On the Transmission Range in Dense Ad Hoc Radio Networks

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Abstract— **In this paper we explore the relationship between the transmission range of nodes in an ad hoc network and the connectivity properties of the network. We consider nodes distributed randomly in $[0,1]^2$ and find the transmission range that would be required of the node for it to be not isolated with very high probability. We then relate this range to the range required to keep the network connected. We also consider a network in which the nodes are not distributed randomly (uniformly) in the operational area. We also report the results of some simulation experiments that have been carried out to further explore the implication of the theoretical results.**

Keywords— **Ad Hoc Networks, Node Isolation, Connectivity, Transmission Range.**

I. Introduction

In radio networks, the transmission range of a node depends on the transmission power. For efficient communication the transmitted power levels of all the nodes should be low so as to conserve power and also to enable a high degree of spatial reuse. However it cannot be too low as that could cause the node to be isolated – the node is not reachable by any other node in the network, and/or the network could become disconnected with multiple components. We consider ad hoc radio networks in which the nodes are distributed in $[0,1]^2$, the area of operation of the network. Let P_i be the probability that a node is isolated, i.e., there is no other node in its transmission range, P_I, the probability that the network contains one or more isolated nodes, and, P_c the probability that the network is connected. For a given area of operation, the probability that the network is connected, P_c, is an increasing function of the number of nodes n in the network and r, the transmission range of the nodes. Now suppose that the transmission range is $r(n)$, a function of n. We would actually like $r(n)$ to be as small as possible and still have no isolated nodes and also have a connected network. In this paper we first obtain bounds on P_i using simple geometric arguments when the number of nodes is fixed and distributed randomly (the x and the y coordinates have a uniform distribution in $[0,1]$). For this case we obtain the bounds on P_i and then the threshold function for P_I as a function of $r(n)$. We then consider the case when the number of nodes in the operational area has a Poisson distribution with mean λ and the nodes are uniformly distributed in $[0,1]^2$ and then obtain the threshold function for P_I as a function of $r(\lambda)$. We also consider the case of a network with n nodes but with one or both the coordinates of the node location being chosen with a nonuniform distribution. The case of both coordinates being nonuniform is investigated using simulation models.

II. Preliminaries

An ad hoc radio network is best modeled as a random graph. Typically, two random graph models have been considered in literature. In the G_p model, for a $n-$node graph an edge is placed between two nodes independently with probability $p(n)$. In the G_m, on an $n-$node graph, m edges are added and each of the $\binom{N}{m}$ are selected with equal probability. Here $N = \binom{n}{2}$. It is obvious that neither of these two are well suited to model ad hoc radio networks because in a packet radio network, a node has a transmission range r i.e., nodes at a distance less than r from the transmitting node can receive the transmission and those at distances greater than r cannot receive the packets. Therefore, as has been discussed by Ramanathan and Lloyd in [5], the following random graph construction is more realistic for radio networks. Node i is placed at (x_i, y_i) where x_i and y_i are chosen according to some distribution, typically iid and uniform. An edge is placed between nodes u and v if the Euclidean distance between them is less than r. This model can be extended to higher dimensions and also use any l_p, $p = 1, 2, \cdots$, norm rather than the l_2 norm that we use in our model of the ad hoc radio network. The random graph constructed as above is called a geometric random graph (GRG).

For the GRG, connectedness is a monotone property (see [2], [7] for definition) in r. For a monotone property, a threshold function is defined as follows. Consider a graph $\mathcal{G}$ and a monotone property Q in $\mathcal{G}$ on a variable α, which may be a function of n, say $\alpha(n)$. A threshold function for the monotone property Q is a function $t(n)$ such that $\frac{\alpha(n)}{t(n)} \to 0$ implies that almost no $\mathcal{G}$ satisfies Q, and $\frac{\alpha(n)}{t(n)} \to \infty$ implies that almost every $\mathcal{G}$ satisfies Q. Threshold functions are not unique. We could have threshold functions of varying degrees of sharpness.

