On restrictions of n-D systems to 1-D subspaces

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***Abstract*— In this paper, we look into restrictions of the solution set of a system of PDEs to 1-d subspaces. We bring out its relation with certain intersection modules. We show that the restriction, which may not always be a solution set of differential equations, is always contained in a solution set of ODEs coming from the intersection module. Next, we focus our attention to restrictions of strongly autonomous systems. We first show that such a system always admits an equivalent first order representation given by an n-tuple of real square matrices called companion matrices. We then exploit this first order representation to show that the system corresponding to the intersection module has a state representation given by the restriction of a linear combination of the companion matrices to a certain invariant subspace. Using this result we bring out that the restriction of a strongly autonomous system is equal to the system corresponding to the intersection module.**

I. Introduction and preliminaries

In n-d systems, restriction of trajectories to smaller subsets of the domain $\mathbb{R}^n$ is of fundamental importance in various issues. For example, the well-known method of characteristic subsets ([11]), dissipativity/path-independence of quadratic functionals ([6]), stability theory ([11], [3]) – all of these issues are inextricably connected with the idea of restriction of n-d systems to certain smaller subsets of $\mathbb{R}^n$. In this paper, we look into restriction of n-d systems to 1-d subspaces. One of the most important results of this paper is that such restrictions can be analyzed by looking into an algebraic entity called intersection module. The remaining part of this section is devoted to some preliminary definitions, notations and results which are essential for the rest of the paper.

The kind of systems we are concerned with in this paper are the ones described by linear partial differential equations (PDEs) with constant real coefficients. Following Willems ([7]), we call such systems *behaviors* and denote them by $\mathfrak{B}$, which are described as

$$\mathfrak{B} := \{w \in \mathcal{W}^{\mathtt{w}} \mid R(\partial_1, \partial_2, ..., \partial_n)\, w = 0\}, \qquad (1)$$

where $R(\partial_1, \partial_2, ..., \partial_n)$ is a matrix with $\mathtt{w}$ number of columns with entries in the n-variable polynomial ring $\mathbb{R}[\partial_1, \partial_2, ..., \partial_n]$. $\mathcal{W}$ (the solution space) is an $\mathbb{R}$-vector space of trajectories which contains the solutions of the differential equations. In this paper, we denote by $\mathfrak{L}^{\mathtt{w}}$ the set of all behaviors as described above with $\mathtt{w}$ number of dependent variables. We also use ∂ to denote the n-tuple $\{\partial_1, \partial_2, ..., \partial_n\}$.

A deceptively simple, but crucial observation is that there is an alternative description of $\mathfrak{B}$: if we denote by $\mathcal{R}$ the row-span of the matrix R over $\mathbb{R}[\partial]$, then $\mathfrak{B}$ can also be written as

$$\mathfrak{B}(\mathcal{R}) := \{w \in \mathcal{W}^{\mathtt{w}} \mid r(\partial)w = 0, \text{ for all } r \in \mathcal{R}\}, \qquad (2)$$

where $r(\partial)$ is a row vector of length $\mathtt{w}$ with entries from $\mathbb{R}[\partial]$. Thus, given a submodule $\mathcal{R}$ of the free module $\mathbb{R}[\partial]^{\mathtt{w}}$, we can associate with $\mathcal{R}$ the behavior $\mathfrak{B}(\mathcal{R})$ given in equation (2). Similarly, given a set of trajectories in $\mathcal{W}^{\mathtt{w}}$, one can define all $r(\partial) \in \mathbb{R}[\partial]^{\mathtt{w}}$, such that the action of $r(\partial)$ on the set of trajectories is zero. In particular, given a behavior $\mathfrak{B}$, we define

$$\mathcal{R}(\mathfrak{B}) := \{r \in \mathbb{R}[\partial]^{\mathtt{w}} \mid r(\partial)w = 0 \text{ for all } w \in \mathfrak{B}\}. \qquad (3)$$

In [4], Oberst shows that $\mathfrak{B}(\bullet)$ and $\mathcal{R}(\bullet)$ are inverses of each other whenever the signal space is a *large injective cogenerator*. This shows that the correspondence between submodules of $\mathbb{R}[\partial]^{\mathtt{w}}$ and behaviors $\mathfrak{B}$ is one-to-one. By this one-to-one correspondence, we call the submodule $\mathcal{R}$ *the equation module* of $\mathfrak{B}$. In this paper, we shall restrict ourselves to the space of infinitely differentiable functions, denoted by $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$, which has been shown in [4] to be a large injective cogenerator.

There is another aspect of the behaviors-submodules correspondence, brought forth in Malgrange's Theorem, involving the quotient module $\mathcal{M} := \mathbb{R}[\partial]^{\mathtt{w}}/\mathcal{R}$. Note that $\mathfrak{B}$ actually has an $\mathbb{R}[\partial]$-module structure, where multiplication by elements from $\mathbb{R}[\partial]$ is identified with differentiation of the elements in $\mathfrak{B}$.

Proposition 1.1: (Malgrange) A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is isomorphic as an $\mathbb{R}[\partial]$-module with

$$\mathrm{Hom}_{\mathbb{R}[\partial]}(\mathbb{R}[\partial]^{\mathtt{w}}/\mathcal{R}(\mathfrak{B}), \mathcal{W}).$$

In this paper we take the categorial view of behaviors; we define morphisms between behaviors to be the ones given by linear PDEs with real coefficients. Malgrange's Theorem (Proposition 1.1) is crucial for us; the $\mathrm{Hom}_{\mathbb{R}[\partial]}(\bullet, \mathcal{W})$ acts as a functor between the categories of finitely generated $\mathbb{R}[\partial]$-modules and behaviors. In [4] it was shown that when $\mathcal{W}$ is a large injective cogenerator, then these two categories are dual to each other.

We shall often talk about elements in the quotient module $\mathcal{M}$ 'acting' on the trajectories in the behavior $\mathfrak{B}$. By this we mean the action of a lift of that element in $\mathbb{R}[\partial]^{\mathtt{w}}$ on the trajectories in $\mathfrak{B}$. Although these lifts are not unique, their actions on $\mathfrak{B}$ are: two distinct lifts always differ by an element in $\mathcal{R}$ and the action of $\mathcal{R}$ on $\mathfrak{B}$ produces the zero trajectory.

It was proved in [5], [9], [8] that a behavior is controllable

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if and only if the quotient module $\mathcal{M}$ is *torsion-free*[1]. With this idea of controllability, one defines an *autonomous* system to be the one which does not contain any non-trivial controllable system within itself. It then follows, as was shown in [9], [5], that an autonomous system is characterized by a quotient module that is a *torsion* module. This algebraic property of the quotient module gives rise to two fundamental invariants of the autonomous system, namely the annihilator ideal of $\mathcal{M}$, which we denote by $\mathrm{ann}(\mathcal{M})$, and the characteristic ideal of the system, which we denote by $\mathcal{I}(\mathfrak{B})$. The characteristic ideal is defined as follows: given a behavior $\mathfrak{B}$ and its corresponding equation module $\mathcal{R}$, let $R \in \mathbb{R}[\partial]^{\mathtt{g}\times\mathtt{w}}$ be a matrix whose rows generate $\mathcal{R}$. Then define the ideal generated by the $(\mathtt{w}\times\mathtt{w})$ minors of R to be the *characteristic ideal* of R. [2]

For an autonomous behavior, there is another invariant, a geometric one, called the characteristic variety and denoted by $\mathbb{V}(\mathfrak{B})$. By the characteristic variety of an autonomous behavior we mean the following set of complex n-tuples.

$$\begin{aligned}\mathbb{V}(\mathfrak{B}) &:= \{\xi \in \mathbb{C}^n \mid f(\xi)=0 \text{ for all } f \in \mathcal{I}(\mathfrak{B})\}\\ &= \{\xi \in \mathbb{C}^n \mid f(\xi)=0 \text{ for all } f \in \mathrm{ann}(\mathcal{M})\}.\end{aligned}$$

The second equality follows by applying Hilbert's Nullstellensatz to the fact that the radicals of $\mathcal{I}$ and $\mathrm{ann}(\mathcal{M})$ are the same. We sum up all these important results in the form of a proposition below.

