widespread popularity are perhaps the two most useful features that they possess. One, systems theoretic insight from energy storage point of view. And, secondly, a recursive formula for the trajectories in the system. Interestingly, both these features follow from the fact that in the state space, the dynamical equation turns out to be first order. However, mathematical models of systems that arise in engineering are often *not* first order. Therefore, in order to make use of the multitude of benefits that the state space theory entails, it is imperative to convert higher order equations to first order equations: to construct state space for a given system.

In [1] it was shown how this construction of state equations can be done from a given higher order system of equations. For 2D systems, that is, systems where the trajectories evolve over 2 independent variables, the notion of state space has not yet achieved unanimity. Broadly, there have been two approaches in this regard: one, where one of the 2 independent variables, namely, time, has been treated as a special variable, and first order representations in this variables have been studied (see among others [2]–[4]). In another one, two independent variables have been treated equally and first order representations in both the variables have been obtained (see among others [5], [6]). Both these approaches are useful in their own applicable areas. However, they both have their limitations, too. For example, none of these approaches are universally applicable to all 2D systems.

In [7], a recursive solution formula, much akin to the one resulted in by 1D state space equations, was presented for autonomous systems described by 2D linear partial difference equations of higher order; in the sequel, we refer to these systems by *autonomous discrete* 2*D systems*. In this paper we show that this approach leads naturally to a state space for *every* autonomous discrete 2D system. Albeit, there is one crucial element unique to 2D systems: here a coordinate transformation on the independent variables is required prior to obtaining the state space. In this paper we elaborate on this construction of the state space, and then we explore some structural properties of it.

A. Notation

We use $\mathbb{R}$ and $\mathbb{C}$ to denote the fields of real and complex numbers, respectively. Consequently, $\mathbb{R}^n$, $\mathbb{C}^n$ denote the n-dimensional vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of integers is denoted by $\mathbb{Z}$, and $\mathbb{Z}^2$ denotes the set of two tuples of elements in $\mathbb{Z}$. In this paper, our main object of study is a particular class of doubly-indexed sequences of elements in $\mathbb{R}^{\mathtt{w}}$, for some positive integer $\mathtt{w}$. We denote the set of doubly-indexed sequences in $\mathbb{R}^{\mathtt{w}}$ by $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, *i.e.*, $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} := \{\mathbb{Z}^2 \to \mathbb{R}^{\mathtt{w}}\}$. The Laurent polynomial ring in two indeterminates σ_1, σ_2, usually written as $\mathbb{R}[\sigma_1^{\pm 1}, \sigma_2^{\pm 1}]$, will be denoted by $\mathcal{A}$, and the same in one indeterminate σ_1, written as $\mathbb{R}[\sigma_1^{\pm 1}]$, will be denoted by $\mathcal{A}_1$. We use $\mathcal{A}^{\mathtt{w}}$ to denote the free module of rank $\mathtt{w}$ over $\mathcal{A}$, where the elements of $\mathcal{A}^{\mathtt{w}}$ are written as $\mathtt{w}$-tuple of rows. For a set S, we use $S^{m\times n}$ to denote the set of $(m\times n)$ matrices with entries from the set S. The single letter σ is often used to denote the tuple (σ_1, σ_2). Further, for an integer tuple $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$, the symbol σ^ν denotes the monomial $\sigma_1^{\nu_1}\sigma_2^{\nu_2}$. In this paper, we follow the bar notation to denote equivalence classes: for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and a submodule $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$, we use $\overline{r(\sigma)}$ to denote the equivalence class of $r(\sigma)$ in the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$.

II. Background

By 2D systems, in this paper, we mean systems described by a set of 2D linear partial difference equations with constant real coefficients. Such partial difference equations are often described using the 2D shift operators σ_1 and σ_2. These shift operators act on a doubly-indexed real-valued sequence $w \in \mathbb{R}^{\mathbb{Z}^2}$ as follows: for $\nu' := (\nu_1', \nu_2'), \nu := (\nu_1, \nu_2) \in \mathbb{Z}^2$

$$(\sigma^{\nu'} w)(\nu_1, \nu_2) = w(\nu_1 + \nu_1', \nu_2 + \nu_2'). \tag{1}$$

This definition can be extended naturally to define the action of $\mathcal{A}$, the Laurent polynomial ring in the shifts, on $\mathbb{R}^{\mathbb{Z}^2}$. And, likewise, the action of the row module $\mathcal{A}^{\mathtt{w}}$ on columns of sequences (trajectories) $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ can be defined: for a row-vector $r(\sigma) = [r_1(\sigma), r_2(\sigma), \cdots, r_{\mathtt{w}}(\sigma)]$ and a column-vector $w = \mathrm{col}(w_1, w_2, \ldots, w_{\mathtt{w}}) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ we define $r(\sigma)w := \sum_{i=1}^{\mathtt{w}} r_i(\sigma)w_i$.

A. The kernel representation

The collection of trajectories $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ that satisfy a given set of partial difference equations is called the *behavior* of the system, and is denoted by $\mathfrak{B}$. In this paper, we often do not distinguish between a system and its behavior and call $\mathfrak{B}$ a system. The above description of the action of $\mathcal{A}^{\mathtt{w}}$

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on $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ gives the following representation of behaviors of 2D partial difference equations:

$$\mathfrak{B} := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid R(\sigma)w = 0\}, \tag{2}$$

where $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$. The above equation (2) is called a *kernel representation* of $\mathfrak{B}$ and written as $\mathfrak{B} = \ker(R(\sigma))$. Note that many different matrices can have the same kernel. Importantly, all matrices having the same row-span over $\mathcal{A}$ result in the same behavior. This leads to the following equivalent definition of behaviors: let $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$ and $\mathcal{R} := \mathrm{rowspan}(R(\sigma))$,

$$\mathfrak{B}(\mathcal{R}) := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid r(\sigma)w = 0 \text{ for all } r(\sigma) \in \mathcal{R}\}. \tag{3}$$

The submodule $\mathcal{R}$ generated by the rows of a kernel representation matrix is called *the equation module* of $\mathfrak{B}$. It was shown in [8] that the submodules of $\mathcal{A}^{\mathtt{w}}$ and 2D behaviors with $\mathtt{w}$ number of manifest variables are in one-to-one correspondence.

