Prob 1 : Calculate the *mp* of $v = [\,1 \;\; 0 \;\; 2 \; -1]^T$ under the action of

$$A = \begin{bmatrix} 6 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ 6 & 1 & 1 & 1 \\ -6 & -1 & 4 & 4 \end{bmatrix}$$

Solution:

- Step 1: We calculate $Av,\ A^2v = A(Av),\ A^3v,\ A^4v$

$$Av = \begin{bmatrix} 7 \\ -2 \\ 7 \\ -2 \end{bmatrix}, \qquad A^2v = \begin{bmatrix} 45 \\ -20 \\ 45 \\ -20 \end{bmatrix}, \qquad \frac{1}{5}\,A^2v = \begin{bmatrix} 9 \\ -4 \\ 9 \\ -4 \end{bmatrix}$$

$$\frac{1}{5}A^3v = \begin{bmatrix} 55 \\ -30 \\ 55 \\ -30 \end{bmatrix}, \qquad \frac{1}{25}A^3v = \begin{bmatrix} 11 \\ -6 \\ 11 \\ -6 \end{bmatrix}, \qquad \frac{1}{25}\,A^4v = \begin{bmatrix} 65 \\ -40 \\ 65 \\ -40 \end{bmatrix}$$

$$\frac{1}{125}\,A^4v = \begin{bmatrix} 13 \\ -8 \\ 13 \\ -8 \end{bmatrix}$$

Note: We have scaled the columns for 'human convenience'. A computer program may not do this.

- Step 2: Now, we do column operations on these columns in order $v,\ Av,\ \frac{1}{5}A^2v,\ \frac{1}{25}A^3v,\ \frac{1}{125}A^4v$ using a 5×5 column operations matrix, and marking the various pivot elements as we go on. We do not need to do 'backward' ('leftward') creation of zeros in pivot columns.

$$\begin{bmatrix} \underline{v} & \underline{Av} & \underline{\tfrac{1}{5}A^2v} & \underline{\tfrac{1}{25}A^3v} & \underline{\tfrac{1}{125}A^4v} \\ \underline{\mathbf{1_1}} & 7 & 9 & 11 & 13 \\ 0 & -2 & -4 & -6 & -8 \\ 2 & 7 & 9 & 11 & 13 \\ -1 & -2 & -4 & -6 & -8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 & 0 \\ 0 & \underline{\mathbf{-2_2}} & -4 & -6 & -8 \\ 2 & -7 & -9 & -11 & -13 \\ -1 & 5 & 5 & 5 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -7 & -9 & -11 & -13 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 & 0 \\ 0 & \underline{\mathbf{-2_2}} & 0 & 0 & 0 \\ 2 & -7 & \underline{\mathbf{5_3}} & 10 & 15 \\ -1 & 5 & -5 & -10 & -15 \end{bmatrix} \qquad \begin{bmatrix} 1 & -7 & 5 & 10 & 15 \\ 0 & 1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 & 0 \\ 0 & \underline{\mathbf{-2_2}} & 0 & 0 & 0 \\ 2 & -7 & \underline{\mathbf{5_3}} & 0 & 0 \\ -1 & 5 & -5 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -7 & 5 & 0 & 0 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fourth column has become zero, as also the fifth , but $\frac{1}{25}A^3v$ comes <u>before</u> $\frac{1}{125}A^4v$. so from the column operations matrix , we obtain

$$\begin{aligned} & & (1)Av + (-2)\frac{1}{5}A^2v + (1)\frac{1}{25}A^3v &= 0_4 \\ &So & A^3v - 10A^2v + 25Av &= 0_4 \\ &or & (A^3 - 10A^2 + 25A)v &= 0_4 \\ &So,\ the\ mp\ is & \underline{s^3 - 10s^2 + 25s} & \end{aligned}$$

Prob 2: Obtain a direct decomposition, for a suitable k:

$$R^4 = ker\ B^k \oplus im\ B^k$$

$$where \qquad B = \begin{bmatrix} 2 & 0 & -5 & 3 \\ 0 & 2 & -3 & 1 \\ -5 & 3 & 2 & 0 \\ -3 & 1 & 0 & 2 \end{bmatrix}.$$

Solution :

- <u>Step 1</u>: We first calculate $ker\ B,\ im\ B$.

