Games and Graphs - Linear Complementarity and the Clique number

Parthe Pandit

Joint work with Prof Ankur Kulkarni

IIT Bombay

April 5, 2016

Students Reading Group, Department of Electrical Engineering, IIT Bombay

"A linear complementarity based characterization of the weighted independence number and the independent domination number in graphs"
http://arxiv.org/abs/1603.05075

Outline

- Introduction
  - Games and the linear complementarity problem (LCP)
  - Graphs and the clique (independence) number problem
- Complexity of LCP and clique number
- Motivation
- Previous work and Our contributions
- Main results
  - LCP formulation for the clique (independence) number
  - Applications of the LCP formulation

Example of NE in Bimatrix Games

Sibling rivalry

$$A(Ian) = \begin{array}{c} \\ Cricket \\ Movie \end{array} \begin{array}{c} \begin{array}{cc} Cricket & Movie \end{array} \\ \left[\begin{array}{cc} \boxed{2} & 0 \\ 0 & \boxed{1} \end{array} \right] \end{array}$$

$$B(eth) = \begin{array}{c} \\ Cricket \\ Movie \end{array} \begin{array}{c} \begin{array}{cc} Cricket & Movie \end{array} \\ \left[\begin{array}{cc} \boxed{1} & 0 \\ 0 & \boxed{2} \end{array} \right] \end{array}$$

Bimatrix Games

- A simultaneous game between two players P_1 and P_2
- Finite set of actions $\mathcal{A}_1$ and $\mathcal{A}_2$ of cardinalities n and m respectively
- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be their $(n \times m)$ payoff[1] matrices:
  a_{ij} is the gain of P_1, if P_1 plays $i \in S_1$ and P_2 plays $j \in S_2$
  b_{ij} is the gain of P_2, if P_1 plays $i \in S_1$ and P_2 plays $j \in S_2$
- P_1 and P_2 play by a strategy $x \in \Delta_n$ and $y \in \Delta_m$ respectively, which are p.m.f.[2] over action spaces $\mathcal{A}_1$ and $\mathcal{A}_2$
- Their respective expected payoffs are $x^\top Ay$ and $x^\top By$
- A strategy profile (x^*, y^*) is called a Nash equilibrium (NE) if neither player benefits by **unilaterally deviating** from it

[1] or loss matrices

[2] Δ_n and Δ_m are the respective spaces of *mixed strategies*

Nash equilibria of Bimatrix games

- A pair of vectors $(x^*, y^*) \in \Delta_n \times \Delta_m$ is a NE is equivalent to,

$$(x^*)^\top A y^* \le x^\top A y^*, \quad \forall\, x \in \Delta_n, \quad (x^*)^\top B y^* \le (x^*)^\top B y, \quad \forall\, y \in \Delta_m,$$

- Let $\quad x' = x^*/(x^*)^\top B y^* \qquad y' = y^*/(x^*)^\top A y^*$
- It can be shown that if (x^*, y^*) is a NE [2, p. 6] then,

$$x', \; y' \ge 0,$$
$$w = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - \mathbf{e} \ge 0,$$
$$w^\top \begin{bmatrix} x' \\ y' \end{bmatrix} = 0,$$

- Conversely, if (x', y') satisfy these equations then $x^* = x'/(\sum_i x'_i)$ and $y^* = y'/\sum_j y'_j$ is a Nash equilibrium.

