Applied Linear Algebra EE635

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1 Example to illustrate calculation of a decomposition using *column operation*

Identity matrix or column vectors e_1, e_2, e_3, e_4, e_5

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{A} \overset{\mathbf{A}}{\begin{bmatrix} \boxed{1}_1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & -2 & 2 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -2 & -2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -2 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1}_4 & 1 \\ 0 & -2 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1}_4 & 0 \\ 0 & -2 & 0 & 2 & 0 \end{bmatrix}$$

Basis for Ker A

Basis for Im A

dim(ker A) + dim(Im A) = 1+4 = 5 = dim R^5 as expected.
But this may *not* be a DIRECT DECOMPOSITION. We can check for independence of Ker A and Im A together OR proceed to calculate Im A^2 and Ker A^2.

Prob. 2

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{A^2} \begin{bmatrix} \boxed{1}_1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 0 & 3 \\ 3 & -5 & -2 & 6 \end{bmatrix}$$

If any of these were zero, the corresponding vector on the left side will give an element for Ker A^2

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -3 & 0 & 3 \\ 3 & -8 & -2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 1 \\ 1 & -3 & 3 & 3 \\ 3 & -8 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1}_1 & 0 & 0 & 0 \\ 0 & \boxed{1}_2 & 0 & 0 \\ 0 & 0 & \boxed{1}_3 & 0 \\ 1 & -3 & 3 & 0 \\ 3 & -8 & 6 & 0 \end{bmatrix}$$

An *additional* basis vector for Ker A^2 (in addition to the vector for Ker A)

Basis for Im A^2

Again, dim(KerA^2) + dim(ImA^2) = 5 so we can check Ker A^2 and Im A^2 for *independence* OR calculate Im A^3 and Ker A^3

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{A^3} \begin{bmatrix} \boxed{1}_1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -3 & 4 \\ 4 & -11 & 9 \\ 9 & -23 & 16 \end{bmatrix}$$

Prob. 3

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1}_1 & 1 & 0 \\ 0 & \boxed{1}_2 & 1 \\ 1 & -4 & 4 \\ 4 & -15 & 9 \\ 9 & -32 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \underbrace{\begin{bmatrix} \boxed{1}_1 & 0 & 0 \\ 0 & \boxed{1}_2 & 0 \\ 1 & -4 & \boxed{8}_3 \\ 4 & -15 & 24 \\ 9 & -32 & 48 \end{bmatrix}}_{\text{Im } A^3}$$

These are independent
dim A^3 = dim A^2

So STOP.

We have a decomposition:

$$\text{Ker } A^2 \oplus \text{Im } A^2$$

with corresponding basis vectors

$$\underbrace{\begin{bmatrix} 1 & -4 \\ -1 & 3 \\ 1 & -2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{Basis for Ker } A^2} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{bmatrix}}_{\text{Basis for Im } A^2}$$

(above basis for Im A^3 will also do)