For the G_p random graph model, $\frac{log_e(n)}{n}$ is a threshold probability function for the disappearance of isolated vertices. (See for example [2], [7].) Further, it is also known that the threshold function for the connectedness of the G_p random graph is the same as that for the disappearance of isolated vertices. Although a similar result is available for G_m random graphs, such a result was not known for GRG until recently.

In [1] Appel and Russo show that in a d-dimensional GRG with l_∞ norm, as $n \to \infty$, $(\frac{1}{2d}\frac{log_e(n)}{n})^{\frac{1}{d}}$ is the threshold value of $r(n)$ when isolated nodes disappear from the graph. In [6], Penrose obtains the generalised result that on a GRG with any l_p metric if one starts with isolated points and keeps adding edges in order of increasing length, then with a very high probability the resulting graph becomes $k+1$ connected at the same $r(n)$ at which the minimum degree of the graph becomes $k+1$ connected for $k \geq 0$. With $k=0$, this result means that the graph becomes connected at the same instant at which isolated vertices disappear from the graph – a result identical to that of the G_p and G_m random graphs. From the above two results we can say that $(\frac{1}{2d}\frac{log(n)}{n})^d$ is the threshold function of $r(n)$ for both connectivity and disappearance of isolated vertices in GRG with l_∞ norm.

III. Transmission Range Thresholds

We first consider an $n-$node ad hoc radio network located in $[0,1]^2$. The nodes are assumed to be uniformly distributed in the square. The transmission range of each node is $r(n)$. This network is modeled as a GRG with l_2 norm and we will denote the corresponding graph by $G^2_{l_2}$. For this network the following holds.

Theorem 1: In $G^2_{l_2}$, the threshold range function $r(n)$ for a vertex being isolated is $c\sqrt{\log_e n/n}$.

Proof: Consider a node x in the network that was randomly placed in $[0,1]$. Let A_x be the area in the square around this point covered by the transmission range of this node. For this node to be isolated, no other node of the network must lie in A_x. Then the probability that the given node is isolated, P_i, is

$$P_i = (1-A_x)^{n-1} \sim (1-A_x)^n$$

Following the method in [7] for probability threshold function for the disappearance of isolated vertices in G_p random graphs, we can write $(1-A_x)^n = e^{n\log_e(1-A_x)} = e^{-nA_x}e^{-nA_x^2(1/2+A_x/3+\cdots)}$ which is simplified if we make $nA_x^2 \to 0$ for large n, i.e., choose $A_x \in o(1/\sqrt{n})$, we will have $P_i \sim e^{-nA_x}$. Let us choose $A_x = \frac{c\log_e(n)}{n}$, where c is a constant. Then

$$P_i \sim n^{-c}$$

which tends to 0 if $c>0$. Observe that $A_x = \pi r^2(n)$ except near the edges and in any case $A_x \geq \pi r^2(n)/4$ with equality being satisfied only when the node is located in a corner. Therefore, choosing

$$\frac{\pi r^2(n)}{4} = \frac{c\ \log_e n}{n} \quad \text{or} \quad r(n) = \sqrt{\frac{c\ \log_e(n)}{\pi\ n}}$$

gives us the transmission range threshold function $r(n) = c\sqrt{\log_e n/n}$ for the probability of a node being isolated. $\square$

Thus $r(n) = \sqrt{\frac{c\ \log_e(n)}{\pi\ n}}$ is the threshold function for the probability of a node being isolated in $G^2_{l_2}$. Now, we argue that the $r(n)$ for which P_i is as asymptotically non zero is the same $r(n)$ for which P_I is asymptotically non zero. Likewise, the $r(n)$ for which $P_i \to 0$ is the same for which $P_I \to 0$. Hence the transmission threshold function $r(n)$ for which the minimum degree of the graph will be one will be the same as that at which the probability of a node being isolated will be zero. This leads us to the ask what would be the transmission range threshold for the network to become connected, i.e., there is exactly one component in the network. The answer to this is given in two parts. First we state the theorem of Penrose [6] which is as follows.