The Linear Complementarity Problem (LCP)

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, $\mathrm{LCP}(M,q)$ is the following problem,

$$\text{Find} \quad x \in \mathbb{R}^n \text{ such that} \qquad \begin{aligned} & x \geq 0, && (1)\\ & y = Mx + q \geq 0, && (2)\\ & y^\top x = 0. && (3) \end{aligned}$$

- Linear complementarity problems arise naturally through the modelling of several problems in optimization and allied areas
- Complementarity constraints (3) implies $x_i y_i = 0$, i.e., $x_i = 0 \vee y_i = 0$, since x and y are non-negative vectors.

$$\boxed{x \geq 0,\ \ y = Mx + q \geq 0, \quad x_j = 0,\ \ \forall j \notin S \text{ and } \ y_j = 0, \quad \forall j \in S.}$$

- Structure of the solution set of $\mathrm{LCP(M,q)}$ is the union of 2^n polyhedra corresponding to every subset $S \subseteq \{1, 2, \ldots, n\}$
- Although an LCP is a continuous optimization problem, it implicitly encodes a problem of combinatorial character

LCP and Convex quadratic programming

Given a symmetric positive semidefinite matrix Q, a matrix A and vectors b and c of appropriate dimensions, consider the following

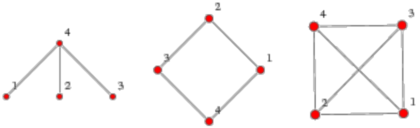
$$
\text{QP} \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{subject to} & \begin{array}{rl} Ax \geq b, & : \lambda \\ x \geq 0, & \end{array} \end{array}
$$

- Let λ denote the vector of Lagrange multipliers corresponding to the constraint "$Ax \geq b$".
- From the KKT conditions, x solves QP *iff* $\exists\, \lambda$ such that,

$$
\begin{pmatrix} x \\ \lambda \end{pmatrix} \geq 0, \quad \begin{pmatrix} Qx + c - A^\top\lambda \\ Ax - b \end{pmatrix} \geq 0, \quad \begin{pmatrix} x \\ \lambda \end{pmatrix}^\top \begin{pmatrix} Qx + c - A^\top\lambda \\ Ax - b \end{pmatrix} = 0.
$$

- This is clearly an LCP in the (x, λ)-space.

- A simple undirected graph $G = (V, E)$ consists of vertices V and edges E which are unordered 2-tuples of distinct vertices.
- Adjacency matrix of a graph is the $|V| \times |V|$ matrix $A = [a_{ij}]$, with $a_{ij} = 1$ iff $(i, j) \in E$
- Trees, Cycles (C_n) and Cliques (K_n)



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure : Graphs and Adjacency Matrices.

- The complement of a graph ($\bar{G}$) the graph with the same vertex set but Edges swapped with non-edges

- A set of vertices $S \subseteq V$ is **independent** if its elements are pairwise disconnected. Independent set S is maximal if it is not a subset of a larger independent set. Maximal independent sets (MIS) can be arrived at using a greedy algorithm.

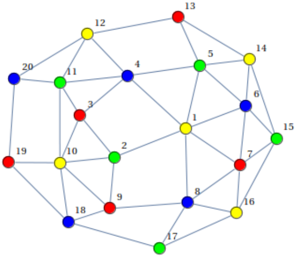


Figure : Arrondissements of Paris. *4-colour theorem*

- A clique is a complete subgraph of the graph, i.e. an independent set of the complement graph.

Independence number

- The maximum and minimum cardinalities of maximal independent sets of a graph G are denoted by $\alpha(G)$[3] and $\beta(G)$[4] respectively. The clique number of a graph is the size of the largest clique, i.e., $\omega(G) := \alpha(\bar{G})$

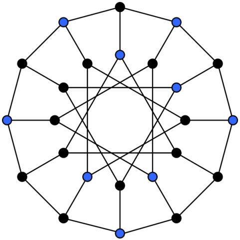


Figure : Maximum Independent set of a Petersen graph.

- Given a vector of vertex weights w. Find $\alpha_w(G)$ – the maximum of sum of vertex weights of independent sets

[3] It is called the **independence number**

[4] Referred to as independent domination number of a graph

Independence number in Coding theory

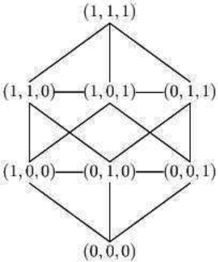
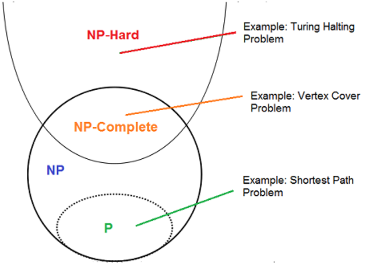


Figure : Graph of an asymmetric error channel[5] in $\mathbb{F}_2^3$ with $d = 1$.

