The axiomatic method: its origin and purpose [1]

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Euclidean geometry and the axiomatic method

Euclid's *Elements* constitutes the earliest extant substantial presentation of a body of material in the axiomatico-deductive $form^1$. Through it the subject of geometry got permanently associated with axiomatico-deductive formulation which was then viewed as a method, so much so that the expression 'more geometrico' (the geometric way) became synonymous with axiomatico-deductive formulation. Thus arose the general belief, especially in methodological quarters, that Euclid's *Elements* and, in particular, Euclid's geometry were merely instances of the application of a previously thought out/discovered/known method, and, thus, that the axiomatico-deductive method existed prior to the axiomatico-deductive formulation of $geometry^2$.

Using Euclid's *Elements* as my principal $evidence^3$, I want to suggest that the true state of affairs is the other way round. The axiomatico-deductive formulation of geometry emerged out of a successful attempt-most probably by some of Euclid's predecessors-to solve some geometrical problems. Once this was done, it was seen by these geometers and also, of course, by Euclid as an instrument of open-ended discovery. Only, then, could the germs of a method be seen in it.

My view of the genesis of the axiomatic method emboldens me to suggest further that in general a method, which is something consciously conceived, arises as the result of reflection on an activity that is already being pursued 'intuitively'. Again, once the method is consciously conceived, it can engender new activity being pursued consciously in accordance with the method, i.e. methodically.

The geometrical problems and their solutions

If the axiomatic method arose as a result of reflection on some geometrical activity being pursued 'intuitively', what could this activity have been? I suggest that this activity was initiated by a problem which, although it is not explicitly posed in the *Elements*, can be solved on the basis of another problem which is explicitly posed and solved in Book II, Proposition 14, of the *Elements*: 'To construct a square equal to a given rectilinear figure.' This problem could well be called the problem of 'squaring a rectilineal figure' by analogy with the name of a well-known problem of Greek geometry: 'squaring the circle.' (Euclid was not able to solve this latter problem, and therefore, perhaps, does not mention it at all in the *Elements*). Let

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us note that Book II ends with Proposition 14; I might say that our teaching and learning of geometry-and of the axiomatic method-ought to begin with this proposition which actually enunciates a problem.

But why is this problem of 'squaring a rectilineal figure' important? The comparison of two straight-line segments to find out whether they are equally long or not, and, if not, to find out which one of the two segments is shorter and which the longer is, practically speaking, a simple matter, if one is allowed to use a string or a *rope*[4]. Euclid solved this problem theoretically, allowing himself the use only of a straight-edge (to draw a straight line joining two given points) and of a pair of compasses (to draw a circle with a given centre and a given segment, of which that centre is an extremity, as a radius of that circle, i.e. without using a pair of compasses as a pair of dividers). In fact, this is reflected in his Postulates 1 and 3 of Book 1. Euclid's solution of this problem of the comparison of two straight-line segments is given as Proposition 3 of Book I: 'Given two unequal straight lines, to cut off from the greater a straight line equal to the less.'

The corresponding problem for plane rectilineal figures is far from easy, even practically speaking. We may, where possible, move one of the two given rectilineal figures and try to place it on the other to see whether the two fit together perfectly, or whether one of them can be fitted entirely within the other. (Common Notions 8 and 9 of Book I reflect this approach. Common Notion 8: 'And things which coincide with one another are equal to one another.' Common Notion 9: 'And the whole is greater than the part.') But very often neither of these two things will happen, even if the figures have some definite and simple shape such as that of a rectangle. However, should both the figures be squares, superposition will always yield a solution; in fact, we need not even superpose the squares: we need only compare their sides. Note that this happy situation is based on the observation that any two right angles fit, and this requirement is what perhaps led the geometers to define a right angle the way Euclid does (Definition 10, Book I: 'When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right'), and led Euclid to put down his Postulate 4, Book I: 'And that all right angles are equal to one another.'

