ℓ^∞-Stability Analysis of Discrete Autonomous Systems Described by Laurent Polynomial Matrix Operators

Chirayu D. Athalye, Debasattam Pal, and Harish K. Pillai

Chirayu D. Athalye, Debasattam Pal, and Harish K. Pillai are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. chirayu@ee.iitb.ac.in, debasattam@ee.iitb.ac.in, hp@ee.iitb.ac.in

Abstract

In this paper we analyze the ℓ^∞-stability of infinite dimensional discrete autonomous systems whose dynamics is governed by a Laurent polynomial matrix $A(\sigma,\sigma^{-1})$ in shift operator σ on vector valued sequences. We give necessary and sufficient conditions for the ℓ^∞-stability of such systems. We also give easy to check tests to conclude or to rule out the ℓ^∞-stability of such systems.

Keywords: ℓ^∞-stability, infinite dimensional autonomous systems, 2-D autonomous systems.

1. Introduction

Infinite dimensional systems – that is, dynamical systems defined over an infinite dimensional state-space – arise as a natural mathematical model for numerous engineering applications. In fact, any system that is modeled by partial differential/difference equations (distributed parameter systems) or by delay-differential equations can be cast as an infinite dimensional dynamical system [1]. Naturally, the question of stability of such systems is an important issue. However, owing to the infinite dimensionality of the state-space, extension of results on stability of finite dimensional systems is often not possible. The question of stability of a certain special class of infinite dimensional systems has been dealt with in the recent interesting work of Feintuch and Francis [2] concerning an infinite chain of vehicles. In [2], the dynamics of the infinite chain of vehicles follows the *nearest-neighbor interaction*: let $q_n(t)$ denote the position of the n^{th} vehicle at time t, then

$$\dot{q}_n = f(q_{n+1}-q_n, q_{n-1}-q_n),$$

where f is the same linear function for all n. Note that, such a dynamical equation can be written succinctly as:

$$\dot{\mathbf{q}} = (a_{-1}\sigma^{-1}+a_0+a_1\sigma)\mathbf{q}, \tag{1}$$

where $\mathbf{q}$ denotes the entire sequence $\{\ldots,q_{-1},q_0,q_1,\ldots\}$ and σ is the (left or right) shift operator with a_{-1},a_0,a_1 being real numbers. The operator $(a_{-1}\sigma^{-1}+a_0+a_1\sigma)$ has the structure of a Laurent polynomial operator in the shift σ. In this paper, we deal with stability of dynamical systems whose dynamics is governed by a generalized discrete version of (1): while (1) involves only scalar trajectories, we consider vector trajectories and instead of just nearest-neighbor interactions, we consider an operator given by a general Laurent polynomial matrix. Thus, the systems we are concerned with are governed by the following type of discrete dynamical equation:

$$\mathbf{x}_{k+1}(\cdot) = A(\sigma,\sigma^{-1})\,\mathbf{x}_k(\cdot), \tag{2}$$

where $A(\sigma,\sigma^{-1})$ is a square Laurent polynomial matrix in shift operator σ, and $\mathbf{x}_k(\cdot)$ is a vector valued sequence defined over integers.

Unlike its finite dimensional counter-part, stability analysis of infinite dimensional systems depends crucially on the normed space chosen as the infinite dimensional state-space. The two most prevalent normed spaces in this regard are $(\ell^2, \|\cdot\|_2)$ and $(\ell^\infty, \|\cdot\|_\infty)$. While working with $(\ell^2, \|\cdot\|_2)$ space is somewhat easier than with $(\ell^\infty, \|\cdot\|_\infty)$ space, in many questions of practical significance, it is $(\ell^\infty, \|\cdot\|_\infty)$ space that becomes the more realistic choice. For example, in the case of infinite chain of vehicles, ℓ^2 perturbation from an equilibrium means: for every $\varepsilon > 0$, *almost all* the vehicles are within ε-neighborhood of their corresponding equilibrium positions. In a practical scenario, this may not be realistic. We, therefore, restrict ourselves entirely to the ℓ^∞-stability analysis of systems governed by (2). Such stability analysis over $(\ell^\infty, \|\cdot\|_\infty)$ space falls under the general setting of stability analysis over an infinite dimensional Banach space, which is a recent topic of interest (see [3, 4]). In this paper we provide elegant necessary and sufficient conditions for the ℓ^∞-stability of systems governed by (2) in terms of spectral radius of $A(e^{i\omega}, e^{-i\omega})$ and operator norm. These necessary and sufficient conditions may not always be easy to check; so, we also provide easily implementable necessary conditions and sufficient conditions for ℓ^∞-stability. These tests can be used to conclude or rule out ℓ^∞-stability.

1.1. Notation

We denote the fields of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. We use the symbol $\mathbb{F}$ to denote $\mathbb{R}$ or $\mathbb{C}$ in statements that hold true for both $\mathbb{R}$ and $\mathbb{C}$. The set of integers is denoted by $\mathbb{Z}$; while the symbols $\mathbb{N}$ and $\mathbb{N}_0$ are used to denote the set of positive integers $\{1, 2, \ldots\}$ and the set of non-negative integers $\{0, 1, 2, \ldots\}$, respectively.

We use I to denote the identity operator. Transpose of a vector $\mathbf{v}$ (a matrix B) is denoted by $\mathbf{v}'$ (B'). The symbol $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$ is used to denote the space of $\mathbb{F}^n$ valued bidirectional sequences; i.e., $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n) := \{\mathbf{a} : \mathbb{Z} \to \mathbb{F}^n\}$. To denote the zero element in $\mathbb{F}^n$ and $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$ we use boldface $\mathbf{0}$; and we expect it to be clear from the context. For $\mathbf{x} \in \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$, $\mathbf{x}(j)$ is used to denote the value of $\mathbf{x}$ at $j \in \mathbb{Z}$; therefore, $\mathbf{x}(j) \in \mathbb{F}^n$, $\forall j \in \mathbb{Z}$. We write $\mathbf{x}(j) = *$, when the exact value of $\mathbf{x}(j)$ is irrelevant. Analogously for $\mathbf{v} \in \mathbb{F}^n$, $\mathbf{v}(j)$ is used to denote the j^{th} component of $\mathbf{v}$.

Laurent polynomial ring in a variable σ with coefficients from $\mathbb{F}$ is denoted as $\mathbb{F}[\sigma, \sigma^{-1}]$. We use i to denote $\sqrt{-1}$, unless specified otherwise. The unit circle, the closed unit disc and the open unit disc in $\mathbb{C}$ centered at the origin are denoted as:

$$S_{\mathbb{C}}(0,1) := \{z \in \mathbb{C} \,:\, |z| = 1\}, \tag{3a}$$

$$B_{\mathbb{C}}(0,1) := \{z \in \mathbb{C} \,:\, |z| \leq 1\}, \tag{3b}$$

$$B^o_{\mathbb{C}}(0,1) := \{z \in \mathbb{C} \,:\, |z| < 1\}. \tag{3c}$$

1.2. Objective, overview and motivation

Consider the *left shift operator* $\sigma : \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n) \to \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$, which is defined as $(\sigma\mathbf{x})(j) := \mathbf{x}(j+1)$. Its inverse is the right shift operator, denoted as σ^{-1}. It follows that a Laurent polynomial matrix $A(\sigma, \sigma^{-1}) = \left(\sum_{j=-m}^{p} A_j \sigma^j\right) \in \mathbb{R}^{n\times n}[\sigma, \sigma^{-1}]$, where $A_j \in \mathbb{R}^{n\times n}$ for $j \in \{-m, \ldots, p\}$, is a well defined operator on $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$; i.e., $A(\sigma, \sigma^{-1}) : \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n) \to \mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$. In this paper, we study the following infinite dimensional discrete autonomous system:

$$\mathbf{x}_{k+1}(\cdot) := A(\sigma, \sigma^{-1})\, \mathbf{x}_k(\cdot), \tag{4}$$

where $A(\sigma, \sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma, \sigma^{-1}]$ and $\mathbf{x}_k \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$, $\forall k \in \mathbb{N}_0$. The trajectories satisfying (4) can be written as:

$$\mathbf{x}_k(\cdot) := A(\sigma, \sigma^{-1})^k\, \mathbf{x}_0(\cdot), \tag{5}$$

where $\mathbf{x}_0 \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$ is an initial condition.