Theorem 2: Given any l_p metric with $1<p<\infty$, a GRG in a $d-$dimensions in $[0,1]^d$, and any integer $k>0$, then the $r(n)$ at which the graph becomes $k+1$ connected is the same at which the minimum degree is $k+1$.

Our interest in the context of ad hoc radio networks and their transmission range is in the l_2 metric and a two dimensional network. Thus, the connectivity threshold is given by the following theorem.

Theorem 3: For $G^2_{l_2}$, the threshold function for connectedness is $c\sqrt{\log_e n/n}$.

Proof: This follows from Theorems 1 and 2 $\square$

We mention here that Gupta and Kumar [4], obtain the the lower and upper bounds for the probability disconnectivity for a radio network on a unit circle in $\mathcal{R}^2$. They show that when n nodes are placed in a disc of unit area in $\mathcal{R}^2$ with a transmission range $r(n)$ for each of the nodes, if $\pi r^2(n) = \frac{\log n + c(n)}{n}$, and $\lim_{n\to\infty} c(n) = c$, then

$$\lim_{n\to\infty} \inf(1-P_c) \geq e^{-c}(1-e^{-c}) \qquad \text{and}$$
$$\lim_{n\to\infty} \sup(1-P_c) \leq 4e^{-c}$$

What this means is that as $n\to\infty$, $e^{-c}(1-e^{-c}) \leq (1-P_c) \leq 4e^{-c}$. Thus, as $n \to \infty$, if $\pi r(n)^2 = \frac{\log n + c(n)}{n}$ the radio network is connected with probability 1 if and only if $c(n) \to \infty$. Thus the threshold function derived by Gupta and Kumar is sharper than the one derived here, we believe that ours is a simpler derivation.

We now consider the case of a random number of nodes distributed randomly in the operational area. Specifically, we assume that the number of active nodes in the operational area has a Poisson distribution with mean λ. This is easily motivated as follows. Let n be very large and at any time each of these nodes is on with probability θ, with θ such that $n\theta = \lambda$. Our interest is in the threshold function for $r(\lambda)$.

Theorem 4: For $G^2_{l_2}$, with the number of nodes in the graph having a Poisson distribution with mean λ, the transmission range threshold function for a node not being isolated is $c\sqrt{\log_e \lambda/\lambda}$.

Proof: The probability that n nodes are active, $P(n) = \frac{e^{-\lambda}\lambda^n}{n!}$. The probability that a given node is isolated is given by

$$\begin{aligned} P_i &= \sum_n P(n)(1-A_x)^{n-1} \\ &= \sum_n \frac{e^{-\lambda}\lambda^n}{n!}(1-A_x)^{n-1} \end{aligned}$$

