On Range Matrices and Wireless Networks in d Dimensions

Madhav Desai and D. Manjunath
Deptartment of Electrical Engineering
Indian Institute of Technology, Bombay
Powai Mumbai INDIA
email: madhav,dmanju@ee.iitb.ac.in

***Abstract*— Suppose that $V = \{v_1, v_2, \ldots v_n\}$ is a set of nodes randomly (uniformly) distributed in the d dimensional cube $[0, x_0]^d$, and $W = \{w(i,j) > 0 : 1 \leq i, j \leq n\}$ is a set of numbers chosen so that $w(i,j) = w(j,i)$. Construct a graph $\mathcal{G}_{n,d,W}$ whose vertex set is V, and whose edge set consists of all pairs $\{u_i, u_j\}$ with $\| u_i - u_j \| \leq w(i,j)$. In the wireless network context, the set V is a set of labeled nodes in the network and W represents the maximum distances between the node pairs for them to be connected. We essentially address the following question: "If G is a graph with vertex set V, what is the probability that G appears as a subgraph in $\mathcal{G}_{n,d,W}$?" Our main contribution is a closed form expression for this probability under the l_∞ norm for any dimension d and a *suitably defined probability density function*. As a corollary to the above answer, we also answer the question, "What is the probability that $\mathcal{G}_{n,d,W}$ is connected?"**

Keywords: Ad Hoc Networks, Geometric Random Graphs, Range Matrices.

I. Introduction

Let n labeled nodes $V = \{v_1, v_2, \ldots v_n\}$ be distributed uniformly in the d-dimensional cube $U_d = [0, x_0]^d$. Let $W = \{w(i,j) \geq 0 : 1 \leq i, j \leq n\}$ be a set of nonnegative numbers such that $w(i,j) = w(j,i)$, which may be viewed as a set representing the upper bounds on the distances between pairs of nodes in V. Let $\mathcal{G}_{n,d,W}$ be the graph with vertex set V, and edge set $\mathcal{E}$ consisting of all pairs $\{v_i, v_j\}$ such that $\| v_i - v_j \| \leq w(i,j)$. In other words, an edge is introduced in the graph $\mathcal{G}_{n,d,W}$ if and only if the corresponding nodes are within a specified distance of each other. $\mathcal{G}_{n,d,W}$ is a *random geometric graph* (RGG). Clearly, randomly deployed wireless networks are best modeled as random geometric graphs. Among the many properties of wireless networks that are of interest are the connectivity [1] and capacity [2], [3], [4] properties and these properties can be studied by studying them on suitably defined RGGs. See [5], [6], [7] for more examples of work on connectivity of random geometric graphs in the context of wireless networks. Random geometric graphs are also of independent interest and have many other applications. They have also been studied in their own right in recent times.

Prior work on connectivity properties of geometric random graphs has considered the restricted case of constructing $\mathcal{G}_{n,d,W}$ with $w(i,j) = \alpha$ for all i, j ($\alpha > 0$ and a function of n) and identifying its properties. α is a function of n. For example, [8], [9] study the properties of $P^{(d)}_{\text{conn}}(n, \alpha)$, the probability that $\mathcal{G}_{n,d,\alpha}$ is connected. In [1], the geometric random graph corresponding to the wireless network is constructed on a unit disk and lower and upper asymptotic bounds for $P^{(2)}_{\text{conn}}(n, \alpha)$ are obtained. It is shown that $P^{(2)}_{\text{conn}}(n, \alpha)$ has a threshold property—as α is varied from 0 to 1, i.e., there is a function $t_n = \sqrt{\frac{\log n}{n}}$ such that

$$\lim_{n \to \infty} P^{(2)}_{\text{conn}}(n, \alpha) = \begin{cases} 1 & \text{if } \alpha / t_n \to \infty \\ 0 & \text{if } \alpha / t_n \to 0 \end{cases}$$

Also note that [8], [1], [9] only have asymptotic analysis as $n \to \infty$. The exact value of $P^{(d)}(n, \alpha)$ is available only for the one-dimensional case ($d = 1$) [10]. See [11], [12] for other problems on geometric random graphs, [13], [14] for early surveys, and [15] for a systematic treatment of the subject. The results in this paper may also be applicable to random interval graphs as in [16].

Our present work is an extension of some of the early work on connectivity of geometric random graphs and a significant step forward in their study in the following senses:

- We relax the restriction that $w(i,j) = \alpha$ for all pairs v_i, v_j and allow them to be specified arbitrarily. This is especially useful in wireless networks in which the nodes may not all have identical properties. We remark here that the requirement that $w(i,j) = w(j,i)$ is not very restrictive. Let i be a small node with a 'weak' transceiver and j a big node with 'strong' transceiver. Although, i transmits with a lower power than j, it is not unreasonable to expect that j will be able to receive and decode the signals from i. The base station and the mobile node of a cellular network is a classic example of this condition. Now if k were a 'weak' node the upper bound on $w(i,k)$ for i and k to be able to communicate directly with each other could be much smaller than in $w(i,j)$.
- We obtain an exact expression for $P^{(d)}_G(\mathbf{x})$, the probability that an arbitrary, labeled network G on the node set V and specified by a distance vector $\mathbf{x}$ (the precise meaning of which will be elaborated in the next section), appears as a subgraph in $\mathcal{G}_{n,d,W}$. In particular, if G is the complete graph on n vertices, $P^{(d)}_G(W)$ is the probability that a random distribution of the n nodes satisfies the

distance relationships specified by a distance matrix W.