Later in Section 2.2 we explain that, $A(\sigma, \sigma^{-1})$ is a continuous linear operator on $\ell^\infty(\mathbb{Z}, \mathbb{F}^n)$. In this paper, we obtain necessary and sufficient conditions for the ℓ^∞-stability of systems given by (4). We also give easy to check necessary conditions and sufficient conditions for the ℓ^∞-stability of such systems. Stability analysis of systems given by (4) is closely related to the stability analysis of discrete 2-D autonomous systems in general (see [5, 6]); and particularly to the stability analysis of time relevant discrete 2-D autonomous systems (see [7]). When time relevant discrete 2-D autonomous systems are brought down to the state space form, the dynamics is exactly same as the one given in (4).

2. Mathematical preliminaries

2.1. Bounded linear operators

Here we briefly mention some preliminaries from functional analysis; reader can refer to [8, 9, 10] for a detailed treatment on these topics. We are interested in the normed subspace $(\ell^\infty(\mathbb{Z}, \mathbb{F}^n), \|\cdot\|_\infty)$ of $\mathbb{F}^\infty(\mathbb{Z}, \mathbb{F}^n)$; for $\mathbf{x} \in \ell^\infty(\mathbb{Z}, \mathbb{F}^n)$,

$$\|\mathbf{x}\|_\infty := \sup\{\|\mathbf{x}(j)\|_\infty \,:\, j \in \mathbb{Z}\}. \tag{6}$$

Let $(X, \|\cdot\|_x)$ be any normed space over $\mathbb{F}$. Let T be a linear operator on a normed space X. The linear operator T is continuous if and only if there exists $\alpha > 0$ such that:

$$\|T(\mathbf{y})\|_x \le \alpha \|\mathbf{y}\|_x, \ \forall \mathbf{y} \in X. \tag{7}$$

Therefore, continuous linear operators are also called as *bounded linear operators*. The space of bounded linear (or continuous linear) operators on X is denoted as $BL(X)$; it is a normed space with the following induced operator norm: for $T \in BL(X)$,

$$\|T\|_x := \sup\{\|T(\mathbf{y})\|_x \,:\, \mathbf{y} \in X \text{ and } \|\mathbf{y}\|_x \le 1\} \tag{8}$$

$$= \inf\{\alpha \in \mathbb{R} \,:\, \|T(\mathbf{y})\|_x \le \alpha\|\mathbf{y}\|_x, \text{ for all } \mathbf{y} \in X\}. \tag{9}$$

The inequality,

$$\|T(\mathbf{y})\|_x \le \|T\|_x \, \|\mathbf{y}\|_x, \ \forall \mathbf{y} \in X \tag{10}$$

is called *the basic inequality*. The operator $T \in BL(X)$ is said to be invertible (in $BL(X)$), if T is bijective and the inverse map, T^{-1}, also belongs to $BL(X)$. For $T \in BL(X)$, *the eigenspectrum* $\Lambda_e(T)_X$, *the spectrum* $\Lambda(T)_X$, *the resolvent set* $\Lambda^c(T)_X$ and *the spectral radius* $\rho(T)_X$ are defined as follows:

$$\Lambda_e(T)_X := \{\lambda \in \mathbb{F} \mid (\lambda I - T) \text{ is not one-one}\}, \tag{11a}$$

$$\Lambda(T)_X := \{\lambda \in \mathbb{F} \mid (\lambda I - T) \text{ is not invertible}\}, \tag{11b}$$

$$\Lambda^c(T)_X := \mathbb{F} \setminus \Lambda(T)_X, \tag{11c}$$

$$\rho(T)_X := \max\{|\lambda| \,:\, \lambda \in \Lambda(T)_X\}. \tag{11d}$$

It follows from the definition that, $\Lambda_e(T)_X \subseteq \Lambda(T)_X$. If X is a finite dimensional vector space, then $\Lambda_e(T)_X = \Lambda(T)_X$.

2.2. Laurent polynomial matrix operator

Consider a Laurent polynomial matrix $A(\sigma,\sigma^{-1}) = \left(\sum_{j=-m}^{p} A_j\,\sigma^j\right) \in \mathbb{R}^{n\times n}[\sigma,\sigma^{-1}]$ in the shift operator σ. For ease of notation, we use L_A to denote the linear operator on $\mathbb{F}^{\infty}(\mathbb{Z},\mathbb{F}^n)$[1] corresponding to the Laurent polynomial matrix $A(\sigma,\sigma^{-1})$. Now, the trajectories satisfying (4) can also be written as:

$$\mathbf{x}_k = L_A^k\,\mathbf{x}_0, \tag{12}$$

where $\mathbf{x}_0 \in \mathbb{R}^{\infty}(\mathbb{Z},\mathbb{R}^n)$ is an initial condition.

Note that, for a given $\mathbf{x} \in \ell^{\infty}(\mathbb{Z},\mathbb{F}^n)$,

$$\begin{aligned}(L_A\,\mathbf{x})\,(r) &= A(\sigma,\sigma^{-1})\,\mathbf{x}(r)\\ &= \sum_{j=-m}^{p} A_j\,\mathbf{x}(r+j)\\ &= \begin{bmatrix} A_{(-m)} & A_{(-m+1)} & \cdots & A_0 & \cdots & A_{p-1} & A_p \end{bmatrix}\begin{bmatrix}\mathbf{x}(r-m)\\ \mathbf{x}(r-m+1)\\ \vdots\\ \mathbf{x}(r)\\ \vdots\\ \mathbf{x}(r+p-1)\\ \mathbf{x}(r+p)\end{bmatrix},\end{aligned} \tag{13}$$

for all $r \in \mathbb{Z}$. Let us define $G \in \mathbb{R}^{n\times(m+p+1)n}$ as,

$$G := [\,A_{(-m)}\; A_{(-m+1)}\;\cdots\; A_0\;\cdots\; A_{p-1}\; A_p\,]. \tag{14}$$

It follows from (13), basic inequality and (6) that; for all $r \in \mathbb{Z}$,

$$\begin{aligned}\|\,(L_A\,\mathbf{x})\,(r)\|_{\infty} &\le \|G\|_{\infty}\,\max\left\{\|\mathbf{x}(r+j)\|_{\infty} \,:\, j \in \{-m,\ldots,p\}\right\}\\ &\le \|G\|_{\infty}\,\|\mathbf{x}\|_{\infty}.\end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned}\|L_A\,\mathbf{x}\|_{\infty} &= \sup\left\{\|\,(L_A\,\mathbf{x})\,(r)\|_{\infty} \,:\, r \in \mathbb{Z}\right\}\\ &\le \|G\|_{\infty}\,\|\mathbf{x}\|_{\infty},\ \forall\,\mathbf{x} \in \ell^{\infty}(\mathbb{Z},\mathbb{F}^n).\end{aligned} \tag{16}$$

As a consequence, $L_A \in BL\left(\ell^{\infty}(\mathbb{Z},\mathbb{F}^n)\right)$ and $\|L_A\|_{\infty} \le \|G\|_{\infty}$.

