Length expansion and invariance in the special theory of relativity

Sadanand D Agashe

Department of Electrical Engineering, Indian Institute of Technology Bombay, Powai, Mumbai-76, India-400076

E-mail: `eesdaia@ee.iitb.ac.in`

Abstract. We look afresh at the deduction of the "Lorentz contraction" of a "rod" from the Lorentz transformation equations of the special theory of relativity. We show that in some situations, which include acceleration of the "rod", length "expansion" and "invariance" are possible. We also show that in some situations, in contrast to "time dilatation", "time contraction" and "time invariance" are possible. Such examples should convince students of relativity that these are purely kinematical phenomena.

1. Introduction

After a brief recapitulation (to fix the notation) of the usual textbook derivation of "Lorentz contraction" from the Lorentz transformation, we consider two novel situations, and show that length "expansion" and even "invariance" are possible. We also show that in contrast to "time dilatation", in some situations, time "contraction" and "invariance" are possible. Such examples should help dispel any lurking suspicion that these are *phenomena* associated with motion of a body. We are dealing here with only the mathematical derivation of the Lorentz contraction from the Lorentz transformation equations, and not with the historical evolution of the Lorentz contraction itself.

2. Usual deduction of the Lorentz contraction

Let S denote a reference frame with coordinates x, y, z, and time t. Let S' be another reference frame whose origin moves with *speed* v $(v > 0)$, relative to S, in the direction of the positive x-axis of S, and let x', y', z' denote the coordinates in S', and t' the time in S'. Let us suppose also that the origins of S and S' coincide at time $t = 0$ which is also the time $t' = 0$. The y'- and z'- axes of S' are assumed to remain parallel to the y- and z- axes of S, respectively. Then, following in Einstein's footsteps, the coordinates x, y, z, and time t in S of an event are related to the coordinates x', y', z', and time t' in S' of the same event by the following Lorentz transformation:

$$x' = \beta(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \beta(t - \frac{vx}{c^2}) \tag{1}$$

where c denotes the speed of light in S *and* S' and $\beta = 1/\sqrt{1 - v^2/c^2}$. Note that $\beta > 1$.

The transformation (1) has an inverse, namely:

$$x = \beta(x' + vt'), \qquad y = y', \qquad z = z', \qquad t = \beta(t' + \frac{vx'}{c^2}). \tag{2}$$

If the origin of S' were to move in the direction of the negative x- axis of S, with *speed* v $(v > 0)$ then in (1) and (2), we would have to replace v by $-v$.

Now, in deducing the Lorentz contraction on the basis of the transformation equations (1) and (2), most authors talk about a rigid rod lying *at rest* on the x'- axis of the moving system S'. The ends $P1$ and $P2$ of this rod can thus be thought of as a *series* of events in S': $P1 \equiv \{(x'_1, 0, 0, t')\}$ and $P2 \equiv \{(x'_2, 0, 0, \bar{t}')\}$, with $x'_2 - x'_1 = l > 0$, say, so that we can call l the constant (in S') length of the rod. Next, one shows that although the rod is at rest in S', it is "observed" to be moving in S with speed v and as it moves in S, its length remains constant in S. However, its length in S is different from its length l in S', and is, in fact, $\frac{1}{\beta}l$, which is smaller than l. Hence the term "contraction". Indeed, using (2), at any time t'_1 of S', the coordinates of $P1$ in S are $(\beta(x'_1 + vt'_1), 0, 0)$, and at any time t'_2 of S', the coordinates of $P2$ in S are $(\beta(x'_2 + vt'_2), 0, 0)$. If t'_1 and t'_2 in S' correspond to a common time t in S, we have

$$t \quad = \quad \beta(t'_1 + \frac{vx'_1}{c^2}) \quad = \quad \beta(t'_2 + \frac{vx'_2}{c^2}). \tag{3}$$

and so, the distance between $P1$ and $P2$ in S at time t, and, thus, the length of the rod in S at time t, are given by

$$\beta(x_2' + vt_2') - \beta(x_1' + vt_1') = \beta(x_2' - x_1') + \beta v(t_2' - t_1') \tag{4}$$

$$= \beta(x_2' - x_1') - \beta v\frac{v}{c^2}(x_2' - x_1') \tag{5}$$

$$= \beta(1 - \frac{v^2}{c^2})(x_2' - x_1') \tag{6}$$

$$= \frac{1}{\beta}l. \tag{7}$$

3. The case of a rod moving uniformly in both reference frames

The above derivation of the relation between the lengths of a rod in two frames of reference in a particular situation is perfectly impeccable. However, the accompanying comments on this "phenomenon" can create some unwarranted impressions in the minds of students. Thus, the author of [1] says:

