Dissipativity analysis of SISO systems using Nyquist-Plot-Compatible supply rates

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Introduction

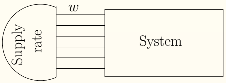


Figure : System and supply rate

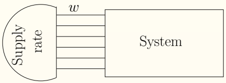
- Electrical two port network, $w = (v, i)$: Power supply $Q = vi$.
- Mechanical system, $w = (f, x)$: Power supply $Q = f\frac{dx}{dt}$.
- γ-contracting system, $w = (d, z)$: $Q = \gamma^2 \parallel d \parallel^2 - \parallel z \parallel^2$.

- Clubbed together as supply rates, these quadratic functions generalize the notion of power supply [Willems and Trentelman, 1998].
- The usual question:

Given a supply rate, characterize all possible systems that are dissipative with respect to the supply rate [Pendharkar and Pillai, 2004, 2009].

- Here, we deal with the converse:

Given a SISO system, how to construct a meaningful and useful supply rate, with respect to which the given system would be dissipative?



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A partial answer with the help of Nyquist-Plot-Compatible (NPC) supply rates.

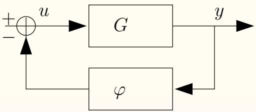
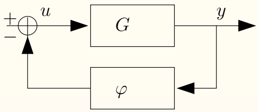


Figure : Feedback interconnection

- Suppose $\|G\| = \gamma_1$ and $\|\varphi\| = \gamma_1$ (the $\mathcal{L}_2$-induced norms).
- G, φ are open-loop stable.

Small gain theorem

$$\gamma_1\gamma_2 < 1 \;\Rightarrow\; \text{closed-loop is finite gain } \mathcal{L}_2\text{-stable.}$$



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Small gain theorem

$$\gamma_1\gamma_2 < 1 \;\Rightarrow\; \text{closed-loop is finite gain } \mathcal{L}_2\text{-stable.}$$

- Suppose, individually, G and φ satisfies $uy > \frac{\mathrm{d}}{\mathrm{d}t}V_i$, where V_1, V_2 are storage functions.
- G, φ are called passive.
- G, φ are open-loop stable.

Passivity theorem

The closed-loop is finite gain $\mathcal{L}_2$-stable.

- Both small gain and passivity are dissipativity properties.
- The above two theorems have been generalized using dissipativity with generalized notion of power supply [Moylan and Hill, 1978], [Megretski and Rantzer, 1997], [Pendharkar and Pillai, 2011].

- The plant is dissipative w.r.t. a quadratic supply rate.
- Controller, too, is dissipative w.r.t. a supply rate determined by the plant's supply rate and the interconnection topology.

Then the interconnected system is guaranteed to be stable.

Motivation

- Both small gain and passivity are dissipativity properties.
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Then the interconnected system is guaranteed to be stable.

Given a plant, we need to find out a suitable supply rate with respect to which the plant is dissipative.

Dissipativity has been dealt with using two variable polynomial matrices [Willems and Trentelman, 1998]

$$\Phi(\zeta,\eta) := \sum_{i,k} \Phi_{ik}\zeta^i\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta].$$

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$$\Phi(\zeta,\eta) := \sum_{i,k} \Phi_{ik}\zeta^i\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta].$$

Then a quadratic differential form (QDF) is a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined as:

$$Q_\Phi(w) := \sum_{i,k} (\frac{\mathrm{d}^i w}{\mathrm{d}t^i})^{\mathrm{T}} \Phi_{ik} (\frac{\mathrm{d}^k w}{\mathrm{d}t^k}).$$

$\Phi(\zeta,\eta)$ induces the QDF Q_Φ called the supply rate.