Note: We are free to use any non-zero entry as pivot . The column operations matrix is 4 X 4.

$$\begin{bmatrix} 2 & 0 & -5 & 3 \\ 0 & 2 & -3 & \underline{\mathbf{1_1}} \\ -5 & 3 & 2 & 0 \\ -3 & 1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{2_2}} & -6 & 4 & 3 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ -5 & 3 & 2 & 0 \\ -3 & -3 & 6 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{2_2}} & 0 & 0 & 3 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ -5 & \underline{\mathbf{-12_3}} & 12 & 0 \\ -3 & -12 & 12 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{2_2}} & 0 & 0 & 3 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ -5 & \underline{\mathbf{-12_3}} & 0 & 0 \\ -3 & -12 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

No more column operations are possible.
Note: It is not necessary to create leftward or rightward zeros in pivot columns, <u>at least right now</u>.

The zero column on the "B side" says that the corresponding columns on the columns-operations matrix side are in $ker\ B$. The non-zero columns on the B side are in $im\ B$.
So,

$$ker\ B = sp\{[1\ 1\ 1\ 1]^T\}$$
$$im\ B = sp\{[2\ 0\ -5\ -3]^T,\ [0\ 0\ -12\ -12]^T,\ [3\ 1\ 0\ 2]^T\}.$$

- <u>Step 2</u>: We now check whether

$$R^4 = ker\ B \oplus im\ B$$

by using column operations .(Note: Nothing is obvious). It is not necessary to write the column operations matrix.

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 2 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 1 & -5 & -12 & 0 \\ 1 & -3 & -12 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 1 & \underline{\mathbf{-2_2}} & 0 & -2 \\ 1 & -7 & -12 & -3 \\ 1 & -5 & -12 & -1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 1 & \underline{\mathbf{-2_2}} & 0 & 0 \\ 1 & -7 & -12 & \underline{\mathbf{4_3}} \\ 1 & -5 & -12 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 1 & \underline{\mathbf{-2_2}} & 0 & 0 \\ 1 & -7 & 0 & \underline{\mathbf{4_3}} \\ 1 & -5 & 0 & 4 \end{bmatrix}$$

The zero column shows that the four vectors do not form a linearly independent set and so,

$$R^4 \neq ker\ B \oplus im\ B\ .$$

- <u>Step 3</u>: So, we have to calculate $ker\ B^2,\ im\ B^2$ and check whether

$$R^4 = ker\ B^2 \oplus im\ B^2\ .$$

It is easier to calculate $im\ B^2$; it is simply the span of vectors obtained by acting by B on the three vectors in $im\ B$.

$$\begin{bmatrix} 2 & 0 & -5 & 3 \\ 0 & 2 & -3 & 1 \\ -5 & 3 & 2 & 0 \\ -3 & 1 & 0 & 2 \end{bmatrix} \qquad \begin{array}{ccc} 2 & 0 & 3 \\ 0 & 0 & 1 \\ -5 & -12 & 0 \\ -3 & -12 & 2 \end{array}$$

namely,

$$\begin{bmatrix} 20 \\ 12 \\ -20 \\ -12 \end{bmatrix}, \quad \begin{bmatrix} 24 \\ 24 \\ -24 \\ -24 \end{bmatrix}, \quad \begin{bmatrix} 12 \\ 4 \\ -12 \\ -4 \end{bmatrix}.$$

We now check whether these vectors are linearly independent. We scale them for human convenience.

$$\begin{bmatrix} 5 & \underline{\mathbf{1_1}} & 3 \\ 3 & 1 & 1 \\ -5 & -1 & -3 \\ -3 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & \underline{\mathbf{1_1}} & 0 \\ \underline{\mathbf{-2_2}} & 1 & \underline{\mathbf{-2_2}} \\ 0 & -1 & 0 \\ 2 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & \underline{\mathbf{1_1}} & 0 \\ \underline{\mathbf{-2_2}} & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

.

