Lorentz Transformation Equations in Galilean Form

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Abstract

Using the definition of "position" given in an earlier paper, we show that the Lorentz transformation equations for position can be put in a particularly simple form which could be said to be "Galilean". We emphasize that two different reference frames use their individual definition of position and distance. This fact gets obscured in the usual "rectangular Cartesian co-ordinate system" approach.

Keywords: Lorentz transformation, inner product, Galilean form.

1 WAS EINSTEIN'S KINEMATICS INCOMPLETE?

Einstein, in his pioneering paper[1], insisted that when talking about motion, we must give a *physical* meaning to "time". He showed how this can be done by introducing his idea of "synchronized clocks". However, he took for granted the concept of "position". He wrote:

> Let us take a system of co-ordinates in which the equations of Newtonian mechanics hold good. ...If a material point is at rest relatively to this system of co-ordinates, its position can be defined relatively thereto by the employment of rigid standards of measurement and the methods of Euclidean geometry, and can be expressed in Cartesian co-ordinates.

Thus, he did not describe how position could be defined in a general, physically meaningful way. Nobody subsequently has done so. A little further in[1], he said:

> Let us in "stationary" space take two systems of co-ordinates, i.e.,two systems, each of three rigid material lines, perpendicular to one another and issuing from a point.

Are the co-ordinate axes, then, material bodies? Do we need to think of what might happen to them when they move? Or are they merely conceptual? In a sense, then, Einstein's Kinematics was incomplete.

2 EINSTEIN'S KINEMATICS COMPLETED

In a recent paper[2], we suggested how Einstein's Kinematics could be completed. We proposed that an "observation system" could have the following ingredients. An observer, S, say, capable of sending and receiving light signals, is equipped with a *single* clock and *three* passive "reflecting" stations, say, S_1, S_2, S_3. Using a radar-like approach, the observer could obtain data sets as follows. He sends a signal in all directions at a time t_0 in his clock, and then records the times of arrivals of the echoes of this signal by reflection in the following four different ways: time t_0' of arrival after reflection at the place P, say, of an event being observed (path SPS); time t_1 of arrival after reflection at the event P first, followed by a reflection at the station $S1$ back to S (path SPS_1S); similarly, time instants t_2,t_3. Thus, he would obtain 5-tuples of data items of time, t_0, t_0', t_1, t_2, t_3. From each 5-tuple, he is to decide *by definition* what could be meaningfully called the "time of occurrence" and "place" of the event. Following Einstein, we chose $t = \frac{t_0+t_0'}{2}$ as the *definition* of the time of occurrence.

There remained the problem of deciding what could meaningfully be called the place of the event. Here, we proposed to go beyond the classical 3-dimensional rectangular Cartesian co-ordinate system idea which was only conceptual; we chose instead to think of the place of an event as an element of a 3-dimensional vector space to be equipped with a suitable scalar or inner product. The place of an event could thus be thought of as a "position vector". Any 3-dimensional vector space, V, say, would do. (Today, we know the possible advantages of such "abstract" *representation.*) Since the reflecting stations deserved "places" of their own, it was natural to represent them by vectors s_1, s_2, s_3, say, forming a basis of V. Again, any basis would do. Of course, S itself would be assigned the zero vector. Now, the reflecting stations could not be allowed to be totally arbitrary; they had to remain at fixed "distances" from S and from one another. But what are distances? These had to be physically determinable. S has only a clock- no measuring rods. As in radar, one could define "distance" in terms of *time interval* through a parameter called the velocity of light, c. By doing some further signalling, involving various reflections, S could obtain a set of 6 time intervals that would correspond to 6 transition times, $SS_1, SS_2, SS_3, S_1S_2, S_2S_3, S_3S_1$. These multiplied by c would be taken as the lengths or "norms" of the 6 vectors $s_1, s_2, s_3, s_1 - s_2, s_2 - s_3, s_3 - s_1$. These 6 numbers would then uniquely determine the scalar or inner product on V, thus making it into an inner product space. Note that although V and a basis for it were arbitrarily chosen, the scalar product was determined by the observation system itself. The problem of obtaining from the 5-tuple of an event a representing vector p, say, in V was then a problem of linear algebra (see[2] for details, with a slightly different notation). There was a "technical" hitch, however. Not any set of 6 numbers would do; this was explored in an "addendum"[3].

