Nonlinear eigenvalue problems - A Review

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Outline

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- Let $F(\lambda)$ be an $m \times n$ nonlinear matrix function.
  The nonlinear eigenvalue problem: Find scalars λ and nonzero vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ such that $F(\lambda)x = 0$ and $y^* F(\lambda) = 0$.
- λ is an eigenvalue, x, y are corresponding right and left eigenvectors.
- In practice, elements of F most often polynomial, rational or exponential functions of λ.

- Vibration problems, for example those that occur in a structure such as a bridge, are often modelled by the generalized eigenvalue problem

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- When damping effects are also included, the problem becomes a quadratic eigenvalue problem (QEP)

$$(\lambda^2 M + \lambda C + K)x = 0,$$

where C is the damping matrix.

Figure: Millennium Bridge

- On its opening day in June 2000, the 320-meter-long Millennium footbridge over the river Thames in London started to wobble under the weight of thousands of people.
- So two days later the bridge was closed.
- To explain the connection between this incident and the quadratic eigenvalue problem (QEP), the subject of this survey, we need to introduce some ideas from vibrating systems, resonance.

• *More details on the Millennium Bridge and the wobbling problem can be found at http://www.arup.com/MillenniumBridge*

(Millenium$_Bridge.mp4$)

- Other applications of the QEP include linear stability of flows in fluid mechanics, electrical circuit simulation and in modelling microelectronic mechanical systems.
- The excellent review paper by Tisseur and Meerbergen [2001] describes many of the applications of the quadratic eigenvalue problem.

• *F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem. SIAM Review, 43:235-286, 2001.*

- Similarly, the polynomial eigenvalue problem $P(\lambda)x = 0$, where $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ and A_j's are $n \times n$ matrices, arise in the study of the vibration analysis of buildings, machines, and vehicles.

- Lancaster, Lambda-Matrices and Vibrating Systems, 1966 (Pergamon Press), 2002 (Dover).

- Gohberg, Lancaster, Rodman, Matrix Polynomials, 1982, (Academic Press, New York), 2009 (SIAM).

- Nonlinear Eigenvalue Problems: A Challenge for Modern Eigenvalue Methods, Volker Mehrmann and Heinrich Voss.

Historical Aspects

- In the 1930s, Frazer, Duncan Collar were developing matrix methods for analyzing flutter in aircraft.
- Worked in Aerodynamics Division of National Physical Laboratory (NPL).
- Wrote Elementary Matrices and Some Applications to Dynamics and Differential Equations, 1938.
- Olga Taussky, in Frazers group at NPL, 1940s. 6×6 quadratic eigenvalue problems from flutter in supersonic aircraft.

(Aircraft Flutter.mp4)

Flutter.mp4

Historical Aspects Cont., (Books)

- Peter Lancaster, 1950s solved quadratic eigenvalue problems of dimension 2 to 20.
- Gohberg, Lancaster, Rodman, Indefinite Linear Algebra and Applications, 2005 (Birkhäuser).
- Gohberg, Lancaster, Rodman, Invariant Subspaces of Matrices with Applications, 1986 (Wiley), 2006 (SIAM).

- Rational eigenvalue problems also arise in wide range of applications such as in calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid-solid structures and in control theory.

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- For example, the rational eigenproblem [V. Mehrmann and H. Voss]

$$G(\lambda)x := (K - \lambda M + \sum_{j=1}^{k} \frac{\lambda}{\lambda - \sigma_j} C_j)x = 0,$$

where $K = K^T$ and $M = M^T$ are positive definite and $C_j = C_j^T$ are matrices of small ranks, arises in the study of the vibrations of fluid solid structures.

- A similar problem

$$G(\lambda)x = -Kx + \lambda Mx + \lambda^2 \sum_{j=1}^{k} \frac{1}{\omega_j - \lambda} C_j x = 0,$$

arises when a generalized linear eigenproblem is condensed exactly [V. Mehrmann and H.Voss].

- Another type of rational eigenproblem is obtained for the free vibrations of a structure . A finite element model takes the form

$$G(\lambda)x := (\lambda^2 M + K - \sum_{j=1}^{k} \frac{1}{1 + b_j \lambda} \Delta K_j)x = 0,$$

Where K and M are positive definite [V. Mehrmann and H.Voss].