So, $im\ B^2 \ = \ sp\left\{[0\ -2\ 0\ 2]^T,\ [1\ 1\ -1\ -1]^T\right\} \subsetneqq im\ B.$

- Step 4: We could now calculate $ker\ B^2$ and check whether

$$R^4 \ = \ ker\ B^2 \oplus im\ B^2,$$

but, it is easier to calculate $im\ B^3$ and check whether

$$im\ B^3 \ = \ im\ B^2.$$

For $im\ B^3$, we calculate

$$\begin{bmatrix} 2 & 0 & -5 & 3 \\ 0 & 2 & -3 & 1 \\ -5 & 3 & 2 & 0 \\ -3 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$

and check the last two vectors for independence:

$$\begin{bmatrix} 6 & 4 \\ -2 & 4 \\ -6 & -4 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 6 & 0 \\ -2 & 16/3 \\ -6 & 0 \\ 2 & -16/3 \end{bmatrix};$$

they <u>are</u> independent. So

$$dim\ im\ B^3 = dim\ im\ B^2\ ,$$

and so,

$$im\ B^3 = im\ B^2.$$

- Step 5: We have to calculate $ker\ B^2$, we already know $ker\ B = sp[1\ 1\ 1\ 1]^T$. We should not calculate B^2. So we have to solve

$$B\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We will need the column operations matrix, which we have already found out. However, B was not fully column reduced, but it is not too late, we can do it now, using the same pivot elements in the same order.

$$\begin{bmatrix} \underline{\mathbf{2_2}} & 0 & 0 & 3 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ -5 & \underline{\mathbf{-12_3}} & 0 & 0 \\ -3 & -12 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{2_2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ -5 & \underline{\mathbf{-12_3}} & 0 & 15/2 \\ -3 & -12 & 0 & 13/2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 & -3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{2_2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ 0 & \underline{\mathbf{-12_3}} & 0 & 0 \\ 2 & -12 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1/4 & 3 & 1 & 3/8 \\ -5/12 & 1 & 1 & 5/8 \\ 0 & 0 & 1 & 0 \\ 5/6 & -2 & 1 & -1/4 \end{bmatrix}.$$

So we have to solve

$$\begin{bmatrix} \underline{\mathbf{2_2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{\mathbf{1_1}} \\ 0 & \underline{\mathbf{-12_3}} & 0 & 0 \\ 2 & -12 & 0 & -1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

giving

$$\begin{aligned} 2y_1 = 1 \ &or\ y_1 = 1/2 \\ y_4 &= 1 \\ -12y_2 = 1 \ &or\ y_2 = -1/12 \end{aligned}$$

Substituting in 4^{th} equation for consistency

$$2(1/2) + (-1/12)(-12) + (-1)1 \stackrel{?}{=} 1.$$

y_3 is free. So solution is

$$x = \begin{bmatrix} -1/4 & 3 & 1 & 3/8 \\ -5/12 & 1 & 1 & 5/8 \\ 0 & 0 & 1 & 0 \\ 5/6 & -2 & 1 & -1/4 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/12 \\ y_3 \\ 1 \end{bmatrix}$$

$$= y_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} (-1/8 - 1/4 + 3/8) \\ (-5/24 - 1/12 + 5/8) \\ 0 \\ (5/12 + 1/6 - 1/4) \end{bmatrix} = y_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

Before proceeding further,we check whether the 'constant' part $[0\ \ 1/3\ \ 0\ \ 1/3]^T$ is such that B acting on it takes it to $ker\ B$.

$$\begin{bmatrix} 2 & 0 & -5 & 3 \\ 0 & 2 & -3 & 1 \\ -5 & 3 & 2 & 0 \\ -3 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We also see(as a check) that $[1\ 1\ 1\ 1]^T$ and $[0\ \ 1/3\ \ 0\ \ 1/3]^T$ form an independent set:

$$\begin{bmatrix} \underline{\mathbf{1}} & 0 \\ 1 & \underline{\mathbf{1/3}} \\ 1 & 0 \\ 1 & 1/3 \end{bmatrix}$$

So, scaling $[0\ \ 1/3\ \ 0\ \ 1/3]^T$,

$$ker\ B^2 \ = \ sp\left\{[1\ 1\ 1\ 1]^T, [0\ 1\ 0\ 1]^T\right\}$$

and

$$im\ B^2 = sp\left\{[0\ \ -1\ 0\ \ -1]^T\ ,\ [1\ 1\ \ -1\ \ -1]^T\right\}$$

Finally, we verify that, these 4 vectors form a linearly independent set.