B. Autonomous systems

In this paper, we concentrate only on *autonomous systems*. Among several equivalent definitions of 2D autonomous systems (see [9], [10]), in this paper, we stick to the following Definition 2.1. In Definition 2.1 we need the notion of *characteristic ideal* of a behavior, whose definition is as follows: Let $\mathfrak{B}$ be given by a kernel representation $\mathfrak{B} = \ker(R(\sigma))$ with $R(\sigma) \in \mathcal{A}^{\mathtt{g}\times\mathtt{w}}$. The *characteristic ideal* of $\mathfrak{B}$, denoted by $\mathcal{I}(\mathfrak{B})$, is defined as the ideal of $\mathcal{A}$ generated by the $(\mathtt{w}\times\mathtt{w})$ minors of $R(\sigma)$. For $\mathtt{g} < \mathtt{w}$, $\mathcal{I}(\mathfrak{B})$ is defined to be the zero ideal.

Definition 2.1: A 2D system is said to be *autonomous* if the characteristic ideal $\mathcal{I}(\mathfrak{B})$ is nonzero. Further, an autonomous behavior is said to be *strongly autonomous* if the quotient ring $\mathcal{A}/\mathcal{I}(\mathfrak{B})$ is a finite dimensional vector space over $\mathbb{R}$.

C. The quotient module

Given a behavior $\mathfrak{B} = \ker(R(\sigma))$, let $\mathcal{R}$ be the submodule of $\mathcal{A}^{\mathtt{w}}$ spanned by the rows of $R(\sigma)$. We define

$$\mathcal{M} := \mathcal{A}^{\mathtt{w}}/\mathcal{R},$$

and call it the *quotient module* of $\mathfrak{B}$. This quotient module $\mathcal{M}$ plays a central role in this paper. We often let elements from $\mathcal{M}$ act on $\mathfrak{B}$. This action is defined as follows: for $m \in \mathcal{M}$, the action of m on $w \in \mathfrak{B}$ is defined to be the action of a lift of m in $\mathcal{A}^{\mathtt{w}}$ on w. For example, let $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ be such that $\overline{r(\sigma)} = m \in \mathcal{M}$, then

$$mw := r(\sigma)w. \tag{4}$$

Note that m may have several distinct lifts in $\mathcal{A}^{\mathtt{w}}$, but all of them have the same action on $w \in \mathfrak{B}$, because two lifts differ by an element in the equation module.

Now note that it follows from Definition 2.1 above that $\mathfrak{B}$ is autonomous *if and only if* the quotient module $\mathcal{M}$ is a *torsion module*, *i.e.*, for every $\overline{m(\sigma)} \in \mathcal{M}$ there exists a $f(\sigma) \in \mathcal{A}$ such that $\overline{f(\sigma)m(\sigma)} = 0 \in \mathcal{M}$. In that case we get the following ideal called the *annihilator ideal* of $\mathcal{M}$.

$$\mathrm{ann}(\mathcal{M}) := \{f(\sigma) \in \mathcal{A} \mid f(\sigma)m = 0 \ \forall m \in \mathcal{M}\}.$$

D. Change of coordinates

Change of coordinates in $\mathbb{Z}^2$ plays a crucial role throughout this paper. By a coordinate change we mean a $\mathbb{Z}$-linear map from $\mathbb{Z}^2$ to itself of the form

$$\begin{array}{rcl} T : \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\ \mathrm{col}(\nu_1,\nu_2) =: \nu & \mapsto & T\nu, \end{array}$$

where $T \in \mathbb{Z}^{2\times 2}$ is a *unimodular* matrix (*i.e.*, $\det(T) = \pm 1$). Note that because of unimodularity, the columns of T span the whole of $\mathbb{Z}^2$ as a $\mathbb{Z}$-module. Such a coordinate transformation T induces the following two maps.

$$\begin{array}{rcl} \varphi_T : \mathcal{A} & \to & \mathcal{A} \\ \sigma^{\nu} & \mapsto & \sigma^{T\nu} \end{array} \quad \Bigg| \quad \begin{array}{rcl} \Phi_T : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} & \to & (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \\ w(\nu) & \mapsto & w(T\nu), \end{array} \tag{5}$$

for all $\nu \in \mathbb{Z}^2$. Unimodularity of T makes both these maps *bijective*. In fact, Φ_T is an automorphism of the $\mathbb{R}$-vector space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, while φ_T is an automorphism of the $\mathbb{R}$-algebra $\mathcal{A}$. As a consequence, an *ideal* $\mathfrak{a} \subseteq \mathcal{A}$ is mapped to another *ideal* $\varphi_T(\mathfrak{a})$. The map φ_T can be extended to a map from $\mathcal{A}^{\mathtt{w}}$ to itself by applying φ pointwise. That is, define $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$ by mapping $[f_1(\sigma), f_2(\sigma), \cdots, f_{\mathtt{w}}(\sigma)]$ to $[\varphi_T(f_1(\sigma)), \varphi_T(f_2(\sigma)), \cdots, \varphi_T(f_{\mathtt{w}}(\sigma))]$. The map $\widehat{\varphi}_T$ is an $\mathcal{A}$-module morphism via the automorphism φ_T, *i.e.*, for $r(\sigma) \in \mathcal{A}^{\mathtt{w}}$ and $f(\sigma) \in \mathcal{A}$,

$$\widehat{\varphi}_T(f(\sigma)r(\sigma)) = \varphi_T(f(\sigma))\widehat{\varphi}_T(r(\sigma)).$$

The bijective property of φ_T extends to the module case: as a result, $\widehat{\varphi}_T(\mathcal{R})$, the image of a *submodule* $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ under $\widehat{\varphi}_T$, is also a *submodule*.

