Agashe's Algorithm for determination of a vecctor whose mp is the same as the mp of the whole vector space

(Comapre with GANTMACHER : Theory of matrices, Vol-I)

November 2014

CONTEXT

V is a finite-dimensional vector space over any field F, $L : V \to V$ a linear function

ALGORITHM

STEP 1

Choose any *non-zero* vector in V, say, v_1 and determine its minimal polynomial (mp), $p_1(s) = s^k + a_{k-1}s^{k-1} + ... + a_1 s + a_0$, say, so that the set of vectors

$$\{v_1, Lv_1, L^2v_1, ..., L^{k-1}v_1\}$$

is an independent set, and

$$L^k v_1 = -a_{k-1}L^{k-1}v_1 - ... - a_1 Lv_1 - a_0 v_1$$

Let $css(v_1)$ denote the span of the independent set above; it is the "cyclic" subspace generated by v_1 under the action of L

STEP 2

If $css(v_1) = V$, equivalently, $k = dim(V)$, **Stop**

We have found a vector v_1 whose mp p_1 is the mp of the whole space (***Prove!***)

If $k < dimV$, equivalently, $css(v_1) \subset V$, go to **Step 3**.

STEP 3

Option (a)

Determine $Ker(p(L))$. If $Ker(p(L)) = V$, stop. v_1 is the required vector.

If $Ker(p(L)) \subset V$, go to **Step 4**.

(This involves more work "right now", but may involve less work "later".)

Option (b)

Go to **Step 4** directly

STEP 4

Determine a vector $v_2^{'}$, which is not in $css(v_1)$, or not in $Ker(p(L))$. (This can be done by choosing any basis for V and choosing a suitable vector from this basis independent of the independent set in Step 1, or a basis of $Ker(p(L))$. (This involves more work!).

"Append" the vectors $v_2^{'}, Lv_2^{'}, L^2v_2^{'}, ...$ sequentially to the independent set in Step 1, checking for independence at every step. Thus, first consider $\{v_1, Lv_1, L^2v_1, ..., L^{k-1}v_1, v_2^{'}\}$. Is it independent? Yes, because $v_2^{'}$ was **CHOSEN** to meet this requirement. Next, consider $\{v_1, Lv_1, L^2v_1, ..., L^{k-1}v_1, v_2^{'}, Lv_2^{'}\}$. Is it independent? If not, $Lv_2^{'}$ is a linear combination of $\{v_1, Lv_1, L^2v_1, ..., L^{k-1}v_1, v_2^{'}\}$. If independent, calculate $L^2v_2^{'}$, and check if for independence with respect to $\{v_1, Lv_1, L^2v_1, ..., L^{k-1}v_1, v_2^{'}, Lv_2^{'}\}$.

After a finite number of steps, you will obtain for a least positive integer $l \geq 1$:

$$L^l v_2^{'} = linear\ combination\ of \{v_1, Lv_1, L^2v_1, ..., L^{k-1}v_1, v_2^{'}, Lv_2^{'}, ..., L^{l-1}v_2^{'}\}$$

This calculation is an example of working with the new vector $v_2^{'}$ MODULO THE SUBSPACE $css(v_1)$, i.e., checking for independence with resprect to $css(v_1)$.

You will thus have obtained two polynomials, $\overline{p_2}$ and $\overline{q_2}$, say, such that

$$\overline{p_2}(L)v_2^{'} = \overline{q_2}(L)v_1$$

$\overline{p_2}$ is of degree l, $\overline{q_2}$ is of degree less than k, and $\overline{p_2}$ is the *unique(monic)* polynomial of *least degree* satisfying the above equation. It may be called the **RELATIVE** mp of $v_2^{'}$ MODULO or 'WITH RESPECT TO' the $css(v_1)$.

CASE 1

If $\overline{q_2}$ is the zero polynomial, $\overline{p_2}$ is the mp of $v_2^{'}$. In that case, take $p_2 = \overline{p_2}$ and $v_2 = v_2^{'}$ for the next step 5, Case 1.

CASE 2

If degree of $\overline{p_2}$ is less than or equal to the degree of $\overline{q_2}$, by polynomial division, obtain two polynomials q_2 and q, with $\deg(q_2) < \deg(\overline{p_2})$, such that

$$\overline{q_2} = q.\overline{p_2} + q_2$$

,and calculate v_2 as

$$v_2 = v_2^{'} - q(L)v_1$$

(Check, if you wish, that $v_2 \neq \mathbf{0}_V, \deg(q_2) < \deg(\overline{p_2})$, $\overline{p_2}(L)v_2 = q_2(L)v_1$).

Proceed to Step 5 with this v_2.

If $\deg(\overline{p_2}) > \deg(\overline{q_2})$, then take $v_2^{'}$ itself as the v_2 for the next step.