$$\begin{matrix} \underline{\mathbf{1_1}} & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 \end{matrix} \quad \rightarrow \quad \begin{matrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 1 & \underline{\mathbf{1_2}} & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & 1 & -2 \end{matrix} \quad \rightarrow \quad \begin{matrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 1 & \underline{\mathbf{1_2}} & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 1 & 1 & \underline{\mathbf{2_3}} & -2 \end{matrix} \quad \rightarrow \quad \begin{matrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 1 & \underline{\mathbf{1_2}} & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & \underline{\mathbf{2_3}} & 0 \end{matrix}$$

Thus $R^4 = ker\ B^2\ \oplus\ im\ B^2$, k $= 2$.

Prob. 3: The *mp* of the matrix C given by

$$C = \begin{bmatrix} -4 & 4 & 3 & 15 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & -4 & -3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

is $(s+1)(s+2)^3$. Calculate a suitable <u>ordered</u> basis so that the matrix is transformed into its Jordan Canonical form. What will be the FROBENIUS Canonical Form of the matrix?

Solution :

- <u>Step 1</u>: The *mp*, say, $p(s)$:

$$p(s) \ = \ (s+1)(s+2)^3$$

is seen to be factorized into two factors $(s+1)$, and $(s+2)^3$, which are coprime, say $p_1(s) \ = \ (s+1)$ and $p_2(s) \ = \ (s+2)^3$.
So,

$$R^4 \ = \ ker(C+I) \oplus ker(C+2I)^3$$

but, since there are only two factors,

$$ker(C+2I)^3 \ = \ im(C+I),$$

that is

$$R^4 \;=\; ker(C+I) \oplus im(C+I)$$

- <u>Step 2</u>: So we simply have to calculate bases for the *ker* and *im* spaces of the matrix (C+I).We do this using column operations matrix.

$$\begin{array}{cc} \mathbf{C+I} & \mathbf{I} \\ \begin{bmatrix} -3 & 4 & 3 & 15 \\ -1 & \underline{\mathbf{1_1}} & 0 & 0 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 2 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_2}} & 4 & 3 & -5 \\ 0 & \underline{\mathbf{1_1}} & 0 & 0 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_2}} & 0 & 0 & 0 \\ 0 & \underline{\mathbf{1_1}} & 0 & 0 \\ 0 & 0 & \underline{\mathbf{-3_3}} & -3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -4 & -3 & 5 \\ 1 & -3 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_2}} & 0 & 0 & 0 \\ 0 & \underline{\mathbf{1_1}} & 0 & 0 \\ 0 & 0 & \underline{\mathbf{-3_3}} & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -4 & -3 & 8 \\ 1 & -3 & -3 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So,

$$ker\ (C+I) = sp\left\{[8\ 3\ -1\ 1]^T\right\}$$

$$im\ (C+I) = sp\left\{[1\ 0\ 0\ 0]^T,\ [0\ 1\ 0\ 0]^T,\ [0\ 0\ -3\ 2]^T\right\}$$

$$= ker\ (C+2I)^3$$

- <u>Step 3</u>: We now check whether the subspace *ker* $(C+2I)^3$ can be decomposed. Its *mp* is$(s+2)^3$. We have to find a vector where *mp* is $p_2(s)$. Since $p_2(s)$ is a <u>power</u> of a <u>prime</u>(irreducible) polynomial, <u>at least one</u> of the basis vectors of the subspace must have $mp = p_2(s)$.

<u>**mp of $[\mathbf{1\ 0\ 0\ 0}]^{\mathbf{T}}$ under C:**</u> Say $v_1 = [1\ 0\ 0\ 0]^T$, $Cv_1 = [-4\ -1\ 0\ 0]^T$, $C^2v_1 = [12\ 4\ 0\ 0]^T$; scaling $\frac{1}{4}C^2v_1 = [3\ 1\ 0\ 0]^T$, $\frac{1}{4}C^3v_1 = [-8\ -3\ 0\ 0]^T$.
To check independence, we will do column operations.

$$\begin{array}{c} \begin{array}{cccc} v_1 & Cv_1 & \frac{1}{4}C^2v_1 & \frac{1}{4}C^3v_1 \end{array} \\ \begin{bmatrix} \underline{\mathbf{1_1}} & -4 & 3 & -8 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 0 & \underline{-\mathbf{1_2}} & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & -3 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 0 & \underline{-\mathbf{1_2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 1 & -4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So,

$$\begin{aligned} & v_1 \ + \ Cv_1 \ + \ \tfrac{1}{4}C^2v_1 \ = \ 0_4 \\ i.e, \qquad & (C^2 \ + \ 4C \ + \ 4I)v_1 = 0_4. \end{aligned}$$

The *mp* of v_1 is $s^2 \ + \ 4s \ + \ 4$ and this has to be power of $(s+2)$: it is indeed $(s+2)^2$.