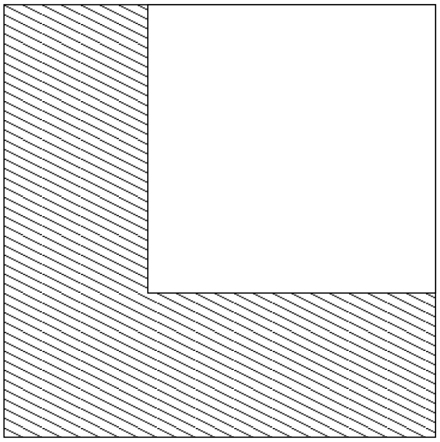
Another important observation would have to be made before one could proceed further with the problem. A given figure can be cut up or decomposed into parts and these parts put together differently to obtain a different-looking figure. (This can be easily seen by cutting up a square into two equal parts and putting these together to obtain a rectangle.) Now, two such figures are not equal (in the sense of Common Notion 8), but there is something special about them, namely, that their 'corresponding' parts are equal in the sense of congruence. At this point, the ancient geometers must have realized that no further progress on the problem of comparison of figures was possible unless one was willing to regard two figures, which were equal in parts, to be equal'. This is, of course, a weakening or widening of the notion of equality of figures, and appears as Common Notion 2 in Book I: 'And if equals are added to equals the wholes are equals.' (The original Greek wording of this Common Notion does not suggest the notion of addition in a numerical sense; rather, it suggests 'putting together'-*prostethe*.) This broadening of the original notion of equality as congruence allows one literally to transform a given figure, i.e., change its form or shape, while retaining its 'size', i.e., while keeping the new figure equal to the original figure. The problem of comparison of two figures could now be 'reduced' to the problem of transformation of one figure into another through the techniques of 'dividing' and 'putting together'. But the fact that squares can be compared with ease would have suggested the following alternative. Suppose, instead of trying to convert one of the given figures into the other, one tries to convert both the figures into squares; and, suppose, it turns out that the converted squares are equal. Could we, then, assert that the two original figures were equal? The astute Greek geometers saw that this was not justified unless the notion of equality was weakened further; thus, we have Common Notion 1 of Book I: 'Things equal to the same thing are also equal to one *another*.'[5]

Having agreed to the broadening of the notion of equality (of figures) through the Common Notions 1 and 2, the problem of comparison of two figures is 'reduced' to the problem of squaring of a figure. Naturally, Euclid takes up the simpler case of a rectilineal figure, and, thus, arrives at the statement of his basic problem in Books I and II , Proposition II. 14; 'To construct a square equal to a given rectilineal figure.'

How does Euclid solve the problem? Or, rather, how did Euclid, or some predecessor, arrive at the solution we find given in the *Elements*? Certainly not by starting, off with the definitions, postulates and common notions, and brilliantly deducing one theorem after another (there are forty-eight propositions in Book I and fourteen in Book II). The problem was solved by reducing it, in turn, to one or more problems. This approach to problem solving was discussed much later by Pappus under the name of 'the Method of Analysis and Synthesis', but we find allusions to it already in Plato. The 'analysis' part involves the formulation of auxiliary or subsidiary problems in what later appears as a 'back tracking' when the solution is finally described in the 'synthesis' part.

Although a triangle would be the simplest rectilineal figure, for obvious reasons Euclid prefers to tackle the rectangle first. So the problem of squaring a rectilineal figure is broken down into two sub-problems: (a) the problem of squaring a rectangle (this construction is given in II.14) and (b) the problem of 'rectangulating' any rectilineal figure (this construction is given in I.45).

Euclid solves (a) essentially by transforming a rectangle into a gnomon (which is an *L*-shaped figure left when a smaller square is taken out of a bigger square; see shaded area in the figure).