Proposition 1.2: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then

1) $\mathfrak{B}$ is autonomous if and only if $\mathcal{M}(\mathfrak{B})$ is a torsion module.
2) If $\mathfrak{B}$ is autonomous then $\sqrt{\mathcal{I}(\mathfrak{B})} = \sqrt{\mathrm{ann}(\mathcal{M}(\mathfrak{B}))}$.
3) If $\mathfrak{B}$ is autonomous then $\mathbb{V}(\mathfrak{B})$ is a proper subset of $\mathbb{C}^n$.

Remark 1.3: For the special case when $n=1$, $\mathbb{R}[\partial]$ turns out to be a PID. So both the characteristic ideal and the annihilator ideal are principal, and hence each is generated by a polynomial. The unique monic generators are, in fact, the characteristic and minimal polynomials of the system, respectively. The above result reasserts the well-known fact for a system of ODEs that the characteristic and minimal polynomials have the same roots with possibly different multiplicities.

II. Restriction of a behavior to a 1-dimensional subspace

Our prime concern in this paper is to analyze an autonomous behavior when restricted to a given direction in its domain space. In this section we make this idea of restriction precise. Then we show how restriction is related to the algebraic idea of intersection submodules.

Definition 2.1: Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, by restriction of $\mathfrak{B}$ to a line $L_v = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = vt, t \in \mathbb{R}\}$ given by a nonzero real vector $v \in \mathbb{R}^n$, we mean

$$\mathfrak{B}|_v := \{w(vt) \mid w \in \mathfrak{B}\} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}).$$

When a trajectory w is restricted to a line L_v, its derivative with respect to the parameter t, appearing in the above definition of L_v, follows the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} w(vt) = ((v_1\partial_1 + v_2\partial_2 + \ldots + v_n\partial_n)w)(vt), \qquad (4)$$

where v_i is the ith entry in the vector v defining the line L_v. We shall write $\langle v, \partial\rangle$ for the linear polynomial $\sum_{i=1}^n v_i\partial_i$. A straightforward extension of equation (4) shows that for $f(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$

$$f(\frac{\mathrm{d}}{\mathrm{d}t}) w(vt) = (f(\langle v, \partial\rangle)w)(vt). \qquad (5)$$

This observation brings out the fact that the action of the $\mathbb{R}$-algebra $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]$ on $w(vt)$ is same as that of the sub-algebra $\mathbb{R}[\langle v, \partial\rangle]$ of $\mathbb{R}[\partial]$ on w followed by restriction to L_v. Our main result Theorem 2.3 is a consequence of this observation. Like the sub-algebra $\mathbb{R}[\langle v, \partial\rangle]$ we consider the free module $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ over $\mathbb{R}[\langle v, \partial\rangle]$ to be sitting inside $\mathbb{R}[\partial]^{\mathtt{w}}$ as a subset. Note that equation (5) can be easily extended to cater for the action of $r(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}$ on $w(vt) \in \mathfrak{B}|_v$:

$$r(\frac{\mathrm{d}}{\mathrm{d}t}) w(vt) = (r(\langle v, \partial\rangle)w)(vt). \qquad (6)$$

Given a behavior $\mathfrak{B}$ and its corresponding equation module $\mathcal{R}$, we look into the following $\mathbb{R}[\langle v, \partial\rangle]$-submodule of $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ obtained by intersecting $\mathcal{R}$ with $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$, we call it the *$v$-intersection submodule of* $\mathcal{R}$ and denote it by $\mathcal{R}_v$:

$$\mathcal{R}_v := \mathcal{R} \cap \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}. \qquad (7)$$

Remark 2.2: Note that the polynomial $\langle v, \partial\rangle$ is transcendental over $\mathbb{R}$. Therefore, the $\mathbb{R}$-algebra $\mathbb{R}[\langle v, \partial\rangle]$ is in fact isomorphic to the polynomial ring in one variable. Hence $\mathbb{R}[\langle v, \partial\rangle]$ is a PID, and therefore, every ideal in $\mathbb{R}[\langle v, \partial\rangle]$ is generated by a single polynomial in $\mathbb{R}[\langle v, \partial\rangle]$.

We define the following quotient module obtained by quotienting $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ by its submodule $\mathcal{R}_v$:

$$\mathcal{M}_v := \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}/\mathcal{R}_v.$$

This is naturally a finitely generated module over the ring $\mathbb{R}[\langle v, \partial\rangle]$. Thus it makes sense to define the annihilator ideal of $\mathcal{M}_v$ as

$$\mathrm{ann}(\mathcal{M}_v) := \{f \in \mathbb{R}[\langle v, \partial\rangle] \mid fm = 0 \text{ for all } m \in \mathcal{M}_v\}.$$

There is another ideal of $\mathbb{R}[\langle v, \partial\rangle]$ related with $\mathcal{R}$, namely $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$, the v-intersection ideal of $\mathrm{ann}(\mathcal{M})$. Theorem 2.3 shows, among other things, that these two ideals are the same. We now state and prove this theorem, which is our main result of this section.

Theorem 2.3: Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and its corresponding equation submodule is $\mathcal{R}$. Let $v \in \mathbb{R}^n$ be a nonzero vector defining the line $L_v \subseteq \mathbb{R}^n$ as in Definition 2.1. Define the 1-d behavior

$$\mathfrak{B}_v := \{\widetilde{w} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})) \mid r(\frac{\mathrm{d}}{\mathrm{d}t})\widetilde{w} = 0 \ \forall r(\langle v, \partial\rangle) \in \mathcal{R}_v\},$$

[1] Controllability roughly means the ability to 'patch-up' two trajectories; see [7], [5] for details.