So next, $v_2 = [0\ 1\ 0\ 0]^T$, $Cv_2 \ = \ [4\ 0\ 0\ 0]^T$, $C^2v_2 \ = \ [-16\ \ -4\ 0\ 0]^T$, scaling $\frac{1}{4}C^2v_2 \ = \ [-4\ \ -1\ 0\ 0]^T$, $\frac{1}{4}C^3v_2 \ = \ [12\ 4\ 0\ 0]^T$, $\frac{1}{16}C^3v_2 \ = \ [3\ 1\ 0\ 0]^T$

$$\begin{array}{c} \begin{array}{cccc} v_2 & Cv_2 & \frac{1}{4}C^2v_2 & \frac{1}{4}C^3v_2 \end{array} \\ \begin{bmatrix} 0 & 4 & -4 & 3 \\ \underline{\mathbf{1_1}} & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & \underline{\mathbf{4_2}} & -4 & 3 \\ \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & \underline{\mathbf{4_2}} & 0 & 0 \\ \underline{\mathbf{1_1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -3/4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So,

$$\begin{aligned} & v_2 \ + \ Cv_2 \ + \ \tfrac{1}{4}C^2v_2 \ = \ 0_4 \\ i.e, \qquad & (C^2 \ + \ 4C \ + \ 4I)v_2 = 0_4. \end{aligned}$$

Hence, *mp* of v_2 is also $(s+2)^2$.

So next $v_3 = [0\ 0\ \ -3\ 2]^T$. It's *mp* had better be $(s+2)^3$!
$Cv_3 \ = \ [21\ \ 10\ \ 6\ \ -4]$, $C^2v_3 \ = \ [-86\ \ -41\ \ -12\ \ 8]$, $C^3v_3 \ = \ [264\ \ 126\ \ 24\ \ -16]$

$$\begin{array}{c} \begin{array}{cccc} v_3 & Cv_3 & C^2v_3 & C^3v_3 \end{array} \\ \begin{bmatrix} 0 & 21 & -86 & 264 \\ 0 & 10 & -41 & 126 \\ \underline{-3} & 6 & -12 & 24 \\ 2 & -4 & 8 & -16 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 21 & -86 & 264 \\ 0 & 10 & -41 & 126 \\ \underline{-3} & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -4 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Here, the usual operations will involve fractions. But,we can use column operation with a difference - instead of creating zeros directly, we can reduce the size of numbers. Thus,

$$C_3 \leftarrow C_3 + 4C_2, \; C_4 \leftarrow C_4 - 12C_2$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 21 & -2 & 12 \\ 0 & 10 & -1 & 6 \\ -3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & -16 \\ 0 & 1 & 4 & -12 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

but we must carry out the operations in <u>one</u> direction only, namely, from v_3 to Cv_3 to C^2v_3. But, we could also scale some columns to avoid fractions. For example,

$$\downarrow$$

$$\begin{bmatrix} 0 & 21 & -860 & 2640 \\ 0 & \underline{\mathbf{10_2}} & -410 & 1260 \\ \underline{\mathbf{-3_2}} & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -40 & 80 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 21 & 1 & -6 \\ 0 & \underline{\mathbf{10_2}} & 0 & 0 \\ \underline{\mathbf{-3_2}} & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 42 & -172 \\ 0 & 1 & 41 & -126 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 21 & 1 & 0 \\ 0 & 10 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 42 & 80 \\ 0 & 1 & 41 & 120 \\ 0 & 0 & 10 & 60 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

So we have,

$$10C^3v_3 + 60C^2v_3 + 120Cv_3 + 80v_3 = 0_4,$$

and so,

$$(C^3 + 6C^2 + 12C + 8I)v_3 = 0_4,$$

as expected!