- Consider a finite block length communication system $\mathcal{C}_{q,d}^n$ with symbols as strings in $\mathbb{F}_q^n$ and vulnerable to d possible errors
- Consider the following graph $G = (\mathbb{F}_q^n, \mathcal{E}_d^n)$ such that for $x, y \in \mathbb{F}_q^n$, $(x, y) \in \mathcal{E}_d^n$ iff "*decoder can mistake x as y*"
- Let $\mathcal{M}_{q,d}^n$ denote the size of the optimal error-correcting code over the channel. Then, $\mathcal{M}_{q,d}^n = \alpha(G)$

[5] $\mathbb{P}(0 \to 1) = 0$

Complexity of LCP and Independence number



This diagram assumes that P != NP

- For rational matrices M and q, solving $\mathrm{LCP}(M,q)$ is NP-complete [1]
- For a general graph G, finding $\alpha(G)$ and $\beta(G)$ are NP-complete problems
- For a general graph with n vertices, there exists no polynomial algorithm[6] that can approximate the independence number within the interval $[n^{1-\epsilon}\alpha(G), \alpha(G)]$ [3], unless $P = NP$

[6] polynomial in n and ϵ

Previous Work

- Independence number is an NP-hard discrete optimization problem to which continuous optimization formulations exist:
  1. Motzkin Strauss theorem (1965)

$$\frac{1}{\alpha(G)} = \min\{x^\top (A+I)x \mid \mathbf{e}^\top x = 1,\ x \geq 0\}$$

  2. Harant et al.

$$\alpha(G) = \max\{\mathbf{e}^\top x - \tfrac{1}{2}x^\top A x \mid 0 \leq x \leq \mathbf{e}\}$$

Our contributions

MAIN RESULT:

- LCP based characterization for *w*-weighted independence number $\alpha_w(G)$ and $\beta(G)$

APPLICATIONS:

- New ILP for finding $\alpha(G)$ and $\beta(G)$
- SDP based upper bound for independence number **stronger than Lovász theta**.
- A new sufficient condition for a graph to be *well-covered*.
- Inapproximability result about linear programs with complementarity constraints (LPCC)

Main LCP

$$\boxed{\text{Find } x \ : \ x_i \geq 0, \ \ \mathcal{C}_i(x) \geq 1, \ \ x_i\big(\mathcal{C}_i(x) - 1\big) = 0, \ \forall \ i \in V.} \qquad (4)$$

For a graph $G = (V, E)$, we study the problem $\mathrm{LCP}(A + I, -\mathbf{e})$, where A is the adjacency matrix of G, I is the identity matrix and $\mathbf{e}$ is the vector of ones

- Let $\mathcal{C}(x) := (A + I)x$ whereby,

$$\mathcal{C}_i(x) := x_i + \sum_{j \in N(i)} x_j$$

- Hence $\mathrm{LCP}(A + I, -\mathbf{e})$ is (4),
- For $x \in \mathbb{R}^{|V|}$, let $\sigma(x) := \{i \in V \mid x_i > 0\}$, the support of x.
- Let G_S denote the subgraph of G induced by $S \subseteq V$ and x_S denote the subvector of x indexed by the set S

Main LCP

$$\boxed{\text{Find } x \ : \ x_i \geq 0, \ \ \mathcal{C}_i(x) \geq 1, \ \ x_i\big(\mathcal{C}_i(x) - 1\big) = 0, \ \forall \ i \in V.} \qquad (4)$$