[2] Note that although the definition relies on a matrix representation of $\mathcal{R}$, the ideal $\mathcal{I}(\mathfrak{B})$ is independent of this representation and depends only on the submodule $\mathcal{R}$, and hence, because of the duality between $\mathfrak{B}$ and $\mathcal{R}$, on the behavior $\mathfrak{B}$.

where $\mathcal{R}_v$ is the v-intersection submodule defined by equation (7). Then

$$\mathfrak{B}|_v \subseteq \mathfrak{B}_v.$$

Further, if $\mathfrak{B}$ is an autonomous behavior, and $\mathcal{M}_v := \mathbb{R}[\langle v, \partial \rangle]^{\mathtt{w}}/\mathcal{R}_v$, then the annihilator ideals of $\mathcal{M}_v$ and $\mathcal{M}$ satisfy the following equation:

$$\operatorname{ann}(\mathcal{M}_v) = \operatorname{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial \rangle].$$

Proof : Suppose $r(\langle v, \partial \rangle) \in \mathcal{R}_v$ and $w(vt) \in \mathfrak{B}|_v$ for some $w \in \mathfrak{B}$. By equation (6), we have

$$r(\frac{\mathrm{d}}{\mathrm{d}t})w(vt) \quad = \quad (r(\langle v, \partial \rangle)w)(vt).$$

But, since $r(\langle v, \partial \rangle) \in \mathcal{R}_v$, $r(\langle v, \partial \rangle)$ is in $\mathcal{R}$ too. Therefore $r(\langle v, \partial \rangle)w$ is the zero trajectory. In particular, $(r(\langle v, \partial \rangle)w)(vt) = 0$ for all t. This means that $r(\frac{\mathrm{d}}{\mathrm{d}t})w(vt) = 0$ for all $r(\langle v, \partial \rangle) \in \mathcal{R}_v$, that is $w(vt) \in \mathfrak{B}_v$.

For the second part, we have to show that the v-intersection of the annihilator ideal, that is, $\operatorname{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial \rangle]$ is equal to the annihilator ideal of the quotient module $\mathcal{M}_v$. We first show that $\operatorname{ann}(\mathcal{M}_v) \supseteq \operatorname{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial \rangle]$. Let $f \in \operatorname{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial \rangle]$. This means that for any $r \in \mathbb{R}[\partial]^{\mathtt{w}}$, $fr \in \mathcal{R}$. In other words, the row span over $\mathbb{R}[\partial]$ of the $(\mathtt{w} \times \mathtt{w})$ matrix $fI_{\mathtt{w}}$ is contained in $\mathcal{R}$. But, since f also belongs to $\mathbb{R}[\langle v, \partial \rangle]$, which is a subalgebra of $\mathbb{R}[\partial]$, the row span of $fI_{\mathtt{w}}$ over $\mathbb{R}[\langle v, \partial \rangle]$ is contained in $\mathcal{R} \cap \mathbb{R}[\langle v, \partial \rangle]^{\mathtt{w}} = \mathcal{R}_v$, which means $f \in \operatorname{ann}(\mathcal{M}_v)$.

Conversely, suppose $f \in \operatorname{ann}(\mathcal{M}_v)$. Then, once again following the same logic, the row span of $fI_{\mathtt{w}}$ over $\mathbb{R}[\langle v, \partial \rangle]$ is contained in $\mathcal{R}_v$. We want to show that the row span of this matrix $fI_{\mathtt{w}}$ over $\mathbb{R}[\partial]$ is contained in $\mathcal{R}$. Since the row span over $\mathbb{R}[\langle v, \partial \rangle]$ of $fI_{\mathtt{w}}$ is contained in $\mathcal{R}_v$, it follows that each of the rows of $fI_{\mathtt{w}}$ is in $\mathcal{R}_v$, and hence, is also in $\mathcal{R}$ because $\mathcal{R}_v \subseteq \mathcal{R}$. Therefore, if we let $R \in \mathbb{R}[\partial]^{\mathtt{g} \times \mathtt{w}}$ be a matrix whose rows span $\mathcal{R}$, then there exists another matrix $E \in \mathbb{R}[\partial]^{\mathtt{w} \times \mathtt{g}}$ such that

$$fI_{\mathtt{w}} = ER.$$

Then it easily follows that the row span of $fI_{\mathtt{w}}$ over $\mathbb{R}[\partial]$ is contained in $\mathcal{R}$. In other words, $f \in \operatorname{ann}(\mathcal{M})$. Also, by assumption, $f \in \mathbb{R}[\langle v, \partial \rangle]$. Thus $f \in \operatorname{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial \rangle]$. $\square$

Remark 2.4: It is not *a priori* clear whether $\mathfrak{B}_v$, defined in the last theorem, is the smallest 1-d behavior containing the restriction $\mathfrak{B}|_v$. $\mathfrak{B}_v$ would indeed be the smallest behavior containing $\mathfrak{B}|_v$ if for all $r(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathcal{R}(\mathfrak{B}|_v) \subseteq \mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}$, $r(\langle v, \partial \rangle) \in \mathcal{R}_v$. However, this may not always be the case because of the following subtlety. Suppose that $r(\frac{\mathrm{d}}{\mathrm{d}t}) \in \mathcal{R}(\mathfrak{B}|_v)$, then $(r(\langle v, \partial \rangle)w)(vt) = 0$. This means that the trajectory $r(\langle v, \partial \rangle)w$ is zero when restricted to L_v. But, this does not imply that $r(\langle v, \partial \rangle)w$ is the zero trajectory, and thus we cannot infer that $r(\langle v, \partial \rangle) \in \mathcal{R}_v$. The above mentioned difficulty does not arise if the line L_v is a so called 'characteristic subspace' (see [11]). However, we shall see in the next section (Section III) that for a certain special class of autonomous systems, $\mathfrak{B}_v$ turns out to be not only the smallest behavior containing $\mathfrak{B}|_v$, but in fact is equal to it.

III. RESTRICTIONS OF STRONGLY AUTONOMOUS SYSTEMS

One of the major distinctions between 1-d and n-d systems comes from the geometry of the characteristic variety; for 1-d autonomous systems the characteristic variety is always a discrete set of finitely many complex numbers, but this is not always true for an n-d autonomous system. In fact, it is the nonzero dimension of the variety that is responsible for making the solution set infinite dimensional. However, there is one special case when the affine variety $\mathbb{V}(\mathfrak{B})$ is a finite set of discrete points in the affine space $\mathbb{C}^n$; in this case $\mathfrak{B}$ is said to be *strongly autonomous*. This is drastically different from the other possible cases. Here, like in 1-d, the solution set turns out to be a finite dimensional vector space over $\mathbb{R}$. Mimicking the 1-d situation, in this case, one can obtain a state representation, although there is one inevitable distinction: here there would be n state matrices accounting for the n first order partial derivatives. This observation is not new, for the case when $n = 2$, this has been shown in [2], while in [10], it has been shown for general n. Here, we are going to provide an alternative proof for the general n case. Our alternative approach will prove to be crucial in bringing out the relation between a state representation of the 1-d behavior $\mathfrak{B}_v$ and the first order representation of the original n-d behavior $\mathfrak{B}$. We are going to make use of the following result from commutative algebra (see [1]) about finite dimensionality of the quotient module as a vector space over $\mathbb{R}$.

Proposition 3.1: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with the corresponding quotient module $\mathcal{M}$. Then the following are equivalent:

1) $\mathfrak{B}$ is strongly autonomous.
2) $\mathbb{V}(\mathfrak{B})$ is a finite set.
3) $\mathcal{I}(\mathfrak{B})$ and $\operatorname{ann}(\mathcal{M})$ are zero dimensional ideals.
4) $\mathcal{M}$ can be viewed as a finite dimensional vector space over $\mathbb{R}$.

Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is a strongly autonomous behavior. Now, for each of the partial derivatives ∂_j, the following map, multiplication by ∂_j in $\mathcal{M}$,

$$\overline{m(\partial)} \mapsto \overline{\partial_j m(\partial)},$$

where $\overline{m(\partial)}$ is the image of an element $m(\partial) \in \mathbb{R}[\partial]^{\mathtt{w}}$ onto the quotient module $\mathcal{M}$, is an $\mathbb{R}[\partial]$-module morphism of $\mathcal{M}$ onto itself. In particular, this map is $\mathbb{R}$-linear. Moreover, by Proposition 3.1, $\mathcal{M}$ is a finite dimensional $\mathbb{R}$-vector space, therefore, the above mentioned map is in fact a linear map of the finite dimensional $\mathbb{R}$-vector space $\mathcal{M}$ onto itself. So, by fixing a basis of $\mathcal{M}$, this linear map can be written as a real square matrix. These matrices, say $\{A_1, A_2, ..., A_n\}$, which are representations of multiplications by $\{\partial_1, \partial_2, ..., \partial_n\}$, respectively, are called *companion matrices* (see [1]). The next result proves that for a strongly autonomous system, one can define a first order system using the companion matrices so that this new system is isomorphic to the original system in the category of behaviors.

Theorem 3.2: Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ a strongly autonomous behavior with the corresponding quotient module $\mathcal{M}$, let $\{A_1, A_2, ..., A_n\}$ be the companion matrices as defined above. Define the following behavior,

$$\mathfrak{B}_z := \ker \begin{bmatrix} \partial_1 I - A_1^{\mathrm{T}} \\ \partial_2 I - A_2^{\mathrm{T}} \\ \vdots \\ \partial_n I - A_n^{\mathrm{T}} \end{bmatrix}.$$

Then $\mathfrak{B}_z$ and $\mathfrak{B}$ are isomorphic as behaviors.

Proof : Let $\mathcal{W}$ be any signal space which is a large injective cogenerator. We first set up a map from $\mathfrak{B}$ to $\mathcal{W}^\gamma$, where $\gamma := \dim_{\mathbb{R}}\mathcal{M} < \infty$. Let $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, ..., \overline{e_\gamma(\partial)}\} \subseteq \mathcal{M}$ be the basis of $\mathcal{M}$ as a vector-space over $\mathbb{R}$, in which the companion matrices are obtained. We lift these $\overline{e_i(\partial)}$'s to $\mathbb{R}[\partial]^{\mathtt{w}}$. Supposing the lifts are $\{e_1(\partial), e_2(\partial), ..., e_\gamma(\partial)\} \subseteq \mathbb{R}[\partial]^{\mathtt{w}}$, we define $z_i := e_i(\partial)w \in \mathcal{W}$. Note that these lifts are not unique, however, their effects on $w \in \mathfrak{B}$ are. This is because two distinct lifts differ by an element in the equation module $\mathcal{R}$, and the action of anything in $\mathcal{R}$ on $\mathfrak{B}$ produces the zero trajectory. Therefore, the above assignment is well-defined. Moreover, this defines a map of behaviors, say φ, from $\mathfrak{B}$ to $\mathcal{W}^\gamma$. We will show that the image of φ is contained in $\mathfrak{B}_z$. First note that the action of ∂_j for any $j \in \{1, 2, ..., n\}$ on z_i for any $i \in \{1, 2, ..., \gamma\}$ translates to w as follows

$$\partial_j z_i = \partial_j(e_i(\partial)w) = (\overline{\partial_j e_i(\partial)})w. \tag{8}$$

Now recall that the $\mathbb{R}$-linear map from $\mathcal{M}$ to $\mathcal{M}$ defined by $\overline{m(\partial)} \mapsto \overline{\partial_j m(\partial)}$ is given by the companion matrix A_j in the chosen basis. Therefore we can write

$$\overline{\partial_j e_i(\partial)} = \sum_{k=1}^{\gamma} a_j^{k,i} \overline{e_k(\partial)},$$

where $a_j^{k,i} \in \mathbb{R}$ is the (k, i)th entry in A_j. Putting this in equation (8) we get

$$\begin{aligned} \partial_j z_i = (\overline{\partial_j e_i(\partial)})w &= \sum_{k=1}^{\gamma} a_j^{k,i} \overline{e_k(\partial)} w \\ &= \sum_{k=1}^{\gamma} a_j^{k,i} z_k \\ &= A_j(:, i)^{\mathrm{T}} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\gamma \end{bmatrix}, \end{aligned}$$

with $A_j(:, i) \in \mathbb{R}^\gamma$ denoting the ith column of A_j. In matrix form we have the following:

$$\partial_j \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\gamma \end{bmatrix} = A_j^{\mathrm{T}} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\gamma \end{bmatrix}. \tag{9}$$

The above holds for all $j \in \{1, 2, ..., n\}$. Thus the image of φ, which we define as $z := \mathrm{col}[z_1, z_2, ..., z_\gamma]$, satisfies equation (9) for all $j \in \{1, 2, ..., n\}$. In other words, $\mathrm{im}(\varphi) \subseteq \mathfrak{B}_z$. Therefore, $\varphi : w \mapsto z$ is in fact a morphism of behaviors from $\mathfrak{B}$ to $\mathfrak{B}_z$.

It remains to show that φ is an isomorphism of behaviors. It is enough to show that φ is a bijection (because we have already shown that it is a morphism of behaviors).

We first show injectivity. Suppose $z = \varphi(w) = 0$, we want to show that $w = 0$. Since $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, ..., \overline{e_\gamma(\partial)}\}$ is a basis for $\mathcal{M}$ considered as an $\mathbb{R}$-vector space, any $\overline{m(\partial)} \in \mathcal{M}$ can be written as an $\mathbb{R}$-linear combination of $\overline{e_i(\partial)}$s, that is, $\overline{m(\partial)} = \sum_{i=1}^{\gamma} \alpha_i \overline{e_i(\partial)}$, $\alpha_i \in \mathbb{R}$. It then follows that $\overline{m(\partial)}w = \sum_{i=1}^{\gamma} \alpha_i e_i(\partial)w = \sum_{i=1}^{\gamma} \alpha_i z_i = 0$, because $z_i = 0$ for all $i \in \{1, 2, ..., \gamma\}$. This further implies that $m(\partial)w = 0$ for all $m \in \mathbb{R}[\partial]^{\mathtt{w}}$, which is true if and only if w is the zero trajectory.