- We derive an exact expression for $P_{\text{conn}}^{(d)}(n,\alpha)$, the probability that $\mathcal{G}_{n,d,W}$ is connected.

In the wireless network context, one would measure the distance using the l_2 norm. In this paper, to allow for mathematical tractability, we will measure the distances using the l_∞ norm.In the rest of this paper, we develop these themes by first setting up some preliminary notation, then presenting the main results and a discussion of these results.

II. Preliminaries and Notation

Let $G = (\hat{V}, \hat{E})$ be a given labeled graph with vertex set $\hat{V} = \{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\}$. The labeled edges of G form the set $\hat{E} = \{\hat{e}_1, \hat{e}_2, \ldots \hat{e}_m\}$, with $|\hat{E}| = m$. and $e_i = \{v_{a_i}, v_{b_i}\}$, $1 \leq a_i < b_i \leq n$. Let $x_i > 0$, for $i = 1, \cdots, m$ be associated with edge e_i. x_i may be interpreted to mean that the length of edge e_i is less than x_i. We will represent a vector in bold font as $\mathbf{x}$, and the components of $\mathbf{x}$ as x_i. Let the $m+1$ dimensional vector $\mathbf{x} = [x_i]_0^m$ represent the set of variables $\{x_0, x_1, \ldots x_m\}$.

Let n labeled nodes $v_1, v_2, \ldots v_n$ be distributed uniformly in $[0, x_0]^d$. As we described in the previous section, construct the geometric graph $\mathcal{G}_{n,d,W}$ from these randomly distributed nodes, by having an edge between vertices corresponding to nodes v_i and v_j if $\| v_i - v_j \|_\infty < w(i,j)$. For $i = 1, \ldots, n$ associate $\hat{v}_i$ of G with v_i of $\mathcal{G}_{n,d,W}$. Let $w(v_{a_i}, v_{b_i}) = x_i$ for $i = 1, 2 \ldots m$. As an example consider nodes A, B, C and D located at $(1/4, 1/4)$, $(1/2, 3/4)$, $(3/4, 1/2)$ and $(1, 1/4)$ respectively. If $w(i,j) = 3/8$ for $i, j = A, B, C, D$, and if G is defined by edges $e_1 = (A, B)$ and $e_2 = (B, C)$ with $x_1 = 3/8$ and $x_2 = 3/8$ then G does not appear in the $\mathcal{G}_{n,d,W}$. On the other hand if G is defined by edges $e_1 = (B, C)$ and $e_2 = (C, D)$ with $x_1 = 3/8$ and $x_2 = 3/8$ then G does appear in $\mathcal{G}_{n,d,W}$ as defined above.

The following inequalities will be satisfied if and only if the labeled graph G appears as a subgraph in $\mathcal{G}_{n,d,W}$.

$$\begin{aligned} \| v_{a_i} - v_{b_i} \|_\infty &\leq x_i, \; i = 1, 2, \ldots m \\ \| v_j \|_\infty &\leq x_0, \; j = 1, 2, \ldots n \end{aligned} \tag{1}$$

Let $P_G^{(d)}(\mathbf{x})$ denote the probability that G appears as a subgraph of $\mathcal{G}_{n,d,W}$. Thus $P_G^{(d)}(\mathbf{x})$ is the probability that the inequalities in (1) are satisfied when n nodes are randomly distributed in d-dimensional space.

The multi-dimensional Laplace transform of a function $f : \mathbf{R}^{+m+1} \rightarrow \mathbf{R}$ is defined as

$$\hat{f}(\mathbf{s}) = \int_{t_0=0}^{\infty} \int_{t_1=0}^{\infty} \cdots \int_{t_m=0}^{\infty} f(\mathbf{t}) e^{-\mathbf{s}\cdot\mathbf{t}} dt_0 dt_1 \ldots dt_m$$

where $\mathbf{s} = [s_i]$ is a vector of complex numbers belonging to the region of convergence of the defining integral, and $\mathbf{s} \cdot \mathbf{t}$ is the inner product of the vectors $\mathbf{s}$ and $\mathbf{t}$. The function $f(\mathbf{t})$ can be recovered from $\hat{f}(\mathbf{s})$ either analytically or by numerical techniques based on Fourier series methods [17] or the Laguerre method [18].

The multi-dimensional Fourier transform of a square-integrable function $f : \mathbf{R}^{m+1} \rightarrow \mathbf{R}$ is defined as

$$\hat{f}(\boldsymbol{\omega}) = \int_{t_0=0}^{\infty} \int_{t_1=0}^{\infty} \cdots \int_{t_m=0}^{\infty} f(\mathbf{t}) e^{-j\boldsymbol{\omega}\cdot\mathbf{t}} dt_0 \; dt_1 \; \ldots \; dt_m$$

where $j = \sqrt{-1}$, $\boldsymbol{\omega} = [\omega_i]$ with $-\infty < \omega_i < \infty$, and $\boldsymbol{\omega} \cdot \mathbf{t}$ is the inner product of the vectors $\boldsymbol{\omega}$ and $\mathbf{t}$. The function $f(\mathbf{x})$ can be recovered from $\hat{f}(\boldsymbol{\omega})$ either analytically or by numerical techniques such as discrete Fourier transform inversion.