Theorem 2.2 brings out precisely how the two maps Φ_T and $\widehat{\varphi}_T$ are related with each other. Given a behavior $\mathfrak{B}$, we define

$$\Phi_T(\mathfrak{B}) := \{v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid v = \Phi_T(w) \text{ for some } w \in \mathfrak{B}\}. \tag{6}$$

Theorem 2.2: Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule with behavior $\mathfrak{B}(\mathcal{R})$, and let $T \in \mathbb{Z}^{2\times 2}$ be unimodular. Then we have

$$\mathfrak{B}(\mathcal{R}) = \Phi_T(\mathfrak{B}(\widehat{\varphi}_T(\mathcal{R}))). \tag{7}$$

III. The representation formula and construction of the state space

It is now well-known that a special class of 2D autonomous systems admits state space representation just like 1D systems. Such systems have been called *strongly autonomous* in the literature. The property that distinguishes the strongly autonomous systems from the rest is that these systems have their quotient modules as finite dimensional vector spaces over $\mathbb{R}$. In that case, it has been shown [11], that there exist two square invertible real matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and a matrix $C \in \mathbb{R}^{\mathtt{w}\times n}$, where n is the dimension of the quotient module $\mathcal{M}$ as a vector space over $\mathbb{R}$, such that the strongly autonomous system admits the following state space representation:

$$\begin{array}{rcl} x(\nu_1+1,\nu_2) & = & A_1 x(\nu_1,\nu_2), \\ x(\nu_1,\nu_2+1) & = & A_2 x(\nu_1,\nu_2), \\ w(\nu_1,\nu_2) & = & Cx(\nu_1,\nu_2). \end{array} \tag{8}$$

The matrices A_1 and A_2 are obtained by representing the multiplication maps by σ_1, σ_2, respectively, in a suitably chosen basis of $\mathcal{M}$ over $\mathbb{R}$. Further, the matrix C is obtained by representing the image of the $\mathtt{w} \times \mathtt{w}$ identity matrix in the chosen basis.

For general autonomous systems, however, the quotient module $\mathcal{M}$ may turn out to not be a finite dimensional vector space. This happens precisely when the characteristic variety of the system has non-zero dimension (that is, when it is a curve). Consequently, the representation by equation (8) no longer works for such systems. It has been shown in [7] how a representation similar to (8) can be obtained in this case. We briefly review that result here; for this will be crucial in the sequel.

For any 2D autonomous system, it has been shown in [7] that, there exists a coordinate transformation matrix $T : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that under the corresponding module map $\widehat{\varphi}_T : \mathcal{A}^{\mathtt{w}} \to \mathcal{A}^{\mathtt{w}}$, the quotient module $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ becomes a finitely generated module over the one-variable Laurent polynomial ring $\mathbb{R}[\sigma_1^{\pm 1}]$ (see Subsection II-D for the definition of $\widehat{\varphi}_T$). For convenience, we denote $\mathbb{R}[\sigma_1^{\pm 1}]$ by $\mathcal{A}_1$.

A. The consequences of $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ being a finitely generated module over $\mathcal{A}_1$

Let us denote $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ by $\widetilde{\mathcal{M}}$. The $\mathcal{A}$-module $\widetilde{\mathcal{M}}$ being finitely generated as a module over $\mathcal{A}_1$ implies that there exists a finite generating set, say $\{g_1, g_2, \cdots, g_n\} \subseteq \widetilde{M}$, such that every element in $\widetilde{\mathcal{M}}$ can be written as an $\mathcal{A}_1$-linear combination of $g_1, g_2, \cdots, g_n$. Using this generating set we can set up a map ψ from the free module $\mathcal{A}_1^n$ to $\widetilde{M}$ as

$$\begin{array}{llll} \psi : \mathcal{A}_1^n & \to & \widetilde{M} & \\ \quad e_i & \mapsto & g_i & \text{for all } 1 \leqslant i \leqslant n, \end{array} \tag{9}$$

where e_i is the standard i^{th} basis row-vector[1] of the free module $\mathcal{A}_1^n$. Note that the map ψ is an $\mathcal{A}_1$-module homomorphism, and is surjective. Using this map ψ we construct the following three matrices: $A(\sigma_1), C(\sigma_1)$, and $X(\sigma_1)$.

1) The flow matrix $A(\sigma_1)$: Consider the map $\mu : \widetilde{M} \to \widetilde{M}$ given by $\mu(m) = \overline{\sigma_2} m$ for any $m \in \widetilde{M}$. Clearly μ is $\mathcal{A}_1$-linear. Now, since $\widetilde{M}$ is finitely generated as an $\mathcal{A}_1$-module, this map μ can be represented by a matrix. This is done as follows: suppose $\{g_1, g_2, \ldots, g_n\} \subseteq \widetilde{M}$ is a generating set for $\widetilde{M}$ as an $\mathcal{A}_1$-module. For $1 \leqslant i \leqslant n$ consider the action of μ on the generator g_i. Since $\{g_1, g_2, \ldots, g_n\}$ is a generating set, the image of g_i under μ can be expressed as an $\mathcal{A}_1$-linear combination of these $g_1, g_2, \ldots, g_n$. That is, for $1 \leqslant i \leqslant n$, there exist $a_{i,1}(\sigma_1), a_{i,2}(\sigma_1), \ldots, a_{i,n}(\sigma_1) \in \mathcal{A}_1$ such that

$$\mu(g_i) = a_{i,1}(\sigma_1) g_1 + a_{i,2}(\sigma_1) g_2 + \cdots + a_{i,n}(\sigma_1) g_n.$$

Using this observation, we define the *flow matrix* $A(\sigma_1)$ as

$$A(\sigma_1) := [a_{i,j}(\sigma_1)]_{1 \leqslant i,j \leqslant n} \in \mathcal{A}_1^{n \times n}. \tag{10}$$

It has been shown in [7] that one can always find a generating set for which the corresponding $A(\sigma_1)$ turns will be invertible in $\mathcal{A}_1$ (that is, $A(\sigma_1)$ is $\mathcal{A}_1$-unimodular).