> One consequence is this: *a body's length is measured to be greatest when it is at rest relative to the observer. When it moves with a velocity v relative to the observer its measured length is contracted in the direction of its motion by the factor* $\sqrt{1 - v^2/c^2}$, *whereas its dimensions perpendicular to the direction of motion are unaffected.*

This might give a beginning student the impression that for a *single* observer, when a body, which has been at rest relative to the observer for a while, starts moving relative to the same observer, its length is contracted in the direction of motion. This usually leads to the further impression that according to the special theory of relativity, the *act* of putting a body in motion *results in* or *causes* its length to contract. To counter such impressions and to emphasize the fact that the transformation equations (1) and (2) relate only *events* in *two* reference frames, and that they cannot deal with the issue of what happens to a body in a *single* reference frame when the state of motion of the body is changed, it may be useful to consider some novel situations, for example, (a) a body which is moving uniformly relative to *both* reference frames and (b) a body which is accelerated relative to the reference frames. We consider situation (a) first.

Let a point $P1$ have a uniform motion in S, given by the series of events $\{(x_0 + ut, 0, 0, t)\}$. Let another point $P2$ have the motion $\{(x_0 + l + u\bar{t}, 0, 0, \bar{t})\}$, with $l > 0$. Thus $P2$ also moves in S with the same speed and in the same direction, and the distance between $P1$ and $P2$ remains constant in S. We could think of $P1$ and $P2$ as the ends of a rigid rod moving in S since its length remains constant in S. Are the motions of $P1$ and $P2$ uniform in S'? Further, does the distance between them remain constant in S' too, so that the rod remains rigid in S'?

Indeed the motions of $P1$ and $P2$ in S', with the times t and $\bar{t}$ in S regarded as parameters, are given by

$$P1 : \{(\beta(x_0 + ut - vt), 0, 0, \beta(t - \frac{v(x_0 + ut)}{c^2}))\} \tag{8}$$

and

$$P2 : \{(\beta(x_0 + l + u\bar{t} - v\bar{t}), 0, 0, \beta(\bar{t} - \frac{v(x_0 + l + u\bar{t})}{c^2}))\}. \tag{9}$$

From these equations we can see that their common speed is given by

$$\frac{u - v}{(1 - \frac{uv}{c^2})}. \tag{10}$$

The S'- distance between $P1$ and $P2$ at a time t' in S' is given by

$$\beta(x_0 + l + u\bar{t} - v\bar{t}) - \beta(x_0 + ut - vt) \tag{11}$$

where t and $\bar{t}$ are related to t' by

$$t' = \beta(t - \frac{v(x_0 + ut)}{c^2}) = \beta(\bar{t} - \frac{v(x_0 + l + u\bar{t})}{c^2}). \tag{12}$$

The distance calculates out to be

$$\frac{1}{\beta(1 - \frac{uv}{c^2})} l. \tag{13}$$

Thus, the rod has constant length in S' too. But is its length in S' necessarily smaller than its length l observed in S? Denoting the factor multiplying l in (13) by $k(u)$, the function k has the following values:

$$k(\frac{c^2}{v}) \quad = \infty, \tag{14}$$

$$k(c) \quad = \frac{1}{\beta(1 - v/c)} = \sqrt{\frac{1 + v/c}{1 - v/c}} > 1, \tag{15}$$

$$k(v) \quad = \frac{1}{\beta(1 - v^2/c^2)} = \beta > 1, \tag{16}$$

$$k(0) \quad = \frac{1}{\beta} < 1, \tag{17}$$

$$k(-c) \quad = \frac{1}{\beta(1 + v/c)} = \sqrt{\frac{1 - v/c}{1 + v/c}} < 1, \tag{18}$$