Given an n-dimensional polytope K, we denote the volume of K by $\mathsf{Vol}_n(K)$. The symbol $\mathcal{S}_n$ will represent the set of permutations of $\{1, 2, \ldots n\}$. We use $\delta(t)$ to denote the *unit distribution* (also called the Dirac-delta function). For an integer $k \geq 0$, the function $u_k : \mathbf{R} \rightarrow \mathbf{R}$ is defined as

$$u_k(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^k & \text{if } t \geq 0 \end{cases}$$

The one-dimensional Laplace transform of $u_k(t)$ is then $k!/s^{k+1}$. The *unit pulse* function $p(t)$ is defined as

$$p(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The one-dimensional Fourier transform of the unit pulse function is then

$$e^{-j\omega/2} \frac{\sin(\omega/2)}{(\omega/2)}$$

Given two functions $f, g : \mathbf{R}^n \rightarrow \mathbf{R}$, $f * g$ represents the convolution of f and g, that is,

$$(f * g)(\mathbf{x}) \;=\; \int_{\mathbf{z} \in \mathbf{R}^n} f(\mathbf{z}) g(\mathbf{x} - \mathbf{z}) d\mu_n(\mathbf{z})$$

where $d\mu_n(\mathbf{z})$ is the volume element in $\mathbf{R}^n$ centered at $\mathbf{z}$. The transform (Laplace or Fourier) of $f * g$ is the product of the corresponding transforms of f and g.

III. Decomposition of $P_G^{(d)}$

We observe that

Proposition 1:

$$P_G^{(d)}(\mathbf{x}) \;=\; \left(P_G^{(1)}(\mathbf{x}) \right)^d.$$

Proof: For the l_∞ norm of a finite dimensional vector $\mathbf{a} = [a_i]$,

$$\| \mathbf{a} \|_\infty \leq K \text{ iff } |a_i| \leq K \text{ for all i}$$

Uniform distribution of the nodes in $[0, x_0]^d$ implies that the coordinates for the nodes are chosen independently and uniformly in $[0, x_0]$. Thus, G appears in $\mathcal{G}_{n,d,W}$ if and only if G appears in $\mathcal{G}_{n,1,W}$ for each of the d (independent) projections of the set V onto $[0, x_0]$ and the proposition follows. □

Our problem is now reduced to that of finding $P_G^{(1)}(\mathbf{x})$. Let v_i be located at $X_i \in [0, x_0]$ and $\mathbf{X} = [X_1, \ldots, X_n]$ be the n-dimensional random vector denoting a realization of the one-dimensional network in $[0, x_0]$. The realizations of $\mathbf{X}$ can be decomposed into $n!$ disjoint events (corresponding to the possible permutations of X_i), each of which is equally probable (because X_i are i.i.d. and uniformly distributed in

$[0, x_0]$). Thus, a realization of $\mathbf{X}$ must fall into one of the $n!$ permutation classes. Let a realization of $\mathbf{X}$ be described by the permutation $\sigma \in \mathcal{S}_n$ ($\mathcal{S}_n$ is the set of all permutation classes), that is, the nodes on $[0, x_0]$ ordered from left to right appear as $v_{\sigma_1}, v_{\sigma_2} \ldots v_{\sigma_n}$. Let $P^{(1)}_{G|\sigma}(\mathbf{x})$ denote the probability that the graph G appears in $\mathcal{G}_{n,1,W}$, given that the ordering of the nodes is described by σ. Then

$$P^{(1)}_G(\mathbf{x}) \;=\; \sum_{\sigma} \frac{P^{(1)}_{G|\sigma}(\mathbf{x})}{n!}$$

Consider the computation of $P^{(1)}_{G|\sigma}(\mathbf{x})$. Without loss of generality, assume that σ is defined by $\sigma_i = i,\ 1 \le i \le n$. Now consider the n nodes in V as they appear on the line. Define $r_i = v_{i+1} - v_i,\ i = 1, 2, \ldots (n-1)$, and $r_0 = v_1$. Let the j^{th} edge in G be $e_j = \{v_{a_j}, v_{b_j}\}$, $1 \le a_j < b_j \le n$. Then, it follows that G will appear as a subgraph of $\mathcal{G}_{n,1,W}$ if and only if

$$v_{b_j} - v_{a_j} \;\le x_j \quad j = 1, 2, \ldots m$$

That is, G will appear as a subgraph of $\mathcal{G}_{n,1,W}$ if and only if

$$\sum_{k=a_j}^{b_j-1} r_k \;\le x_j \quad j = 1, 2, \ldots m \tag{2}$$

Note that in addition, the $r_k's$ satisfy

$$\sum_{k=0}^{n-1} r_k \;\le\; x_0 \tag{3}$$

and of course

$$r_k \;\ge\; 0,\ 0 \le k \le n-1 \tag{4}$$

The bounded polytope in n-dimensions, $A^{\sigma}_G(\mathbf{x})$, defined by the inequalities of Eqs. (2)–(4) describes the set of choices of $r_k, 0 \le k < n$ for which G will appear. The n-dimensional polytope $A_n(\mathbf{x})$ consisting of all possible valid choices for r_k is defined by the inequalities of Eq. (3) and Eq. (4). Then, following the development in [10], we can write

$$\begin{aligned} P^{(1)}_{G|\sigma}(\mathbf{x}) \quad &= \frac{\mathsf{Vol}_n(A^{\sigma}_G(\mathbf{x}))}{Vol_n(A_n(\mathbf{x}))} \\ &= n! \frac{\mathsf{Vol}_n(A^{\sigma}_G(\mathbf{x}))}{x_0^n} \\ &= n! \frac{F^{\sigma}_n(\mathbf{x})}{x_0^n} \end{aligned} \tag{5}$$

where $F^{\sigma}_n(\mathbf{x}) \triangleq \mathsf{Vol}_n(A^{\sigma}_G(\mathbf{x}))$.