Remark 2.1. *Note that:*

1. $\|L_A\|_{\infty}$ *is the induced operator norm, as defined in Section 2.1, of* $L_A \in BL\left(\ell^{\infty}(\mathbb{Z},\mathbb{F}^n)\right)$.
2. $\|G\|_{\infty}$ *is the* ∞*-norm of matrix* $G \in \mathbb{R}^{n\times(m+p+1)n}$*, which is equal to the maximum absolute row sum.*

Remark 2.2. *The algebra* $\mathbb{R}^{n\times n}[\sigma,\sigma^{-1}]$ *is isomorphic to the sub-algebra of* $BL\left(\ell^{\infty}(\mathbb{Z},\mathbb{F}^n)\right)$*, where multiplication operation is given by composition of maps.*

[1]Though $A(\sigma,\sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma,\sigma^{-1}]$, later for ℓ^{∞}-stability analysis of the system given by (4), we consider $A(\sigma,\sigma^{-1})$ as an operator over $\mathbb{C}^{\infty}(\mathbb{Z},\mathbb{C}^n)$ also.

For $G \in \mathbb{R}^{n\times(m+p+1)n}$, there exists $\mathbf{y} \in \mathbb{R}^{(m+p+1)n}$ with $\|\mathbf{y}\|_\infty = 1$ such that:

$$\|G\mathbf{y}\|_\infty = \|G\|_\infty. \tag{17}$$

Using this $\mathbf{y} \in \mathbb{R}^{(m+p+1)n}$, one can easily construct[2] $\mathbf{x}^* \in \ell^\infty(\mathbb{Z},\mathbb{F}^n)$ with $\|\mathbf{x}^*\|_\infty = 1$ such that:

$$\|L_A\mathbf{x}^*\|_\infty = \|G\|_\infty. \tag{21}$$

As a consequence, $\|L_A\|_\infty = \|G\|_\infty$.

Remark 2.3. *One can view L_A as a doubly infinite banded block matrix given by,*

$$L_A(j,k) := \begin{cases} A_{j-k}, & \text{if } (j-k) \in \{-m,..,0,..,p\} \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

For example, when $A(\sigma,\sigma^{-1}) = A_{-1}\sigma^{-1} + A_0 + A_1\sigma$, L_A would be as follows:

$$\begin{array}{c} \\ \\ L_A = \end{array} \begin{array}{c} \qquad\qquad\qquad (k=0) \\ \qquad\qquad\qquad \downarrow \\ \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_{-1} & A_0 & A_1 & 0 & 0 & \cdots \\ \cdots & 0 & A_{-1} & A_0 & A_1 & 0 & \cdots \\ \cdots & 0 & 0 & A_{-1} & A_0 & A_1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \leftarrow (j=0) \end{array}$$

3. ℓ^∞-stability

Definition 3.1. *The system given by* (4) *is said to be ℓ^∞-stable, if*

$$\lim_{k\to\infty} \|\mathbf{x}_k\|_\infty = 0, \ \forall \mathbf{x}_0 \in \ell^\infty(\mathbb{Z},\mathbb{R}^n). \tag{23}$$

[2] For $k \in \{0,\ldots,m+p\}$, define $\mathbf{v}_{(-m+k)} \in \mathbb{R}^n$ as follows:

$$\mathbf{v}_{(-m+k)} := \begin{bmatrix} \mathbf{y}(kn+1) \\ \mathbf{y}(kn+2) \\ \vdots \\ \mathbf{y}(kn+n) \end{bmatrix}. \tag{18}$$

Define $\mathbf{x}^* \in \ell^\infty(\mathbb{Z},\mathbb{F}^n)$ as follows:

$$\mathbf{x}^*(j) := \begin{cases} \mathbf{v}_j, & \text{if } j \in \{-m,..,0,..,p\} \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{19}$$

Note that, $\|\mathbf{x}^*\|_\infty = 1$ and $(L_A\mathbf{x}^*)(0) = G\mathbf{y}$. Therefore,

$$\|G\|_\infty = \|(L_A\mathbf{x}^*)(0)\|_\infty \le \|L_A\mathbf{x}^*\|_\infty \le \|L_A\|_\infty \le \|G\|_\infty. \tag{20}$$

3.1. Spectrum of L_A as an element of $BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n))$

In order to find necessary and sufficient conditions for ℓ^∞-stability of the system given by (4), we first prove one result related to $\Lambda(L_A)_{\ell^\infty}$ in Theorem 3.5. This result will be used later to prove our main result of this section, Theorem 3.12, which gives necessary and sufficient conditions for ℓ^∞-stability of the system given by (4).

Consider the left shift operator $\sigma : \mathbb{F}^\infty(\mathbb{Z},\mathbb{F}^n) \to \mathbb{F}^\infty(\mathbb{Z},\mathbb{F}^n)$. When $n=1$, i.e. for scalar valued sequences, the spectrum of σ as an element of $BL(\ell^\infty(\mathbb{Z},\mathbb{C}))$ has been shown to be equal to $S_{\mathbb{C}}(0,1)$ in [2]. As stated in the Lemma 3.2 below, same result holds when $n>1$, and its proof follows on the similar lines. This result is required for proving Theorem 3.5.

Lemma 3.2. *Consider σ as an operator on $\ell^\infty(\mathbb{Z},\mathbb{C}^n)$. Then,*

$$\Lambda_e(\sigma)_{\ell^\infty} = \Lambda(\sigma)_{\ell^\infty} = S_{\mathbb{C}}(0,1)\,. \tag{24}$$

Corollary 3.3. *Consider σ^{-1} as an operator on $\ell^\infty(\mathbb{Z},\mathbb{C}^n)$. Then,*

$$\Lambda_e(\sigma^{-1})_{\ell^\infty} = \Lambda(\sigma^{-1})_{\ell^\infty} = S_{\mathbb{C}}(0,1)\,. \tag{25}$$

Following Lemma is also used in the proof of Theorem 3.5; this Lemma can be easily proved using the uniqueness of inverse.

Lemma 3.4. *Let $(X,\|\cdot\|)$ be a normed space. Suppose $T_1,T_2 \in BL(X)$ satisfy the following conditions:*

1. *T_1 and T_2 are invertible in $BL(X)$.*
2. *T_1 and T_2 commute with each other.*

Then, T_1^{-1} and T_2^{-1} also commute with each other.

Let us define a two variable Laurent polynomial $p(\cdot,\cdot)$ and a set Ω as follows:

$$p(\xi,\eta) := \det\left(\xi I - A(\eta,\eta^{-1})\right) \tag{26}$$

$$\begin{aligned}\Omega &:= \{\lambda \in \mathbb{C} \mid \exists\, \omega \in [0,2\pi) \text{ such that } p(\lambda,e^{i\omega}) = 0\} \\ &= \bigcup_{\omega\in[0,2\pi)} \Lambda\left(A(e^{i\omega},e^{-i\omega})\right)_{\mathbb{C}^n}\end{aligned} \tag{27}$$

Theorem 3.5. *Let L_A be the operator corresponding to the Laurent polynomial matrix $A(\sigma,\sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma,\sigma^{-1}]$. Consider L_A as an operator on $\ell^\infty(\mathbb{Z},\mathbb{C}^n)$. Then,*

$$\Lambda(L_A)_{\ell^\infty} = \Omega = \bigcup_{\omega\in[0,2\pi)} \Lambda\left(A(e^{i\omega},e^{-i\omega})\right)_{\mathbb{C}^n}. \tag{28}$$

Proof. <u>Claim-1</u>: $\Omega \subseteq \Lambda(L_A)_{\ell^\infty}$.