To show surjectivity we invoke Malgrange's theorem (Proposition 1.1). Recall that the theorem says $\mathfrak{B}$ and $\mathrm{Hom}_{\mathbb{R}[\partial]}(\mathcal{M}, \mathcal{W})$ are isomorphic as $\mathbb{R}[\partial]$-modules. We now give a map from $\mathfrak{B}_z$ to $\mathfrak{B}$ via $\mathrm{Hom}_{\mathbb{R}[\partial]}(\mathcal{M}, \mathcal{W})$. For $z = \mathrm{col}[z_1, z_2, ..., z_\gamma] \in \mathfrak{B}_z$, let us define the following $\mathbb{R}$-linear map $\psi_z : \mathcal{M} \to \mathcal{W}$ which acts on the basis vectors as $\psi_z(\overline{e_i(\partial)}) := z_i$. We claim that this is also an $\mathbb{R}[\partial]$-module morphism. It is enough to check this for multiplication by ∂_j. First note that we get the following by using the companion matrix A_j:

$$\psi_z(\partial_j \overline{e_i(\partial)}) = \psi_z(\sum_{k=1}^{\gamma} a_j^{k,i} \overline{e_k(\partial)}).$$

Because ψ_z is $\mathbb{R}$-linear, we get

$$\begin{aligned} \psi_z(\partial_j \overline{e_i(\partial)}) &= \psi_z(\sum_{k=1}^{\gamma} a_j^{k,i} \overline{e_k(\partial)}) \\ &= \sum_{k=1}^{\gamma} a_j^{k,i} \psi_z(\overline{e_k(\partial)}) = \sum_{k=1}^{\gamma} a_j^{k,i} z_k. \end{aligned}$$

But z satisfies the PDE $\partial_j z = A_j^{\mathrm{T}} z$, that means $\sum_{k=1}^{\gamma} a_j^{k,i} z_k = \partial_j z_i$. Therefore,

$$\begin{aligned} \psi_z(\partial_j \overline{e_i(\partial)}) &= \sum_{k=1}^{\gamma} a_j^{k,i} z_k \\ &= \partial_j z_i = \partial_j \psi_z(\overline{e_i(\partial)}). \end{aligned}$$

Thus ψ_z is indeed an element in $\mathrm{Hom}_{\mathbb{R}[\partial]}(\mathcal{M}, \mathcal{W})$. Define now $\psi : \mathfrak{B}_z \to \mathrm{Hom}_{\mathbb{R}[\partial]}(\mathcal{M}, \mathcal{W})$ as $\psi(z) := \psi_z$. Following Malgrange's theorem, we obtain $w \in \mathfrak{B}$ from ψ_z by making the following assignment: for $w = \mathrm{col}[w_1, w_2, ..., w_{\mathtt{w}}]$ define $w_i := \psi_z(\overline{s_i})$ for $i \in \{1, 2, ..., \mathtt{w}\}$, where $\overline{s_i}$ is the image of the standard ith basis vector of $\mathbb{R}[\partial]^{\mathtt{w}}$, that is, $(0, 0, ..., 1, ..., 0)$ with 1 at the ith position, onto $\mathcal{M}$. In order to show surjectivity of the map $\varphi : w \mapsto z = \mathrm{col}[e_1(\partial)w, e_2(\partial)w, ..., e_\gamma(\partial)w]$, it is enough to show that for each $i \in \{1, 2, ..., \gamma\}$, the action of $e_i(\partial)$ on the w defined above is z_i, that is, $\overline{e_i}(\partial)\mathrm{col}[\psi_z(\overline{s_1}), \psi_z(\overline{s_2}), ..., \psi_z(\overline{s_{\mathtt{w}}})] = z_i$. Owing to Malgrange's theorem we can lift everything from $\mathcal{M}$ into $\mathbb{R}[\partial]^{\mathtt{w}}$.

So we have to show $e_i(\partial)\mathrm{col}[\psi_z(\overline{s_1}), \psi_z(\overline{s_2}), ..., \psi_z(\overline{s_{\mathtt{w}}})] = z_i$. Suppose $e_i(\partial) = [\alpha_1(\partial), \alpha_2(\partial), ..., \alpha_{\mathtt{w}}(\partial)]$. Then

$$\begin{aligned} e_i(\partial)w &= e_i(\partial)\begin{bmatrix} \psi_z(\overline{s_1}) \\ \psi_z(\overline{s_1}) \\ \vdots \\ \psi_z(\overline{s_{\mathtt{w}}}) \end{bmatrix} \\ &= \sum_{k=1}^{\mathtt{w}} \alpha_k(\partial)\psi_z(\overline{s_k}) \\ &= \psi_z\left(\sum_{k=1}^{\mathtt{w}} \alpha_k(\partial)\overline{s_k}\right) \\ &\quad \text{(since } \psi_z \text{ is a module morphism)} \\ &= \psi_z\left(\sum_{k=1}^{\mathtt{w}} \overline{\alpha_k(\partial)s_k}\right) \\ &= \psi_z\left(\overline{\sum_{k=1}^{\mathtt{w}} \alpha_k(\partial)s_k}\right) \\ &= \psi_z(\overline{e_i(\partial)}) = z_i. \end{aligned}$$

Hence surjectivity follows. □

Theorem 3.2 shows that the action of partial derivatives ∂_i are represented by companion matrices. In Theorems 3.5, 3.6 we show that a state representation of $\mathfrak{B}_v$ can be obtained as a linear combination of these companion matrices. Utilizing this result, we show that for the strongly autonomous case, $\mathfrak{B}_v$ is always equal to the restriction $\mathfrak{B}|_v$.

Recall from Remark 2.2 that $\mathbb{R}[\langle v, \partial\rangle]$ is a PID, and therefore, if the v-intersection ideal $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$ is nonzero, then it has a unique monic generator.

Lemma 3.3: Let $\mathfrak{B}$ be a strongly autonomous behavior. Let $\{A_1, A_2, ..., A_n\} \subseteq \mathbb{R}^{\gamma\times\gamma}$ be as in Theorem 3.2. Further, let $v = \mathrm{col}[v_1, v_2, ..., v_n] \in \mathbb{R}^n$ be nonzero. Then the following hold.

1) The v-intersection ideal $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$ is nonzero, and thus, has a unique monic generator $\mu_v(\langle v, \partial\rangle) \in \mathbb{R}[\langle v, \partial\rangle]$.
2) The eigenvalues (without counting multiplicities) of the matrix $\sum_{i=1}^n v_i A_i$ are given by the roots of $\mu_v(s) \in \mathbb{R}[s]$, where s, a transcendental over $\mathbb{R}$, is a place holder for $\langle v, \partial\rangle$.

Proof : 1) Recall from the discussion preceding Theorem 3.2 that the maps given by multiplication by $\{\partial_1, \partial_2, ..., \partial_n\}$ in $\mathcal{M}$ are represented by companion matrices $\{A_1, A_2, ..., A_n\}$, respectively. A simple extension of this idea shows that multiplications by a polynomial $f(\partial) \in \mathbb{R}[\partial]$ in $\mathcal{M}$ is similarly represented by the matrix polynomial $f(A_1, A_2, ..., A_n)$. (The companion matrices commute with each other, and thus it makes sense to talk about the matrix polynomial $f(A_1, A_2, ..., A_n)$; see [1] for detailed discussions about the companion matrices and polynomials in them.) Now, suppose $f(\partial) \in \mathbb{R}[\partial]$ is a nonzero polynomial such that the corresponding matrix polynomial $f(A_1, A_2, ..., A_n)$ is the zero matrix. It then follows that the map given by multiplication by $f(\partial)$ in $\mathcal{M}$ is the zero map. In other words, for all $\overline{m(\partial)} \in \mathcal{M}$, we have $\overline{f(\partial)m(\partial)} = 0$ meaning $f(\partial) \in \mathrm{ann}(\mathcal{M})$. We now define $A := \sum_{i=1}^n v_i A_i$ and consider the minimal polynomial of A, say $\mu(s) \in \mathbb{R}[s]$, that is the smallest degree monic polynomial for which $\mu(A)$ is the zero matrix. Since $\mu(A)$ is the zero matrix, by putting $\sum_{i=1}^n v_i A_i$ for A we get $\mu(\sum_{i=1}^n v_i A_i)$ to be equal to the zero matrix. It then follows from the above discussion that the polynomial $\mu(\sum_{i=1}^n v_i\partial_i) = \mu(\langle v, \partial\rangle) \in \mathbb{R}[\partial]$ must be in $\mathrm{ann}(\mathcal{M})$. Also $\mu(\langle v, \partial\rangle)$ is a polynomial in $\mathbb{R}[\langle v, \partial\rangle]$. Therefore, $\mu(\langle v, \partial\rangle) \in \mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$. Since every real square matrix has a nonzero minimal polynomial, we must have $\mu(s) \neq 0$, and hence $\mu(\langle v, \partial\rangle)$, too, is nonzero. Thus, the v-intersection of $\mathrm{ann}(\mathcal{M})$, that is $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$, contains a nonzero polynomial $\mu(\langle v, \partial\rangle)$, and therefore, is a nonzero ideal.

