On the Connectivity in Finite Ad Hoc Networks

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***Abstract*— Connectivity and capacity analysis of ad hoc networks has usually focused on asymptotic results in the the number of nodes in the network. In this paper we analyze finite ad hoc networks. With the standard assumption of uniform distribution of nodes in $[0, z]$, $z > 0$, for a one-dimensional network, we obtain the exact formula for the probability that the network is connected. We then extend this result to find bounds for the connectivity in a two-dimensional network in $[0, z]^2$.**

***Index Terms*— One Dimensional Ad Hoc Networks, Finite Ad Hoc Networks, Connectivity.**

I. INTRODUCTION

The transmission range, and hence the transmission power, that is necessary to keep the network connected is an important parameter in the design of ad hoc multihop radio networks. One of the first results available for the connectivity of a multihop radio network was in [1]. More recently, in [2] the asymptotic (in the number of nodes in the network) lower and upper bounds for the probability of disconnectivity of an ad hoc radio network with nodes randomly placed in a unit circle in $\mathcal{R}^2$ are obtained. They consider the following problem: let n nodes be randomly placed in a disc of unit area in $\mathcal{R}^2$ and $r(n)$ the transmission range of each of the nodes. Let p_c be the probability that the resulting ad hoc network is connected. Let $\pi r^2(n) = (\log n + \alpha(n))/n$, and $\lim_{n\to\infty} \alpha(n) = \beta$. Then

$$\lim_{n\to\infty} \inf(1 - p_c) \geq e^{-\beta}(1 - e^{-\beta}) \qquad \text{and}$$
$$\lim_{n\to\infty} \sup(1 - p_c) \leq 4e^{-\beta}$$

What this means is that as $n \to \infty$, $e^{-\beta}(1-e^{-\beta}) \leq (1-p_c) \leq 4e^{-\beta}$. Thus, as $n \to \infty$, if $\pi r^2(n) = (\log n + \alpha(n))/n$ the radio network is connected with probability 1 if and only if $\alpha(n) \to \infty$. In [5], using recent results from geometric random graphs [4], the threshold function (see [3] for a formal definition) for the transmission range to make a random ad hoc network in $[0, 1]^2$ connected was obtained as $\sqrt{(\log n)/n}$. This result is similar in spirit to that in [2]. Another issue of interest is the capacity of ad hoc networks. See [6], [7] for some recent analytical results on capacity. As is evident, all of the above are asymptotic results as n, the number of nodes in the network, becomes large. We do not know of any analytical result for finite ad hoc networks.

The results that we report in this paper concern networks with a finite number of nodes. We first consider a one dimensional network in $[0, z]$, $z > 0$, and obtain the exact probability for the network to be connected and then use it to obtain some loose bounds for the probability that a finite two dimensional network is connected.

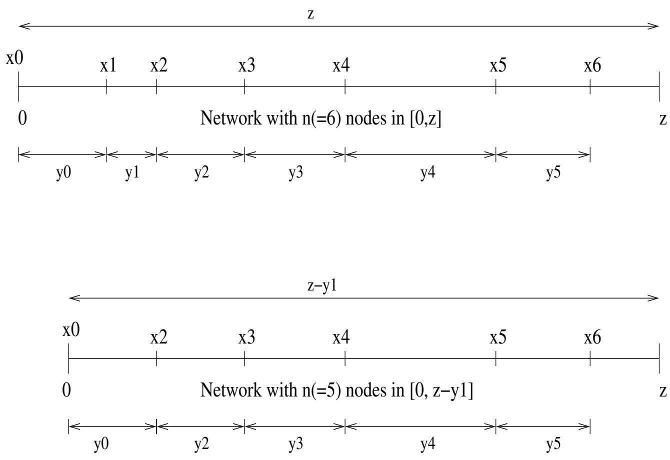

Fig. 1. Illustrating Eqn. 3. For the n node network to be connected, the $n-1$ node network without the node at x_1 should also be connected and y_1 should be at most r. Thus the $n-1$ node connected network will be spread over $z - y_1$ with y_1 taking values in $(0, r)$.

II. PROBABILITY OF CONNECTIVITY

Let n nodes be uniformly in distributed $[0, z]$ forming a one dimensional, ad hoc, multihop, radio network. One dimensional networks have been studied in the context of cellular networks [8] and circuit switched networks [9], [10], [11] and provide useful insights when analysis of networks in higher dimensions becomes hard.