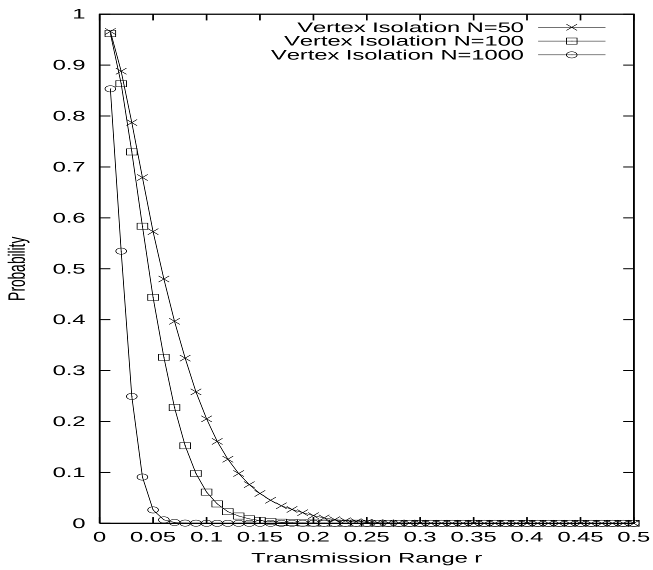

Fig. 1. Probability of a node being isolated *vs* r

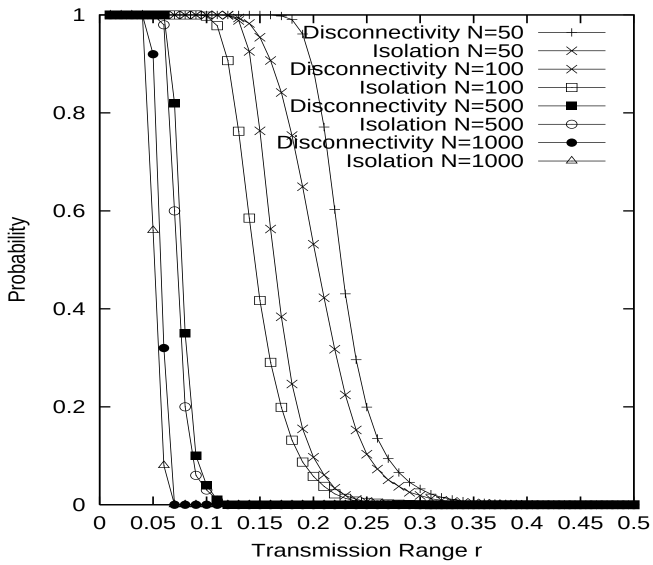

Fig. 2. Probability of the network being disconnected *vs* r

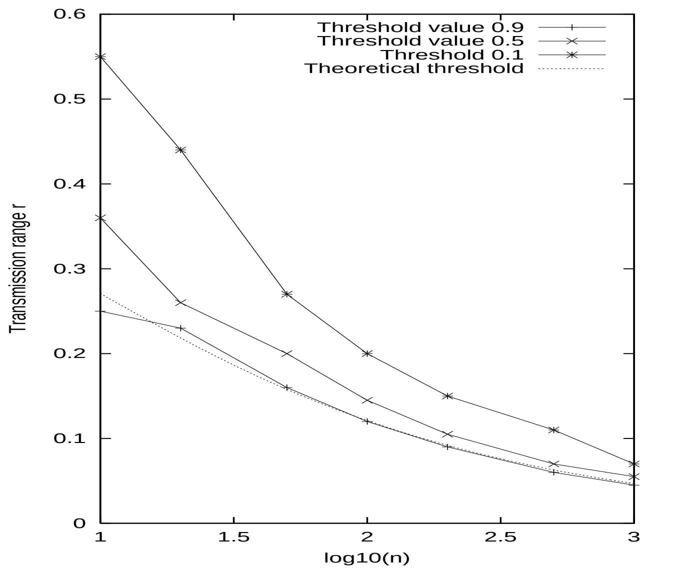

Fig. 3. Threshold values of r *vs* n

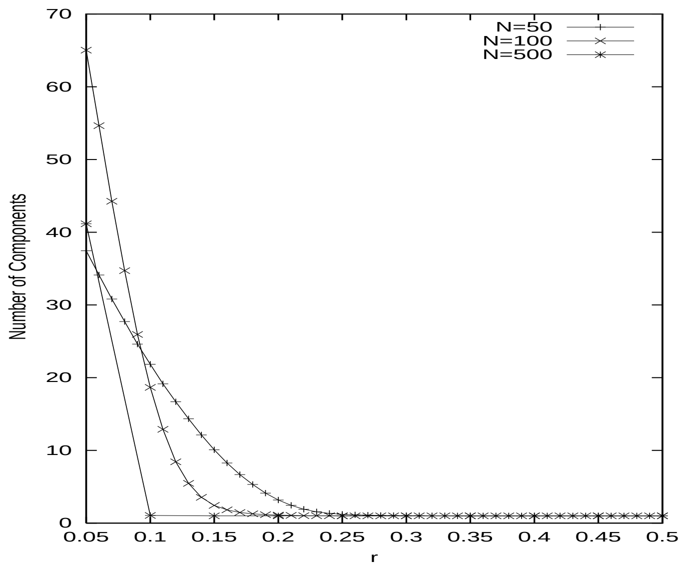

Fig. 4. Number of components in the network *vs* r

$$= \frac{e^{-\lambda A_x}}{1 - A_x}$$

As before, we choose $r(\lambda) = c \log_e n/n$ and the rest of the proof will be identical to that of Theorem 1. □

IV. Simulation Results

We now present results from simulation models to study the connectedness and related properties of ad hoc radio networks. In addition to considering the case of random distribution in the operational area, we also investigate non uniform distribution of the nodes in the operational area.