2) The last part, that is part 1), of this proof actually shows that the minimal polynomial of the matrix $A := \sum_{i=1} v_i A_i$, which we have called $\mu(s)$, is such that $\mu(\langle v, \partial\rangle) \in \mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$. This means that if $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$ is generated by a monic polynomial $\mu_v(\langle v, \partial\rangle)$, then $\mu_v(s)$ divides $\mu(s)$.

On the other hand, let $p(\langle v, \partial\rangle) \in \mathbb{R}[\langle v, \partial\rangle]$ be a nonzero polynomial in $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$. This means that the map $\mathcal{M} \ni \overline{m(\partial)} \mapsto \overline{p(\langle v, \partial\rangle)m(\partial)}$, is the zero map. Therefore, the matrix polynomial $p(\sum_{i=1}^n v_i A_i) = p(A)$ is the zero matrix. Thus the minimal polynomial $\mu(s)$ divides $p(s)$; in particular, when $p(\langle v, \partial\rangle) = \mu_v(\langle v, \partial\rangle)$, the monic generator of $\mathrm{ann}(\mathcal{M}) \cap \mathbb{R}[\langle v, \partial\rangle]$, then $\mu(s)$ divides $\mu_v(s)$. Therefore, $\mu(s) = \mu_v(s)$. But the eigenvalues of A (without counting multiplicities) are given by the roots of $\mu(s)$. It then follows that the eigenvalues of A are given by the roots of $\mu_v(s)$. □

An immediate corollary to the above lemma follows from looking at the characteristic variety. The variety of the v-intersection ideal of $\mathrm{ann}(\mathcal{M})$ is nothing but the Zariski closure of the projection of the variety $\mathbb{V}(\mathfrak{B})$ on the complex 1-dimensional subspace $L_v^{\mathbb{C}} := \{\xi \in \mathbb{C}^n \mid \xi = vs, s \in \mathbb{C}\}$. Since $\mathbb{V}(\mathcal{I})$ is zero dimensional, the above mentioned projection is already a closed set. Thus the eigenvalues of the matrix $\sum_{i=1}^n v_i A_i$ is given by the projection of $\mathbb{V}(\mathfrak{B})$ on the 1-dimensional complex subspace $L_v^{\mathbb{C}}$. We state this observation as a corollary below.

Corollary 3.4: Let $\mathfrak{B}$ be a strongly autonomous behavior with $\{A_1, A_2, ..., A_n\} \subseteq \mathbb{R}^{\gamma\times\gamma}$ be as in Theorem 3.2. Let $v = \mathrm{col}[v_1, v_2, ..., v_n] \in \mathbb{R}^n$ be nonzero. Define the projection of the complex variety $\mathbb{V}(\mathfrak{B})$ on the complex 1-dimensional subspace $L_v^{\mathbb{C}} := \{\xi \in \mathbb{C}^n \mid \xi = vs, s \in \mathbb{C}\}$ as

$$\Pi_v(\mathbb{V}(\mathfrak{B})) := \{v^{\mathrm{T}}\xi \mid \xi \in \mathbb{V}(\mathfrak{B})\} \subseteq \mathbb{C}.$$

Then the set of eigenvalues (without counting multiplicities) of the matrix $\sum_{i=1}^n v_i A_i$ is equal to $\Pi_v(\mathbb{V}(\mathfrak{B}))$.

One important question raised in Remark 2.4 was: when is the restriction of a behavior equal to the behavior obtained from the intersection submodule? For the case when the behavior is strongly autonomous, we will show now, that this happens for every nonzero v. Our first observation is that the quotient module $\mathcal{M}_v = \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}/\mathcal{R}_v$ can be embedded inside the original quotient module $\mathcal{M}$ as an $\mathbb{R}$-subspace. This follows from the following diagram of set-maps.

$$\begin{array}{ccc} \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}} & \twoheadrightarrow & \mathcal{M}_v \\ \downarrow & & \downarrow \iota \\ \mathbb{R}[\partial]^{\mathtt{w}} & \twoheadrightarrow & \mathcal{M}. \end{array}$$

We define the map ι via the inclusion $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}} \hookrightarrow \mathbb{R}[\partial]^{\mathtt{w}}$: for an element in $\mathcal{M}_v$ we take a lift in $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$, consider it inside $\mathbb{R}[\partial]^{\mathtt{w}}$ by the inclusion map, and then project it onto $\mathcal{M}$. From the definitions of $\mathcal{R}_v$ and $\mathcal{M}_v$ it easily follows that ι is well-defined, and injective. Crucially, when $\mathcal{M}$ and $\mathcal{M}_v$ are considered as $\mathbb{R}$-vector spaces then ι becomes an $\mathbb{R}$-linear map of finite dimensional $\mathbb{R}$-vector spaces, and therefore, gives an embedding of $\mathcal{M}_v$ into $\mathcal{M}$ as a subspace. Note that by this embedding, $\mathcal{M}_v$ is identified with the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ onto $\mathcal{M}$.

Our next result shows that the image of ι is a $(\sum_{i=1}^n v_i A_i)$-invariant subspace. In fact, it is the smallest such subspace containing the image of the matrix $I_{\mathtt{w}}$ under the projection $\mathbb{R}[\partial]^{\mathtt{w}} \twoheadrightarrow \mathcal{M}$. This observation constitutes the following theorem. From now on, we are going to omit the use of ι and consider $\mathcal{M}_v$ to be a subspace of $\mathcal{M}$. Moreover, we are going to identify $\mathcal{M}$ with $\mathbb{R}^\gamma$, by identifying the basis vectors of $\mathcal{M}$ with the standard basis vectors of $\mathbb{R}^\gamma$, that is, if $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, ..., \overline{e_\gamma(\partial)}\}$ is a basis for $\mathcal{M}$, then the identification is done by

$$\overline{e_j(\partial)} \mapsto \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} \\ \\ \\ \leftarrow j\text{th position} \\ \\ \\ \end{array} . \tag{10}$$