$$k(-\infty) = 0, \tag{19}$$

with

$$k(-c) < k(0) < 1 < k(v) < k(c). \tag{20}$$

The case $u = 0$ corresponds to the rod being at rest in S and its length in S' is observed to be smaller than its length in S, but if $u = v$, the rod is at rest in S', its length in S' is observed to be *larger* than its length in S. Further, there is a particular value of u, namely:

$$\bar{u} = \frac{c^2}{v} \left[1 - \sqrt{1 - v^2/c^2}\right], \tag{21}$$

such that $k(\bar{u}) = 1$, and $0 < \bar{u} < v$. Thus, there is a speed $\bar{u}$ for which the rod is observed to be moving in *both* S and S', but its length is observed to be the *same* in both.

So, we can have not only a *contraction* but also an *expansion* and even *invariance*!

4. Length of a rod changes and does not change!

To drive home further the point that one should talk, not of a *change* in length, but of a *difference* in length in two reference frames, we may point out that in the deductions above, it is not necessary that the points $P1$ and $P2$, or the rod whose ends they might be, be in uniform motion (or at rest) for *all* time t. It is enough if there is uniform motion (or rest) over a sufficiently long time-interval or duration. Therefore, one can imagine a motion of a rod which somehow remains constant in length, say, l, in S all the time; let it "start" in a state of rest, say, then smoothly accelerate to a state of uniform motion at the appropriate velocity $\bar{u}$, stay in that state for a while, and then smoothly accelerate again to a state of uniform motion with speed v, so that it is finally at rest in S'. As seen from S', the length of the rod will start with a value that is less than l, changing smoothly to l after some time, and then changing smoothly to a value greater than l finally. Thus, the rod will be seen to change its length in S' as it moves maintaining a constant length in S, getting accelerated in *both* in S and S'.

We may conclude, therefore, that the "change" in length is purely a *kinematical* fact, arising out of the manner in which the two systems S and S' and their coordinates and times are related, and we need not look for any *dynamical* reason for the change in either system. It is high time, therefore, that teachers of relativity stopped highlighting *contraction* as an important phenomenon in special relativity.

We consider next situation (b) of accelerated motion.

5. Accelerated motion of the rod

Consider the following question: is it possible for two points $P1$ and $P2$ which have *continuously* accelerated motion in S while maintaining a constant distance between them in S, to maintain a constant distance between them in S' *also*? We explore the special situation when the motion in S is along the x- axis (and so, in S' the motion is along the x'- axis), and so, we suppress the y- and z- coordinates in the calculations below.

Let the motions of two points $P1$ and $P2$ in S be given by two functions $x_1(t)$ and $x_2(t)$ with $x_2(t) = x_1(t) + l$ for all t, so that the distance between them remains contant in S. Let the two functions $x'_1(t')$ and $x'_2(t')$ describe their motions in S'. We seek conditions under which the difference $x'_2(t') - x'_1(t')$ will be constant. Now, let t_1 and t_2 denote the times in S, corresponding to a common time t' in S', for the two motions, so that we have, from (2):

$$t_1 = \beta(t' + \frac{vx'_1(t')}{c^2}) \ , \quad x_1(t_1) = \beta(x'_1(t') + vt') \tag{22}$$

$$t_2 = \beta(t' + \frac{vx'_2(t')}{c^2}) \ , \quad x_2(t_2) = \beta(x'_2(t') + vt') \tag{23}$$

and so,

$$t_2 - t_1 = \frac{\beta v}{c^2}(x'_2(t') - x'_1(t')). \tag{24}$$

Thus, the distance $x_2'(t') - x_1'(t')$ between the two points in S' will be constant, say, l', if and only if $(t_2 - t_1)$ is constant, say α, where $\alpha = \beta v l'/c^2$. But then

$$x_2(t_2) - x_1(t_1) = \beta(x_2'(t') - x_1'(t')) \tag{25}$$

$$= \frac{c^2\alpha}{v}. \tag{26}$$

Since $t_2 - t_1 = \alpha$, we can write the above as

$$x_2(t_1 + \alpha) - x_1(t_1) = \frac{c^2\alpha}{v}, \tag{27}$$

and since

$$x_2(t_1 + \alpha) = x_1(t_1 + \alpha) + l, \tag{28}$$

finally

$$x_1(t_1 + \alpha) - x_1(t_1) = \frac{c^2\alpha}{v} - l. \tag{29}$$

Now (29) must hold for all times t_1. (To see why, imagine choosing first the time instant t_1, then using (1), determining the corresponding t' in S', and finally determining the corresponding t_2 in S using (2)). Thus, we have proved the following necessary condition.