We may of course verify directly that $(s+2)^3 = s^3 + 6s^2 + 12s + 8$ is the mp of $[0 \; 0 - 3 \; 2]$. We can check whether,

$$\begin{bmatrix} 264 \\ 126 \\ 24 \\ -16 \end{bmatrix} + 6 \begin{bmatrix} -86 \\ -41 \\ -12 \\ 8 \end{bmatrix} + 12 \begin{bmatrix} 21 \\ 10 \\ 6 \\ -4 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$LHS = \begin{bmatrix} 264-516+252 \\ 126-246+120 \\ 24-72+72-24 \\ -16+48-48+16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Also verify that $(s+2)$ and $(s+2)^2 = s^2+4s+4$ are <u>not</u> mp's of $[0\ 0 -3\ 2]$.

$$[21\ 10\ 6-4]^T + 2[0\ 0-3\ 2]^T \ = \ [21\ 10\ 0\ 0]^T \ \neq [0\ 0\ 0\ 0]^T$$

$$[-86-41-12\ 8]^T + 4[21\ 10\ 6-4]^T + 4[0\ 0-3\ 2]^T = [-2-1\ 0\ 0]^T \neq [0\ 0\ 0\ 0]^T$$

Since $(C+2I)^3 v_3 = 0_4$, the basis vectors will be $(C+2I)^2 v_3, (C+2I)v_3, v_3$, i.e., $[-2{-1}\ 0\ 0]^T, [21\ 10\ 0\ 0]^T,\ [0\ 0\ {-3}\ 2]^T$ <u>in that order</u>.

Putting, together this basis for $ker(C+2I)^3$ <u>and</u> the basis for $ker(C+I)$, we obtain an <u>ordered</u> basis for R^4:
$[-2\ -1\ 0\ 0]^T, [21\ 10\ 0\ 0]^T, [0\ 0-3\ 2]^T, [8\ 3\ \ -1\ 1]^T$,
such that with respect to this basis, the matrix C is represented by (transformed into) its JORDAN CANONICAL FORM:

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ -- & -- & -- & -- \\ 0 & 0 & 0 & -1 \end{array}\right],$$

the 3X3 Jordan Block corresponding to the eigenvalue -2 appearing"'first"', and the 1X1 Jordan Block corresponding to the eigenvalue -1 appearing "'second"'.
The *mp* of C, namely, p(s), is of degree 4, which equals the size of the matrix (and the dimensions of the vector space R^4). So, the FROBENIUS CANONICAL FORM of the matrix will be a single block, namely:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & -20 & -18 & -7 \end{bmatrix},$$

since $p(s) \ = \ s^4 \ + \ 7s^3 \ + \ 18s^2 \ + \ 20s \ + \ 8$.

Prob.4: Using the STEINITZ procedure, extend the set $\{v_1, v_2, v_3\}$ to a <u>basis</u> for R^4 .

$$v_1 = [1\ 1\ 1\ 1]^T, v_2 = [3\ 3-1-1]^T, v_3 = [3\ 5\ 1\ 1]^T.$$

Solution: Incidentally, we will check that the given set is linearly independent.

We will select additional vectors - in this case, only one - from <u>any</u> known basis, for simplicity, say, the basis of the 'unit' vectors, $(e^4_1, e^4_2, e^4_3, e^4_4\ )$.

We use column operations to check dependence. There is no need to use the column operations matrix. It helps to create zeros to the left and right.

$$\begin{array}{rrr|rrrr} v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \\ \underline{\mathbf{1_1}} & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}$$

$$\downarrow$$

$$\begin{array}{rrr|rrrr}
\underline{\mathbf{1_1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \underline{\mathbf{2_2}} & -1 & 1 & 0 & 0 \\
1 & -4 & -2 & -1 & 0 & 1 & 0 \\
1 & -4 & -2 & -1 & 0 & 0 & 1
\end{array}$$

$$\downarrow$$

$$\begin{array}{rrr|rrrr}
\underline{\mathbf{1_1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \underline{\mathbf{2_2}} & 0 & 0 & 0 & 0 \\
2 & \underline{\mathbf{-4_3}} & -2 & -2 & 1 & 1 & 0 \\
2 & -4 & -2 & -2 & 1 & 0 & 1
\end{array}$$

$$\downarrow$$

$$\left[\begin{array}{rrr|rrrr}
\underline{\mathbf{1_1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \underline{\mathbf{2_2}} & 0 & 0 & 0 & 0 \\
0 & \underline{\mathbf{-4_3}} & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 & -1 & 1
\end{array}\right]$$