We turn our attention to the computation of $F^{\sigma}_n(\mathbf{x})$. First, some more notation: Denote the edge set of G by $E(G)$ and for $e_j \in E(G)$, let $\mathbf{h}^{\sigma,e_j} = [h^{\sigma,e_j}_i]$ be the n dimensional vector defined by

$$h^{\sigma,e_j}_k \;= \begin{cases} 1 & \text{if } a_j \le k \le b_j - 1 \\ 0 & \text{otherwise} \end{cases}$$

That is, $\mathbf{h}^{\sigma,e_j}$ can be considered to be a *representation vector* for the set

$$\{r_k : h^{\sigma,e_j}_k = 1\}$$

which consists of precisely those r_k which are spanned by the edge e_j. We will say that $r_k \in \mathbf{h}^{\sigma,e_j}$ if $h^{\sigma,e_j}_k = 1$.

Let $\mathbf{r} = [r_k]$, and let $\mathbf{1}_n$ denote the n dimensional vector all of whose entries are 1. Then Eqs. (2)–(4) may be rewritten as

$$\mathbf{r}.\mathbf{h}^{\sigma,e_j} \;\le\; x_j \quad 1 \le j \le m \tag{6}$$

$$\mathbf{r}.\mathbf{1}_n \;\le\; x_0 \tag{7}$$

$$\mathbf{r} \;\ge\; 0 \tag{8}$$

Note that $A^{\sigma}_G(\mathbf{x})$ is the set of all $\mathbf{r}$ that satisfy Eqs. (6)–(8). Then, we can write

$$F^{\sigma}_n(\mathbf{x}) \;=\; \int_{\mathbf{r} \in A^{\sigma}_G(\mathbf{x})} d\mu \tag{9}$$

where $d\mu \;=\; dr_0 dr_1 \ldots dr_{n-1}$ is the differential volume element.

We will now decompose the integral in Eq. (9). Suppose that $r_1 = t \ge 0$. Then the feasible set for $\{r_0, r_2, \ldots r_{n-1}\}$ can be obtained from Eqs. (6)–(8) in the following manner: let $\mathbf{r}^1$ denote the restriction of the n dimensional vector to $n-1$ dimensions obtained by dropping the entry corresponding to $k = 1$. For $1 \le j \le m$, obtain $\mathbf{h}^{\sigma,e_j,1}$ from $\mathbf{h}^{\sigma,e_j}$ similarly (by dropping the appropriate entry). The feasible region for $\mathbf{r}^1$ is then

$$\begin{aligned} \mathbf{r}^1.\mathbf{h}^{\sigma,e_j,1} \;&\le\; \begin{cases} x_j - t & \text{if } r_1 \in \mathbf{h}^{\sigma,e_j,1} \\ x_j & \text{otherwise} \end{cases} \\ \mathbf{r}^1.\mathbf{1}_{n-1} \;&\le\; x_0 - t \\ \mathbf{r}^1 \;&\ge\; 0 \end{aligned} \tag{10}$$

We denote the set of feasible $\mathbf{r}^1$ by $A^{\sigma,1}_G(\mathbf{x})$ and define $F^{\sigma}_{n-1}(\mathbf{x}) = \mathsf{Vol}_{n-1}(A^{\sigma,1}_G(\mathbf{x}))$. It follows from Eq. (10) that

$$F^{\sigma}_n(\mathbf{x}) \;=\; \int_{t \ge 0} F^{\sigma}_{n-1}(\mathbf{x} - t\mathbf{z}^{\sigma,1}) dt \tag{11}$$

where $\mathbf{z}^{\sigma,k}$, $k = 0, 1, 2, \ldots (n-1)$ is the $m+1$ dimensional vector whose entries are defined by

$$z^{\sigma,k}_i = \begin{cases} 1 & \text{if } r_k \in \mathbf{h}^{\sigma,e_i} \\ 0 & \text{otherwise} \end{cases}$$

We set $\mathbf{h}^{\sigma,e_0} \triangleq \mathbf{1}_n$. If we take the multi-dimensional Laplace transform of both sides in Eq. (11), we obtain

$$\hat{F}^{\sigma}_n(\mathbf{s}) \;=\; \frac{\hat{F}^{\sigma}_{n-1}(\mathbf{s})}{\mathbf{s}.\mathbf{z}^{\sigma,1}}$$

This process of decomposition may be continued by developing the volume along $r_2, r_3 \ldots r_{n-1}$ in the same manner, and is terminated by the condition

$$\hat{F}^{\sigma}_1(\mathbf{s}) = \frac{1}{s_0^2 \prod_{k=1}^{m} s_k}.$$

because at this terminal condition, we are left with only the constraints given by Eqs. (7) and (8). Because $\hat{F}^{\sigma}_1$ is independent of σ, we will denote it by $T(\mathbf{s})$

$$T(\mathbf{s}) = \frac{1}{s_0^2 \prod_{k=1}^{m} s_k}$$

In other words, we can write

$$\hat{F}_n^{\sigma}(\mathbf{s}) = \frac{T(\mathbf{s})}{\prod_{k=1}^{n-1}(\mathbf{s}.\mathbf{z}^{\sigma,k})}$$

Putting the discussion in this section together, we can state the main result of this paper.