- Consider the delay differential equations $(DDEs)$:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, $A_i \in \mathbb{R}^{n \times n}$, $i = 0 : m$ are real entries, and $0 < \tau_1 < \tau_2 \cdots < \tau_m$ represent the time-delays.

- The substitution of a sample solution of the form $e^{\lambda t} v$,with $v \neq 0$, leads us to the characteristic equation

$$H(\lambda)v := (\lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda \tau_i})v = 0.$$

where $H(\lambda)$ is holomorphic in λ.

Comments: NEPs Can be very difficult to solve:

- nonlinear,
- large problem size,
- poor conditioning,
- lack of good numerical methods.

- Motivated by these applications, we consider the following problem.
- **Problem.** Let $F(\lambda)$ be an $n \times n$ nonlinear matrix function. Compute $\lambda \in \mathbb{C}$ and nonzero vectors x and y in $\mathbb{C}^n$ such that $F(\lambda)x = 0$ and $y^* F(\lambda) = 0$.

NLEVP Toolbox

Collection of Nonlinear Eigenvalue Problems : T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, F. T., 2010.

- Quadratic, polynomial, rational and other nonlinear eigenproblems.
- Provided in the form of a MATLAB Toolbox.
- Problems from real-life applications + specifically constructed problems.
- For example: Loaded string problem gives the EVP

$$G(\lambda)x = \left(A - \lambda B + \frac{\lambda}{\lambda - \sigma} E\right) x = 0.$$

Then using the NLEVP toolbox, we can find out the coefficients A, B, E exactly for our numerical computations.

http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html

Outline

Example

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 0 & 8 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $\det Q = -6\lambda^5 + 11\lambda^4 - 12\lambda^3 + 12\lambda^2 - 6\lambda + 1 \neq 0$, for some λ.
Six eigenpairs $(\lambda_m, x_m), m = 1:6$, given by

m	1	2	3	4	5	6
λ_m	1/3	1/2	1	i	-i	∞
x_m	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Solution Method for PEP

Now consider $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i, \; A_i \in \mathbb{C}^{n\times n}, \; A_m \neq 0$

Regular: if $\det(P(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$.

Spectrum: $\mathbf{Sp}(P) := \{\lambda \in \mathbb{C} : \det(P(\lambda)) = 0\}$.

- $P(\lambda)$ has mn eigenvalues. Zero eigenvalues when A_0 is singular and infinite eigenvalues when A_m is singular.

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Spectrum: $\mathbf{Sp}(P) := \{\lambda \in \mathbb{C} : \det(P(\lambda)) = 0\}$.

- $P(\lambda)$ has mn eigenvalues. Zero eigenvalues when A_0 is singular and infinite eigenvalues when A_m is singular.
- The standard approach for solving and investigating polynomial eigenvalue problem $P(\lambda)x = 0$ is to transform the given polynomial into a linear matrix pencil $L(\lambda) = \lambda X + Y$ with the same eigenvalue and then solve with this pencil.

Definition

Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree m with $m \geq 1$. A pencil $\lambda X + Y$ with $X, Y \in \mathbb{C}^{mn \times mn}$ is called a linearization of $P(\lambda)$ if their exist unimodular matrix polynomials $E(\lambda), F(\lambda)$ ($\det E(\lambda)$ is a nonzero constant, independent of λ) such that
$E(\lambda)(\lambda X + Y)F(\lambda) = \text{diag}\left(P(\lambda), I_{(m-1)n}\right)$ for all $\lambda \in \mathbb{C}$.

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Consider $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$. Then we have

$$C(\lambda)z := \left(\lambda \begin{bmatrix} A_2 & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ -I_n & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ \lambda x \end{bmatrix} =: (\lambda X + Y)z = 0.$$

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- $C(\lambda)$ is referred as the first companion linearization of $Q(\lambda)$.
- Solve generalized eigenvalue problem (GEP).
- Recover eigenvectors of $Q(\lambda)$ from those of $\lambda X + Y$.