The stage was set to admit another observation system, an observer S', say, with a clock and reflecting stations S'_1, S'_2, S'_3. This observer could choose a vector space, V', say, not necessarily the same as V of S, basis vectors s'_1, s'_2, s'_3 to represent its stations, and using the same "velocity of light" constant c, determine a scalar product on it, and finally obtain the representing vector p', say, and time t', say, of the same event for which S had obtained p as position vector and t as time. To relate p, t with p', t', one assumed, as usual, that the system S' was in uniform motion relative to system S with velocity v. This motion would be observed by S, and thus, v would be a vector in V. S would observe the motions of S', S'_1, S'_2, S'_3 to be given by the vectors $d_0 + tv, d_0 + tv + d_1, d_0 + tv + d_2, d_0 + tv + d_3$, say, d_0, d_1, d_2, d_3 being all vectors in V. To go further, one needed some relation between the clocks of S and S' in the following sense. Suppose that as S' moves, the clocks of S and S' *at* S' show values t and t', respectively. We need some relation between these two "times". We assume, with Einstein, linearity of this relation: $t' = \beta_1 t$, where β_1 is some constant. This is the only assumption of linearity that we make. We then prove(see[2]) that the following linear relations hold between the times and places of the events in the two systems:

$$t' = \beta_1 \left[t - \frac{(p - d_0 - tv, v)_S}{c^2 - v^2} \right] \tag{1}$$

where the symbol $(u, w)_S$ denotes the scalar product of the vectors u and w in V, and

$$p' = T(p - (d_0 + tv)) \tag{2}$$

where T is a linear transformation on V onto V' such that it maps each vector d_i of V to the vector s'_i of V', i.e., the vectors in V representing the relative positions of the stations of S' are mapped to the vectors in V' representing the stations of S'. This completes a summary of our derivation of the Lorentz transformation in[2]. We call it "Einstein's Lorentz transformation" because we have followed an Einsteinian approach - except with respect to the meaning of "position".

3 LORENTZ TRANSFORMATION IN GALILEAN FORM

Although we allowed the possibility that the representation vector spaces V and V' could be different, they could be chosen to be the same. Further, the vectors d_i representing the relative positions in V of the stations of S' could be chosen to be the basis vectors for S'. Thus, we could choose $s'_i = d_i$. Of course, the scalar products could be different, as they are dictated by the observational data. The transformation T then becomes the identity transformation, and for the vectors representing the place of the event in S and S', we obtain the following simple *Galilean* relation:

$$p' = p - (d_0 + tv) \tag{3}$$

which can be seen as the Galilean position vector of P relative to S'. It must be emphasized, however, that there are still two representations because there are possibly two different scalar products for S and S', and these scalar products relate to two different calculations of distances in the two systems. Both are *Euclidean* in the sense they are both based on a scalar product. The normal "Euclidean" co-ordinate systems use the distance $\sqrt{x^2 + y^2 + z^2}$, which is related to a special scalar product.

Let us consider in this context the Einstein form of the Lorentz equations:

$$x' = \beta(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \beta(t - \frac{vx}{c^2}). \tag{4}$$

One could argue that one could put them in the Galilean form by using new variables $\bar{x}, \bar{y}, \bar{z}$:

$$\bar{x} = x - vt, \qquad \bar{y} = y, \qquad \bar{z} = z \tag{5}$$

and explicitly defining a new distance for S' given in terms of norm by

$$||(p, q, r)||_{S'}{}^2 = \beta^2 p^2 + q^2 + r^2, \tag{6}$$

choosing the constant β_1 equal to $1/\beta$, but this would have looked like a mathematical "trick". In contrast, here we have envisaged the possibility of a different scalar product and distance in the very notion of representation by a vector.

4 REMARKS

The Galilean form for the position vector has the advantage that the Einsteinian factor β does not appear in it, making manipulations easier. Also, length contraction and time dilatation disappear. However, β does remain in a slightly different appearance in the equation relating the times. The additional factor β_1 could be chosen as unity. The main point of this paper is, however, that the concept of "position" has to be defined and that there is a choice in representing position. The present paper could be read as a postscript to the earlier papers[2] and[3].

It would be interesting to apply the coordinate-free vector representation of position to Maxwell's equations and to see how the Galilean form of the Lorentz transformation works out.

REFERENCES

[1] Einstein, A.: On the Electrodynamics of Moving Bodies, pp. 37-65, in H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The principle of relativity: a collection of original memoirs on the special and general theory of relativity, with notes by A. Sommerfeld, translated by Perrett, W. and Jeffery, G. B.*, (Methuen, London, 1923); reprinted (Dover, New York, 1952).

[2] Agashe, S. D.: Einstein's "Zur Electrodynamik..." (1905) revisited, with some consequences. Found. Phys. 36, 955-1011 (2006).

[3] Agashe, S. D.: Addendum to "Einstein's "Zur Elektrodynamik ..." (1905) revisited, with some consequences" by S. D. Agashe. Found. Phys. 37, 306-309 (2007).