A New Observer

S. D. Agashe*
Adjunct Professor
Department of Electrical Engineering
Indian Institute of Technology Bombay
Mumbai, India-400076

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1 The Parametrization approach

A "parametrization" approach was introduced in [1] to solve the problem of determining *all* inputs which achieve a specified "state-transfer" of a linear controllable system. That approach does not require computation of the state-transition matrix, and reduces the state-transfer problem to an "interpolation" problem, namely, the problem of "determining" all sufficiently differentiable functions which, along with their derivatives upto some order,have specific values at the initial and final time-instants of the control interval. Subsequently, the approach was used [2] to solve an optimal control problem. In this paper,we solve the problem of determining the state of a linear observable system, knowing the output history over a time-interval. The solution uses the well-known duality between controllability and observability, and the parametrization of inputs in the state-transfer problem.

2 The New Observer

As shown in [1], for a given single-input controllable system

$$\dot{x} = Ax + ub \tag{1}$$

*email: eesdaia@ee.iitb.ac.in

with $x(t) \in R^n$, the problem of determining *all* inputs which steer the system from any specified initial state $x(t_i)$ to any specified final state $x(t_f)$ can be reduced to the problem of determining all n-times differentiable functions $\phi(t)$ which, along with their $(n-1)$ derivatives $D\phi, D^2\phi, ..., D^{(n-1)}\phi$, have specific values, at the initial and final time instants, that are determined by $x(t_i)$ and $x(t_f)$ respectively, as follows:

$$D^{(n-1)}\phi(t_i)b + D^{(n-2)}\phi(t_i)Ab + ... + D\phi(t_i)A^{(n-2)}b + \phi(t_i)A^{(n-1)}b = x(t_i), \quad (2)$$
$$D^{(n-1)}\phi(t_f)b + D^{(n-2)}\phi(t_f)Ab + ... + D\phi(t_f)A^{(n-2)}b + \phi(t_f)A^{(n-1)}b = x(t_f), \quad (3)$$

and the corresponding input is given by

$$u = p(D)\phi$$

where $p(s)$ is the characteristic polynomial of A (and also the minimum polynomial of b under the action of A). Such functions ϕ were referred to as parametrizing functions, and the problem of determining these functions was referred to as the interpolation problem. There are, of course, such functions; indeed, there are polynomials of degree at most $(2n-1)$ which satisfy the conditions (2) and (3). (The well-known "Hermite Polynomial Interpolation Problem" is precisely this problem.) Note that the solution of the interpolation problem *does not* require calculation of the state-transition matrix function e^{At}.

Now, let

$$\begin{aligned} \dot{z} &= Az \\ y &= c^T z \end{aligned} \quad (4)$$

be a single-output observable autonomous system. Then, the pair (A^T, c) is a controllable pair, and so is also the pair $(-A^T, c)$. Consider an associated controllable system,

$$\dot{x} = -A^T x + uc \quad (5)$$

with input u over a time-interval $[t_i, t_f]$. Then, we have

$$\begin{aligned} D(x^T z) &= [-A^T x + uc]^T z + x^T A z \\ &= u(c^T z) \\ &= uy, \end{aligned}$$

and so, integrating over $[t_i, t_f]$,

$$x^T(t_f)z(t_f) - x^T(t_i)z(t_i) = \int_{t_i}^{t_f} u(\tau)y(\tau)d\tau.$$

Note that, the input u steers the system (5) from $x(t_i)$ at t_i to $x(t_f)$ at t_f.

Let us now choose $x(t_i) = 0$, so that u is an input which steers the system from state $\mathbf{0}$ at t_i to $x(t_f)$ at t_f, and we can choose $x(t_f)$ to be any vector that we like. Such an input can be calculated from a suitable parametrizing function ϕ. We, therefore, have the result:

Theorem 1. *Let e_j denote the j-th unit vector of R^n and let ϕ_j be the corresponding parametrizing function. Then,*

$$e_j^T z(t_f) = \int_{t_i}^{t_f} p(D)\phi_j(\tau)y(\tau)d\tau \tag{6}$$

where z and y are solutions of (4) over the time-interval $[t_i, t_f]$, and so, $z(t_f)$ can be determined knowing the integrals in (6), and thus, knowing the output history over the observation interval $[t_i, t_f]$.

Remark 1. *The reconstruction of the state $z(t_f)$ is "instantaneous", and not asymptotic, as $t_f \to \infty$. So, the state can be recovered in real time. Note also that the functions $\phi_j(\tau)$ depend on the choice of t_i and t_f, and these could be regarded as parameters. In particular, the time-instant t_f could be chosen as the "running" or "current" time t, and the time-instant t_i could be chosen as $t-1$, say, corresponding to an observation interval $[t-1, t]$ of duration 1. For a specific class of functions, such as polynomials or trigonometric polynomials, the functions $\phi_j(\tau)$ can be explicitly determined as functions of τ, with t_i and t_f as parameters. (See Example below.)*

In particular, if we choose $\phi_j(\tau)$ to be a polynomial of degree atmost $(2n-1)$, then the integrals in (6) involve "the moments"

$$\int_{t_i}^{t_f} \tau^k y(\tau)d\tau$$

for $k = 0, ..., (2n-1)$. These moments could be computed on-line, in real time. Then, the state $z(t)$ is given by

$$z(t) = M(t) \begin{bmatrix} \int_{t-1}^{t} y(\tau)d\tau \\ \int_{t-1}^{t} \tau y(\tau)d\tau \\ \vdots \\ \int_{t-1}^{t} \tau^{2n-1} y(\tau)d\tau \end{bmatrix}$$

where $M(t)$ is an appropriate polynomial matrix which can be determined knowing A and c. The integrals in (6) could also be generated by "hardware".