Theorem 1: Let $G = (V, E)$ be a graph on n vertices with m edges. The probability that G appears in $\mathcal{G}_{n,d,W}$ is given by

$$P_G^{(d)}(\mathbf{x}) = \left(P_G^{(1)}(\mathbf{x})\right)^d$$

where

$$P_G^{(1)}(\mathbf{x}) = \frac{Q(\mathbf{x})}{x_0^n}$$

and the Laplace transform of $Q(\mathbf{x})$ is given by

$$\hat{Q}(\mathbf{s}) = \sum_{\sigma\in\mathcal{S}_n} \frac{T(\mathbf{s})}{\prod_{k=1}^{n-1}(\mathbf{s}.\mathbf{z}^{\sigma,k})} \tag{12}$$

This provides a systematic procedure for the calculation of the required probability.

We will illustrate this result with a few examples in the sequel. But before proceeding, we outline one more result which can be extracted from the preceding discussion. Note that $P_G^{(d)}(\mathbf{x})$ is a *probability distribution function* for the condition

$$\| v_i - v_j \|_\infty \leq \ w_{ij} \text{ when } (v_i, v_j) \in E(G)$$

where $w_{ij} = x_k$ where k is the index of the edge (v_i, v_j) in G. The density function $p_G^{(d)}(\mathbf{x})$ may be defined as follows: the probability that, for all edges $e_k = (v_i, v_j)$ $(k = 1, 2, \ldots m)$,

$$x_k - dx_k \leq \| v_i - v_j \|_\infty \leq \ x_k + dx_k$$

is $p_G^{(d)}(\mathbf{x})dx_1 dx_2 \ldots dx_m$. Thus,

$$P_G^{(d)}(\mathbf{x}) = \int_{\mathbf{0}\leq\mathbf{t}\leq\mathbf{x}} p_G^{(d)}(\mathbf{t})d\mu(\mathbf{t})$$

Then, we have

Corollary 1: Let $G = (V, E)$ be a graph on n vertices with m edges. The probability density function for the appearance of G in $\mathcal{G}_{n,d,W}$ is given by

$$\int_\psi p_G^{(d)}(\mathbf{x}) = \left(\int_\psi p_G^{(1)}(\mathbf{x})\right)^d$$

where ψ is any subset of the d-dimensional set over which the graph is obtained and

$$p_G^{(1)}(\mathbf{x}) = \frac{q(\mathbf{x})}{x_0^n}$$

and the Laplace transform of q is given by

$$\hat{q}(\mathbf{s}) = \sum_{\sigma\in\mathcal{S}_n} \frac{1}{s_0} \frac{1}{\prod_{k=1}^{n-1}(\mathbf{s}.\mathbf{z}^{\sigma,k})} \tag{13}$$

It is usually easier to invert the Laplace transform in Eq. (13), when compared to the problem of inverting Eq. (12). Consequently, computing the density function is easier than computing the distribution function. More importantly, the density can be used to obtain other properties. For example, $P_G^{(d)}$ can be calculated when the $w(i, j)$ are specified as ranges rather than as upper bounds.

A. On Using the Multi-Dimensional Fourier Transform Instead of the Laplace Transform

Instead of using Laplace transforms, we could have used multi-dimensional Fourier transforms. Let $\boldsymbol{\omega} = [\omega_k]_0^m$. Then the multi-dimensional Fourier transform of $Q(\mathbf{x})$ can be written as

$$\hat{Q}(\boldsymbol{\omega}) = B(\boldsymbol{\omega}) \sum_{\sigma\in\mathcal{S}_n} \prod_{k=1}^{n-1} \mathcal{D}(\boldsymbol{\omega} E{\cdot}\mathbf{z}^{\sigma,k}) \tag{14}$$

where

$$\mathcal{D}(y) = e^{-jy/2}\frac{\sin(y/2)}{y/2}$$

and

$$B(\boldsymbol{\omega}) = \mathcal{D}(\omega_0)^2 \prod_{k=1}^{m} \mathcal{D}(\omega_k)$$

We have used the property that the nodes are all distributed in $[0, 1]^d$ to derive the Fourier transform. To find $P_G^{(d)}(\mathbf{x})$, we must invert the Fourier transform $\hat{Q}(\boldsymbol{\omega})$ at the point $\mathbf{x}$. This can be accomplished using numerical techniques, for example, by discrete Fourier transform inversion.

B. A Trivial Example

Suppose we distribute two nodes v_1, v_2 on the unit square. Set $w(1, 2) = \alpha < 1$. Let G be the graph with one edge between v_1 and v_2. What is the probability that this edge appears in the graph? This corresponds to $m = 1$, with $x_0 = 1$, $x_1 = \alpha$. Then, it is easy to see that

$$\hat{Q}(s_0, s_1) = \frac{2}{s_0^2 s_1(s_0 + s_1)}$$

The inverse transform can be calculated easily as

$$Q(x_0, x_1) = x_0^2 u(x_0)u(x_1) - (x_0 - x_1)^2 u(x_1)u(x_0 - x_1)$$

and the final probabilities can be expressed as

$$P_G^{(1)}(2, \alpha) = 2\alpha - \alpha^2$$

and

$$P_G^{(2)}(2, \alpha) = (2\alpha - \alpha^2)^2$$

These can be verified to be the correct probabilities [10].