A gnomon is clearly a difference of two squares, and we thus have the new problem of constructing a square equal to the difference of two squares. This problem can be solved perhaps if we succeed in solving the problem of constructing a square equal to the 'sum' of two squares; this is precisely what the famous Pythagorean proposition amounts to, and it is Proposition I.47, the last but one proposition in Book I, the last (48th) proposition being the converse of the Pythagorean proposition. Of course, Pythagoras' Theorem in the special case of the isosceles right-angled triangle was known to many civilizations before Euclid, and perhaps even before Pythagoras, and its 'truth' could be visually ascertained. It must have been natural to conjecture that the theorem was true for any arbitrary right-angled triangle, but this already presupposes a broadened notion of equality of figures. Indeed, Euclid makes use of this broadened notion in his proof of Pythagoras' Theorem by dividing the square on the hypotenuse into two rectangles and showing the 'equality' of these rectangles with the squares on the corresponding sides. Now, getting convinced about the 'correctness' of the Pythagorean construction for the sum of two squares required further backtracking and ultimately must have led to the inverted or backward construction of Book I, or something similar to it, perhaps by some predecessors of Euclid. This involves, in particular, getting convinced that the diagonal of a parallelogram splits it into two equal triangles, and that under certain conditions two triangles are equal. (Incidentally, Common Notion 3 is 'demanded' or postulated in claiming that the gnomon is 'equal' to an appropriate square.)

In his solution of problem (b), i.e., converting a rectilineal figure into a rectangle (in fact, Euclid gives a stronger construction I.45: 'to construct in a given rectilineal angle a parallelogram equal to a given rectilineal figure', and to effect that the construction I.44: 'to a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle'), Euclid uses the obvious fact that a rectilineal figure can be easily decomposed into triangles, so that one is led next to the problem solved in I.44.

To summarize, I wish to suggest that investigations into the problem of comparison of two rectilineal figures led the Greeks before Euclid to the realization that some 'concessions' had to be made with regard to the notion of equality, which led to the formulation and investigation of some subsidiary problems, leading finally to a number of postulates, common notions and definitions. Having done this, they then reversed the whole process of thinking, making it appear to posterity that, almost by a miracle, from the small 'acorns' of a few innocent-looking definitions and postulates mighty ‘oaks’ such as Pythagoras’ Theorem and II.14 could be grown. I have indicated this with reference to Books I and II, but the same could be said about the other geometrical books.

It should be noted, however, that the other non-geometrical books of Euclid’s *Elements*, namely, those on natural numbers and general magnitudes do not invoke any postulates explicitly but are based only on definitions. So they could well have been the result of an application in the forward direction of the axiomatic method discovered by investigations in the reverse direction into some geometrical problems. Of course, geometers after Euclid-and even Euclid himself-did carry out further geometrical investigations in the forward direction, proving many interesting new theorems. Eventually, Lobachevskii, and Bolyai followed, non-Euclidean lines of exploration. This last step, after some initial resistance, later turned into reluctance, and a considerable delay of about fifty years led to our modern conception of the axiomatic method as the method of mathematics, involving notions of ’definition’, ‘axiom’ and ‘proof’.

The purposes of the axiomatic method

Having discussed the possible genesis of the axiomatic method in rather great detail, I would like to turn to the several purposes or uses, to which it has been put subsequently.

The Mathematical Use

As mentioned just above, the axiomatic method was put to use in mathematics no sooner than it was discovered, and thus it was recognized to be a powerful instrument of open-ended discovery or derivation. This had several consequences. Firstly, the process of ’derivation’ or ’deduction’ came under close scrutiny giving rise to the subject of logic, and I would venture the guess that Aristotle’s investigations into logic were stimulated more by mathematics, particularly geometry, than by rhetoric or sophistic discourse. Eventually, this led to the feeling that, logic was an engine of deduction which required only the turning of a handle to churn out new propositions from old. Now, deduction done by mathematicians-at least the human ones-are not so mechanical as that, but it is possible to automate the process of deduction, and this is, indeed, what has been done recently by ’theorem-proving programs’.