Figure 1 shows the probability of a node being isolated in an $n-$node, randomly distributed network as a function of r, the transmission range. Observe that as n becomes larger the probability of a node being isolated has a sharper transition from nearly 1 to 0. This suggests a probable thresholding behaviour for this probability as a function of the transmission range. This threshold function is characterised in Theorem 1. In Figure 2 we explore the relationship between the behaviour of the probability of a network having an isolated node and the probability that the network is disconnected. We plot these two probabilities for different sized networks as a function of r. Observe that these two probabilities exhibit the thresholding behaviour alluded to above. In addition, as expected by Theorems 2 and 3, the r for which the two probabilities become zero are identical. In Figure 3 we plot the threshold function $\sqrt{\log_e n/(\pi n)}$ *vs* n and also the r at which the probabilities of the $n-$node network being disconnected is 0.1, 0.5 and 0.9. Observe the goodness of the threshold function.

In percolation models and in random graph models, it is of interest to know the threshold at which a large (or an in-

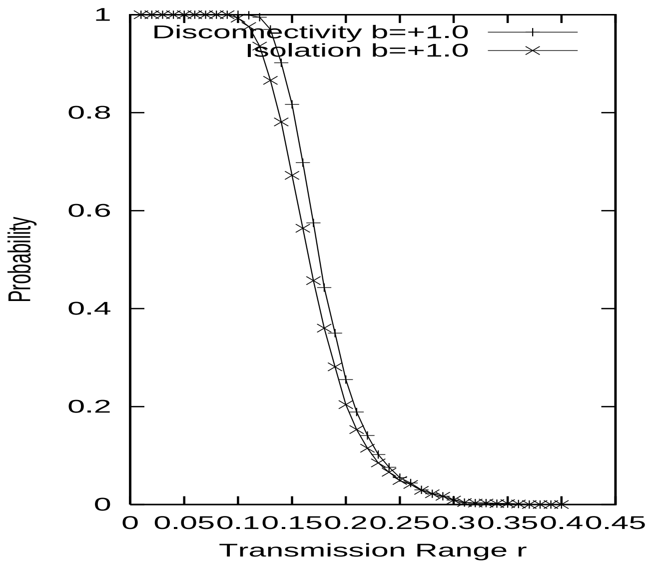

Fig. 5. Probability of disconnectivity and isolation *vs* r. Both the coordinates are non-uniformly distributed with $\beta = +1$ and $n = 100$.

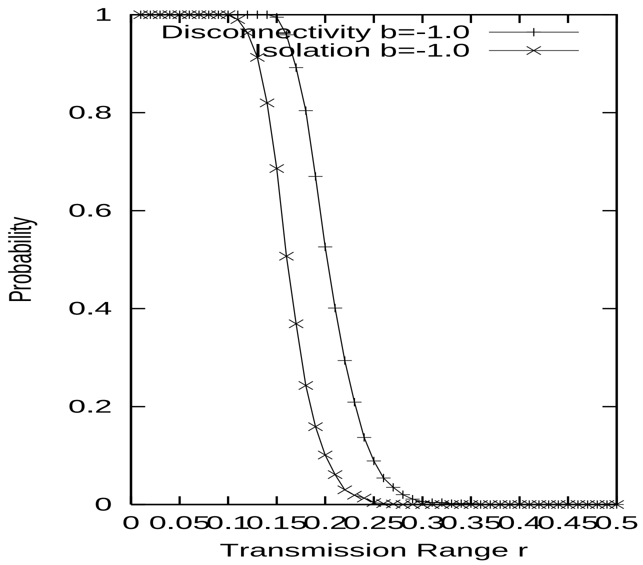

Fig. 6. Probability of disconnectivity and isolation *vs* r. Both the coordinates are non-uniformly distributed with $\beta = -1.0$ and $n = 100$.

finite) cluster or a giant component appears in the network [3]. To investigate similar behaviour of the ad hoc radio network, we investigate the mean number of components in the network as a function of r for different values of n. This is shown in Figure 4. Observe that the r at which the number of components in the graph drops suddenly is the same as that at which the probability that the network becomes disconnected drops to zero.