[1] As a convention, we consider the elements in $\mathcal{A}_1^n$ to be written as row-vectors.

2) The output matrix $C(\sigma_1)$: Let $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w} \times n}$ be such that $\psi(C(\sigma_1)) = \overline{I_{\mathtt{w}}}$, the image of $I_{\mathtt{w}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{A}^{\mathtt{w} \times \mathtt{w}}$ under the canonical map $\mathcal{A}^{\mathtt{w}} \twoheadrightarrow \widetilde{M}$. We call $C(\sigma_1)$ the *output matrix*.

3) The matrix of relations $X(\sigma_1)$: The finitely generated $\mathcal{A}_1$-module $\widetilde{M}$ may *not* be *free*, that is, the generators may satisfy nontrivial relations among themselves over $\mathcal{A}_1$. In that case, recalling the map $\psi : \mathcal{A}_1^n \to \widetilde{M}$ defined by equation (9), we must have $\ker(\psi)$ to be a nontrivial submodule of $\mathcal{A}_1^n$. Since $\mathcal{A}_1^n$ is a Noetherian module, this submodule $\ker(\psi)$ must be finitely generated. Let $X(\sigma_1) \in \mathcal{A}_1^{n' \times n}$ be a matrix whose rows generate $\ker(\psi)$, *i.e.*

$$\text{rowspan}(X(\sigma_1)) = \ker(\psi). \tag{11}$$

We call this matrix $X(\sigma_1)$ a *matrix of relations* of $\{g_1, g_2, \ldots, g_n\}$.

B. The representation formula

In [7] it was shown how the above-mentioned three matrices can be used to give a representation formula for trajectories in a 2D discrete autonomous system. We quote this result from [7] as Theorem 3.1 below. First, note that, using the discrete version of Noether's normalization lemma ([7, Theorem 4.3]), the following was proven in [7]: Suppose $\mathfrak{B}$ is an autonomous behavior whose equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ is such that the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is not a finite dimensional vector space over $\mathbb{R}$. Then there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular, such that $\widetilde{\mathcal{M}} := \mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated module over $\mathcal{A}_1$. In this setting, then, the above-mentioned three matrices $A(\sigma_1)$, $C(\sigma_1)$ and $X(\sigma_1)$ can be created.

Theorem 3.1: In the situation mentioned above, suppose the following 1-variable Laurent polynomial matrices: $X(\sigma_1) \in \mathcal{A}_1^{n' \times n}$, $C(\sigma_1) \in \mathcal{A}_1^{\mathtt{w} \times n}$, $A(\sigma_1) \in \mathcal{A}_1^{n \times n}$, with $A(\sigma_1)$ invertible in $\mathcal{A}_1^{n\times n}$, are created as in Subsections III-A.1, III-A.2, III-A.3, respectively. Then, for any $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$, $w \in \mathfrak{B}$ if and only if there exists $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ which satisfies

$$X(\sigma_1) x = 0$$

and for all $\nu = \text{col}(\nu_1, \nu_2) \in \mathbb{Z}^2$

$$w(\nu) = \left(C(\sigma_1) A(\sigma_1)^{(T\nu)_2} x \right) ((T\nu)_1),$$

where $T\nu = \text{col}((T\nu)_1, (T\nu)_2)$.

C. The state space $\mathscr{X}$

In this paper, we call the following set (which is a linear vector space over $\mathbb{R}$)

$$\mathscr{X} := \ker X(\sigma_1) \subseteq (\mathbb{R}^n)^{\mathbb{Z}} \tag{12}$$

a *state space* of the given 2D discrete autonomous system $\mathfrak{B}$. The reason for calling the above set as a state space will become apparent shortly. First, note that the $\mathscr{X}$ remains invariant under the map $A(\sigma_1) : (\mathbb{R}^n)^{\mathbb{Z}} \to (\mathbb{R}^n)^{\mathbb{Z}}$. Indeed, for this follows from the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \widetilde{M} \\
A(\sigma_1) \downarrow & & \downarrow \mu \\
\mathcal{A}_1^n & \overset{\psi}{\twoheadrightarrow} & \widetilde{M}
\end{array}
\tag{13}
$$

where $A(\sigma_1) : \mathcal{A}_1^n \ni r(\sigma_1) \mapsto r(\sigma_1)A(\sigma_1) \in \mathcal{A}_1^n$.

This fact enables us in setting up a dynamical system on $\mathscr{X}$ in the following manner: Let us first define by $\mathcal{W}$ the space of 1D trajectories whose domain is $\mathbb{Z}$ and codomain is $\mathscr{X}$. That is, $\mathcal{W} := \{\mathbf{x} : \mathbb{Z} \to \mathscr{X}\} = \mathscr{X}^{\mathbb{Z}}$. Since $\mathscr{X}$ is $A(\sigma_1)$-invariant, the restriction $A(\sigma_1)$ turns out to be a map from $\mathscr{X}$ to itself. In this way we can set up a first order dynamical system on $\mathscr{X}$ as

$$\mathfrak{B}_{\text{state}} := \{\mathbf{x} \in \mathcal{W} \mid \mathbf{x}(k+1) = A(\sigma_1)\mathbf{x}(k) \forall k \in \mathbb{Z}\}. \tag{14}$$

This first order dynamical system $\mathfrak{B}_{\text{state}}$ defined over the space $\mathscr{X}$ is equivalent to the original 2D discrete autonomous system $\mathfrak{B}$. This comes as a consequence of Theorem 3.1. Before proving this result we need to make the following formalization showing how trajectories in $\mathcal{W}$ can be viewed as 2D trajectories.