C. A Non-Trivial Example

Distribute three nodes v_1, v_2, v_3 in the unit cube. Let $w(i, j) = \alpha < 1$ for $i \neq j$. What is the probability that the complete graph on three vertices (a triangle) appears in $\mathcal{G}_{3,3,W}$?

As before, we can determine $\hat{Q}(s_0, s_1, s_2, s_3)$ as

$$\begin{aligned}\hat{Q}(s_0, s_1, s_2, s_3) \quad &= \quad (6s_0 + 4s_1 + 4s_2 + 4s_3) \times \left(s_0^2 s_1 s_2 s_3\right. \\ &\qquad \left.(s_0 + s_1 + s_2)(s_0 + s_2 + s_3)(s_0 + s_1 + s_3)\right)^{-1}\end{aligned}$$

Inverting this Laplace transform analytically is a little tedious, but routine. In this example, we could instead have taken the Fourier transform approach and used discrete Fourier analysis to compute the required probability.

IV. Simplifications by Exploiting Symmetries

We can exploit two types of symmetries in order to simplify the evaluation of $P_G^{(d)}(\mathbf{x})$. The first type corresponds to symmetries in the vector $\mathbf{x}$. The second corresponds to symmetries in the graph G. We illustrate this using two examples.

First, suppose that G is an arbitrary graph, but $\mathbf{x}$ is chosen so that $x_i = \alpha$ for $i > 0$. This is an important sub-case of the geometric random graph. In this situation, we will denote $P_G^{(d)}(\mathbf{x})$ by $P_G^{(d)}(\alpha)$ (this corresponds to $x_0 = 1$ and $x_i = \alpha$ for $i > 0$). Then, given a permutation σ, when we evaluate $P_{G|\sigma}^{(1)}$ as described in the previous section, we can make the following observation: if an edge e_i spans another edge e_j for the ordering given by σ, then the edge e_j can be dropped from consideration (the corresponding inequality in Eq. (6) can be omitted). Thus, $P_{G|\sigma}^{(d)}(\mathbf{x})$ can be obtained by inverting a simpler Laplace transform. For instance, consider the example at the end of the previous section (the triangle). In this case, $P_{G|\sigma}^{(d)}(\alpha)$ can be obtained by inverting the Laplace transform

$$\frac{1}{s_0^2 s_1 (s_0 + s_1)^2}$$

at the point $(1, \alpha)$.

To illustrate the second type of symmetry, consider the triangle once again. In this case, each $G|\sigma$ is isomorphic when considered as an unlabeled graph. If in addition, we have the symmetry in $\mathbf{x}$ described in the previous paragraph, then $P_G^{(1)}(\mathbf{x})$ can be obtained by inverting the Laplace transform

$$\frac{6}{s_0^2 s_1 (s_0 + s_1)^2}$$

at the point $(1, \alpha)$. This can be generalized as follows: If G is the complete graph on n vertices, and if the distance matrix W is chosen so that $w(i,j) = \alpha$ for all i, j, then $P_G^{(1)}(\mathbf{x})$ can be obtained by inverting the Laplace transform

$$\frac{n!}{s_0^2 s_1 (s_0 + s_1)^{n-1}}$$

at the point $(1, \alpha)$. This particular transform can be inverted symbolically. Omitting all the details, we present the final result below. The required probability is

$$\begin{aligned} P_G^{(1)}(\alpha) &= [(u_1(x_0)u_0(x_1)) \times (n(n-1)u_{n-2}(x_0)\delta(x_1 - x_0))]_{1,\alpha} \\ &= n\alpha^{n-1} - (n-1)\alpha^n \end{aligned} \tag{15}$$

V. On the Probability that $\mathcal{G}_{n,d,W}$ is Connected

As a direct application of Theorem 1, we obtain the exact probability that $\mathcal{G}_{n,d,W}$ is connected for a special case. Suppose W is defined so that $w(i,j) = \alpha$ for all pairs i, j. We now ask the question "What is the probability that $\mathcal{G}_{n,d,W}$ is connected?" Let us denote this probability by $P_{\mathrm{conn}}^{(d)}(n, \alpha)$. We describe a formula (and hence a procedure) for calculating this probability.