So e_1, e_2 are dependent on $\{v_1, v_2, v_3\}$, but both e_3, e_4 are independent of $\{v_1, v_2, v_3\}$. So, any one of them can be adjoined.
Answer: $\{v_1, v_2, v_3, e_3\}$, $\{v_1, v_2, v_3, e_4\}$

A short-cut suggested by Bhavin Patel: If the unit vector basis is used, by looking at the pivot element columns, we can immediately see which unit vector will be independent of these columns. Thus, in the above example, both e_3 and e_4 will be independent, so we do not have to do column operations on the unit vectors.

Steinitz replacement procedure: What is given above is not a replacement procedure. In the replacement procedure, given a basis $\{v_1, v_2, \ldots, v_n\}$ and an independent set $\{w_1, w_2, \ldots, w_k\}$, with $k < n$, one finds out what members of the basis can be replaced by vectors from the independent set to form a new basis.

Prob.5: Obtain the "least squares" solution to following inconsistent system using column operations only. What is the norm of the error vector for this solution?

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution: We do column operations with a difference. Instead of creating zeros to the left and right of a chosen pivot element, we choose a pivot column and orthogonalize the remaining columns, including the right hand side vector, using the pivot column. We also need the column operations matrix.

Step 1: Choose any non-zero column of the coefficient matrix(multiplying $\underline{x}$) as the pivot column, say, the first column. If we denote the pivot column by v, then another column w is replaced by,

$$w - \left(\tfrac{v \cdot w}{v \cdot v}\right) v.$$

This new column is then orthogonal to v. We will write the inner products involved above at the bottom.

$$\begin{array}{rrr|r|rrrr}
\underline{(\mathbf{1})} & & & & & & & \\
1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 2 & 4 & 2 & 0 & 0 & 0 & 1 \\
(\mathbf{4}) & (\mathbf{2}) & (\mathbf{6}) & (\mathbf{6}) & & & &
\end{array}$$

Step 2:

$$
\begin{array}{cccc|c|cccc}
 & \underline{(\mathbf{2})} & & & & & & & \\
1 & -3/2 & -1/2 & -1/2 & 1 & -1/2 & -3/2 & -3/2 \\
1 & -1/2 & -3/2 & 1/2 & 0 & 1 & 0 & 0 \\
1 & 1/2 & -1/2 & -1/2 & 0 & 0 & 1 & 0 \\
1 & 3/2 & 5/2 & 1/2 & 0 & 0 & 0 & 1 \\
 & (\mathbf{5}) & (\mathbf{5}) & (\mathbf{1}) & & & &
\end{array}
$$

(We could scale the columns, but then we have to keep track of the scale factors.)

Step 3:

$$
\begin{array}{ccc|c|cccc}
 & & \underline{(\mathbf{3})} & & & & & \\
1 & -3/2 & 1 & -1/5 & 1 & -1/2 & -1 & -7/5 \\
1 & -1/2 & -1 & 3/5 & 0 & 1 & -1 & -1/5 \\
1 & 1/2 & -1 & -3/5 & 0 & 0 & 1 & 0 \\
1 & 3/2 & 1 & 1/5 & 0 & 0 & 0 & 1 \\
 & & (\mathbf{4}) & (\mathbf{0}) & & & &
\end{array}
$$

The column corresponding to the right hand side vector has zero inner-product with the third pivot column, so no operation is necessary. It is the error vector(or it's negative, depending on how you define "error").
Using the last column of the column-operation matrix, we have,

$$
(-7/5)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (-1/5)\begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix} + (0)\begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \\ -3/5 \\ 1/5 \end{bmatrix}
$$

So the solution is,

$$
\underline{x} = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \end{bmatrix}
$$

$(Norm\ \ of\ \ error)^2 = 1/25 + 9/25 + 9/25 + 1/25 = 20/25$

$\therefore Norm\ \ of\ \ error = 2/\sqrt{5}.$