For a graph $G = (V, E)$, we study the problem $\mathrm{LCP}(A + I, -\mathbf{e})$, where A is the adjacency matrix of G, I is the identity matrix and $\mathbf{e}$ is the vector of ones

- Let $\mathcal{C}(x) := (A + I)x$ whereby,

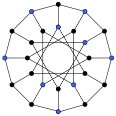
$$\mathcal{C}_i(x) := x_i + \sum_{j \in N(i)} x_j$$

- Hence $\mathrm{LCP}(A + I, -\mathbf{e})$ is (4),
- For $x \in \mathbb{R}^{|V|}$, let $\sigma(x) := \{i \in V \mid x_i > 0\}$, the support of x.
- Let G_S denote the subgraph of G induced by $S \subseteq V$ and x_S denote the subvector of x indexed by the set S

Does the $\mathrm{LCP}(A + I, -\mathbf{e})$ have a game theoretic interpretation?

YES.

$\mathrm{LCP}(A + I, -\mathbf{e})$ and the Public Goods Game



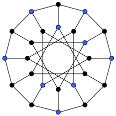
- Let there exist a social network $G = (V, E)$ of people. And let every player put in effort x_i with marginal cost c and obtain a benefit $b(x_i + \sum_{j \in N_i} x_j) = b(\mathcal{C}_i(x))$, i.e. players benefit from their neighbours and their own efforts
- Eg: Going to EE office to submit assignment : $x_i \in \{0, 1\}$
- Payoff of player i is $U_i(x) = b(\mathcal{C}_i(x)) - cx_i$
- Let $b : \mathbb{R} \to \mathbb{R}$, be concave monotone[7], i.e., $b(0) = 0,\ b' > 0,\ b'' < 0$. Let $b'(1) = c$, w.l.o.g.
- Solutions to $\mathrm{LCP}(A + I, -\mathbf{e})$ correspond to NE in this game

[7] reasonable assumption

Does the $\mathrm{LCP}(A + I, -\mathbf{e})$ have a game theoretic interpretation?

YES.

$\mathrm{LCP}(A + I, -\mathbf{e})$ and the Public Goods Game



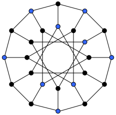
- Let there exist a social network $G = (V, E)$ of people. And let every player put in effort x_i with marginal cost c and obtain a benefit $b(x_i + \sum_{j \in N_i} x_j) = b(\mathcal{C}_i(x))$, i.e. players benefit from their neighbours and their own efforts
- Eg: Going to EE office to submit assignment : $x_i \in \{0, 1\}$
- Payoff of player i is $U_i(x) = b(\mathcal{C}_i(x)) - cx_i$
- Let $b : \mathbb{R} \to \mathbb{R}$, be concave monotone[7], i.e., $b(0) = 0$, $b' > 0$, $b'' < 0$. Let $b'(1) = c$, w.l.o.g.
- Solutions to $\mathrm{LCP}(A + I, -\mathbf{e})$ correspond to NE in this game

[7] reasonable assumption

Does the $\mathrm{LCP}(A+I,-\mathbf{e})$ have a game theoretic interpretation?

YES.

$\mathrm{LCP}(A+I,-\mathbf{e})$ **and the Public Goods Game**



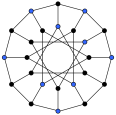
- Let there exist
  a social network $G=(V,E)$ of people. And let every player put in effort x_i with marginal cost c and obtain a benefit $b(x_i+\sum_{j\in N_i} x_j)=b(\mathcal{C}_i(x))$, i.e. players benefit from their neighbours and their own efforts
- Eg: Going to EE office to submit assignment : $x_i \in \{0,1\}$
- Payoff of player i is $U_i(x)=b(\mathcal{C}_i(x))-cx_i$
- Let $b:\mathbb{R}\to\mathbb{R}$, be concave monotone[7], i.e., $b(0)=0$, $b'>0$, $b''<0$. Let $b'(1)=c$, w.l.o.g.
- Solutions to $\mathrm{LCP}(A+I,-\mathbf{e})$ correspond to NE in this game

[7] reasonable assumption

Does the $\mathrm{LCP}(A+I,-\mathbf{e})$ have a game theoretic interpretation?