STEP 5

CASE 1

The mp of $v_2 (= v_2^{'})$ is $p_2 = \overline{p_2}$, as if v_1 and $css(v_1)$ did not exist. (This could be called the ASBOLUTE mp of v_2, in contrast with the RELATIVE mp of $\overline{p_2}$ MODULO $css(v_1)$, in Step 4, case 2).

Calculate its cyclic subspace, i.e., $css(v_2)$.

It can be shown that $css(v_2)$ is 'DISJOINT' from $css(v_1)$, i.e.,

$$css(v_2) \cap css(v_1) = \{\mathbf{0}_V\}.$$

Further the vector $(v_1 + v_2)$ has the mp $LCM(p_1, p_2)$, which is also the mp of the sum

$$css(v_2) \oplus css(v_1).$$

This sum has a dimension greater than that of $css(v_1)$. Thus, we have obtained a "BIGGER" subspace, and also a vector whose mp is the mp of this bigger subspace. Note that $css(v_1 + v_2)$ **MAY NOT** equal $css(v_2) \oplus css(v_1)$, but mp of $css(v_2) \oplus css(v_1)$ is $LCM(p_1, p_2)$ which is also the mp of $(v_1 + v_2)$.

CASE 2(follows step 4, case 2)

Calculate the ABSOLUTE mp of p_2 of v_2, as if v_1 and $css(v_1)$ did not exist. It can be shown that, the degree of p_2 is greater than the degree of $p_1, (\deg(p_2) > \deg(p_1))$, and so $css(v_2)$ has a dimension greater than that of the dimension of $css(v_1)$.

If $css(v_2) = V$, **Stop**.

If not, determine a vector $v_3^{'}$, which is not in $css(v_2)$, and proceed as in Step 4, with $v_3^{'}$ in place of $v_2^{'}$, and check for independence MODULO $css(v_2)$.

(As a check on your calculations, p_2 should turn out to be a multiple of $\overline{p_2}$, and p_1 should turn out to be the same multiple of q_2.)

STEP 6

If $css(v_1) \oplus css(v_2) = V$, **STOP**.

If not, determine a vector $v_3^{'}$ which is not in $css(v_1) \oplus css(v_2)$, and proceed as in Step 4 with this $v_3^{'}$ in place of $v_2^{'}$, and checking for independence MODULO $css(v_1) \oplus css(v_2)$, instead of MODULO $css(v_1)$.

Calculate polynomials $\overline{p_3}, \overline{q_{31}}, \overline{q_{32}}$ such that

$$\overline{p_3}(L)v_3^{'} = \overline{q_{31}}(L)v_1 + \overline{q_{32}}(L)v_2,$$

and determine v_3 from $v_3^{'}$ in a way similar to the way in which you obtained $v_2^{'}$ from v_2, and the mp p_3 of v_3.

Calculate $css(v_3)$. It will be disjoint from $css(v_1) \oplus css(v_2)$. Further,

$$LCM(LCM(p_1, p_2), p_3)$$

will be mp of $(v_1 + v_2 + v_3)$ as also the mp of the still BIGGER subspace

$$css(v_1) \oplus css(v_2) \oplus css(v_3).$$

Do YOU SEE that after a finite number of steps, you will have obtained a vector whose mp is the same as the mp of the whole space V ?

Problems

Try the algorithm as a numerical example.
$V = \mathbb{R}^4_{col}$, L is the action of the matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{pmatrix}$$

Problem

1. Start with $v_1 = e_1^4 = (1\ 0\ 0\ 0)^T$
2. Start with $v_1 = (1\ -1\ 1\ -1)^T$
3. Start with $v_1 = (2\ -3\ 5\ -9)^T$

GOOD LUCK!

REMARK

If one is willing to do more wotk at the beginning, at Step 1, instead of choosing v_1 arbitratily and perhaps, choosing it as one of the 'unit' vectors, one can choose a basis for V, calculate the mp of each of the basis vectors. One can then choose for V, a basis vector whose mp has the highest degree.

If willing to do some work, you can check the mp's for co-primeness, or the corresponding cyclic subspaces for disjointness, and use the following theorems.

THEOREM 1

If w_1, w_2 have mp's p_1, p_2, and p_1, p_2 are co-prime, $(w_2 + w_2)$ has mp $(p_1 p_2)$; $css(w_1)$ and $css(w_2)$ are disjoint, and $(p_1 p_2)$ annihilate $css(w_1) \oplus css(w_2)$.

THEOREM 2

If w_1, w_2 are such that $css(w_1)$ and $css(w_2)$ are disjoint, then $(w_1 + w_2)$ has for mp the LCM of the mp's of w_1 and w_2, and the subspace annihilated by this mp contains $css(w_1) \oplus css(w_2)$.

Flow Chart for the Algorithm