The key observation is the following: $\mathcal{G}_{n,d,W}$ is connected if and only if it contains *some* tree on n vertices as a subgraph. That is,

$$P_{\mathrm{conn}}^{(d)}(n, \alpha) = \mathrm{Prob}(T \subset \mathcal{G}_{n,d,W} \text{ for some tree T}) \tag{16}$$

By the inclusion-exclusion principle, Eq. (16) can be further broken down as follows. Let $\mathcal{T}$ be the set of labeled trees on n vertices (there are n^{n-2} such trees [19]). If $T_{i_1}, T_{i_2} \ldots T_{i_k}$ are trees in $\mathcal{T}$, then we use $T_{i_1,i_2,\ldots i_k}$ to denote $T_{i_1} \cup T_{i_2} \cup \ldots T_{i_k}$.

$$\begin{aligned} P_{\mathrm{conn}}^{(d)}(n, \alpha) &= \sum_{T_i} P_{T_i}^{(d)}(\alpha) \\ &\quad - \sum_{i<j} P_{T_{i,j}}^{(d)}(\alpha) \\ &\quad + \sum_{i<j<k} P_{T_{i,j,k}}^{(d)}(\alpha) \ldots \end{aligned} \tag{17}$$

In this formula, we use $P_G^{(d)}(\alpha)$ to denote the probability that G appears in the random graph under the restriction that $w(i,j) = \alpha$ for all i, j. Each term of the form $P_{T_{i_1,i_2,\ldots i_k}}^{(d)}(\alpha)$ can be computed using the procedure outlined in the earlier section. Clearly, this entire exercise is computationally feasible only for small values of n. For larger values of n, the thresholding behavior of $P_{\mathrm{conn}}^{(d)}(n, \alpha)$ is well understood from the asymptotic analysis and the calculation of the exact probability is not as critical.

The formula in Eq. (17) can serve as the basis of some approximation schemes.

1) To obtain a lower bound on the probability $P_{\mathrm{conn}}^{(d)}(n, \alpha)$, we could use a smaller collection of trees to estimate the probability. Given a collection of k trees, the probability that at least one of these trees appears in the random graph is obtained by 2^k basic computations of $P_G^{(d)}(\cdot)$.
2) To obtain an estimate of $P_{\mathrm{conn}}^{(d)}(n, \alpha)$ using Eq. (17), we note that keeping the first $2k+1$ terms ($k \geq 0$) gives us an upper bound for the probability, while keeping the first $2k$ terms ($k > 0$) gives us a lower bound for the probability. It is likely that the first few terms are good enough to yield a reasonable estimate for the probability.

These approximation schemes need to be explored further, but are beyond the scope of the current work.

A. An Example

We illustrate the formula in this section for $n = 3$. In this case, there are three possible labeled trees T_1, T_2, and T_3 (which are isomorphic when considered as unlabeled graphs), and $T_i \cup T_j$ is the triangle K_3 for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Since T_i are isomorphic, it is easy to see that

$$P_{T_1}^{(d)}(\alpha) = P_{T_2}^{(d)}(\alpha) = P_{T_3}^{(d)}(\alpha)$$

Thus, we have

$$\begin{aligned} P_{\mathrm{conn}}^{(d)}(\alpha) &= 3P_{T_1}^{(d)}(\alpha) - 2P_{K_3}^{(d)}(\alpha) \qquad (18) \\ &= 3P_{T_1}^{(d)}(\alpha) - 2(3\alpha^2 - 2\alpha^3)^d \qquad (19) \end{aligned}$$

We know that

$$P_{T_1}^{(d)}(\alpha) = \left(P_{T_1}^{(1)}(\alpha)\right)^d$$

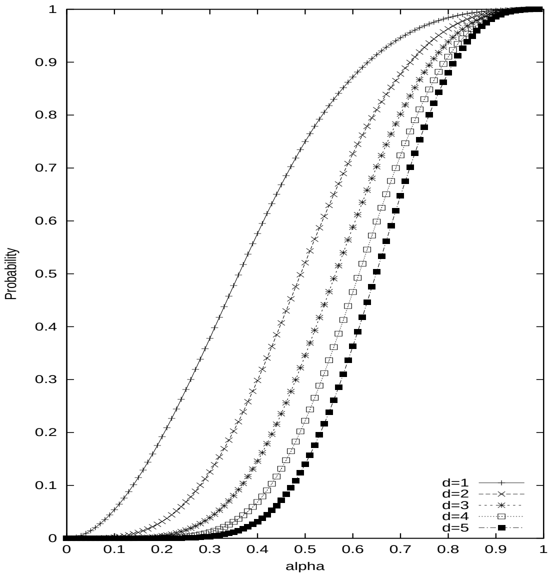

Fig. 1. Probability of connectivity for three nodes, in d dimensions

and we can use the method outlined earlier to derive

$$P_{T_1}^{(1)}(\alpha) = 2\alpha^2 - \frac{4\alpha^3}{3} + \frac{1}{3} - \frac{2(1-\alpha)^3 u_0(1-\alpha)}{3} + \\ + \frac{(1-2\alpha)^3 u_0(1-2\alpha)}{3}$$

and thus, obtain a formula for $P_{\rm conn}^{(d)}(3, \alpha)$. This formula is plotted in Figure 1 for $d = 1, 2, \ldots 5$. Note that if we fix α, then, as d increases, the probability of connectivity decreases.

VI. On the Use of Other Norms

We have derived the exact probability of the appearance of a graph in the random graph under the l_∞ norm in $\mathbf{R}^d$. This can be translated to lower and upper bounds on $P_G^{(d)}(\mathbf{x})$ if a distance norm which is norm-equivalent to the l_∞ norm is used to define $\mathcal{G}_{n,d,W}$. For example, it easy to see that

$$\left(P_G^{(d)}(\frac{\mathbf{x}}{\sqrt{d}})\right)_{l_\infty} \leq \left(P_G^{(d)}(\mathbf{x})\right)_{l_2} \leq \left(P_G^{(d)}(\mathbf{x})\right)_{l_\infty}$$

These bounds are asymptotically optimal with respect to n, the number of vertices in the graph. In other words, these bounds will predict the correct threshold behavior for the probabilities of interest.