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$\Phi(\zeta,\eta)$ induces the QDF Q_Φ called the supply rate.

$$vi = [v \quad i]\Phi(\zeta,\eta)\begin{bmatrix} v \\ i \end{bmatrix};$$

$$\Phi(\zeta,\eta) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

$$f\frac{\mathrm{d}x}{\mathrm{d}t} = [f \quad x]\Phi(\zeta,\eta)\begin{bmatrix} f \\ x \end{bmatrix};$$

$$\Phi(\zeta,\eta) = \begin{bmatrix} 0 & \eta/2 \\ \zeta/2 & 0 \end{bmatrix}.$$

Definition (linear differential behavior) [Polderman and Willems, 1998]

$\mathfrak{B}$ is said to be a linear differential behavior, denoted by $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if it is a set of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ trajectories satisfying a system of linear differential equations with constant coefficients.

$$\Updownarrow$$

Existence of a polynomial matrix $R(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\mathfrak{B} := \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0\}.$$

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Consider a transfer function $G = \frac{Y(s)}{U(s)}$. The corresponding behavior with $w = (y, u)$ is given by:

$$\mathfrak{B}_G = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2) \mid [U(\tfrac{\mathrm{d}}{\mathrm{d}t}) \quad -Y(\tfrac{\mathrm{d}}{\mathrm{d}t})] \begin{bmatrix} y \\ u \end{bmatrix} = 0 \right\}.$$

A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be controllable if for every $w', w'' \in \mathfrak{B}$, there exists a $w \in \mathfrak{B}$ and a $\tau > 0$ such that

$$w(t) = w'(t) \text{ for all } t \leqslant 0 \text{ and } w(t) = w''(t) \text{ for all } t \geqslant \tau.$$

For the SISO system, controllability is equivalent to $Y(s)$ and $U(s)$ being coprime.

- $\mathfrak{B}$ is controllable if and only if there exists a polynomial matrix $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that

$$\mathfrak{B} := \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\} = \operatorname{im} M(\tfrac{\mathrm{d}}{\mathrm{d}t}).$$

Definition (dissipativity)

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is said to be dissipative on $\mathbb{R}$ with respect to $\Phi(\zeta, \eta)$ if

$$\int_{\mathbb{R}} Q_{\Phi}(w)\mathrm{d}t \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

- $\mathfrak{B}$ is called strictly dissipative if the above inequality is strict.

Proposition [Willems and Trentelman, 1998]

- Consider $\mathfrak{B} = \text{im } M(\frac{\mathrm{d}}{\mathrm{d}t})$, and
- $\Phi(\zeta, \eta)$ a two variable polynomial matrix.

Then $\mathfrak{B}$ is dissipative with respect to $\Phi(\zeta, \eta)$ on $\mathbb{R}$ if and only if

$$M^{\mathrm{T}}(-j\omega)\Phi(-j\omega, j\omega)M(j\omega) \geqslant 0 \text{ for all } \omega \in \mathbb{R}.$$

- $\|G\| \leqslant \gamma \Leftrightarrow \mathfrak{B}_G$ is dissipative w.r.t

$$\Phi_{\mathrm{sg}} := \begin{bmatrix} \gamma^2 & 0 \\ 0 & -1 \end{bmatrix}$$

- G is passive $\Leftrightarrow \mathfrak{B}_G$ is dissipative w.r.t

$$\Phi_{\mathrm{pa}} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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Then $\mathfrak{B}$ is strictly dissipative with respect to $\Phi(\zeta, \eta)$ on $\mathbb{R}$ if and only if

$$M^{\mathrm{T}}(-j\omega)\Phi(-j\omega, j\omega)M(j\omega) > 0 \text{ for almost all } \omega \in \mathbb{R}.$$

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Nyquist-Plot-Compatible supply rates

- Special supply rates, dissipativity with respect to which can be directly read off from systems' Nyquist plots.

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Definition: NPC supply rates

A supply rate $\Phi(\zeta,\eta) \in \mathbb{R}^{2\times 2}[\zeta,\eta]$ is said to induce a trichotomy of the complex plane $\mathbb{C}$ if corresponding to $\Phi(\zeta,\eta)$ there exists a 3-tuple of disjoint sets $\{\mathcal{A}_\Phi^+, \mathcal{A}_\Phi^0, \mathcal{A}_\Phi^-\}$, such that

$$\mathcal{A}_\Phi^+ \cup \mathcal{A}_\Phi^0 \cup \mathcal{A}_\Phi^- = \mathbb{C}.$$

Plus, for every $\mathfrak{B}_G$ having image representation matrix $M(\frac{\mathrm{d}}{\mathrm{d}t})$, we have the following:

for all real frequency $\omega \geqslant 0$

1. Nyquist plot of G at ω is contained in $\mathcal{A}_\Phi^+ \iff M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) > 0.$
2. Nyquist plot of G at ω is contained in $\mathcal{A}_\Phi^0 \iff M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) = 0.$
3. Nyquist plot of G at ω is contained in $\mathcal{A}_\Phi^- \iff M^T(-j\omega)\partial\Phi(j\omega)M(j\omega) < 0.$

If a supply rate satisfies all these properties, then it is called a Nyquist-Plot-Compatible (NPC) supply rate.