Take an arbitrary $\lambda \in \Omega$. For this λ, there exists $\omega_0 \in [0,2\pi)$ such that, $p(\lambda,e^{i\omega_0}) = 0$. Let $\mathbf{v} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an eigenvector of $A(e^{i\omega_0},e^{-i\omega_0})$ corresponding to eigenvalue λ. Define $\mathbf{x} \in \ell^\infty(\mathbb{Z},\mathbb{C}^n)$ as,

$$\mathbf{x}(j) := (e^{i\omega_0})^j\,\mathbf{v},\ \forall j \in \mathbb{Z} \tag{29}$$

From equation (13) and the fact that $\mathbf{v}$ is an eigenvector of $A(e^{i\omega_0},e^{-i\omega_0})$ corresponding to eigenvalue λ, it follows that:

$$L_A\,\mathbf{x} = \lambda\,\mathbf{x}\,. \tag{30}$$

Therefore, $\lambda \in \Lambda_e(L_A)_{\ell^\infty} \subseteq \Lambda(L_A)_{\ell^\infty}$. This proves Claim-1.
Claim-2: $\Omega^c \subseteq \Lambda^c(L_A)_{\ell^\infty}$.

For every $z \in \mathbb{C}$, $(zI - L_A)$ is a well defined operator in $BL(\ell^\infty(\mathbb{Z}, \mathbb{C}^n))$ corresponding to the Laurent polynomial matrix $\left(zI - A(\sigma, \sigma^{-1})\right) \in \mathbb{C}^{n \times n}[\sigma, \sigma^{-1}]$. We define Adj $\left(zI - A(\sigma, \sigma^{-1})\right)$ to be the transpose of the cofactors' matrix of $\left(zI - A(\sigma, \sigma^{-1})\right)$, and $p(z, \sigma) := \det\left(zI - A(\sigma, \sigma^{-1})\right)$. Let $L_{\text{adj}\,(zI-A)}$ be the operator corresponding to the Laurent polynomial matrix Adj $\left(zI - A(\sigma, \sigma^{-1})\right) \in \mathbb{C}^{n \times n}[\sigma, \sigma^{-1}]$. Define L_z as, $L_z := (zI - L_A)\, L_{\text{adj}\,(zI-A)}$. By Remark 2.2 it follows that, L_z is the operator corresponding to the Laurent polynomial matrix

$$(zI - A(\sigma, \sigma^{-1}))\ \text{Adj}\left(zI - A(\sigma, \sigma^{-1})\right) = p(z, \sigma) I. \tag{31}$$

Now, take an arbitrary $z \in \Omega^c$. For this z, we can factorize the Laurent polynomial $p(z, \sigma)$ as follows:

$$p(z, \sigma) = \alpha \prod_{j=1}^{p} (\sigma - a_j) \prod_{k=1}^{m} (\sigma^{-1} - b_k), \tag{32}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $a_j \in \mathbb{C}$ for $j = 1, \ldots, p$ and $b_k \in \mathbb{C}$ for $k = 1, \ldots, m$. Note that, $a_1, \ldots, a_p$ and $b_1, \ldots, b_m$ are the roots of the Laurent polynomial $p(z, \sigma)$, where $z \in \Omega^c$. Therefore,

$$|a_j| \neq 1, \quad \text{for } j = 1, \ldots, p \tag{33a}$$

$$|b_k| \neq 1, \quad \text{for } k = 1, \ldots, m. \tag{33b}$$

This can be proved by contradiction. Suppose either condition in (33a) or (33b) is violated. Then, there exists $\omega_0 \in [0, 2\pi)$ such that $p(z, e^{i\omega_0}) = 0$. This means $z \in \Omega$, which is a contradiction. Following are the consequences of (33a), (33b), Lemma 3.2 and Corollary 3.3:

1. $(\sigma - a_j)$ is invertible in $BL(\ell^\infty(\mathbb{Z}, \mathbb{C}^n))$ for $j = 1, \ldots, p$.
2. $(\sigma^{-1} - b_k)$ is invertible in $BL(\ell^\infty(\mathbb{Z}, \mathbb{C}^n))$ for $k = 1, \ldots, m$.

It follows from Lemma 3.4 that: $(\sigma - a_1)^{-1}, \ldots, (\sigma - a_p)^{-1}, (\sigma^{-1} - b_1)^{-1}, \ldots, (\sigma^{-1} - b_k)^{-1}$ commute as bounded linear operators on $\ell^\infty(\mathbb{Z}, \mathbb{C}^n)$. Therefore,

$$\alpha^{-1} \left(\prod_{j=1}^{p} (\sigma - a_j)^{-1} \prod_{k=1}^{m} (\sigma^{-1} - b_k)^{-1} \right) = L_z^{-1}. \tag{34}$$

It then follows that:

$$\left((zI - L_A)\, L_{\text{adj}\,(zI-A)}\right) \left(\alpha^{-1} \left(\prod_{j=1}^{p} (\sigma - a_j)^{-1} \prod_{k=1}^{m} (\sigma^{-1} - b_k)^{-1} \right) \right) = L_z\, L_z^{-1} = I. \tag{35}$$

In other words, for an arbitrary $z \in \Omega^c$, the bounded linear operator $(zI - L_A)$ is invertible in $BL(\ell^\infty(\mathbb{Z}, \mathbb{C}^n))$ with its inverse being

$$(zI - L_A)^{-1} = L_{\text{adj}\,(zI-A)}\, \alpha^{-1} \left(\prod_{j=1}^{p} (\sigma - a_j)^{-1} \prod_{k=1}^{m} (\sigma^{-1} - b_k)^{-1} \right). \tag{36}$$

Therefore, $z \in \Lambda^c(L_A)_{\ell^\infty}$. This proves Claim-2. $\square$

Remark 3.6. *1. Let L_Φ be* the block Laurent operator[3] *on $\ell^2(\mathbb{Z},\mathbb{C}^n)$ obtained from the Fourier coefficient matrices (see [11, 12]) of a continuous function $\Phi : S_{\mathbb{C}}(0,1) \to \mathbb{C}^{n\times n}$. The set*

$$\{\Phi : S_{\mathbb{C}}(0,1) \to \mathbb{C}^{n\times n} \mid \Phi \text{ is continuous}\}, \tag{37}$$

forms a Banach algebra[4]. Using Fourier expansion and Banach algebra techniques, it has been shown in [11, Theorem 3.2] that,

$$\Lambda(L_\Phi)_{\ell^2} = \bigcup_{\omega\in[0,2\pi)} \Lambda\left(\Phi(e^{i\omega})\right)_{\mathbb{C}^n}. \tag{38}$$

2. *Recall from Remark 2.3 that, the operator L_A corresponding to the Laurent polynomial matrix $A(\sigma,\sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma,\sigma^{-1}]$ is in fact a* banded block Laurent operator. *In Theorem 3.5, we have extended the above result ([11, Theorem 3.2]) for banded block Laurent operators on $\ell^\infty(\mathbb{Z},\mathbb{C}^n)$. Note that, Banach algebra techniques used in [11] are not applicable in this case.*

3.2. Necessary and sufficient conditions for ℓ^∞-stability

We give below some known results from functional analysis ([9, Theorem 7.3-4] and [8, Theorems 9.3, 12.5 and 12.6], respectively) for easy reference later in the proof of Lemma 3.11. This lemma is used in the proof of Theorem 3.12, which gives necessary and sufficient conditions for ℓ^∞-stability of the system given by (4).