The second, and rather unfortunate, consequence was that the postulates and common notions, with the exception of Euclid’s ’parallel postulate’, were regarded as being ’true’ in some sense and so irreplaceable. Logic was then seen as an engine to derive new, ’less obvious’ truths from old, ‘more obvious’, ‘self-evident’ truths. I doubt if the Greek geometers themselves regarded their postulates and common notions as ‘self-evident’ or ‘true’. Three of the five postulates are, not about propositions, that is, about any state of affairs in this world or in some other world. Rather, they are assumptions about what can be done in an ideal world. Of the other two postulates, equality of all right angles could have had some empiricism about it, but was finally assumed in order for some constructions to work. Finally, the 'parallel postulate' was necessitated by the somewhat empirical fact that parallel straight lines cut by a transversal produced equal angles, but this, too, was necessitated by the conception of a square, say, as having all angles equal and right (Definition 22). (Euclid's I.46 shows how to construct a square: 'On a given straight line to describe a square'.) The common notions were all required in order to surmount the problem of equality and comparability of (rectilineal) figures.

Of course, there was a happy side to the view that the postulates and common notions were self-evident. Thanks to the non-self-evident nature of the 'parallel postulate', it eventually emboldened geometers to abandon it, to replace it by something equally non-self-evident and then, working the engine of deduction, squeeze out some startling and "almost false" consequences. But this development, in its turn, had the effect that henceforth axioms (to use a single word for postulates and common notions) were deemed to be completely arbitrary and unprovable assertions, and, in an extreme view, even meaningless and having no relation with truth or reality whatsoever. This was accompanied by the view, that definitions also were completely arbitrary, and one merely defined some terms (the 'defined' terms) 'in terms of' some other terms (the 'undefined' or undefinable? terms). Now clearly, for Euclid, definitions were far from arbitrary, though he stretched himself too far, trying to define almost every geometrical term. But it must be noted that nowhere did he or any of his predecessors, say that terms like 'part', 'breadthless length', 'extremity', etc. were undefined in the modern mathematical sense of being devoid of any connotations. They were undefinable in a relative sense; they were simply left undefined in Euclid's formulation. There was nothing either undefined (meaningless) or undefinable about them.

However, towards the end of the nineteenth century there did arise a widespread view of mathematics that it consists of setting out some 'undefined terms' and some 'unproved propositions' at the 'beginning'; and then, after giving some definitions of defined terms as and when one fancies, of proving or deriving some other assertions on the basis of or from the unproved assertions using sheer logic or rules of inference. The American mathematician Benjamin Peirce said, 'Mathematics is the science which draws necessary conclusions'; and Russell confessed (with tongue-in-cheek humour) [3]: 'Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.' (One realizes, of course, that mathematics is a creative or imaginative activity, and not a routine, mechanical activity, because necessary conclusions do not 'follow' easily or automatically from the unproved assertions; rather, they have to be conjectured and then ‘drawn out’ by hard work.) This open-ended view of the axiomatic method in mathematics leads one to believe that one is free to start with arbitrary undefined terms and arbitrary unproved assertions, and then to make arbitrary definitions in order to draw the conclusions, too, somewhat arbitrarily, i.e., as and when they occur to the mathematician, so that the whole thing is a stupendous exercise in arbitrariness! Of course, Russell himself realized that this was not so, for he said (about twenty years after his earlier quip) [4]:

> Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers, from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analysing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterises mathematical philosophy as opposed to ordinary mathematics. But it should be understood that the distinction is one, not in the subject matter, but in the state of mind of the investigator. ... The distinction between mathematics and mathematical philosophy is one which depends upon the interest inspiring the research, and upon the stage which the research has reached; not upon the propositions with which the research is concerned.

I might add that many great mathematicians of the last hundred years or so have contributed a lot to ’mathematical philosophy’ in Russell’s sense, because they have contributed to the process of axiomatization of mathematics in the original Euclidean sense. Further, it must be added that usually one stipulates one or more of the following requirements for an ’arbitrary’ set of axioms, namely, that they must be ’consistent’, ’independent’, ’complete’, ’categorical’.