The location of node i, x_i, $i = 1, \cdots, N$ is independently chosen. The network is therefore represented by a random vector $\mathbf{N} = [x_1, x_2, \cdots, x_n]$. Let $p_c(n, z, r)$ denote the probability that the network $\mathbf{N}$ in $[0, 1]$ is connected when the transmission radius is r. Let $\hat{\mathbf{N}} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n]$ be the nodes ordered according to their positions on $[0, z]$, i.e., $\hat{x}_1 < \hat{x}_2 < \cdots, < \hat{x}_n$. Define $\hat{x}_0 = 0$.

Let $r \geq 0$ be the transmission range of all the nodes. For $\mathbf{N}$ to be connected we need $\hat{x}_{i+1} - \hat{x}_i < r$ for $i = 1, \cdots, (n-1)$. To obtain the probability that the network is connected, consider the following interpretation. The set of all realizable networks is contained in the polytope A_n, where A_n is defined by $0 \leq \hat{x}_1 \leq \hat{x}_2 \leq \cdots, \leq \hat{x}_n \leq z$. The set of connected networks is contained in the polytope, $A_c(n, z, r)$, defined by $\hat{x}_{i+1} - \hat{x}_i < r$ for $i = 1, \cdots, (n-1)$. Let $V_n(z)$ and $V_c(n, z, r)$ be the volumes of the polytopes A_n and $A_c(n, z, r)$ respectively. It is easily seen that the probability that the network is connected, $p_c(n, z, r)$, will be $V_c(n, z, r)/V_n(z)$.

We obtain $p_c(n, z, r)$ as follows. Define $y_i = \hat{x}_{i+1} - \hat{x}_i$,

$i = 0, \cdots n-1$. Let $U_n(z)$ be the volume of the set

$$\left\{ y_0, y_1, \ldots y_{n-1} : y_i \geq 0 \text{ for } i \geq 0, \sum_0^{n-1} y_i \leq z \right\},$$

and let $U_c(n, z, r)$ be the volume of the set

$$\left\{ y_0, y_1, \ldots y_{n-1} : y_i \geq 0 \text{ for } i \geq 0, y_i \leq r \text{ for } i > 0, \sum_0^{n-1} y_i \leq z \right\}.$$

Then, notice that since $\hat{x}$ and y are related by a linear invertible transformation, we have $U_n(z) = KV_n(z)$ and $U_c(n, z, r) = KV_c(n, z, r)$ for some constant $K > 0$. Thus

$$p_c(n, z, r) = \frac{U_c(n, z, r)}{U_n(z)} \tag{1}$$

Now, $U_n(z)$ can be calculated very easily and is given by the following recurrence relation

$$U_n(z) = \int_0^z U_{n-1}(z-t)dt = \frac{z^n}{n!} \tag{2}$$

From Figure 1, $U_c(n, z, r)$ can be written as follows

$$U_c(n, z, r) = \int_0^r U_c(n-1, z-t, r)dt \tag{3}$$

Eqn. 3 essentially means that the n node network is connected if the $n-1$ node network formed after taking out the first node is connected and that the node that was taken out is within r of the rightmost node in the $n-1$ node network. This, in turn means that the $n-1$ node network will span over at most $z-r$. The right hand side of Eqn. 3 can be written as the convolution of $U_c(n-1, z, r)$ and $h(z)$ where

$$h(z) = \begin{cases} 1 & 0 \leq z \leq r \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$U_c(n, z, r) = h * U_c(n-1, z, r) = \cdots = h^{((n-1)*)} * U_c(1, z, r) \tag{4}$$

where $h^{((n-1)*)}(\cdot)$ is the $(n-1)-$fold convolution of $h(\cdot)$ with itself. It is easy to see that $U_c(1, z, r) = zu(z)$, where $u(z)$ is the unit step function.

Let $\mathcal{L}(\cdot)$ denote the Laplace transform. We then have the following

$$\begin{aligned} \mathcal{L}(h(t)) &= \frac{1-e^{sr}}{s} & (5) \\ \mathcal{L}(U_c(n, z, r)) &= \left(\frac{1-e^{sr}}{s}\right)^{n-1} \frac{1}{s^2} \\ &= \frac{(1-e^{sr})^{n-1}}{s^{(n+1)}} \\ &= \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k e^{srk}}{s^{(n+1)}} & (6) \end{aligned}$$

Taking the inverse Laplace transform,

$$U_c(n, z, r) = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k (z-kr)^n u(z-kr)}{n!} \tag{7}$$