Proposition 3.7. *Let $(X, \|\cdot\|_x)$ be a Banach space over $\mathbb{C}$. Then, for every $T \in BL(X)$; $\Lambda(T)_X$ is a closed and bounded subset of $\mathbb{C}$. Moreover, the spectral radius of T satisfies the inequality: $\rho(T)_X \leq \|T\|_x$.*

Proposition 3.8 (Resonance Theorem)**.** *Let $(X, \|\cdot\|_x)$ be a normed space over $\mathbb{C}$, and E be a subset of X. Let X' denote the space of bounded linear functionals on X. Then, the set E is bounded in X if and only if $f(E)$ is bounded in $\mathbb{C}$, for all $f \in X'$.*

Proposition 3.9. *Let $(X, \|\cdot\|_x)$ be a Banach space. The set of all invertible operators is open in $BL(X)$; and the map $T \mapsto T^{-1}$ is continuous on this set with respect to the topology induced by operator norm on $BL(X)$.*

Proposition 3.10 (Neumann Expansion)**.** *Let $(X, \|\cdot\|_x)$ be a Banach space over $\mathbb{C}$, and $T \in BL(X)$. Let $z \in \mathbb{C}$ be such that, $|z|^n > \|T^n\|_x$ for some $n \in \mathbb{N}$. Then, $z \in \Lambda^c(T)_X$ and:*

$$(zI - T)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{z^{k+1}}. \tag{39}$$

Lemma 3.11. *Let L_A be the operator corresponding to the Laurent polynomial matrix $A(\sigma,\sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma,\sigma^{-1}]$. Consider L_A as an operator on $\ell^\infty(\mathbb{Z},\mathbb{C}^n)$. Define,*

$$E := \{z \in \mathbb{C} \,:\, |z| > \rho(L_A)_{\ell^\infty}\} \subseteq \Lambda^c(L_A)_{\ell^\infty}. \tag{40}$$

Then, for every $z \in E$, there exists $\alpha > 0$ such that:

$$\|L_A^k\|_\infty \,\leq\, \alpha\,|z|^{k+1},\ \forall k \in \mathbb{N}. \tag{41}$$

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[3]A bounded linear operator L on a separable Hilbert space $\left(\ell^2(\mathbb{Z},\mathbb{F}^n), \|\cdot\|_2\right)$ can be represented by a doubly infinite block matrix $L = [L(j,k)]_{j,k=-\infty}^{\infty}$, where $L(j,k) \in \mathbb{F}^{n\times n}$, $\forall (j,k) \in \mathbb{Z}^2$. A bounded linear operator L is said to be a *block Laurent operator*, if its matrix elements $L(j,k)$ depend only on the difference $(j-k)$.

[4]Banach algebra is a Banach space which is also a ring.

Proof. Let $(BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n)))'$ denote the space of bounded linear functionals on the normed linear space $BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n))$. Further, let $f \in (BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n)))'$. We define $\beta_f : \Lambda^c(L_A)_{\ell^\infty} \to \mathbb{C}$ as follows,

$$\beta_f(z) := f\left((zI - L_A)^{-1}\right), \ \forall z \in \Lambda^c(L_A)_{\ell^\infty}. \tag{42}$$

As a consequence of the fact that f is a continuous linear functional and Proposition 3.9, we have β_f as an analytic (holomorphic) function[5] on $\Lambda^c(L_A)_{\ell^\infty} \subset \mathbb{C}$.

We define set D as follows,

$$D := \{z \in \mathbb{C} \ : \ |z| > \|L_A\|_\infty\}. \tag{43}$$

As $\ell^\infty(\mathbb{Z},\mathbb{C}^n)$ is a Banach space, by Proposition 3.7, $\rho(L_A)_{\ell^\infty} \leq \|L_A\|_\infty$. Therefore, $D \subseteq E$. For every $f \in (BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n)))'$, the corresponding β_f is analytic on E. However, if $z \in D$, then by Neumann expansion (see Proposition 3.10):

$$(zI - L_A)^{-1} = \sum_{k=0}^{\infty} \frac{L_A^k}{z^{k+1}}. \tag{44}$$

Therefore, by continuity and linearity of f, we obtain the following Laurent expansion of β_f over D:

$$\beta_f(z) = \sum_{k=0}^{\infty} \frac{f(L_A^k)}{z^{k+1}}, \ \forall z \in D. \tag{45}$$

By uniqueness of the Laurent expansion[6] and the fact that β_f is analytic on E, it follows that: the expansion of β_f given in (45) is valid over E.

Now fix an arbitrary $z \in E$. For this z, the series $\sum_{k=0}^{\infty} \frac{f(L_A^k)}{z^{k+1}}$ is summable in $\mathbb{C}$, $\forall f \in (BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n)))'$. As a consequence, for this arbitrarily fixed $z \in E$, the sequence $\left(f(L_A^k/z^{k+1})\right)$ is bounded in $\mathbb{C}$, $\forall f \in (BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n)))'$. Therefore, by Resonance Theorem (see Proposition 3.8), the set $\{\frac{L_A^k}{z^{k+1}} \mid k \in \mathbb{N}\}$ is bounded in $BL(\ell^\infty(\mathbb{Z},\mathbb{C}^n))$; and hence there exists $\alpha > 0$ such that:

$$\|L_A^k\|_\infty \ \leq \ \alpha\, |z|^{k+1}, \ \forall k \in \mathbb{N}. \tag{46}$$

□

Theorem 3.12. *Following are equivalent:*

1. *The system given by* (4) *is ℓ^∞-stable.*
2. $\rho\left(A(e^{i\omega}, e^{-i\omega})\right)_{\mathbb{C}^n} < 1, \ \forall \omega \in [0, 2\pi)$.
3. $\lim_{k\to\infty} \|L_A^k\|_\infty = 0$.

Proof. (1) ⇒ (2): Suppose not, i.e. there exists $\psi \in [0, 2\pi)$ for which $\rho\left(A(e^{i\psi}, e^{-i\psi})\right)_{\mathbb{C}} \geq 1$.

Let $(\lambda_1, \mathbf{v}_1)$ be an eigenpair of $A(e^{i\psi}, e^{-i\psi})$ such that, $\rho\left(A(e^{i\psi}, e^{-i\psi})\right)_{\mathbb{C}^n} = |\lambda_1|$; therefore, $|\lambda_1| \geq 1$. Take $\mathbf{y}_0 \in \ell^\infty(\mathbb{Z},\mathbb{C}^n)$, which is defined as,

$$\mathbf{y}_0(j) := (e^{i\psi})^j\, \mathbf{v}_1, \ \forall j \in \mathbb{Z}. \tag{47}$$

[5] This can be proved on the similar lines of Theorem 7.5-2 in [9] and Theorem 5.1-C in [10].

[6] See [13, 14] for the result about uniqueness of Laurent expansion.

From equation (13) and the fact that $\mathbf{v}_1 \in \mathbb{C}^n$ is an eigenvector of $A(e^{i\psi}, e^{-i\psi})$ corresponding to eigenvalue λ_1, it follows that:

$$L_A\,\mathbf{y}_0 = \lambda_1\,\mathbf{y}_0\,. \tag{48}$$

As $|\lambda_1| \geq 1$,

$$\lim_{k\to\infty} \|L_A^k\,\mathbf{y}_0\|_\infty \;=\; \lim_{k\to\infty} |\lambda_1|^k\,\|\mathbf{v}_1\|_\infty \;\neq\; 0\,. \tag{49}$$