VII. Discussion

The geometric random graph $\mathcal{G}_{n,d,W}$ is of interest in the analysis of mobile wireless networks because it allows us to consider nodes with non uniform transmitter and receiver ranges.

Asymptotic analysis of the connection probability $P_{conn}^{(d)}$ is available. However, in some situations it may be necessary to determine the probability that a desirable local configuration involving a small number of transmitters and receivers is obtained. This is especially true if in applications like sensor networks, there are a small number of nodes with special capabilities, e.g., data collection centers for processing the data collected by the sensors or beacon nodes that help in the self organization of the network. Examples for possible G are as follows.

1) If G were a star, the central node could represent a node with special privileges like a satellite uplink and the other nodes are within one hop of this node.
2) There could be a set of nodes with relay capabilities, e.g., those that can form cluster heads. These nodes will be at the top of the routing hierarchy and the graph representing the connectivity among these nodes may need to be fully connected (like in the telephone network) and in this case G would be a fully connected subgraph. Alternately we could just require that G be a connected graph.

As has been mentioned earlier, most previous work make two important assumptions—the node ranges are all equal and the nodes are uniformly distributed in the operational area. In this paper we have considered the case of non uniform transmission ranges. Networks with non uniform distribution of the nodes in the operational area are considered in [20].

Acknowledgment

D. Manjunath's work was supported in part by a grant for the 'WAN QoS Testbed' from the Ministry of Information Technology, Government of India.

References

[1] P. Gupta and P. R. Kumar, "Critical power for asymptotic connectivity in wireless networks," in *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming*, W. M. McEneany, G. Yin, and Q. Zhang, Eds. Birkhauser, Boston, 1998, pp. 547–566.

[2] ——, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 388–404, March 2000.

[3] ——, "Towards an information theory of large networks: An achievable rate region," *IEEE Transactions on Information Theory*, vol. 49, no. 8, pp. 1877–1894, 2003.

[4] L. L. Xie and P. R. Kumar, "A network information theory for wireless communication: Scaling laws and optimal operation," *IEEE Transactions on Information Theory*, vol. 50, no. 5, pp. 748–767, May 2004.

[5] P. Santi and D. M.Blough, "The critical transmitting range for connectivity in sparse wireless ad hoc networks," *IEEE Transactions on Mobile Computing*, vol. 2, no. 1, pp. 25–39, March 2003.

[6] C. Bettstetter, H. Hartenstein, and X. Perez-Costa, "Stochastic properties of the random waypoint mobility model," *ACM/Kluwer Wireless Networks, Special Issue on Modeling and Analysis of Mobile Networks*, vol. 10, no. 5, pp. 555–567, September 2004.

[7] C. Bettstetter and C. Hartmann, "Connectivity of wireless multihop networks in a shadow fading environment," *ACM/Kluwer Wireless Networks, Special Issue on Modeling and Analysis of Mobile Networks*, vol. 11, no. 4, July 2005.

[8] M. J. B. Appel and R. Russo, "The minimum vertex degree of a graph on uniform points in $[0,1]^d$," *Advances in Applied Probability*, vol. 29, no. 3, pp. 582–594, 1997.

[9] M. D. Penrose, "On k-connectivity for a geometric random graph," *Random Structures and Algorithms*, vol. 15, no. 2, pp. 145–164, 1999.

[10] M. Desai and D. Manjunath, "On the connectivity in finite ad-hoc networks," *IEEE Communication Letters*, vol. 6, no. 10, pp. 237–239, 2002.

[11] M. D. Penrose, "The longest edge of the random minimal spanning tree," *Annals of Applied Probability*, vol. 7, pp. 340–361, 1997.

[12] J. Diaz, M. D. Penrose, M. Serna, and J. Petit, "Approximating layout problems on random graphs," *Journal of Algorithms*, vol. 39, pp. 78–117, 2001.

[13] P. Jaillet, “On properties of geometric random graphs in the plane,” *Annals of Operations Research*, vol. 1995, pp. 1–20, 61.

[14] J. Diaz, J. Petit, and M. Serna, “Random geometric graph in $[0,1]^2$,” in *Randomization and Approximation Techniques in Computer Science, Proceedings of the Second International Workshop, RANDOM'98, Barcelona, Spain*, M. Luby, J. D. P. Rolim, and M. J. Serna, Eds. Lecture Notes in Computer Science 1518, Springer-Verlag (1998), 1998.

[15] M. D. Penrose, *Random Geometric Graphs*. Oxford University Press, 2003.

[16] E. Godehardt and J. Jaworski, “On the connectivity of a random interval graph,” *Random Structures and Algorithms*, vol. 9, pp. 137–161, 1996.

[17] M. V. Moorthy, “Numerical inversion of two-dimensional laplace transforms—fourier series representation,” *Applied Numerical Mathematics*, vol. 17, pp. 119–127, 1995.

[18] W. T. Weeks, “Numerical inversion of laplace transforms using laguerre functions,” *Journal of the ACM*, vol. 13, pp. 419–426, 1966.

[19] D. B. West, *Graph Theory*. Prentice-Hall India, New Delhi, 1998.

[20] S. K. Iyer, D. Manjunath, and B. Gupta, “On the topological properties of the one dimensional exponential random geometric graph,” Submitted, 2005.