V. Non Uniform Distribution of Nodes

In this section, we explore the effect on connectivity of non uniform distribution of nodes in the operational area. Specifically, consider that the x and y coordinates have the following density function with $-1 \leq \beta \leq +1$. Here β represents the deviation from the uniform with negative β biasing the node locations towards the periphery and a positive β biasing it towards the centre of the operational area.

$$f_X(x) = \begin{cases} 4\beta x + 1 - \beta & \text{for } 0 \leq x \leq 0.5 \\ -4\beta x + 1 + 3\beta & \text{for } 0.5 \leq x \leq 1.0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In Figures 5 and 6 we plot the probability of the network being disconnected and the probability of the network having an isolated node in a 100 node network for $\beta = +1$ and $\beta = -1$ respectively. For both these cases, we assume that both the coordinates are chosen independently and with the same distribution. The case $\beta = +1$ corresponds to the all the nodes being clustered around the center of the operational area. The case of $\beta = -1$ corresponds to the nodes being distributed in clusters near the edges of the operational area.

From these figures we see that for $\beta = +1$, the probability of isolation and the probability of disconnectivity are almost equal and they reach zero at the same value of r i.e., the probability of isolation is a good lower bound on the probability of disconnectivity. However, for $\beta = -1$, the difference between the probability of isolation and the disconnectivity increases and the r at which the probability of finding an isolated node in the network reaches zero is different from the r at which the probability of the graph being disconnected becomes zero. This is because as the nodes are biased away from the centre of the unit square, nodes don't remain isolated but form clusters near the edges of the operational are and these clusters are disconnected from one another. Hence though the probability of isolation is small, the probability of disconnectivity is high. So the probability of isolation will not be a good lower bound on the probability of disconnectivity.

The analytical modeling of the network for the nonuniform distribution of the nodes is considerably complex. In the Appendix, we obtain the lower bound on the probability of a node being isolated.

VI. Discussion and Conclusion

In this paper we have reported on the transmission ranges required for a node to not become isolated in an ad hoc network and also to keep the network connected. Using results from geometric random graphs and some simple geometric arguments, we were able to obtain the range threshold functions to keep the network connected. We also constructed simulation models to further explore the theoretical results.

It would be useful to be able to similarly analyse networks with a random number of nodes, like our model of a Poisson distribution for the number of active nodes, and also the case of nonuniform distribution of nodes in the operational area.

Appendix

We analyse a network with a nonuniform distribution of nodes by considering the case where only the distribution along one axis is nonuniform. To make the analysis tractable, we consider the case where the $x-$coordinate is chosen according to the density function $f_X(x) = 6(x-x^2)$ and the $y-$coordinate is uniform in $[0,1]$. This is just a smoother version of the function in Eqn. 1 with $\beta = +1$. In the following we derive the probability of finding an isolated node in the network.

We obtain a bound on the probability of this node being isolated by dividing the region in which this node could lie into the following four kinds of regions

1. A_1: $0 \leq x_1, y_1 \leq r$ and $x_1 \leq \sqrt{(}r^2 - y_1^2)$
2. A_2: $0 \leq x_1, y_1 \leq r$ and $x_1 \geq \sqrt{(}r^2 - y_1^2)$
3. A_3: $0 \leq x_1 \leq r$ and $r \leq y_1 \leq (1-r)$
4. A_4: $r \leq x_1, y_1 \leq (1-r)$

We obtain the conditional probability that a node placed in that region is isolated. We then uncondition by finding the probability of the being located in that region.