- Strict dissipativity $\Leftrightarrow$ Nyquist plot of G being contained in $\mathcal{A}_\Phi^+$ for almost all positive frequencies.
- We refer to $\mathcal{A}_\Phi^+$ as NPC-region, and $\mathcal{A}_\Phi^0$ as NPC-boundary associated with the NPC supply rate Φ.

Small-gain: $\Phi_{\text{sg}} = \begin{bmatrix} r^2 & 0 \\ 0 & -1 \end{bmatrix}$.

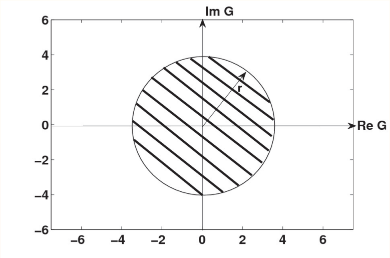

Figure : Associated region of small-gain supply rate.

- Are these all? How do we get more such NPCs?
- Define

$$\Omega := \{\Phi(\zeta, \eta) \in \mathbb{R}^{2\times 2}[\zeta, \eta] \mid \Phi \text{ is NPC}\}.$$

Examples of various standard NPC supply rates

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Passivity: $\Phi_{\text{pa}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

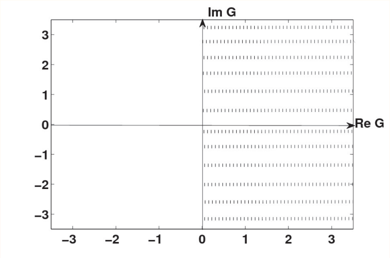

Figure : Associated region of passivity supply rate.

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Examples of various standard NPC supply rates

- Strict dissipativity ⇔ Nyquist plot of G being contained in $\mathcal{A}_\Phi^+$ for almost all positive frequencies.
- We refer to $\mathcal{A}_\Phi^+$ as NPC-region, and $\mathcal{A}_\Phi^0$ as NPC-boundary associated with the NPC supply rate Φ.

Negative-imaginary [Petersen and Lanzon, 2010]: $\Phi_{\text{ni}} = \begin{bmatrix} 0 & \eta \\ \zeta & 0 \end{bmatrix}$.

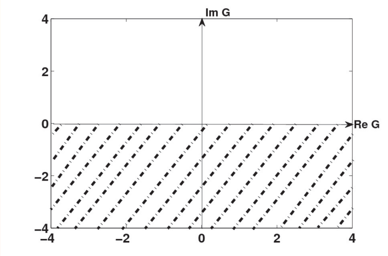

Figure : Associated region of negative-imaginary supply rate.

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- Define

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More NPC supply rates

Proposition

- $\Phi \in \Omega$.
- $T \in \mathbb{R}^{2\times 2}$ non-singular.

the supply rate $T^T \Phi T \in \Omega$.

Lemma

- $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is non-singular.
- $\Sigma_{br} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\Phi = T^T \Sigma_{br} T$ has $\mathcal{A}_\Phi^+$ one of the following:

1. If $b = d$, then $\mathcal{A}_\Phi^0$, is a line parallel to the imaginary axis. Further, if $ab - cd > 0$ (or, if $ab - cd < 0$) then $\mathcal{A}_\Phi^+$ is the RHS (LHS) of the line $\mathcal{A}_\Phi^0$.
2. If $b \neq d$ then, $\mathcal{A}_\Phi^0$, is a circle with center on the real axis. Further, the corresponding $\mathcal{A}_\Phi^+$ is the interior (or the exterior) of the circle if $b^2 - d^2 < 0$ ($b^2 - d^2 > 0$).