YES.

$\mathrm{LCP}(A+I,-\mathbf{e})$ **and the Public Goods Game**



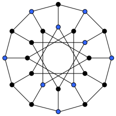
- Let there exist a social network $G=(V,E)$ of people. And let every player put in effort x_i with marginal cost c and obtain a benefit $b(x_i+\sum_{j\in N_i} x_j)=b(\mathcal{C}_i(x))$, i.e. players benefit from their neighbours and their own efforts
- Eg: Going to EE office to submit assignment : $x_i \in \{0,1\}$
- Payoff of player i is $U_i(x)=b(\mathcal{C}_i(x))-cx_i$
- Let $b:\mathbb{R}\to\mathbb{R}$, be concave monotone[7], i.e., $b(0)=0,\ b'>0,\ b''<0$. Let $b'(1)=c$, w.l.o.g.
- Solutions to $\mathrm{LCP}(A+I,-\mathbf{e})$ correspond to NE in this game

[7] reasonable assumption

Does the $\mathrm{LCP}(A + I, -\mathbf{e})$ have a game theoretic interpretation?

YES.

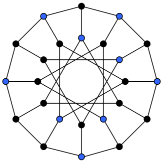
$\mathrm{LCP}(A + I, -\mathbf{e})$ **and the Public Goods Game**



- Let there exist
  a social network $G = (V, E)$ of people. And let every player put in effort x_i with marginal cost c and obtain a benefit $b(x_i + \sum_{j \in N_i} x_j) = b(\mathcal{C}_i(x))$, i.e. players benefit from their neighbours and their own efforts
- Eg: Going to EE office to submit assignment : $x_i \in \{0, 1\}$
- Payoff of player i is $U_i(x) = b(\mathcal{C}_i(x)) - cx_i$
- Let $b : \mathbb{R} \to \mathbb{R}$, be concave monotone[7], i.e., $b(0) = 0$, $b' > 0$, $b'' < 0$. Let $b'(1) = c$, w.l.o.g.
- Solutions to $\mathrm{LCP}(A + I, -\mathbf{e})$ correspond to NE in this game

[7] reasonable assumption

Intermediate Lemmas

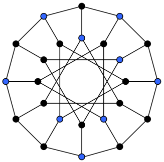


Lemma

For a graph $G = (V, E)$, *if* $x \in \mathbb{R}^n$ solves $\mathrm{LCP}(A + I, -\mathbf{e})$ *then,*

1. $x \neq 0$ *and* $0 \leq x \leq \mathbf{e}$,
2. $x \in \{0, 1\}^n$ *iff* $\sigma(x)$ *is a maximal independent set,*
3. *If* G *is a forest, then* $\sigma(x) = V$ *only if* $K_1 \cup K_2$,
4. *If* x *solves* $\mathrm{LCP}(G)$, *then* x_S *solves* $\mathrm{LCP}(G_S)$. *Exit of free-riders doesn't affect the equilibrium.*
5. *If* S *is a maximal independent set of* G, *and* x *is a solution such that* $S \subseteq \sigma(x)$, *then* $\mathbf{e}^\top x \leq |S|$,

Intermediate Lemmas



Lemma

For a graph $G = (V, E)$, if $x \in \mathbb{R}^n$ solves $\mathrm{LCP}(A + I, -\mathbf{e})$ then,

1. $x \neq 0$ *and* $0 \leq x \leq \mathbf{e}$,
2. $x \in \{0, 1\}^n$ *iff* $\sigma(x)$ *is a maximal independent set,*
3. *If* G *is a forest, then* $\sigma(x) = V$ *only if* $K_1 \cup K_2$,
4. *If* x *solves* $\mathrm{LCP}(G)$, *then* x_S *solves* $\mathrm{LCP}(G_S)$. *Exit of free-riders doesn't affect the equilibrium.*
5. *If* S *is a maximal independent set of* G, *and* x *is a solution such that* $S \subseteq \sigma(x)$, *then* $\mathbf{e}^\top x \leq |S|$,