Proposition 3.2: Let $\mathbf{x}$ be a typical trajectory in $\mathcal{W}$, that is, $\mathbf{x}(k) \in \mathscr{X}$ for all $k \in \mathbb{Z}$. Since $\mathscr{X} \subseteq (\mathbb{R}^n)^{\mathbb{Z}}$, it follows that $\mathbf{x}(k) \in (\mathbb{R}^n)^{\mathbb{Z}}$ for all $k \in \mathbb{Z}$. Therefore, for all $h \in \mathbb{Z}$, we have

$$\left(\mathbf{x}(k)\right)(h) \in \mathbb{R}^n.$$

Define $\widetilde{\mathbf{x}} \in (\mathbb{R}^n)^{\mathbb{Z}^2}$ as

$$\widetilde{\mathbf{x}}(h,k) := \left(\mathbf{x}(k)\right)(h).$$

Thus, $\mathbf{x}$ can be identified with a 2D discrete trajectory.

Using Proposition 3.2 we define the following 2D discrete system:

$$\mathfrak{B}_{\text{aux}} := \left\{\widetilde{C(\sigma_1)\mathbf{x}} \mid \mathbf{x} \in \mathfrak{B}_{\text{state}}\right\}. \tag{15}$$

We are now in a position to prove the equivalence of $\mathfrak{B}$ and $\mathfrak{B}_{\text{state}}$.

Theorem 3.3: Suppose $\mathfrak{B}$ is a 2D discrete autonomous system with equation module $\mathcal{R}$. Let T, $A(\sigma_1)$, $C(\sigma_1)$ and $X(\sigma_1)$ be as in Theorem 3.1. Further, let $\mathscr{X}$, $\mathfrak{B}_{\text{state}}$ and $\mathfrak{B}_{\text{aux}}$ be as defined in equations (12), (14) and (15), respectively. Then we have

$$\mathfrak{B} = \Phi_T\left(\mathfrak{B}_{\text{aux}}\right).$$

Proof. We first write down explicitly the trajectories in $\mathfrak{B}_{\text{state}}$.

$$\mathbf{x}(k) = A(\sigma_1)^k \mathbf{x}(0), \tag{16}$$

for all $k \in \mathbb{Z}$, where $\mathbf{x}(0) \in \mathscr{X}$ is arbitrary. Suppose we denote this initial condition by x. It then follows from equation (16) that the trajectories in $\mathfrak{B}_{\text{aux}}$ are given by

$$v(h,k) = \left(C(\sigma_1)A(\sigma_1)^k x\right)(h), \tag{17}$$

with $x \in \mathscr{X}$ arbitrary. Therefore, for arbitrary $\nu = (\nu_1, \nu_2)$, we have

$$\Phi_T(v) = v(T\nu) = \left(C(\sigma_1)A(\sigma_1)^{(T\nu)_2}x\right)((T\nu)_1),$$

where $T\nu = \operatorname{col}((T\nu)_1, (T\nu)_2)$. It then follows from Theorem 3.1 that for every $w \in \mathfrak{B}$ there exists $v \in \mathfrak{B}_{\text{aux}}$ such that $w = \Phi_T(v)$, and conversely, for every $v \in \mathfrak{B}_{\text{aux}}$ there exists $w \in \mathfrak{B}$ such that $w = \Phi_T(v)$. Hence, $\mathfrak{B} = \Phi_T\left(\mathfrak{B}_{\text{aux}}\right)$. □

We illustrate this result in the following example.

Example 3.4: Consider the scalar behavior

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma_2^2\sigma_1^2 + 5\sigma_1\sigma_2 + 6 \\ \sigma_1^3\sigma_2^2 + 2\sigma_2\sigma_1^2 - \sigma_1\sigma_2 - 2 \end{bmatrix}.$$

The equation module is the ideal $\mathfrak{a} = \langle \sigma_2^2\sigma_1^2 + 5\sigma_1\sigma_2 + 6, \sigma_1^3\sigma_2^2 + 2\sigma_2\sigma_1^2 - \sigma_1\sigma_2 - 2\rangle$. The quotient module $\mathcal{M} = \mathcal{A}/\mathfrak{a}$ can be checked to be not a finitely generated module over $\mathcal{A}_1$. However, under the coordinate transformation $T = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ the equation ideal $\mathfrak{a}$ gets mapped to the ideal

$$\varphi_T(\mathfrak{a}) = \langle \sigma_2^2 + 5\sigma_2 + 6, \sigma_1\sigma_2 + 2\sigma_1 - \sigma_2 - 2\rangle.$$

Now, the quotient $\mathcal{A}/\varphi_T(\mathfrak{a})$ is a finitely generated module over $\mathcal{A}_1$. Generators can be chosen to be $\{\overline{1}, \overline{\sigma_2}\}$. Here, $n = 3$ and

- $X(\sigma_1) = \begin{bmatrix} 2(\sigma_1 - 1) & (\sigma_1 - 1) \end{bmatrix}$,
- $A(\sigma_1) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$,
- $C(\sigma_1) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Therefore, the state space is given by

$$\begin{aligned} \mathscr{X} &= \ker X(\sigma_1) \\ &= \left\{x \in \left(\mathbb{R}^2\right)^{\mathbb{Z}} \;\middle|\; \begin{bmatrix} 2(\sigma_1 - 1) & (\sigma_1 - 1) \end{bmatrix} x = 0\right\}. \end{aligned}$$

Consequently, trajectories in $\mathfrak{B}_{\text{state}}$ are given by

$$\mathbf{x}(k) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^k x,$$

where $x \in \ker X(\sigma_1)$. Thus, the 2D system $\mathfrak{B}_{\text{aux}}$ turns out to be the collection of trajectories of the following form

$$\begin{aligned} v(h,k) &= \widetilde{C(\sigma_1)\mathbf{x}}(h,k) \\ &= \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^k x\right)(h). \end{aligned}$$

Therefore, the trajectories in $\Phi_T(\mathfrak{B}_{\text{aux}})$ are of the form

$$w(\nu_1, \nu_2) = \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^{(\nu_1+\nu_2)} x\right)(2\nu_1 + \nu_2),$$

where $x \in \ker X(\sigma_1)$. This is precisely the form of trajectories in $\mathfrak{B}$ by Theorem 3.1.