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- Recently, another class of linearizations of a matrix polynomial - referred to as Fiedler linearizations has been introduced in [Fernando, Dopico , Mackey, Antoniou, Vologiannidis, (2009, 2010)]

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- It has been shown in Freiling, Mehrmann Xu (2002) and Ran Rodman (1988, 1989) that the problem may be well-conditioned under structured perturbations, but ill-posed under unstructured perturbations.
- So it is important to study structure preserving linearizations, see D. Steven Mackey Thesis, 2006 and R. Alam and Bibhas Adhikari, 2011.
- Developed theory concerning the sensitivity and stability of linearizations [Higham, Mackey, 2006, Higham, Li, 2007, Grammont, Higham, 2011].

Outline

- As we have seen Rational Eigenvalue Problems arise in a variety of physical applications, e.g., Vibration of Fluid solid structures, calculations of quantum dots, ...

- Matrix function takes the form

$$G(\lambda) = P(\lambda) - \sum_{i=1}^{k} \frac{s_i(\lambda)}{q_i(\lambda)} E_i,$$

  - $P(\lambda)$ is a matrix polynomial,
  - s_i and q_i are scalar polynomials,
  - E_i are constant matrices.

Numerical Methods for REP

Iterative Methods: Newton Method, nonlinear Arnoldi, Jacobo-Davidson,...

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Iterative Methods: Newton Method, nonlinear Arnoldi, Jacobo-Davidson,...

- Suitable when a few eigenpairs are desired.
- Convergence analysis is a challenging task.

Direct Method: For example consider

$$G(\lambda) := \begin{bmatrix} 1 & \frac{1}{\lambda-2} \\ 0 & 1 \end{bmatrix} \text{ and } P(\lambda) = (\lambda-2)G(\lambda) = \begin{bmatrix} \lambda-2 & 1 \\ 0 & \lambda-2 \end{bmatrix},$$

$\mathrm{Eig}(G) = \phi$ and $\mathrm{Eig}(P) = \{2\}$

- Not practical, when $q_i(\lambda)$ has several poles and PEP may introduce spurious eigenvalues.

Solution via Minimal realization

Linearizations for rational problems: [Su and Bai, Simax[2011]]

- Rewrite

$$G(\lambda) = P(\lambda) - \sum_{i=1}^{k} \frac{s_i(\lambda)}{q_i(\lambda)} E_i,$$

as minimal realization of $G(\lambda)$ of the form

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1} B, \tag{2}$$

where $A, E \in \mathbb{C}^{r \times r}$ and E is nonsingular.

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where $A, E \in \mathbb{C}^{r\times r}$ and E is nonsingular.

- Example: if $P(\lambda) = \lambda A_1 + A_0$, $G(\lambda)x = 0$ becomes a linear eigenproblem

$$\mathcal{C}(\lambda) := \left(\lambda \begin{bmatrix} A_1 & 0 \\ 0 & -E \end{bmatrix} + \begin{bmatrix} A_0 & C \\ B & A \end{bmatrix}\right) \begin{bmatrix} x \\ (\lambda E - A)^{-1} B x \end{bmatrix} = 0.$$

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The pencil $\mathcal{C}(\lambda)$ is referred to as a companion form of $G(\lambda)$ and the eigenvalues and eigenvectors of $G(\lambda)$ could be computed by solving the generalized eigenvalue problem for the pencil.

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- An advantage of realization based approach to solving problem is that the size of the companion form $\mathcal{C}(\lambda)$ of $G(\lambda)$ could be much smaller than that of a pencil obtained by converting the REP to PEP followed by linearization especially when the coefficient matrices of $G(\lambda)$ have low ranks see, Su Bai, 2011 paper.

Outline

- Recall the example, $G(\lambda) = \begin{bmatrix} 1 & 1/\lambda - 2 \\ 0 & 1 \end{bmatrix}$ has no eigenvalue.

  Therefore, it is necessary to enlarge the spectrum of a rational matrix function so that the spectrum is nonempty.