Main Result

- Let $M(G)$ and $m(G)$ indicate the maximum and minimum ℓ_1 norm of solutions of $\mathrm{LCP}(G)$.
- From Lemma 3, characteristic vectors maximal independent sets are solutions to $\mathrm{LCP}(G)$. Hence we have

$$\alpha(G) \le M(G), \qquad \beta(G) \ge m(G)$$

Theorem

For a graph $G = (V, E)$, if $w \in \mathbb{R}^{|V|}$ is a non-negative vector

$$\alpha_w(G) = M_w(G) = \max\{w^\top x \mid x \text{ solves } \mathrm{LCP}(A + I, -\mathbf{e})\}.$$

$$\beta(G) \ge m(G) = \min\{\mathbf{e}^\top x \mid x \text{ solves } \mathrm{LCP}(A + I, -\mathbf{e})\},$$

equality for $\beta(G)$ is achieved if G is a forest.

Proof by Induction

$$M(G) \leq \alpha(G)$$

We prove this using induction on the number of vertices n of G.

- For the graph G_1 consisting of a single vertex, the adjacency matrix is the scalar 0 and $\mathrm{SOL}(G_1) = \{1\}$. Thus the statement holds for the base case.
- Assume the induction hypothesis for all graphs with $n < k$ vertices
- For $n = k$, let x^* be the LCP solution with maximum ℓ_1 norm
  - **Case I**: $\sigma(x^*) = V$. From Lemma 5, we have $\mathbf{e}^\top x \leq |S|$ for any maximal independent set of G, whereby $M(G) \leq \alpha(G)$
  - **Case II**: $x_i^* = 0$ for some i. Consider the subgraph G_{-i} by omitting i and its edges. Using Lemma 4, we have that x_{-i} solves $\mathrm{LCP}(G_{-i})$ whereby

$$M(G) = \mathbf{e}^\top x = \mathbf{e}_{-i}^\top x_{-i} \leq M(G_{-i}) \leq \alpha(G_{-i}) \leq \alpha(G)$$

This concludes the proof for $\alpha(G) = M(G)$. The weighted case follows in a similar manner.

NEW ILP for $\alpha(G)$ and $\beta(G)$

- We derive a new integer linear program (ILP) for $\alpha(G)$ which is more efficient than the previously known formulation.

$$\alpha(G) = \max_{\{0,1\}^n} \left\{ \sum_{i \in V} x_i \mid x_i \ + \ x_j \leq 1, \forall \ (i,j) \in E \right\}, \qquad (edge - ILP)$$

$$\alpha(G) = \max_{\{0,1\}^n} \left\{ \sum_{i \in V} x_i \mid 0 \leq \mathcal{C}_i(x) - 1 \leq r(1 - x_i), \forall i \in V \right\}, \quad (ILP^*)$$

where $r = d_i - 1$ is an upper bound on $\mathcal{C}_i(x) - 1$.

- The constraint in the ILP^* above is a proxy for i^{th} complementarity constraint for binary vectors
- The number of constraints in the ILP^* is invariant to number of edges which could be $\mathcal{O}(n^2)$ for densely connected graphs.