Theorem 3.5: Let s_j denote the jth standard basis vector of $\mathbb{R}[\partial]^{\mathtt{w}}$, that is a row vector of $\mathtt{w}$ entries with all zeros except 1 at the jth position. Let the image of s_j in $\mathcal{M}$ be given by an $\mathbb{R}$-linear combination as

$$\overline{s_j} = \sum_{k=1}^{\gamma} b_{k,j} \overline{e_k(\partial)},$$

where $\{\overline{e_1(\partial)}, \overline{e_2(\partial)}, ..., \overline{e_\gamma(\partial)}\}$ are the basis vectors of $\mathcal{M}$ and $\gamma = \dim_{\mathbb{R}}(\mathcal{M})$. Denote by B the $(\gamma \times \mathtt{w})$ real matrix whose (k, j)th entry is $b_{k,j}$ of the above expression. Define, as in Lemma 3.3, $A := \sum_{i=1}^n v_i A_i$. Then, with the above mentioned identification of $\mathcal{M}$ with $\mathbb{R}^\gamma$ by equation (10), we have

$$\mathcal{V}_v := \text{colspan}\begin{bmatrix} B & AB & A^2B & ... & A^{\gamma-1}B \end{bmatrix} = \mathcal{M}_v.$$

Proof : First, recall that $\mathcal{M}_v$, considered as a subspace of $\mathcal{M}$ via the inclusion $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}} \hookrightarrow \mathbb{R}[\partial]^{\mathtt{w}}$, is equal to the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ in $\mathcal{M}$. Now, under the identification of $\mathcal{M}$ with $\mathbb{R}^\gamma$ given by equation (10), the $\mathbb{R}$-linear span of $\{s_1, s_2, ..., s_{\mathtt{w}}\}$, when projected to $\mathcal{M}$, goes to $\text{colspan}(B)$. But, clearly, each of the vectors $\{s_1, s_2, ..., s_{\mathtt{w}}\}$ is contained in $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$. Therefore, $\mathcal{M}_v$, which is the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ in $\mathcal{M}$, contains $\text{colspan}(B)$. Moreover, $\mathcal{M}_v$ is also A-invariant. This is because multiplication by A in $\mathcal{M}$ amounts to multiplication by $\langle v, \partial\rangle$ in $\mathbb{R}[\partial]^{\mathtt{w}}$, but $\mathcal{M}_v$ is the image of $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ in $\mathcal{M}$ and $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ is invariant under multiplication by $\langle v, \partial\rangle$. From elementary linear algebra, $\mathcal{V}_v = \text{colspan}\begin{bmatrix} B & AB & A^2B & ... & A^{\gamma-1}B \end{bmatrix}$ is the smallest A-invariant subspace containing $\text{colspan}(B)$. Since $\mathcal{M}_v$ is A-invariant and contains $\text{colspan}(B)$, it easily follows that $\mathcal{M}_v \supseteq \mathcal{V}_v$.

Conversely, any element in $\mathcal{M}_v$, say $\overline{m(\langle v, \partial\rangle)}$, when lifted to $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ looks like

$$m(\langle v, \partial\rangle) = \sum_{i=1}^{\mathtt{w}} f_i(\langle v, \partial\rangle) s_i,$$

where $f_i \in \mathbb{R}[\langle v, \partial\rangle]$. This can be further expanded according to ascending degrees of $\langle v, \partial\rangle$ as

$$\begin{aligned} m(\langle v, \partial\rangle) \quad = \quad & \sum_{i=1}^{\mathtt{w}} a_{0,i} s_i + \langle v, \partial\rangle \sum_{i=1}^{\mathtt{w}} a_{1,i} s_i + \\ & \langle v, \partial\rangle^2 \sum_{i=1}^{\mathtt{w}} a_{2,i} s_i + ... + \langle v, \partial\rangle^k \sum_{i=1}^{\mathtt{w}} a_{k,i} s_i \end{aligned}$$

for some $k \in \mathbb{N}$ with $a_{j,i} \in \mathbb{R}$. Projecting this to $\mathcal{M}$ we get

$$\begin{aligned} \overline{m(\langle v, \partial\rangle)} \quad = \quad & \sum_{i=1}^{\mathtt{w}} a_{0,i} \overline{s_i} + \overline{\langle v, \partial\rangle} \sum_{i=1}^{\mathtt{w}} a_{1,i} \overline{s_i} + \\ & \overline{\langle v, \partial\rangle}^2 \sum_{i=1}^{\mathtt{w}} a_{2,i} \overline{s_i} + ... + \overline{\langle v, \partial\rangle}^k \sum_{i=1}^{\mathtt{w}} a_{k,i} \overline{s_i}. \end{aligned}$$

When the identification of $\mathcal{M}$ with $\mathbb{R}^\gamma$ is done via equation (10), the first term in the above expression takes the form

$$B \begin{bmatrix} a_{0,1} \\ a_{0,2} \\ \vdots \\ a_{0,\mathtt{w}} \end{bmatrix} \in \text{colspan}(B).$$

Similarly, the second term looks like

$$AB \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,\mathtt{w}} \end{bmatrix} \in \text{colspan}(AB),$$

and this trend continues. Thus

$$\overline{m(\langle v, \partial\rangle)} \in \text{colspan}(B) + \text{colspan}(AB) + \text{colspan}(A^2B) + ...$$

Now, by Caley-Hamilton theorem, the right-hand side of the above expression is equal to

$$\begin{aligned} \text{colspan}(B) + \text{colspan}(AB) + ... + \text{colspan}(A^{\gamma-1}B) = \\ \text{colspan}\begin{bmatrix} B & AB & A^2B & ... & A^{\gamma-1}B \end{bmatrix} \end{aligned}$$

because $\dim_{\mathbb{R}} \mathcal{M} = \gamma$. It follows that

$$\begin{aligned} \overline{m(\langle v, \partial\rangle)} \quad & \in \quad \text{colspan}\begin{bmatrix} B & AB & A^2B & ... & A^{\gamma-1}B \end{bmatrix} \\ & = \quad \mathcal{V}_v. \end{aligned}$$

This proves that $\mathcal{V}_v \supseteq \mathcal{M}_v$. $\square$

We now use the last observation, Theorem 3.5, in our next result to show that when $\mathfrak{B}$ is strongly autonomous, its restriction to L_v is equal to $\mathfrak{B}_v$. The behavior-module duality, discussed in Section I, tells us that the quotient module corresponding to the behavior given by the v-intersection submodule is nothing but $\mathcal{M}_v$. To see this consider the following commutative diagram:

$$\langle v, \partial\rangle s_j \quad \mapsto \quad \tfrac{\mathrm{d}}{\mathrm{d}t} s_j \qquad \forall j \in \{1, 2, ..., \mathtt{w}\}$$

$$\begin{array}{ccc} \varphi : \mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}} & \longrightarrow & \mathbb{R}[\tfrac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}} \\ \downarrow & & \downarrow \\ \widetilde{\varphi} : \mathcal{M}_v & \longrightarrow & \mathbb{R}[\tfrac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}/\varphi(\mathcal{R}_v), \end{array}$$