Proposition 1 *If the motions $x_1(t)$ of $P1$ and $x_2(t) = x_1(t) + l$ of $P2$ are such that the distance between them in S' is a constant, say, l', then there is number α such that for all t:*

$$x_1(t + \alpha) - x_1(t) \quad = \quad \frac{c^2\alpha}{v} - l \quad = \quad \beta l' - l. \tag{30}$$

Further, $l' = \frac{c^2\alpha}{\beta v}$.

Comment: If the function x_1 is continuously differentiable, from (30), we see immediately that the derivative of x_1, i.e., the *velocity* , of $P1$, is a periodic function of the time in S, with period α.

In the special case when x_1 is a uniform motion with speed u, we have

$$\alpha = l/\left(\frac{c^2\alpha}{v} - u\right), \tag{31}$$

and so

$$l' = \frac{1}{\beta(1 - \frac{uv}{c^2})} l, \tag{32}$$

which agrees with (13).

Note that in the accelerated case we can no longer talk about the entire rod remaining rigid, i.e., all the points of the rod including its end points maintaining a constant distance between each other in S', because (30) cannot hold for all points; we can only talk about a pair of points whose distance apart is l satisfying (30).

Proposition 1 suggests the following surprising proposition.

Proposition 2 *If the motion $x(t)$ in S of a particle is such that its velocity is periodic with period α, say, and the displacement during any one period is d, the corresponding motion in S' is also with periodic velocity, with the period α' given by $\alpha' = \beta(\alpha - \frac{vd}{c^2})$.*

We have, for all t, $x(t+\alpha)-x(t)=d$. Let t'_1 correspond to t and t'_2 to $t+\alpha$. Then,

$$t'_1=\beta(t-\frac{vx(t)}{c^2}),\quad t'_2=\beta(t+\alpha-\frac{vx(t+\alpha)}{c^2}), \tag{33}$$

so that

$$t'_2-t'_1=\beta(\alpha-\frac{vd}{c^2}). \tag{34}$$

If $x'_1(t'_1)$ and $x'_2(t'_2)$ correspond to $x(t)$ and x$(t+\alpha)$, then

$$x'_1(t'_1)=\beta(x(t)-vt),\qquad\qquad x'_2(t'_2)=\beta(x(t+\alpha)-v(t+\alpha)) \tag{35}$$

so

$$x'_2(t'_2)-x'_1(t'_1)=\beta(d-v\alpha). \tag{36}$$

Thus, the motion in S' has a periodic velocity with period $\beta(\alpha-\frac{vd}{c^2})$ and displacement during a period given by $\beta(d-v\alpha)$.

Corollary 3 *If a particle has a periodic motion in S with period α, its motion in S' is periodic too, with period $\beta\alpha$.*

6. Time Dilatation

What about "time dilatation"? Just as we have given examples of "length expansion" and "length invariance", we give examples of "time contraction" and "time invariance". Indeed, consider again a particle moving with velocity u along the x-axis in S. Suppose that the path of this particle is crossed by two other particles, by one particle at time t_1 and by the other particle at time t_2, so that the time interval in S between the crossings is (t_2-t_1). What will be the time interval in S' between these crossings? Since the two events in S are (ut_1,t_1) and (ut_2,t_2), we can quickly calculate that the time interval in S' is given by $\beta(1-\frac{uv}{c^2})(t_2-t_1)$. So, from our earlier discussion, it is clear that depending on the value of u, the S'-interval will be greater than, equal to, or, smaller than, the S-interval. One should, therefore, talk, not of time dilatation or slowing down of clocks, but of difference in time reckonings in two reference frames.

References

[1] Resnick Robert 1968 *Introduction to special relativity* (New York: John Wiley) p.62