The Cartesian Purpose

The use to which Descartes sought to put the axiomatic method was the establishment of indubitable truths. A proposition about whose truth we are ’doubtful’ (such as ’I exist’) is sought to be established on the basis of some intuitively clear or indubitable propositions (such as ’I think’). Thus, the axiomatic method is an instrument for dispelling doubt and for creating certainty. Of course, the process of finding out whether a seemingly doubtful proposition can, indeed, be indubitably established is one of back-tracking, quite similar to the back-tracking in mathematics, where a conjectured theorem is sought to be proved. But the difference is that, in mathematics we do not bother about the 'truth' of the axioms, whereas in the Cartesian approach the 'first principles' have to be indubitable and thus true.

Organization of Knowledge

Another use that has been found for the axiomatic method is that of 'organizing a body of knowledge' or 'systematizing a discipline'. Here, it is supposed that we already have a set of truths somehow obtained, but these truths are perhaps too many or seemingly unrelated to each other. We then try to create some system or order by trying to discover whether a small subset of them can serve as a set of axioms from which all the rest can be derived. One may, of course, question the utility of such an enterprise. The whole exercise of organization is to start with the knowledge base that is already there. This base would include terms whose meanings we already know and assertions whose truth we are already confident of. But, if this is so, why bother to define the already known terms in terms of 'undefined' terms, and to derive the already trustworthy assertions in terms of some selected assertions? Perhaps one is trying to apply Ockham's razor here, i.e., one is trying to obtain simplicity. But simplicity in the form of a small number of axioms is won at the cost of complexity of derivations of the other truths from the axioms.

Discovering Unknown Causes or Hypotheses

In this application of the axiomatic method, one starts with a known body of truths with terms whose meanings are known. One then tries to discover a set of undefined and unknown terms, a set of definitions of the known terms in terms of the undefined and unknown terms; and, finally, a set of assertions whose truth is unknown in such a way that the known truths, when reformulated using the definitions in terms of the undefined terms, can all be derived from the axioms. This is, of course, the game of (scientific) theory construction. What is the point of such a game? Well, after the axiomatization, using the axiomatic method in the forward direction as an instrument of discovery, one may stumble across new consequences of the axioms, which, when reformulated using the known terms, give us propositions whose truth can then be ascertained. Their truth is not guaranteed, because the axioms are not necessarily (known to be) true. But the task of ascertaining the truth of new propositions can produce new truths which, otherwise, we may not have bothered to look for. The axioms could be called causes, hypotheses or principles of the body of knowledge or the science that one is dealing with. Success in this approach at the initial stages depends upon the size of the body of knowledge one starts with; usually, it does not pay to be too ambitious, but one may gradually enlarge the body of knowledge and simultaneously modify the undefined terms, definitions and axioms, that is, the theory.

I may, finally add that perhaps one should not be too much preoccupied with 'truths'. Taking the cue from the initial axiomatization of geometry, one should perhaps be equally concerned with problems, and should try to discover an axiomatization in the course of the attempt to find acceptable solutions.

Notes

1. Although the name 'Euclid' is almost synonymous with the word 'geometry', it should be noted that Euclid's *Elements* deals not only with geometry but also with (the natural) numbers, certain incommensurable geometrical magnitudes (and thus indirectly with a special class of irrational numbers), and a theory of general magnitudes. The *Elements* is divided into thirteen Books. Books I to IV, VI, and X to XIII deal with geometrical topics. Books VII to IX are concerned with natural numbers. Book V-a very interesting one but, unfortunately, rather overlooked by physicists and philosophers of science-contains a theory of general magnitudes, which is in many respects similar to algebra and lays the foundation of a theory of measurement. Each Book contains a number of propositions, which are either assertions (or theorems, in modern terminology), or problems. The theorems (for example, in Book I, Proposition 5: 'In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further the angles under the base will be equal to one another') are followed by a demonstration of the correctness of the assertion (proof), ending in the proverbial 'Q.E.D.' (in the Latin version). The problems (for example, in Book I, Proposition 1: 'On a given finite straight line to construct an equilateral triangle') are followed by a construction and a demonstration that the construction, indeed, solves the problem, ending with the less familiar 'Q.E.F.'. Some Books (I to VII, X and XI) have some definitions stated at the beginning. Only Book I has some postulates and common notions following the definitions. (In today's terminology, these can be called 'specific axioms' and 'general axioms' respectively.)