BOUNDS ON $\alpha(G)$

- Semidefinite programs are convex optimization problems which are solvable in polynomial time

$$\min_{X \succeq 0}\{C \bullet X \mid A_i \bullet X \leq b_i,\ i = 1, 2, \ldots, m\},$$

where $C \bullet X := tr(C^\top X)$

- SDP relaxation of $\max_{x \in \{0,1\}} \{c^\top x \mid Ax \geq b\}$ is obtained as follows
  - Multiply every equation by x_i and $1 - x_i$.
  - Replace product terms $x_i x_j$ by X_{ij} and x_i^2 by x_i[8].
  - $\therefore X = xx^\top$ and $\operatorname{diag}(X) = x$
  - ILP is now of the form

$$\min\{C \bullet X \mid A_i \bullet X \leq b_i,\ i = 1, 2, \ldots, m;\ rank(X) = 1\}$$

  - Relaxing the *rank* constraint gives a semidefinite program

[8] since $x_i \in \{0, 1\}$

BOUNDS ON $\alpha(G)$

- Lovasz theta ($\vartheta(G)$) is perhaps the most famous SDP bound for $\alpha(G)$

$$\begin{aligned} \vartheta(G) = \max_{X \succeq 0} \quad & \mathbf{e}\mathbf{e}^\top \bullet X \\ \text{s.t.} \quad & tr(X) = 1, \\ & X_{ij} = 0, \ (i,j) \in E(G), \end{aligned} \qquad (\vartheta\text{-SDP})$$

- SDP relaxation of the *ILP** using *Lift-and-Project* method gives a new variant of the Lovász theta $\vartheta^*(G) \le \vartheta(G)$.

$$\alpha(G) \le \vartheta^*(G) \le \vartheta(G),$$

where equality is attained for perfect graphs.

Application III

WELL-COVEREDNESS

- A graph is *well-covered* if all its maximal independent sets are of the same cardinality, i.e., $\alpha(G) = \beta(G)$.
- Clearly, we have that a graph G is well covered if $\mathbf{e}^\top x$ is constant for all vectors x that solve $\mathrm{LCP}(G)$.
- Moreover, this is also necessary condition if the graph is a *well-covered* forest since if G is a forest then,

$$\beta(G) = \min\{\mathbf{e}^\top x \mid x \text{ solves } \mathrm{LCP}(G)\}$$

Examples of Well covered graphs

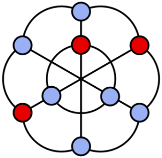


Figure : A well covered graph

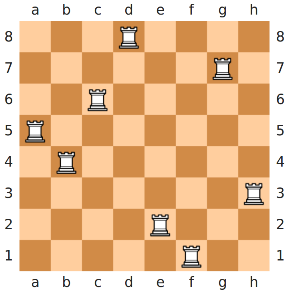


Figure : *Rooks graph*: A non-attacking placement of 8 rooks on a chessboard. If fewer than 8 rooks are placed in a non-attacking way on the board, they can always be extended to 8 rooks that are non-attacking.

Application IV

Complexity of Linear programs with complementarity constraints (LPCC)

$$
\text{LPCC} \qquad \underset{x,y}{\text{maximize}} \quad c^\top x + d^\top y
$$
$$
\text{subject to} \quad \begin{aligned} Bx + Cy &\geq b, \\ Mx + Ny + q &\geq 0, \\ x &\geq 0, \\ x^\top(Mx + Ny + q) &= 0. \end{aligned}
$$

- Haastad in 1996 showed that for a graph G, there is no polynomial time algorithm that can approximate the independence number within a factor of $n^{1-\epsilon}$ of the actual value, unless $P = NP$
- Theorem 1 reduces the independence number to an LPCC with $d, B, C, N = 0$, $M = A$ and c, $-q = \mathbf{e}$. Hence LPCCs are inapproximable even if the data matrices are binary

Current Work

- Non-constructive lower bounds on Error Correcting Codes
- Existence of Specialized equilibria in Public Goods Games

S.-J. Chung.
Np-completeness of the linear complementarity problem.
Journal of Optimization Theory and Applications, 60(3):393–399, 1989.

R. W. Cottle, J.-S. Pang, and R. E. Stone.
The Linear Complementarity Problem.
Academic Press, Inc., Boston, MA, 1992.

J. Håstad.
Clique is hard to approximate within $n^{1-\epsilon}$.
In *Foundations of Computer Science, 1996. Proceedings., 37th Annual Symposium on*, pages 627–636. IEEE, 1996.