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A Survey of Real Time Integration Methods for Systems of Ordinary Differential Equations.

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ABSTRACT

Many of the problems in real time systems can be reduced to the problem of solving differential equations. It is therefore important to obtain their solutions in real time. This seminar report is a survey of different methods available for this purpose. The characteristics of these methods, like stability, convergence and accuracy are studied. Observations are made regarding the behaviour of these methods, when applied to some typical equations.

MOTIVATION

For many real time applications, a fast and accurate method of finding solutions to differential equations is an important task. This motivates to study various methods for finding solution to differential equations.

I. INTRODUCTION

Most of the physical phenomenons are described as Differential Equations. Many problems in control systems can be reduced to the problem of solving differential equations that satisfy certain given conditions. For solving these equations on a digital system, requires Numerical Methods. For some closed form solutions the value of dependent variable can be evaluated from expressions, but it takes long time and is not suitable for real time processing and control.

A general first order differential equation can be written as

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with, initial condition $y(x_0) = y_0$.

The various methods for solving such Differential Equations can be categorized into two forms [3].

1) A series for y in terms of x, from which the value of y can be obtained by direct substitution.

2) A set of tabulated values of x and y.

The methods whose solution is of type 1) are TAYLOR and PICARD'S methods. The methods like EULER, RUNGE-KUTTA, ADAMS-BASHFORTH etc belong to type (2). These methods are called step by step or marching methods. Some of these type (2) methods use NEWTON'S Finite Differences and are called PREDICTOR-CORRECTOR methods.

To solve n^{th} order Differential Equations $'n'$ arbitrary constants are required. If these conditions are specified at the initial point only, then these are called initial value problems. In Boundary Value problems the conditions are specified at different values of the independent variable.

II. METHODS OF DIRECT SUBSTITUTION.

There are two methods that are discussed under this category.

They are a) Solution by TAYLOR Series and b) Solution By PICARD'S Method of Successive Approximation.

a) Solution by TAYLOR Series

The solution $y(x)$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + ... \tag{2}$$

This method has the advantage of calculating the value of $y(x)$ at any given instant directly. This can be used to obtain an estimate of Local truncation errors [1]. Thus, it provides valuable insight into the other methods like EULER'S method etc.

Local Truncation Error: It is the error created during a single step of a method.

This method is not useful for practical applications, as the physical equations are more complicated than simple polynomials and obtaining n^{th} order derivative is not always an easy task.

b) Solution by PICARD'S Method of Successive Approximation.

The Differential Equation is integrated to obtain as an integral equation.

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \tag{3}$$

By substituting $y = y_0$ in RHS of above equation, the first approximation is observed. Using this approximated value for estimating second approximation in same equation and proceeding for n^{th} approximation the solution is obtained, as n approaches infinity.

This method has an advantage that for second step a recursive function can be defined. Thus, it reduces programming difficulty.

In this method integrating becomes a difficult task making it a non-practical solution.

Both of these methods cannot be used for real time applications, as they require much iteration and take long time to be computed on a processor.

III. STEP BY STEP METHODS.

These methods give solution in the form of a set of tabulated values. Each time the new value at $n+1$ is computed with the previous value i.e. at n. So, these are also called Marching Methods.

In this, first Direct Methods and then PREDICTOR-CORRECTOR methods are described. In the direct methods EULER'S methods and the improvements over it, are analyzed and then RUNGE -KUTTA methods.

THE DIRECT METHODS

These methods utilise information at a single point x_n, and predicts the value at y_{n+1} for the next point x_{n+1}.

a) EULER'S methods

In this method, the differential equation is integrated to obtain the integral equation.

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx \tag{4}$$

As $f(x, y)$ is not known, an assumption is made

(i) If $f(x, y)$ is equated to $f(x_0, y_0)$ in the interval $x_0 < x < x_1$, gives the EULER'S explicit equation

$$y_1 = y_0 + hf(x_0, y_0) \quad \Rightarrow \quad y_{n+1} = y_n + hf(x_n, y_n) \tag{5}$$

(ii) If $f(x, y)$ is equated to $f(x_1, y_1)$ in the interval $x_0 < x < x_1$, gives the EULER'S implicit equation

$$y_1 = y_0 + hf(x_1, y_1) \quad \Rightarrow \quad y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \tag{6}$$

(iii) If $f(x,y)$ is taken as $\frac{f(x_0,y_0)+f(x_1,y_1)}{2}$ in the interval $x_0 < x < x_1$, gives the TRAPEZOIDAL method.