Using the real or the imaginary part of $\mathbf{y}_0 \in \ell^\infty(\mathbb{Z}, \mathbb{C}^n)$ one can construct $\mathbf{x}_0 \in \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ such that:

$$\lim_{k\to\infty} \|L_A^k\,\mathbf{x}_0\|_\infty \;\neq\; 0\,. \tag{50}$$

Hence a contradiction to the statement (1).
(2) $\Rightarrow$ (3): As a consequence of Theorem 3.5,

$$\rho\left(A(e^{i\omega}, e^{-i\omega})\right)_{\mathbb{C}^n} < 1,\ \forall\,\omega \in [0, 2\pi) \tag{51}$$

$$\Updownarrow$$

$$|\lambda| < 1,\ \forall\,\lambda \in \Lambda(L_A)_{\ell^\infty} \tag{52}$$

It follows from Proposition 3.7 and Weierstrass extreme value theorem that, there exists $\lambda_1 \in \Lambda(L_A)_{\ell^\infty}$ such that $\rho(L_A)_{\ell^\infty} = |\lambda_1|$. Therefore,

$$|\lambda| < 1,\ \forall\,\lambda \in \Lambda(L_A)_{\ell^\infty} \quad\Longrightarrow\quad \rho(L_A)_{\ell^\infty} < 1\,. \tag{53}$$

If $\rho(L_A)_{\ell^\infty} < 1$, then $\exists\, z \in E$ for which $|z| < 1$, where E is defined as in (40). It follows from Lemma 3.11 that, for such $z \in E$ with $|z| < 1$, there exists $\alpha > 0$ such that:

$$\|L_A^k\|_\infty \;\leq\; \alpha\,|z|^{k+1}. \tag{54}$$

As $|z| < 1$, taking limit as $k \to \infty$ we get:

$$\lim_{k\to\infty}\ \|L_A^k\|_\infty = 0\,. \tag{55}$$

(3) $\Rightarrow$ (1): The trajectories satisfying (4) can be written as:

$$\mathbf{x}_k = L_A^k\,\mathbf{x}_0, \tag{56}$$

where $\mathbf{x}_0 \in \mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$ is an initial condition. For ℓ^∞-stability analysis, we restrict initial condition $\mathbf{x}_0$ to the subspace $\ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ of $\mathbb{R}^\infty(\mathbb{Z}, \mathbb{R}^n)$. Now, using basic inequality we get:

$$\|\mathbf{x}_k\|_\infty \;\leq\; \|L_A^k\|_\infty\,\|\mathbf{x}_0\|_\infty,\ \forall\,k \in \mathbb{N}\,. \tag{57}$$

Therefore, taking limit as k tends to infinity we get:

$$\lim_{k\to\infty} \|\mathbf{x}_k\|_\infty \;\leq\; \lim_{k\to\infty} \|L_A^k\|_\infty\,\|\mathbf{x}_0\|_\infty\,. \tag{58}$$

Therefore, if $\lim_{k\to\infty}\ \|L_A^k\|_\infty = 0$, then the system given in (4) is ℓ^∞-stable. $\square$

Remark 3.13. *1. Condition-2 in Theorem 3.12 can be checked using LMI approach given in [7].*

2. Condition similar to condition-2 in Theorem 3.12 is a sufficient condition for the ℓ^2-stability of time relevant 2-D systems (see [7, 15]).

4. Stability Theorems

In this section we give some tests for checking ℓ^∞-stability of the system given by (4). We first discuss block circulant matrices which are to be used later in this section. Consider a block circulant matrix $C \in \mathbb{R}^{nk\times nk}$ given by,

$$C := \begin{bmatrix} B_0 & B_1 & \cdots & \cdots & B_{k-2} & B_{k-1} \\ B_{k-1} & B_0 & \cdots & \cdots & B_{k-3} & B_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_2 & B_3 & \cdots & \cdots & B_0 & B_1 \\ B_1 & B_2 & \cdots & \cdots & B_{k-1} & B_0 \end{bmatrix}, \tag{59}$$

where $B_j \in \mathbb{R}^{n\times n}$ for $j = 0,1,\ldots,(k-1)$. We state below a result from Section 2.1 in [16] for easy reference later in this section.

Proposition 4.1. *Let $\{\mu_j : j \in \{0,1,\ldots,k-1\}\}$ denote the set of complex k^{th} roots of unity. For $j \in \{0,1,\ldots,k-1\}$, let $H_j \in \mathbb{R}^{n\times n}$ be defined as, $H_j := \sum_{m=0}^{k-1} \mu_j^m B_m$. Then, $\Lambda(C)_{\mathbb{C}^{nk}} = \bigcup_{j=0}^{k-1} \Lambda(H_j)_{\mathbb{C}^n}$.*

Consider a Laurent polynomial matrix $A(\sigma,\sigma^{-1}) = \left(\sum_{j=-m}^{p} A_j \sigma^j\right)$, where $A_j \in \mathbb{R}^{n\times n}$ for $j \in \{-m, \ldots, p\}$. Corresponding to each such Laurent polynomial matrix, one can associate a block circulant matrix $C_A \in \mathbb{R}^{(m+p+1)n\times(m+p+1)n}$. For example, when $A(\sigma,\sigma^{-1}) = A_{-1}\sigma^{-1} + A_0 + A_1\sigma$, the block circulant matrix C_A would be:

$$C_A = \begin{bmatrix} A_{-1} & A_0 & A_1 \\ A_1 & A_{-1} & A_0 \\ A_0 & A_1 & A_{-1} \end{bmatrix}. \tag{60}$$

4.1. Necessary conditions

We give necessary conditions for ℓ^∞-stability of the system given by (4), which are simple to check and can be used to rule out the ℓ^∞-stability.

Theorem 4.2. *Suppose the system given by (4) is ℓ^∞-stable, where $A(\sigma,\sigma^{-1}) = \sum_{j=-m}^{p} A_j \sigma^j$. Then, $\rho(A_{-m})_{\mathbb{C}^n} < 1$, $\rho(A_p)_{\mathbb{C}^n} < 1$ and $\rho(C_A)_{\mathbb{C}^{(m+p+1)n}} < 1$.*

Proof. We have $A(\sigma,\sigma^{-1}) = \sum_{j=-m}^{p} A_j \sigma^j$. Define $\tilde{A}(\sigma) \in \mathbb{R}^{n\times n}[\sigma]$ and $\hat{A}(\sigma^{-1}) \in \mathbb{R}^{n\times n}[\sigma^{-1}]$ as,

$$\begin{aligned} \tilde{A}(\sigma) &:= \sigma^m A(\sigma,\sigma^{-1}) \\ &= \sum_{j=0}^{p+m} \tilde{A}_j \sigma^j, \end{aligned} \tag{61}$$

$$\begin{aligned} \hat{A}(\sigma^{-1}) &:= \sigma^{-p} A(\sigma,\sigma^{-1}) \\ &= \sum_{j=-(p+m)}^{0} \hat{A}_j \sigma^j. \end{aligned} \tag{62}$$

Now, consider discrete autonomous systems defined as follows:

$$\mathbf{x}_{k+1}(\cdot) := \tilde{A}(\sigma)\,\mathbf{x}_k(\cdot) \tag{63}$$

$$\mathbf{x}_{k+1}(\cdot) := \hat{A}(\sigma^{-1})\,\mathbf{x}_k(\cdot). \tag{64}$$

Let $L_{\tilde{A}}$ and $L_{\hat{A}}$ be the operators corresponding to the polynomial matrices $\tilde{A}(\sigma)$ and $\hat{A}(\sigma^{-1})$ respectively.

It follows from (61) and (62) that, $\tilde{A}_0 = A_{-m}$ and $\hat{A}_0 = A_p$. Also for all $\omega \in [0,\, 2\pi)$, we have:

$$\rho\left(A(e^{i\omega},\, e^{-i\omega})\right)_{\mathbb{C}^n} = \rho\left(\tilde{A}(e^{i\omega})\right)_{\mathbb{C}^n} = \rho\left(\hat{A}(e^{-i\omega})\right)_{\mathbb{C}^n}.$$

Therefore from Theorem 3.12, ℓ^∞-stability of the systems given by (4), (63) and (64) are equivalent.