To “recover” the state of an observable linear system driven by an input, such as :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^T x \end{aligned} \tag{7}$$

one could construct a “model”- a “physical” one or a “computational” one- given by

$$\begin{aligned} \dot{w} &= Aw + bu \\ \bar{y} &= c^T w \end{aligned} \tag{8}$$

so that the difference $z = x - w$ between x and w satisfies an autunomous equation

$$\begin{aligned} \dot{z} &= Az \\ (y - \bar{y}) &= c^T z \end{aligned} \tag{9}$$

From $y - \bar{y}$, we can obtain $z(t)$ and then $x(t)$ as

$$x(t) = z(t) + w(t) \tag{10}$$

3 Example

Let

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} z \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} z \end{aligned} \tag{11}$$

be the observable system whose state $z(t)$ is to be determined. Let the associated controllable system, whose input is to be determined using the parametrization approach, be

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Choosing the parametrizing functions ϕ_1 and ϕ_2 to be polynomials of degree at most three, to steer the state $\mathbf{0}$ at time-instant t_i to e_1 and e_2, respectively, at time-instant t, we obtain.

$$\phi_1(\tau) = \frac{(\tau - t_i)^2(3t - t_i - 2\tau)}{(t - t_i)^3},$$

and

$$\phi_2(\tau) = \frac{(\tau - t_i)^2(\tau - t)}{(t - t_i)^2},$$

and so,

$$z(t)=\frac{1}{(t-t_i)^3}\begin{bmatrix}2t_i^2(3t-t_i)-18t_it+6(t+t_i) & -12t_it+18(t+t_i)-12 & 6(t+t_i)-18 & -4\\ -2t-4t_i+3t_i^2+6tt_i-2tt_i^2 & 6-6t-12t_i+2t_i^2+4tt_i & 9-2t-4t_i & 2\end{bmatrix}w \tag{12}$$

where

$$w=\begin{bmatrix}\int_{t_i}^t y(\tau)d\tau\\ \int_{t_i}^t \tau y(\tau)d\tau\\ \int_{t_i}^t \tau^2 y(\tau)d\tau\\ \int_{t_i}^t \tau^3 y(\tau)d\tau\end{bmatrix}.$$

If we choose $t_i=t-1$, we get

$$z(t)=\begin{bmatrix}(-4+30t-24t^2+4t^3) & (-30+48t-12t^2) & (-24+12t) & -4\\ (7-20t+13t^2-2t^3) & (20-26t+6t^2) & (13-6t) & 2\end{bmatrix}\begin{bmatrix}\int_{t-1}^t y(\tau)d\tau\\ \int_{t-1}^t \tau y(\tau)d\tau\\ \int_{t-1}^t \tau^2 y(\tau)d\tau\\ \int_{t-1}^t \tau^3 y(\tau)d\tau\end{bmatrix} \tag{13}$$

We could choose trigonometric polynomials for the interpolation. Thus, let the observation interval be $[t-\pi,t]$ and let

$$\phi_1(\tau)=a_1cos(\tau-t)+a_2cos2(\tau-t)+b_1sin(\tau-t)+b_2sin2(\tau-t).$$

Then we find, applying the condition (2),

$$\phi_1(\tau)=\frac{1}{2}cos(\tau-t)+\frac{1}{2}cos2(\tau-t)$$

and

$$\phi_2(\tau)=\frac{1}{2}sin(\tau-t)+\frac{1}{4}sin2(\tau-t).$$

The state reconstruction is given by

$$z(t)=\begin{bmatrix}\frac{1}{2}cost+\frac{3}{2}sint & -cos2t+3sin2t & \frac{1}{2}sint-\frac{3}{2}cost & -sin2t-3cos2t\\ cost-sint & cos2t+\frac{1}{4}sin2t & cost+sint & -\frac{1}{4}cos2t+sin2t\end{bmatrix}\begin{bmatrix}\int_{t-\pi}^t (cos\tau)y(\tau)d\tau\\ \int_{t-\pi}^t (cos2\tau)y(\tau)d\tau\\ \int_{t-\pi}^t (sin\tau)y(\tau)d\tau\\ \int_{t-\pi}^t (sin2\tau)y(\tau)d\tau\end{bmatrix}. \tag{14}$$

Reconstruction (10) might be preferred to (8) and (9) because of the boundedness of the trigonometric functions. Interestingly, the integrals in (10) are, almost, Fourier series coefficients of $y(\tau)$.

References

[1] S. D. Agashe and B. K. Lande, *A new approach to the state-transfer problem*, Journal of the Franklin Institute, Vol. 333B, Issue 1, pp.15-21, 1996.

[2] S. A. Deshpande and S. D. Agashe, *Application of a parametrization method to problem of optimal control*, Journal of the Franklin Institute, Vol. 348, Issue 9, pp.2390-2405, 2011.