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_{n+1},y_{n+1}) + f(x_n,y_n)\right) \tag{7}$$

These are slow methods. These methods are useful for a few time steps and a small value of step to obtain reasonable accuracy [1]. These are not useful for practical purpose due to restriction on size of h.

(i) EULER'S explicit method

It is globally a first order method. It gives a simple expression. The initial value y_0 also has significant influence on the number of steps required to obtain solution with reasonable accuracy. The specific number of substitutions is one.

Stability

$$y_{n+1} = y_n + hf(x_n,y_n)$$

Taking Z transform.

$$zY(z) = Y(z) + h\mathcal{Z}\{f(x_n,y_n)\} \tag{8}$$

and, let $\mathcal{Z}\{f(x_n,y_n)\}$ be equal to some function of z such that the above equation can be written as

$$\frac{Y(z)}{X(z)} = \frac{N(z)}{D(z)} \tag{9}$$

If the roots of $D(z)$ (poles) lie inside a unit circle of the Z plane, the system is stable and the final result converges to the actual solution.

(ii) EULER'S implicit method: On both sides of the expression it has the term to be evaluated. These methods are not normally useful for computer applications. These have to be first converted to explicit form to obtain the values. The initial value y_0 plays an important role in determining the number of steps required to obtain result for a given tolerance. The specific number of substitutions is one.

Stability

These methods are generally stable [5].

(iii) TRAPEZOIDAL method: It is globally second order method. The specific number of substitutions is one. The curve joining the points (x_n,y_n) and (x_{n+1},y_{n+1}) is approximated by a straight line, so that the area under the curve is Trapezoidal.

Stability

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_{n+1}, y_{n+1}) + f(x_n, y_n)\right)$$

$$\Rightarrow\ y_{n+1} - \frac{h}{2} f(x_{n+1}, y_{n+1}) = y_n - \frac{h}{2} f(x_n, y_n) \tag{10}$$

Let $y_n - \frac{h}{2} f(x_n, y_n)$ be equal to 'l', then

$$y_{n+1} - \frac{h}{2} f(x_{n+1}, y_{n+1}) = l \tag{11}$$

As $x_{n+1} = x_n + h$; and y_{n+1} is $f(x_n + h, y_n + h)$, for stability

$$\left|\frac{2 + lh}{2 - lh}\right| < 1 \tag{12}$$

This is more convergent method compared to the EULER'S method. This improvement in stability is obtained with increase in computational effort.

Comparison of these methods

These methods impose the condition of small h, and much iteration for obtaining a solution with reasonable accuracy. These are not efficient for practical purposes as the order of error is directly proportional to h. EULER'S explicit method can be used with an advantage in certain situations where the slope of the solution doesn't vary much.

(b) RUNGE-KUTTA METHODS

These are one step methods. In these methods the Differential Equation is written in the form of

$$y_{n+1} = y_n + \phi(x_n, y_n, h)h \tag{13}$$

Here $\phi(x_n, y_n, h)$ is called as an Incremental Function that can be interpreted as the representative slope over the interval. This Incremental function can be written as following

$$\phi(x_n, y_n, h) = \mathrm{a}_1\mathrm{k}_1 + \mathrm{a}_2\mathrm{k}_2 + \mathrm{a}_3\mathrm{k}_3 + \ldots + a_i k_i \tag{14}$$

Where $\mathrm{a_i}$ is some constant and the $\mathrm{k_i}$ is recurrence relationship [5] i.e. k_1 is used in the expression of k_2 and k_2 in expression of k_3 and so on. For a particular order of RUNGE-KUTTA method, that many terms 'i' are used in the Incremental function. An Nth order RUNGE-KUTTA method has N terms in its Incremental function.

i) Second order RUNGE-KUTTA methods

The second order equation can be written as

$$y_{n+1} = y_n + (a_1 k_1 + a_2 k_2)h \tag{15}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + ph, y_n + qh) \tag{16}$$

The constants $a_1, a_2, \ p, q$ are obtained by comparing the above equation with TAYLOR series expansion till the second order term. Then three equations and four unknowns are obtained. So by choosing a_2 arbitrarily, a_1, p, q are determined. For different values of a_2, different second order RUNGE-KUTTA methods are obtained.