2. Although both Euclid's name and the subject of geometry have become synonymous with the axiomatic method, unfortunately we do not find any elaboration of this method which says something about the genesis, evolution or purpose of the method, either in Euclid's *Elements* or in any extant work by his predecessors (such as Plato and Aristotle, among others). There is, for example, no preface to the *Elements*. Plato, of course, alludes frequently to the 'method of the geometers', and Aristotle has written in detail on the 'demonstrative sciences'.

3. My main source is the second revised edition of *The Thirteen Books of Euclid's Elements'* (3 vols.) translated from the text of Heiberg with introduction and commentary by Sir Thomas L. Heath and published by Cambridge University Press in 1925. The book was reprinted by Dover Publications, Inc., in 1956. The contents of the *Elements* have been put together in the appendix in Ian Mueller's *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* published by M.I.T. Press in 1981.

4. The comparison of two line segments to find out which one is the longer and which the shorter is perhaps the earliest example of the idea of the comparison of two objects with respect to a given quality to detect which one of the two has 'more' and which one 'less' of the quality. I have argued in another paper being presented at this workshop ('The Genesis and Purpose of Quantification and Measurement') that this idea of comparison with respect to a quality is more primitive than the precursor of the notion of, quantity. The Greeks, and in particular Plato talked repeatedly of the notion of 'the more' and the 'less', or 'the greater' and 'the lesser'.

5. Thus far, I have 'accounted' for three of the five Euclidean Postulates and four of the five Euclidean Common Notions in Book I. (Mueller lists one more Postulate and four more Common Notions, but these are not regarded as genuinely Euclidean and so are enclosed within square brackets.) This leaves only one more Common Notion (Common Notion 3): 'And if equals are subtracted from equals the remainders are equal' and two more Postulates; Book I, Postulate 2 is: 'To produce a finite straight line continuously in a straight line' and Book I, Postulate 5 is the so-called 'Parallel Postulate': 'If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.' Postulate 2 is obviously required in most constructions where a point is to be obtained by the intersection of two straight lines or of a straight line and a circle. As regards Postulate 5, Euclid 'postpones' the use of this postulate as far as possible; it, is involved for the first time in proving Proposition 29: 'A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.' In fact, this Proposition could well have been taken as a postulate in place of Postulate 5. (The converse of this Proposition is contained in Propositions 27 and 28 which are proved without invoking Postulate 5, and this is incidentally the first occasion for Euclid to talk about parallel lines). I have put the verb 'postpones' in quotation marks, because, according to the view that I am putting forward here, this was not a deliberate postponement by Euclid on

account of some inherent abhorrence of the parallel Postulate, as alleged by many critics, but rather it was the last step along one line of progress in Euclid's 'backtracking' journey from Book II, Proposition 14 to the Definitions, Postulates and Common Notions.

6. One would immediately think of the Kinetic Theory of Gases as an example.

References

1. Euclid's *Elements*. See note 3.

2. Mueller, Ian. See note 3.

3. Russell, Bertrand A.W., 'Recent Work on the Principles of Mathematics' in *International Monthly*, vol. 4, 1901, pp. 83-101. Reprinted as 'Mathematics and the Meta-physicians' in *Mysticism and Logic and Other Essays*, London, Longmans Green, 1918. Issued as a paperback by Penguin Books Ltd. p. 75.

4. —-, *Introduction to Mathematical Philosophy*, London, George Allen and Unwin Ltd,, 1919. Reprinted by Simon and Schuster.