□

Some interesting points to notice about Theorem 3.3 are:

- $\mathfrak{B}_{\text{aux}}$ is completely determined by the space $\mathscr{X}$, and the flow matrix $A(\sigma_1)$ and the output matrix $C(\sigma_1)$.
- The evolution on the space $\mathscr{X}$ is 1D and first order.
- The initial conditions for this 1D evolution are arbitrary elements of $\mathscr{X}$.
- The equivalence of $\mathfrak{B}$ with $\mathfrak{B}_{\text{aux}}$ is via a coordinate change T.

These points provide ample justification to call equation (14) a state equation, and likewise, $\mathscr{X}$ a state space of $\mathfrak{B}_{\text{aux}}$. By the equivalence proved in Theorem 3.3, it is also justified to call $\mathscr{X}$ a state space of the original 2D discrete autonomous system $\mathfrak{B}$. In the next section we investigate deeper into this state space $\mathscr{X}$ and bring out certain interesting structural properties of it.

IV. Some structural properties of the state space $\mathscr{X}$

Since the state space $\mathscr{X}$ is given by the kernel of the operator matrix $X(\sigma_1)$, it can be treated as a 1D behavior itself. This 1D behavior has $\ker\,\psi$ for its equation module, which, in turn, is generated as an $\mathcal{A}_1$-module by the rows of the matrix $X(\sigma_1)$. Note that the map $\psi : \mathcal{A}_1^n \to \widetilde{\mathcal{M}}$ depends inextricably on the choice of the generating set for $\widetilde{\mathcal{M}}$ as an $\mathcal{A}_1$-module. As a result, the module $\ker\,\psi$, and therefore, the state space $\mathscr{X}$, too, depend on the generating set. This may cause the state space $\mathscr{X}$ to be non-unique, even for a fixed coordinate transformation T. However, if T is kept fixed, the various different state spaces $\mathscr{X}$ that are obtained due to different choices of generating sets for $\widetilde{\mathcal{M}}$ as an $\mathcal{A}_1$-module, all turn out to be *isomorphic* to each other (in the sense of [12]).

Two 1D systems, say $\mathfrak{B}_1$ and $\mathfrak{B}_2$, are said to be isomorphic in this sense if there exists an operator matrix $F(\sigma_1)$ with Laurent polynomial entries in the shift operator σ_1, such that $F(\sigma_1)$ is injective on $\mathfrak{B}_1$ and $F(\sigma_1)(\mathfrak{B}_1) = \mathfrak{B}_2$. In general, it may be difficult to find out the operator $F(\sigma_1)$. However, it was shown in [12], that two 1D discrete systems $\mathfrak{B}_1, \mathfrak{B}_2$, with equation modules $\mathcal{R}_1 \subseteq \mathcal{A}_1^{\mathtt{w}_1}$ and $\mathcal{R}_2 \subseteq \mathcal{A}_1^{\mathtt{w}_2}$, respectively, if and only if the quotient modules $\mathcal{A}_1^{\mathtt{w}_1}/\mathcal{R}_1$ and $\mathcal{A}_1^{\mathtt{w}_2}/\mathcal{R}_2$ are isomorphic as $\mathcal{A}_1$-modules. The fact that various different $\mathscr{X}$'s are isomorphic as 1D behaviors follows immediately from this result. Suppose $\widetilde{\mathcal{M}}$ is finitely generated as an $\mathcal{A}_1$-module, and let $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_{n'}\}$ be two distinct sets of generators for $\widetilde{\mathcal{M}}$ as and $\mathcal{A}_1$-module. As done in Subsection III-A.3, suppose we find out two matrices $X_1(\sigma_1)$ and $X(\sigma_1)$, which are the matrices of relations of $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_{n'}\}$, respectively. It then follows that both $\mathcal{A}_1^n/\mathrm{rowspan}(X_1(\sigma_1))$ and $\mathcal{A}_1^{n'}/\mathrm{rowspan}(X_2(\sigma_1))$ are isomorphic to $\widetilde{\mathcal{M}}$ as $\mathcal{A}_1$-modules. Therefore, clearly, $\mathcal{A}_1^n/\mathrm{rowspan}(X_1(\sigma_1))$ and $\mathcal{A}_1^{n'}/\mathrm{rowspan}(X_2(\sigma_1))$ are isomorphic to each other as $\mathcal{A}_1$-modules. Hence, the two state spaces $\mathscr{X}_1 = \ker\,X_1(\sigma_1)$ and $\mathscr{X}_2 = \ker\,X_2(\sigma_1)$ are isomorphic as 1D systems.

When $\mathscr{X}$ is viewed as a 1D system, it is natural to ask whether this 1D system is controllable or not. Our first observation regarding this issue is:

Proposition 4.1: The state space $\mathscr{X}$ corresponding to the quotient module $\widetilde{\mathcal{M}}$ is autonomous as a 1D system if and only if the original 2D system is strongly autonomous.