1) If a_2 = 1, then it is MID POINT method.

$$y_{n+1} = y_n + k_2 h$$

$$\Rightarrow \quad y_{n+1} = \quad y_n + f\left(x_n + h/2, y_n + h/2\right) \tag{17}$$

This is called MID POINT method or improved polygon method. In this the value of y is predicted at the midpoint of interval (x_n, x_{n+1}) . Using this predicted value the slope at the mid point is calculated and is taken as average slope in that interval. Using this slope value y_{n+1} is extrapolated.

2) If $a_2 = 1/2$, then it is TRAPEZOIDAL method.

$$y_{n+1} = y_n + (k_1 + k_2)h/2$$

$$\Rightarrow \quad y_{n+1} = \quad y_n + h\left(\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}\right) \tag{18}$$

This is TRAPEZOIDAL method.

3) If $a_2 = 2/3$, then it is RALSTON'S method.

RALSTON'S method provides a minimum bound on the truncation error. The specific number of substitutions is two.

$$y_{n+1} = y_n + \left(k_1/3 + 2k_2/3\right)h$$

$$\Rightarrow \quad y_{n+1} = \quad y_n + h\left(\frac{f(x_n, y_n)}{3} + \frac{2f\left(x_n + \frac{3h}{4}, y_n + \frac{3f(x_n, y_n)h}{4} \right)}{3} \right) \tag{19}$$

Comparison of the above methods

These methods are more accurate [5]. They can be obtained by modifying the Euler's methods. The modification is done on the slope value that is considered, as it is the fundamental source of error. This is obtained at the cost of computational effort in determining the slope. They are also more stable and converge to the solution at a faster rate. As the error in these methods is of the next higher order of h we can take larger values of h when compared to that of EULER'S method.

(ii) Third order RUNGE-KUTTA method

In this method Incremental Function has three terms. One common expression used is

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)h \tag{20}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right) \tag{21}$$

$$k_3 = f(x_n + h, y_n - hk_1 + 2hk_2)$$

These method have local and global errors of the order of h^4 and h^3 respectively. These methods are better, and yield exact result if the solution is cubic.

(iii) Fourth order RUNGE-KUTTA method

In these fourth order Runge-Kutta methods, the constants of Incremental Function are evaluated by comparing with TAYLOR series expansion till forth order term. On comparing these, eight equations and ten unknown values are obtained. So, two values are selected at random and thus there are many methods. Two methods that are mostly used are

1) CLASSICAL method

In this method the equation used is

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{22}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right) \tag{23}$$

$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2\right)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

2) GILL'S method

In this method the equation used is

$$y_{n+1} = y_n + \frac{1}{3}\left(2k_1 + \left(1 - \frac{1}{\sqrt{2}}\right)k_2 + \left(1 + \frac{1}{\sqrt{2}}\right)k_3 + 4k_4\right)h \tag{24}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right) \tag{25}$$

$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \left(1 - \frac{1}{\sqrt{2}}\right)hk_2 + \frac{\sqrt{2}-1}{2}k_1 h\right)$$

$$k_4 = f\left(x_n + h, y_n + \left(1 + \frac{1}{\sqrt{2}}\right)hk_3 - \frac{1}{\sqrt{2}}k_2 h\right) \tag{25}$$

In both this methods the error is of the order of h^5. These are more accurate and are used to calculate initial values for many practical problems [5]. In these methods larger values of h can be used when compared to EULER'S method, for the same tolerance and stability. These methods even converge faster towards the solution at the expense of greater computational effort. GILL'S method is generally not used, as the computations are much harder. The most generally used fourth order RUNGE-KUTTA method is CLASSICAL method. This method is equivalent to the SIMPSON'S $1/3$ rule when the Differential Equation is a function of 'x'only.