where s_j is, once again, the standard jth basis vector in $\mathbb{R}[\partial]^{\mathtt{w}}$. Because $\langle v, \partial\rangle$ is transcendental over $\mathbb{R}$, the map φ in the above diagram is an isomorphism, and $\varphi(\mathcal{R}_v)$ is a submodule of $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}$. Moreover, from the definition of the the behavior $\mathfrak{B}_v$, the equation module corresponding to it is equal to this $\varphi(\mathcal{R}_v)$. Now, observe that $\widetilde{\varphi}$ defined via φ by taking lifts in $\mathbb{R}[\langle v, \partial\rangle]^{\mathtt{w}}$ is well-defined, and not only that, it is in fact an isomorphism of modules over 1-variable polynomial rings. Thus the quotient module corresponding to $\mathfrak{B}_v$ can be identified with $\mathcal{M}_v$, with the role of $\frac{\mathrm{d}}{\mathrm{d}t}$ played by $\langle v, \partial\rangle$. Since multiplication by $\langle v, \partial\rangle$ in $\mathcal{M}$ is represented by the matrix $A := \sum_{i=1}^{n} v_i A_i$, and $\mathcal{M}_v$ is A-invariant, $A|_{\mathcal{M}_v}$ must be the representation of multiplication by $\langle v, \partial\rangle$. But we just showed that $\mathcal{M}_v$ is isomorphic to $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}/\varphi(\mathcal{R}_v)$, therefore, it follows that multiplication by $\frac{\mathrm{d}}{\mathrm{d}t}$ in $\mathbb{R}[\frac{\mathrm{d}}{\mathrm{d}t}]^{\mathtt{w}}/\varphi(\mathcal{R}_v)$ is represented by $A|_{\mathcal{M}_v}$. By following exactly the same line of arguments as in the proof of Theorem 3.2, it can be concluded that a state representation of $\mathfrak{B}_v$ is given by the matrix $A|_{\mathcal{M}_v}$. Our next result makes use of this observation to infer that $\mathfrak{B}_v$ is contained in $\mathfrak{B}|_v$.

Theorem 3.6: Given a strongly autonomous behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and a nonzero vector $v \in \mathbb{R}^n$, let $\mathcal{R}_v$ be the v-intersection submodule and $\mathfrak{B}_v$ be the corresponding 1-d behavior. Then the restriction of $\mathfrak{B}$ to the line L_v is equal to $\mathfrak{B}_v$, that is,

$$\mathfrak{B}|_v = \mathfrak{B}_v.$$

Proof : Let $B \in \mathbb{R}^{\gamma \times \mathtt{w}}$ be the matrix as in the statement of Theorem 3.5. It is implicit in the proof of Theorem 3.2, where we show surjectivity, that, if $z = \mathrm{col}[z_1, z_2, ..., z_\gamma]$ is the state variable as defined in the same proof, then the manifest variable w is obtained from z by

$$w = B^{\mathrm{T}} z. \tag{11}$$

By Theorem 3.5, $\mathcal{M}_v$ is the smallest $A := \sum_{i=1}^{n} v_i A_i$-invariant subspace containing $\mathrm{colspan}(B)$, so, if we take a basis of $\mathcal{M}_v$ and extend it to a basis of $\mathcal{M}$, in this new basis the matrices B and A would look like:

$$B = \begin{array}{r} \gamma_1 \updownarrow \\ \gamma - \gamma_1 \updownarrow \end{array} \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$A = \begin{array}{rc} \gamma_1 \updownarrow & \\ \gamma - \gamma_1 \updownarrow & \\ & \end{array} \begin{array}{c} \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix} \\ \overset{\leftrightarrow}{\gamma_1} \quad \overset{\leftrightarrow}{\gamma - \gamma_1} \end{array},$$

where $\gamma_1 = \dim_{\mathbb{R}}(\mathcal{M}_v)$. The structure of B, in this new basis, is as above because $\mathrm{colspan}(B) \subseteq \mathcal{M}_v$, while that of A is due to the fact that $\mathcal{M}_v$ is A-invariant. Notice that in this new basis $A|_{\mathcal{M}_v} = A_{1,1}$. It then follows from the discussion preceding the statement of the theorem that $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{z} = A_{1,1}^{\mathrm{T}}\widetilde{z}$ is a state representation for $\mathfrak{B}_v$. Moreover, in the new basis, the images of the standard basis vectors $\{s_1, s_2, ..., s_{\mathtt{w}}\}$ of $\mathbb{R}[\partial]^{\mathtt{w}}$ in $\mathcal{M}_v$ is given by $\mathrm{colspan}(B_1)$. Therefore, as in equation (11), the manifest variable $\widetilde{w}$ of $\mathfrak{B}_v$ is obtained from $\widetilde{z}$ by $\widetilde{w} = B_1^{\mathrm{T}}\widetilde{z}$. So every trajectory in $\mathfrak{B}_v$ can be obtained by solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{z} = A_{1,1}^{\mathrm{T}}\widetilde{z}, \quad \widetilde{w} = B_1^{\mathrm{T}}\widetilde{z}.$$

These solutions look like

$$\widetilde{w}(t) = B_1^{\mathrm{T}} \exp(A_{1,1}^{\mathrm{T}} t)\widetilde{z}(0).$$

On the other hand every solution in $\mathfrak{B}|_v$ looks like

$$w(vt) = B^{\mathrm{T}} \exp(\sum_{i=1}^{n} v_i A_i^{\mathrm{T}} t) z(\mathbf{0}) = B^{\mathrm{T}} \exp(A^{\mathrm{T}} t) z(\mathbf{0}),$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^n$. It easily follows from the structures of B and A that

$$w(vt) = B^{\mathrm{T}} \exp(A^{\mathrm{T}} t) z(\mathbf{0}) = \begin{bmatrix} B_1^{\mathrm{T}} \exp(A_{1,1}^{\mathrm{T}} t) & 0 \end{bmatrix} z(\mathbf{0}).$$

Therefore, by choosing $z(\mathbf{0}) = \begin{bmatrix} \widetilde{z}(0) \\ * \end{bmatrix}$, $* \in \mathbb{R}^{\gamma - \dim_{\mathbb{R}}(\mathcal{M}_v)}$ being arbitrary, we get

$$w(vt) = B_1^{\mathrm{T}} \exp(A_{1,1}^{\mathrm{T}} t)\widetilde{z}(0) = \widetilde{w}(t).$$

Hence we conclude that $\mathfrak{B}_v \subseteq \mathfrak{B}|_v$. That $\mathfrak{B}_v \supseteq \mathfrak{B}|_v$ has already been proved in Theorem 2.3. Thus equality follows. $\square$

IV. Concluding remarks

In this paper, we have investigated the restriction of n-d systems to 1-d subspaces. We have shown the strong connection between the restricted solutions and an algebraic entity, which we have called a v-intersection submodule. We have shown that the intersection submodule naturally gives rise to a 1-d system which always contains the restricted trajectories. We then looked into a special kind of autonomous system, namely strongly autonomous systems, whose solution sets are finite dimensional vector spaces, and showed that such systems always admit first order representations involving an n-tuple of real square matrices called companion matrices. Then we made use of this result to show that the 1-d behavior corresponding to the intersection submodule admits a state-space representation given by the restriction of a linear combination of the companion matrices to an invariant subspace. Utilizing this result we showed that, for the strongly autonomous case, the restriction of the behavior is in fact equal to the behavior of the intersection submodule.

There is a strong connection between the idea of restriction with the well-known method of characteristics and stability theory. The results presented in this paper provides an algebraic approach to deal with restriction of systems, which will be utilized to address the above mentioned issues in subsequent papers.

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