Claim-1: If $\rho(A_{-m})_{\mathbb{C}^n} \geq 1$, then the system given by (4) is ℓ^∞-unstable.

As $\tilde{A}_0 = A_{-m}$, we have $\rho(\tilde{A}_0)_{\mathbb{C}^n} \geq 1$. Let $(\tilde{\lambda}_1, \mathbf{v}_1)$ be an eigenpair of $\tilde{A}_0$ (as an operator over $\mathbb{C}^n$) such that, $\rho(\tilde{A}_0) = |\tilde{\lambda}_1|$. Take $\mathbf{y}_0 \in \ell^\infty(\mathbb{Z}, \mathbb{C}^n)$, which is defined as,

$$\mathbf{y}_0(j) := \begin{cases} \mathbf{v}_1, & \text{if } j = 0 \\ \mathbf{0}, & \text{if } j \neq 0 \end{cases} \tag{65}$$

$\tilde{A}(\sigma)$ contains only non-negative powers of σ; therefore, it follows that:

$$\left(L_{\tilde{A}}^k\,\mathbf{y}_0\right)(j) = \begin{cases} *, & \text{if } j = -1, -2, \ldots.., -k(p+m) \\ \tilde{A}_0^k\,\mathbf{v}_1, & \text{if } j = 0 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Observe that for all $k \in \mathbb{N}$,

$$\left(L_{\tilde{A}}^k\,\mathbf{y}_0\right)(0) = \tilde{A}_0^k\,\mathbf{v}_1 = \tilde{\lambda}_1^k\,\mathbf{v}_1. \tag{66}$$

As $|\tilde{\lambda}_1| \geq 1$, we have $\lim\limits_{k\to\infty} \left(L_{\tilde{A}}^k\,\mathbf{y}_0\right)(0) \neq \mathbf{0}$. Therefore,

$$\lim_{k\to\infty} \|L_{\tilde{A}}^k\,\mathbf{y}_0\|_\infty \neq 0. \tag{67}$$

Using the real or the imaginary part of $\mathbf{y}_0 \in \ell^\infty(\mathbb{Z}, \mathbb{C}^n)$ one can construct $\mathbf{x}_0 \in \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ such that:

$$\lim_{k\to\infty} \|L_{\tilde{A}}^k\,\mathbf{x}_0\|_\infty \neq 0. \tag{68}$$

This shows that, if $\rho(\tilde{A}_0)_{\mathbb{C}^n} \geq 1$, then the system given by (63) is ℓ^∞-unstable. This proves Claim-1, as ℓ^∞-stability of the systems given by (4) and (63) are equivalent, and $\tilde{A}_0 = A_{-m}$.

Claim-2: If $\rho(A_p)_{\mathbb{C}^n} \geq 1$, then the system given by (4) is ℓ^∞-unstable.

The proof of Claim-2 follows on the similar lines of the proof of Claim-1.

Claim-3: If the system given by (4) is ℓ^∞-stable, then $\rho(C_A)_{\mathbb{C}^{(m+p+1)n}} < 1$.

The ℓ^∞-stability of the systems given by (4) and (63) are equivalent. From Theorem 3.12, the system given by (63) is ℓ^∞-stable if and only if $\rho\left(\tilde{A}(e^{i\omega})\right)_{\mathbb{C}^n} < 1,\ \forall\, \omega \in [0,\, 2\pi)$.

It follows from Proposition 4.1 that,

$$\Lambda(C_{\tilde{A}})_{\mathbb{C}^{(m+p+1)n}} = \bigcup_{j=0}^{p+m} \Lambda\left(\tilde{A}(e^{i2\pi j/(p+m+1)})\right)_{\mathbb{C}^n} \subseteq \bigcup_{\omega\in[0,2\pi)} \Lambda\left(\tilde{A}(e^{i\omega})\right)_{\mathbb{C}^n}.$$

Therefore we can conclude that,

$$\rho\left(\tilde{A}(e^{i\omega})\right)_{\mathbb{C}^n} < 1,\ \forall\, \omega \in [0,\, 2\pi) \implies \rho(C_{\tilde{A}})_{\mathbb{C}^{(m+p+1)n}} < 1,$$

where $C_{\tilde{A}}$ is the block circulant matrix corresponding to the polynomial matrix $\tilde{A}(\sigma)$. This proves the Claim-3, as $C_{\tilde{A}} = C_A$. □

Note that, if j is neither equal to $(-m)$ nor equal to p, then $\rho(A_j)_{\mathbb{C}^n}$ is not required to be strictly less than 1 for the ℓ^∞-stability; below is an example to illustrate this.

Example 4.3. *Consider a* 2×2 *Laurent polynomial matrix,*

$$A(\sigma,\sigma^{-1}) = \begin{bmatrix} \sigma & 0.5\sigma^2 \\ (0.08\sigma - 0.2) & (-0.1\sigma + 0.4\sigma^2) \end{bmatrix}.$$

In this case, $\det(sI - A(e^{i\omega}, e^{-i\omega})) = (s-0.9e^{i\omega})(s-0.4e^{2i\omega})$*. Therefore* $\rho(A(e^{i\omega},e^{-i\omega}))_{\mathbb{C}^n} < 1$, $\forall\omega\in[0,2\pi)$*. If we write* $A(\sigma,\sigma^{-1}) = A_0 + A_1\sigma + A_2\sigma^2$*, then we get:*

$$A_0 = \begin{bmatrix} 0 & 0 \\ -0.2 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0.08 & -0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.4 \end{bmatrix}. \tag{69}$$

Note that: $\rho(A_1)_{\mathbb{C}^n} = 1$*, though* $\rho(A(e^{i\omega},e^{-i\omega}))_{\mathbb{C}^n} < 1$, $\forall\omega\in[0,2\pi)$.

4.2. Sufficient conditions

In Theorem 4.5 we give sufficient conditions for ℓ^∞-stability of the system given by (4). These conditions, in terms of coefficient matrices of Laurent polynomial matrix $A(\sigma,\sigma^{-1})$, are simple to check and can be used to conclude the ℓ^∞-stability.

We give below some definitions which will be used in the statement of Theorem 4.5. For $A(\sigma,\sigma^{-1}) = \sum_{j=-m}^{p} A_j\sigma^j$, let $\tilde{A}(\sigma)$ be defined as in (61). The block circulant matrix $C_{\tilde{A}} \in \mathbb{R}^{(m+p+1)n\times(m+p+1)n}$ corresponding to $\tilde{A}(\sigma)$ turns out to be,

$$C_{\tilde{A}} = \begin{bmatrix} \tilde{A}_0 & \tilde{A}_1 & \cdots & \cdots & \tilde{A}_{(m+p-1)} & \tilde{A}_{(m+p)} \\ \tilde{A}_{(m+p)} & \tilde{A}_0 & \cdots & \cdots & \tilde{A}_{(m+p-2)} & \tilde{A}_{(m+p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{A}_2 & \tilde{A}_3 & \cdots & \cdots & \tilde{A}_0 & \tilde{A}_1 \\ \tilde{A}_1 & \tilde{A}_2 & \cdots & \cdots & \tilde{A}_{(m+p)} & \tilde{A}_0 \end{bmatrix}.$$

We define $F_0, F_1 \in \mathbb{R}^{(m+p+1)n\times(m+p+1)n}$ as follows:

$$F_0 := \begin{bmatrix} \tilde{A}_0 & \tilde{A}_1 & \cdots & \cdots & \tilde{A}_{(m+p-1)} & \tilde{A}_{(m+p)} \\ 0 & \tilde{A}_0 & \cdots & \cdots & \tilde{A}_{(m+p-2)} & \tilde{A}_{(m+p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \tilde{A}_0 & \tilde{A}_1 \\ 0 & 0 & \cdots & \cdots & 0 & \tilde{A}_0 \end{bmatrix},\ F_1 := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 \\ \tilde{A}_{(m+p)} & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{A}_2 & \tilde{A}_3 & \cdots & \cdots & 0 & 0 \\ \tilde{A}_1 & \tilde{A}_2 & \cdots & \cdots & \tilde{A}_{(m+p)} & 0 \end{bmatrix}.$$

It follows from definitions of F_0 and F_1 that, $C_{\tilde{A}} = F_0 + F_1$. We give below Corollary 1 from [17] for easy reference later in the proof of Theorem 4.5.