Proof. First note that when $\mathscr{X}$ is viewed as a 1D system, its corresponding quotient module turns out to be $\widetilde{\mathcal{M}}$ viewed as a module over $\mathcal{A}_1$. Now, the 1D system $\mathscr{X}$ is autonomous if and only if its quotient module $\widetilde{\mathcal{M}}$ is a finite dimensional vector space over $\mathbb{R}$ [13]. But, since $\widehat{\varphi}_T$ is an automorphism of the module $\mathcal{A}^{\mathtt{w}}$ over the $\mathbb{R}$-algebra $\mathcal{A}$, $\widetilde{\mathcal{M}}$ is a finite dimensional vector space over $\mathbb{R}$ if and only if $\mathcal{M}$ is a finite dimensional vector space over $\mathbb{R}$, which is equivalent to the 2D system $\mathfrak{B}$ being strongly autonomous. □

Thus, by Proposition 4.1, if $\mathfrak{B}$ is assumed to be not strongly autonomous then $\mathscr{X}$ must be non-autonomous as a 1D system. When $\mathscr{X}$ is not autonomous, then it will contain variables that are free, like inputs. This makes $\mathscr{X}$ an infinite dimensional vector space over $\mathbb{R}$. Understandably, it is easier to apply the representation formula of Theorem 3.1 if $\mathscr{X}$ turns out to be free, that is, $\mathscr{X}$ contains only free variables. In other words, $\mathscr{X} = \left(\mathbb{R}^{n'}\right)^{\mathbb{Z}}$ for some positive integer n'.

In order to resolve this issue of free $\mathscr{X}$ we shall first see how a clever choice of generators of $\widetilde{\mathcal{M}}$ results in a more useful form for the matrix of relations $X(\sigma_1)$. Note that we can make $X(\sigma_1)$ *full row-rank* over the field of fractions $\mathrm{qt}(\mathcal{A}_1)$. This is true because $\mathcal{A}_1$ is a PID[2]. Lemma 4.3 below utilizes another consequence of $\mathcal{A}_1$ being a PID: $X(\sigma_1)$ admits a *Smith form.* For our purpose, the Smith canonical form in full generality is not required, a weaker version suffices. We state this result as Proposition 4.2 below. See [13] for a proof.

Proposition 4.2: Let $X(\sigma_1) \in \mathcal{A}_1^{m\times n}$ be a full row-rank matrix. Then there exist square matrices $U(\sigma_1) \in \mathcal{A}_1^{m\times m}$ and $V(\sigma_1) \in \mathcal{A}_1^{n\times n}$, with the property that $\det(U(\sigma_1))$ and $\det(V(\sigma_1))$ are units in $\mathcal{A}_1$, such that

$$U(\sigma_1)X(\sigma_1)V(\sigma_1) = \left[\begin{array}{cc} D(\sigma_1) & 0 \end{array}\right],$$

where $D(\sigma_1) \in \mathcal{A}_1^{m\times m}$ is square with nonzero determinant.

Lemma 4.3: Let $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ be a submodule such that $\mathcal{M} = \mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is a finitely generated module over $\mathcal{A}_1$. Then there exists a set of generators of $\mathcal{M}$ as an $\mathcal{A}_1$-module, which admits a matrix of relations $X(\sigma_1)$ of the following form

$$X(\sigma_1) = \left[\begin{array}{cc} D(\sigma_1) & 0 \end{array}\right], \tag{18}$$

where $D(\sigma_1)$ is a square matrix with nonzero determinant.

Proof. Let $\{\overline{g_1'(\sigma)}, \overline{g_2'(\sigma)}, \ldots, \overline{g_n'(\sigma)}\}$ be an arbitrary set of generators for $\mathcal{M}$ as an $\mathcal{A}_1$-module, and let $X'(\sigma_1) \in \mathcal{A}_1^{m\times n}$ be its matrix of relations. As mentioned earlier, $X'(\sigma_1)$ can be assumed to be full row-rank. Then by Proposition 4.2 there exist square matrices $U(\sigma_1) \in \mathcal{A}_1^{m\times n}$ and $V(\sigma_1) \in \mathcal{A}_1^{n\times n}$, both having units for determinants, such that

$$U(\sigma_1)X'(\sigma_1)V(\sigma_1) = \left[\begin{array}{cc} D(\sigma_1) & 0 \end{array}\right], \tag{19}$$

where $D(\sigma_1) \in \mathcal{A}_1^{m\times m}$ with nonzero determinant. Since $\det(V(\sigma_1))$ is a unit in $\mathcal{A}_1$, it follows that $V(\sigma_1)$ has an inverse in $\mathcal{A}_1^{n\times n}$. Define

$$\left[\begin{array}{c} g_1(\sigma) \\ g_2(\sigma) \\ \vdots \\ g_n(\sigma) \end{array}\right] := V(\sigma_1)^{-1} \left[\begin{array}{c} g_1'(\sigma) \\ g_2'(\sigma) \\ \vdots \\ g_n'(\sigma) \end{array}\right]. \tag{20}$$

Clearly, $\mathcal{G} := \{\overline{g_1(\sigma)}, \overline{g_2(\sigma)}, \ldots, \overline{g_n(\sigma)}\}$ is a generating set for $\mathcal{M}$ as an $\mathcal{A}_1$-module. It then follows that a matrix of relations for this new set of generators is given by

$$X(\sigma_1) := U(\sigma_1)X'(\sigma_1)V(\sigma_1).$$

[2] $\mathcal{A}_1$ being a PID implies the submodule $\ker(\psi)$ of the free module $\mathcal{A}_1^n$ is free, and hence there exists a full row-rank matrix $X(\sigma_1)$ whose rows will generate the free module $\ker(\psi)$ over $\mathcal{A}_1$.