Mixing NPC supply rates

- There are many systems whose Nyquist plots need not be in any obvious NPC region.
- Can we go beyond NPC supply rates for these situations.
- What happens if Nyquist plot of a system is contained in the union of two (or, more, but finitely many) known NPC regions?

Theorem

- $G(s)$ is a SISO LTI system.
- $\mathfrak{B}_G = \operatorname{im} M(\frac{d}{dt})$ is its image representation.
- Let Φ_1 and Φ_2 be NPC supply rates.

Then the following two statements are equivalent:

1. G has Nyquist plot contained in $\mathcal{A}^+_{\Phi_1} \cup \mathcal{A}^+_{\Phi_2}$ for almost all $\omega \geqslant 0$.
2. There exist $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta, \eta) := p(\zeta)\Phi_1(\zeta, \eta)p(\eta) + q(\zeta)\Phi_2(\zeta, \eta)q(\eta).$$

- $\mathfrak{B}_G$ is strictly dissipative with respect to the $\Phi(\zeta,\eta)$ defined above $\Leftrightarrow$ $p(\xi), q(\xi)$ satisfy

$$M^T(-j\omega)p(-j\omega)\partial\Phi_1(j\omega)p(j\omega)M(j\omega) + M^T(-j\omega)q(-j\omega)\partial\Phi_2(j\omega)q(j\omega)M(j\omega) > 0$$

for almost all $\omega \in \mathbb{R}$, or, equivalently,

$$\begin{bmatrix} p(-j\omega) \\ q(-j\omega) \end{bmatrix} \begin{bmatrix} \Gamma(-j\omega,j\omega) & 0 \\ 0 & \Pi(-j\omega,j\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} > 0$$

for almost all $\omega \in \mathbb{R}$, where Γ and Π are defined as

$$\begin{aligned} \Gamma(-j\omega,j\omega) &:= M^T(-j\omega)\partial\Phi_1(j\omega)M(j\omega) \\ \Pi(-j\omega,j\omega) &:= M^T(-j\omega)\partial\Phi_2(j\omega)M(j\omega). \end{aligned}$$

- This is true $\Leftrightarrow$ the auxiliary behavior, $\mathfrak{B}_{\text{aux}} := \text{im} \begin{bmatrix} p(\frac{d}{dt}) \\ q(\frac{d}{dt}) \end{bmatrix}$ is strictly dissipative with respect to

$$\Phi_{\text{aux}}(\zeta,\eta) = \begin{bmatrix} \Gamma(\zeta,\eta) & 0 \\ 0 & \Pi(\zeta,\eta) \end{bmatrix}.$$

- It has been shown in [Pendharkar and Pillai, 2004 and 2009] that it is possible to find a $\mathfrak{B}_{\text{aux}}$ $\Leftrightarrow$ the worst inertia of Φ_{aux} is *not* $(2,0)$.

Definition: worst inertia [Pendharkar and Pillai, 2004 and 2009]

- $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is para-Hermitian
- $P(\xi)$ is nonsingular as a polynomial matrix, i.e., $\det(P(\xi)) \not\equiv 0$.
- $\omega \in \mathbb{R}$ is such that $j\omega$ is not a zero of $P(\xi)$, i.e., $\det(P(j\omega)) \neq 0$.
- The inertia of $P(j\omega)$ is defined as the 2-tuple: $(\sigma_-(P(j\omega)), \sigma_+(P(j\omega)))$ where

$$\begin{aligned}\sigma_-(P(j\omega)) &= \text{no. of negative eigenvalues of } P(j\omega) \text{ and}\\ \sigma_+(P(j\omega)) &= \text{no. of positive eigenvalues of } P(j\omega).\end{aligned}$$

- If $P(j\omega)$ is singular, then the inertia is undefined at that point.
- Worst inertia is $(\nu_{\max}, \mathtt{w} - \nu_{\max})$, where

$$\nu_{max} := \max_{\omega\in\mathbb{R}}\{\sigma_-(P(j\omega))\}$$

Example: mixing of small-gain and passivity

- $G = \frac{3}{s^2+3s+2}$.
- The Nyquist plot (for positive frequencies) is contained in the union of the unit circle ($r = 1$ in Φ_{sg}) and the right half plane.
- There exists $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta, \eta) = p(\zeta)\Phi_{\text{sg}}p(\eta) + q(\zeta)\Phi_{\text{pa}}q(\eta). \quad (1)$$