Proposition 4.4. *Let* $A, B \in \mathbb{C}^{n\times n}$ *and let,*

$$\gamma = \left(\|A\|_2 + \|B\|_2 + \sqrt{(\|A\|_2 - \|B\|_2)^2 + 4\min(\|AB\|_2, \|BA\|_2)}\right).$$

Then, $\rho(A+B)_{\mathbb{C}^n} \leq \frac{\gamma}{2}$.

Let us define γ_1 and γ_2 as follows:

$$\gamma_1 := \|F_0\|_2 + \|F_1\|_2, \tag{70a}$$

$$\gamma_2 := \sqrt{(\|F_0\|_2 - \|F_1\|_2)^2 + 4\min(\|F_0F_1\|_2, \|F_1F_0\|_2)}. \tag{70b}$$

Theorem 4.5. *Each of the following is a sufficient condition for ℓ^∞-stability of the system given by* (4)*:*

1. $\|G\|_\infty < 1$*, where* $G \in \mathbb{R}^{n\times(m+p+1)n}$ *is defined as in* (14).
2. $(\|F_0+F_1\|_p^2 + \|F_0-F_1\|_p^2) < 1$*, for some* $p \in [1,\infty]$.
3. $(\gamma_1+\gamma_2) < 2$.

Proof. 1. Recall from section-2.2 that, $\|L_A\|_\infty = \|G\|_\infty$. Now, if $\|L_A\|_\infty < 1$, then

$$0 \;\le\; \lim_{k\to\infty} \|L_A^k\|_\infty \;\le\; \lim_{k\to\infty} (\|L_A\|_\infty)^k \;=\; 0. \tag{71}$$

Therefore, by Theorem 3.12: if $\|G\|_\infty < 1$, then the system given by (4) is ℓ^∞-stable.
2. It is enough to show that, the given condition implies ℓ^∞-stability of the system given by (63).

Let L_F be the operator corresponding to the Laurent polynomial matrix $F(\sigma,\sigma^{-1}) = F_0 + \sigma F_1$. Consider a discrete autonomous system defines as,

$$\mathbf{z}_{k+1}(\cdot) := F(\sigma,\sigma^{-1})\,\mathbf{z}_k(\cdot), \tag{72}$$

where $F(\sigma,\sigma^{-1}) \in \mathbb{R}^{(m+p+1)n\times(m+p+1)n}[\sigma,\sigma^{-1}]$ and $\mathbf{z}_k \in \mathbb{R}^\infty(\mathbb{Z},\mathbb{R}^{(m+p+1)n})$, $\forall\, k \in \mathbb{N}_0$.

If one views operators L_F and $L_{\tilde{A}}$ as doubly infinite banded block matrices (as explained in Remark 2.3); then it follows that, the banded block Laurent operator L_F is obtained by grouping finite number of blocks of $n\times n$ matrices in the banded block Laurent operator $L_{\tilde{A}}$. Therefore, the trajectories satisfying (72) and (63) can be obtained from each other as follows:

$$\mathbf{z}_k(j) = \begin{bmatrix} \mathbf{x}_k(r_j) \\ \mathbf{x}_k(r_j+1) \\ \vdots \\ \vdots \\ \mathbf{x}_k(r_j+p+m) \end{bmatrix}, \quad \forall\, j \in \mathbb{Z}, \tag{73}$$

and for all $k \in \mathbb{N}_0$; where $r_j := j(p+m+1)$, $\forall\, j \in \mathbb{Z}$. Also,

$$\lim_{k\to\infty} \|L_F^k\|_\infty = 0 \quad\Longleftrightarrow\quad \lim_{k\to\infty} \|L_{\tilde{A}}^k\|_\infty = 0. \tag{74}$$

Therefore by Theorem 3.12, ℓ^∞-stability of the systems given by (72) and by (63) are equivalent.

Using interpolation formula to express DTFT in terms of DFT ([18, section-7.1]), we have:

$$F_0 + e^{i\omega}F_1 = (F_0+F_1)\,q(\omega) + (F_0-F_1)\,q(\omega-\pi), \tag{75}$$

for all $\omega \in [0,2\pi)$, where interpolation function q (in this case of two samples) is defined as,

$$q(\omega) := \frac{\sin(\omega)}{2\sin(\omega/2)}\, e^{-i\omega/2}, \quad \forall\, \omega \in [0,2\pi). \tag{76}$$

Applying triangle inequality to (75), we get:

$$\|F_0 + e^{i\omega}F_1\|_p \;\le\; \|F_0+F_1\|_p\, |q(\omega)| + \|F_0-F_1\|_p\, |q(\omega-\pi)|,$$

for all $\omega \in [0,2\pi)$ and for all $p \in [1,\infty]$. Note that, $|q(\omega)|^2 + |q(\omega-\pi)|^2 = 1$, $\forall\, \omega \in [0,2\pi)$. Therefore, using Cauchy-Schwarz inequality we get the following implication:

$$\begin{gathered} \left(\|F_0+F_1\|_p^2 + \|F_0-F_1\|_p^2\right) < 1 \\ \Downarrow \\ \|F_0 + e^{i\omega}F_1\|_p < 1,\ \forall\, \omega \in [0,2\pi). \end{gathered} \tag{77}$$

Therefore if $\left(\|F_0+F_1\|_p^2+\|F_0-F_1\|_p^2\right)<1$, for some $p\in[1,\infty]$; then $\rho(F_0+e^{i\omega}F_1)_{\mathbb{C}^{(m+p+1)n}}<1, \forall\,\omega\in[0,2\pi)$. Now it follows from Theorem 3.12 that, $\left(\|F_0+F_1\|_p^2+\|F_0-F_1\|_p^2\right)<1$, for some $p\in[1,\infty]$ is a sufficient condition for ℓ^∞-stability of the system given by (72); and hence it is a sufficient condition for ℓ^∞-stability of the system given by (63).

3. It is enough to show that, given condition implies ℓ^∞-stability of the system by (63).

As $\|e^{i\omega}F_1\|_2=\|F_1\|_2, \forall\,\omega\in[0,2\pi)$; we have the following implication as a consequence of Proposition 4.4.

$$\begin{gathered}(\gamma_1+\gamma_2)<2\\ \Downarrow\\ \rho(F_0+e^{i\omega}F_1)_{\mathbb{C}^n}<1,\ \forall\,\omega\in[0,2\pi)\end{gathered}\tag{78}$$

Now it follows from Theorem 3.12 that, $(\gamma_1+\gamma_2)<2$ is a sufficient condition for ℓ^∞-stability of the system given by (72); and hence it is a sufficient condition for ℓ^∞-stability of the system given by (63). ☐

5. Conclusion

We have given necessary and sufficient conditions for the ℓ^∞-stability of discrete autonomous systems described by Laurent polynomial matrix operators. In the process, we have partially extended the spectrum result about block Laurent operators in [11]. We have also given easy to check necessary conditions and sufficient conditions which can be used to rule out the ℓ^∞-stability and to conclude the ℓ^∞-stability, respectively, of such systems.

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