Indeed, $X'(\sigma_1)V(\sigma_1)$ is clearly a matrix of relations for $\mathcal{G}$. Since $\det(U(\sigma_1))$ is a unit in $\mathcal{A}_1$, it also has an inverse in $\mathcal{A}_1^{m\times m}$. It then follows that the row-span of $X'(\sigma_1)V(\sigma_1)$ is the same as that of $U(\sigma_1)X'(\sigma_1)V(\sigma_1)$. Therefore, $X(\sigma_1) := U(\sigma_1)X'(\sigma_1)V(\sigma_1)$ is a matrix of relations for $\mathcal{G}$. The statement of the lemma then follows from equation (19). □

Using Lemma 4.3 we can now obtain the following interesting description for the state space $\mathscr{X}$. Suppose $X(\sigma_1) = \begin{bmatrix} D(\sigma_1) & 0 \end{bmatrix} \in \mathcal{A}_1^{m\times n}$ with $D(\sigma_1) \in \mathcal{A}_1^{m\times m}$ having non-zero determinant. Now partition $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ as $x = (x_1, x_2)$, where $x_1 \in (\mathbb{R}^m)^{\mathbb{Z}}$ and $x_2 \in (\mathbb{R}^{n-m})^{\mathbb{Z}}$. Then $x \in \ker(X(\sigma_1))$ if and only if

$$D(\sigma_1)x_1 = 0,$$

and x_2 is free. Now, since $D(\sigma_1)$ is square with nonzero determinant, it follows that $\ker(D(\sigma_1))$ is a finite dimensional vector space over $\mathbb{R}$. In other words, there exists a fixed set of finitely many 1D trajectories $\{z_1, z_2, \ldots, z_r\} \subseteq (\mathbb{R}^{n'})^{\mathbb{Z}}$, such that $x \in \mathscr{X}$ if and only if it is of the form

$$x = \begin{bmatrix} a_1z_1 + a_2z_2 + \cdots + a_rz_r \\ x_2 \end{bmatrix}, \tag{21}$$

where $\{a_1, a_2, \ldots, a_r\} \subseteq \mathbb{R}$ and $x_2 \in (\mathbb{R}^{n-n'})^{\mathbb{Z}}$. This leads to the following description of $\mathscr{X}$.

Theorem 4.4: Suppose $\mathfrak{B}$ is an autonomous behavior whose equation module $\mathcal{R} \subseteq \mathcal{A}^{\mathtt{w}}$ is such that the quotient module $\mathcal{A}^{\mathtt{w}}/\mathcal{R}$ is not a finite dimensional vector space over $\mathbb{R}$. Then there exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that the state space corresponding to $\widetilde{\mathcal{M}} = \mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is given by

$$\mathscr{X} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (\mathbb{R}^m)^{\mathbb{Z}} \times \left(\mathbb{R}^{n-m}\right)^{\mathbb{Z}} \;\middle|\; D(\sigma_1)x_1 = 0 \right\},$$

where n, m are two positive integers with $n \geqslant m$ and $D(\sigma_1) \in \mathcal{A}_1^{m\times m}$ has nonzero determinant.

Proof. The result follows directly from Lemma 4.3 □

Remark 4.5: In order for the state space obtained in Theorem 4.4 to be applicable to Theorem 3.3 it is crucial too check whether the flow matrix remains invertible in the new generating set. However, the way the new generators are obtained in Lemma 4.3 this is guaranteed to happen. Note that when a new set of generators, say $\{\overline{g_1'(\sigma)}, \overline{g_2'(\sigma)}, \ldots, \overline{g_n'(\sigma)}\}$, is obtained from an old one, say $\{\overline{g_1(\sigma)}, \overline{g_2(\sigma)}, \ldots, \overline{g_n(\sigma)}\}$, by equation (20), the corresponding matrix representations of the map μ turn out to obey the following equation:

$$A'(\sigma_1) = V(\sigma_1)A(\sigma_1)V(\sigma_1)^{-1}.$$

This is analogous to a similarity transformation done on the state-space in 1D systems. Observe that $A'(\sigma_1)$ is invertible if and only if $A(\sigma_1)$ is.

Theorem 4.4 explicitly brings out the extent of free-ness of $\mathscr{X}$; the $x_2 \in (\mathbb{R}^{n-m})^{\mathbb{Z}}$ trajectories constitute the free part. Once again, viewing $\mathscr{X}$ as a 1D system, x_2 can be thought of as input. In [14] this number $(n-m)$ has been called the *input cardinality* of the concerned 1D behavior.

In light of Theorem 4.4, $\mathscr{X}$ will turn out to be free if the determinant of $D(\sigma_1)$ is a unit in $\mathcal{A}_1$. This means $\ker(D(\sigma_1)) = 0$. Algebraically, $D(\sigma_1)$ having a unit for determinant is equivalent to $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ being a free module over $\mathcal{A}_1$ because $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R}) \cong \mathcal{A}_1^n/\mathrm{rowspan}(X(\sigma_1))$ as $\mathcal{A}_1$-modules. This is known to be equivalent to $\mathscr{X}$ being controllable as a 1D system [13]. We state this result as Theorem 4.6 below.

Theorem 4.6: Let $\mathfrak{B}$ be an autonomous 2D discrete system with equation module $\mathcal{R}$. Then the following are equivalent:

1) $\mathfrak{B}$ admits a free state space $\mathscr{X}$.
2) There exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that the corresponding module $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a free module over $\mathcal{A}_1$.
3) There exists $T \in \mathbb{Z}^{2\times 2}$ unimodular such that the corresponding module $\mathcal{A}^{\mathtt{w}}/\widehat{\varphi}_T(\mathcal{R})$ is a finitely generated module over $\mathcal{A}_1$ and the corresponding matrix of relations $X(\sigma_1)$ is left-prime.

V. Concluding remarks

In this paper we have looked into a novel representation of 2D discrete autonomous systems that is very much similar to the well-known state space equations for 1D systems. This representation follows from the main result of [7]. Following this we show the construction of a state space for 2D discrete autonomous systems. Unlike 1D systems, the state space in this case may turn out to be infinite dimensional. We provide various structural properties of this state space in this paper.

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