Let us consider a node located at point (x_1, y_1). If (x_1, y_1) lies in the region $0 \leq x_1,\ y_1 \leq r$ and $x_1 \leq \sqrt{(}r^2 - y_1^2)$, then the area around it in the unit square in which no other node of the network must lie for it to be isolated is given by

$$A_1 = \int_0^{x_1+\sqrt{(r^2-y_1^2)}} \int_0^{y_1+\sqrt{(r^2-(x-x_1)^2)}} 6(x-x^2)dydx + \\ + \int_{x_1+\sqrt{(r^2-y_1^2)}}^{x_1+r} \int_{y_1-\sqrt{(r^2-(x-x_1)^2)}}^{y_1+\sqrt{(r^2-(x-x_1)^2)}} 6(x-x^2)dydx$$

If (x_1, y_1) lies in the region $0 \leq x_1, y_1 \leq r$ and $x_1 \geq \sqrt{(}r^2 - y_1^2)$, then the area around it in which no other node should exist is given by

$$A_2 = \int_0^{x_1-\sqrt{(r^2-y_1^2)}} \int_{y_1-\sqrt{(r^2-(x-x_1)^2)}}^{y_1+\sqrt{(r^2-(x-x_1)^2)}} 6(x-x^2)dydx \\ + \int_{x_1-\sqrt{(r^2-y_1^2)}}^{x_1-\sqrt{(r^2-y_1^2)}} \int_0^{y_1+\sqrt{(r^2-(x-x_1)^2)}} 6(x-x^2)dydx \\ + \int_{x_1+\sqrt{(r^2-y_1^2)}}^{x_1+r} \int_{y_1-\sqrt{(r^2-(x-x_1)^2)}}^{y_1+\sqrt{(r^2-(x-x_1)^2)}} 6(x-x^2)dydx$$

If $0 \leq x_1 \leq r$ and $r \leq y_1 \leq (1-r)$ then,

$$A_3 = \int_0^{x_1} \int_{y_1-\sqrt{(r^2-(x-x_1)^2)}}^{y_1+\sqrt{(r^2-(x-x_1)^2)}} 6(x-x^2)dydx$$

If (x_1, y_1) lies in the region $r \leq x_1, y_1 \leq (1-r)$ then,

$$A_4 = \int_{x_1-r}^{x_1+r} \int_{y_1-\sqrt{r^2-(x-x_1)^2}}^{y_1+\sqrt{r^2-(x-x_1)^2}} 6(x-x^2)dydx$$

Hence the probability of isolation of points when they are located in the specific areas mentioned above can be found out to be

$$P_1 \geq \int_0^r \int_0^{\sqrt{(r^2-y_1^2)}} (1-(n-1)A_1 dx \quad dy$$

$$P_1 \geq \frac{\pi r^2}{4} - (n-1)(0.2166r^8 + 0.0046r^7 - 5.6068r^6 - 5.2788r^5)$$

$$P_2 \geq \int_0^r \int_{\sqrt{(r^2-y_1^2)}}^r (1-(n-1)A_2 dx \quad dy$$

$$P_2 \geq r^2(1-\frac{\pi}{4}) - (n-1)(-100.47526r^6 + 7.67222r^5 + 1.612r^4)$$

$$P_3 \geq \int_0^r \int_r^{1-r} (1-(n-1)A_3 dx \quad dy$$

$$P_3 \geq r(1-2r) - (n-1)(1.96r^6 + 4.489r^5 + 7.334r^4 - 1.9r^3)$$

$$P_4 \geq \int_r^{1-r} \int_r^{1-r} (1-(n-1)A_4 dx \quad dy$$

$$P_4 \geq (1-2r)^2 - (n-1)(37.69911r^5 - 18.84956r^4 - \\ 6.28319r^3 + 3.14159r^2)$$

Hence the probability of finding an isolated node in the graph is

$$P_i \geq 4(P_1 + P_2 + P_3) + P_4$$

$$P_i \geq 1 - (n-1)(3.1415r^2 - 13.8831r^3 + 16.9344r^4 + \\ + 65.2319r^5 - 416.4880r^6 + 0.01852r^7 + 0.8662r^8)$$

Thus we have derived a lower bound on the probability of isolation when nodes are distributed with a specific density function.

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