The required p, q are

$$\begin{aligned} p(\xi) &= 2.449\xi^3 + 2.449\xi^2 + 0.3709\xi + 2.0781 \\ q(\xi) &= 1.3163\xi^3 - 2.65256\xi^2 - 0.36314\xi - 2.236. \end{aligned}$$

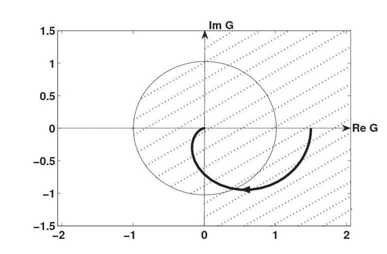

Figure : Mixing of small-gain and passivity

- $G = \frac{2s-1}{s^3+2s^2+2s}$.
- The Nyquist plot (for positive frequencies) is contained in the union of the right half plane and the lower half plane.
- There exists $p, q \in \mathbb{R}[\xi]$ such that $\mathfrak{B}_G$ is strictly dissipative with respect to

$$\Phi(\zeta,\eta) = p(\zeta)\Phi_{\text{pa}}p(\eta) + q(\zeta)\Phi_{\text{ni}}q(\eta). \qquad (2)$$

The required p, q are

$$\begin{aligned} p(\xi) &= -2.69282\xi^3 - 1.30718\xi^2 - 2.0\xi \\ q(\xi) &= -2.0\xi^4 - 2.0\xi^3 + 0.0784\xi^2 - 2.0784\xi. \end{aligned}$$

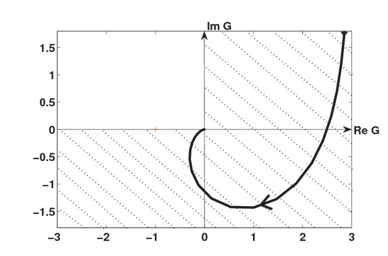

Figure : Mixing of passivity and negative imaginary

Algorithm to find the weighting polynomials

- Define

$$S(\xi) := \begin{bmatrix} \Gamma(-\xi,\xi) & 0 \\ 0 & \Pi(-\xi,\xi) \end{bmatrix}.$$

- Note that, Statement (1) not satisfied means $S(j\omega)$ has worst inertia (2,0). Then $S(j\omega)$ is negative semi-definite for all $\omega \in \mathbb{R}$. No p, q exists.
- If $S(j\omega)$ has worst inertia (0,2) then $S(j\omega)$ is positive semi-definite (losing its rank only at finitely many frequencies). Thus any pair of $p, q \in \mathbb{R}[\xi]$ will work.

What happens when the worst inertia is (1,1)?

Proposition [Pendharkar and Pillai, 2004 and 2009]

There exist polynomial matrices $K \in \mathbb{R}^{2\times 2}[\xi]$ and $L \in \mathbb{R}^{\bullet\times 2}[\xi]$, with K square and nonsingular, such that

$$P(\xi) = K^T(-\xi)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} K(\xi) + L^T(-\xi)L(\xi).$$

Algorithm to find the weighting polynomials

- Choose $p, q \in \mathbb{R}[\xi]$ such that

$$K(\xi)\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix}$$

gives the image representation of a behavior whose $\mathcal{H}_\infty$-norm is less than 1.

- Such p, q can be found thus:
  - Let $\widetilde{G}(s) = \frac{n(s)}{d(s)}$ be such that $\|G\|_{\mathcal{H}_\infty} < 1$.
  - Define

$$\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} = \text{adj}(K(\xi))\begin{bmatrix} n(\xi) \\ d(\xi) \end{bmatrix}.$$

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$$\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} = \text{adj}(K(\xi)) \begin{bmatrix} n(\xi) \\ d(\xi) \end{bmatrix}.$$

We have a given a simple algorithm to carry out this factorization under following assumptions:

1. The number of crossover frequencies is only two.
2. The roots of the polynomials $\Gamma(-j\omega, j\omega)$ and $\Pi(-j\omega, j\omega)$ are known precisely.

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Thank you