Eigenvalues and Eigenpoles

- Suppose that the Smith-McMillan form $\mathbf{SM}(G(\lambda))$ of $G(\lambda)$ is given by

$$\mathbf{SM}(G(\lambda)) = \mathrm{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \cdots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0_{n-k,n-k}\right),$$

where the scalar polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are monic, are pairwise coprime and, $\phi_i(\lambda)$ divides $\phi_{i+1}(\lambda)$ and $\psi_{i+1}(\lambda)$ divides $\psi_i(\lambda)$, for $i = 1, 2, \ldots, k-1$.

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- Define the **zero polynomial** $\phi_G(\lambda)$ and the **pole polynomial** $\psi_G(\lambda)$ of $G(\lambda)$ are given by
$$\phi_G(\lambda) := \prod_{j=1}^{k} \phi_j(\lambda) \quad \text{and} \quad \psi_G(\lambda) := \prod_{j=1}^{k} \psi_j(\lambda). \tag{3}$$

Definition (Zeros and poles, Vardulakis)

A complex number λ is said to be a **zero** (**pole**) of $G(\lambda)$ if $\phi_G(\lambda) = 0$ ($\psi_G(\lambda) = 0$).

- We denote the normal rank of $G(\lambda)$ by $\mathrm{nrank}(G)$. Then $\mathrm{nrank}(G) = \max_\lambda \mathrm{rank}(G(\lambda))$ where the maximum is taken over all λ which are not poles.

Eigenvalues **Eig**$(G) := \{\lambda_0 \in \mathbb{C} : \mathrm{rank}(G(\lambda_0)) < \mathrm{nrank}(G).\}$

Eigenpoles **Eip**$(G) := \{\lambda_0 \in \mathbb{C} : \lambda_0$ is a pole of $G(\lambda)$ and there exists $v(\lambda) \in \mathbb{C}^n[\lambda]$ with $v(\lambda_0) \neq 0$ such that $\lim_{\lambda \to \lambda_0} G(\lambda)v(\lambda) = 0.\}$

- For example, consider

$$G(\lambda) := \begin{bmatrix} 1 & \frac{1}{\lambda-2} \\ 0 & 1 \end{bmatrix}.$$

Then $G(\lambda)$ is proper and **Eig**$(G) = \{\emptyset\}$. However, $\lambda = 2$ is an eigenpole of $G(\lambda)$, since $u(\lambda) = \begin{bmatrix} 1 & -(\lambda-2) \end{bmatrix}^T$ and $\lim_{\lambda \to 2} u(\lambda) = e_1$ and $G(\lambda)u(\lambda) \to 0$ as $\lambda \to 2$. Note that 2 is a pole of G. So **Eip**$(G) = \{2\}$.

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- On the other hand, considering $G(\lambda) := \mathrm{diag}(\lambda, 1/\lambda)$, it follows that $G(\lambda)$ is nonproper $\textbf{Eig}(G) = \{\emptyset\}$. However, $\textbf{Eip}(G) = \{0, \infty\}$.

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- Spectrum **Sp**(G) := **Eig** $(G)\cup$**Eip**(G).

- Note that **Eig**$(G) \subsetneq$ **Sp**(G).

Theorem

Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$. *Then we have*

$$\begin{aligned} \mathbf{Eig}(G) &= \{\lambda \in \mathbb{C} : \phi_G(\lambda) = 0 \text{ and } \psi_G(\lambda) \neq 0\}, \\ \mathbf{Eip}(G) &= \{\lambda \in \mathbb{C} : \phi_G(\lambda) = 0 \text{ and } \psi_G(\lambda) = 0\}. \end{aligned}$$

Thus $\mathbf{Sp}(G) = \{\lambda \in \mathbb{C} : \phi_G(\lambda) = 0\}$ *and* $\mathbf{Eip}(G) = \mathbf{Sp}(G) \cap \mathit{Poles}(G)$.

- For finding out the pencils and linearizations for rational matrix function we need LTI State Space System, Rosenbrock system matrix and the linearizations for Rosenbrock system matrix.
- For this purpose, in my Ph.D, thesis I introduced three classes of linearizations, which we refer to as Fiedler pencils, Generalized Fiedler pencils and Generalized Fiedler pencils with repetition of the Rosenbrock system polynomial.
- Then under some conditions one can show that the pencils of Rosenbrock system matrix are also linearization of rational matrix function $G(\lambda)$.

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Thank You