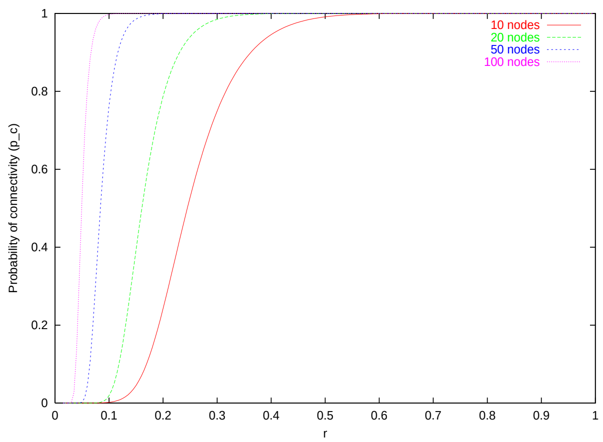

Fig. 2. Probability of the network being connected as a function of r, the transmission radius. We have used $z = 1$ and show $p_c(n, 1, r)$ for different n.

We thus get

$$\begin{aligned} p_c(n, z, r) &= \frac{U_c(n, z, r)}{U_n(z)} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{(z-kr)^n}{z^n} u(z-kr) & (8) \end{aligned}$$

Now consider a two dimensional network in which n nodes are randomly distributed in the positive quadrant $[0, z]^2$. Let N_x be a one dimensional network obtained by projecting N on to the x axis and let N_y be the network obtained by projecting the nodes on to the $y-$axis. A lower bound on the probability that N is disconnected can be obtained by observing that $\text{Prob}(N \text{ is disconnected}) \geq \text{Prob}(N_x \text{ is disconnected } OR \ N_y \text{ is disconnected})$. It then follows that

$$\begin{aligned} \text{Prob}(N \text{ is disconnected}) &\geq 2(1-p_c(n, z, r)) - \\ &\quad -(1-p_c(n, z, r))^2 \\ &= 1-(p_c(n, z, r))^2 \end{aligned}$$

or

$$\text{Prob}(N \text{ is connected}) \leq (p_c(n, z, r))^2 \tag{9}$$

III. Discussion

In Figure 2 we plot $p_c(n, r, z)$ as a function of r for different values of n using $z = 1.0$. Observe that the transition from being almost surely disconnected to almost surely becomes sharper with increasing n as described by [5]. Further, the threshold values, the value of r at which $p_c(\cdot)$ exceeds 0.5 are, for different n, 0.245 for $n = 10$, 0.161 for $n = 20$, 0.0840 for $n = 50$ and 0.0493 for $n = 100$. Compare this with the asymptotic threshold of $\log n/n$ from [2], [5] from which the corresponding threshold values of r are 0.230, 0.150, 0.0782 and 0.0461 respectively. There is a consistent 6-7% difference between the exact value from analysis and that obtained from asymptotic analysis.

In Figure 3 we plot the bounds on the threshold values of r for the two dimensional network as a function of n, the number

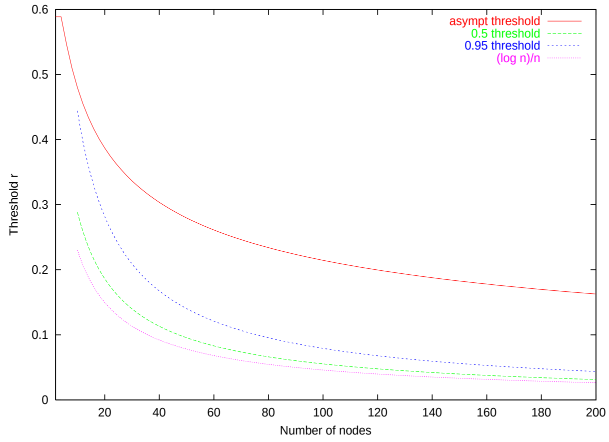

Fig. 3. Comparing the bounds for threshold functions (defined at at 0.5 and 0.95) and the asymptotic threshold function, $\sqrt{\log_n}/n$.

of nodes. We plot the value of r for which the upper bound evaluates to 0.5 and 0.95, which we call the bounds on the 0.5 and 0.95 thresholds. We also show the asymptotic threshold as a function of n, $\sqrt{(\log n)/n}$. From the figure, the bounds seem to converge to the $\log n/n$ graph which is also shown. This latter observation can also be justified as follows. As n becomes large $p_c(z, n, r)$ evaluated from Eqn. 8 as a function of r approaches a step function with a step at $r = (\log n)/n$. The bound for the two dimensional network being just a square of this function will also be a step with a step at the same value. Thus, we note that the simple upper bound of Eqn. 9 is not asymptotically tight.

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