THE PREDICTOR-CORRECTOR METHODS

These methods consider the values of the function at $x_n, x_{n-1}, x_{n-2}, x_{n-3}, \ldots$ for computing the value at x_{n+1}. A PREDICTOR formula is used to predict the value of y_{n+1} at

$x = x_{n+1}$ and then a corrector formula is applied to improve that value [5]. The predicted value is represented either by $y*$ or y^p , p representing predicted value.

a) HEUN'S method

In HEUN'S method, first an estimate of y_1 is made and then it is substituted in the actual equation. The specific number of substitutions is two. It is also a predictor-corrector method. It first predicts $y^p{}_{n+1}$ and then uses it in the EULER'S implicit equation. This is different from TRAPEZOIDAL method. This is more stable and accurate when compared to EULER'S method.

$$y^p{}_{n+1} = y_n + hf(x_n, y_n) \tag{26}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_{n+1}, y^p{}_{n+1}) + f(x_n, y_n)\right) \tag{27}$$

b) ADAMS-MOULTON method

In this the predictor formula is called ADAMS-BASHFORTH formula. Using this predicted value in correcting formula called ADAM-MOULTON formula, the solution is obtained. By neglecting 4th and higher order differences, the expressions can be made simple for computations and the error is of the order h^5 . The error can be further reduced till required accuracy is obtained, by repeatedly using the corrector formula.

NEWTON'S Backward Difference Interpolation can be written as

$$f(x,y) = f_0 + m\nabla f_0 + \frac{m(m+1)}{2}\nabla^2 f_0 + \frac{m(m+1)(m+2)}{6}\nabla^3 f_0 + ... \tag{28}$$

where

$$m = \frac{x - x_0}{h} \quad \text{and} \quad f_0 = f(x_0, y_0)$$

Using this in

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)dx \tag{1}$$

and taking values up to fourth order differences, the predictor formula can be obtained as

$$y_1^p = y_0 + \left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4\right] f_0 \tag{29}$$

$f(x,y)$ at f_1 is

$$f(x,y) = f_1 + mf_1 + m\nabla f_0 + \frac{m(m+1)}{2}\nabla^2 f_1 + \ldots \quad (30)$$

Using this in (1) the corrector formula is obtained.

$$y_1^c = y_0 + \left[1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3\right] f_1^p \quad (31)$$

These equations can be written as

$$y_{n+1}^p = y_n + \frac{h}{24}\left[55 f(x_n, y_n) - 59 f(x_{n-1}, y_{n-1}) + 37 f(x_{n-2}, y_{n-2}) - 9 f(x_{n-3}, y_{n-3})\right] \quad (32)$$

$$y_{n+1}^c = y_n + \frac{h}{24}\left[9 f(x_{n+1}, y^p{}_{n+1}) + 19 f(x_n, y_n) - 5 f(x_{n-1}, y_{n-1}) - 9 f(x_{n-2}, y_{n-2})\right] \quad (33)$$

When values till first order of differences are considered the method is the HEUN'S method. This is very easy for computation purposes compared with other methods to obtain same error tolerance. The accuracy can be increased by using the corrector formula again and again [2]. This method has a drawback that it requires starting values. These values can be obtained from starter method like TAYLOR'S series, or EULER'S, or RUNGE-KUTTA methods. The specific number of computations is two, if the corrector formula is used only once. This is the major advantage over the RUNGE-KUTTA method.

This process converges provided the step size is chosen so that

$$\left| h \frac{\partial f(x,y)}{\partial y} \right| < 1 \quad (34)$$

over the region of interest. Usually a k-step ADAMS-BASHFORTH equation is paired up with a (k-1) step ADAM-MOULTON equation but this is not necessary. It is possible to pair up any k-step ADAMS-BASHFORTH with any l- step ADAM-MOULTON equation. As the above methods have a constant step size, there is limitation on the usage of these methods for equations whose solution has abrupt changes. Adaptive step size control methods like adaptive RUNGE-KUTTA methods solve this problem.

IV. OBSERVING THE BEHAVIOUR OF THESE METHODS IN OBTAINING THE SOLUTION OF SOME ORDINARY DIFFERENTIAL EQUATIONS.

a) $\frac{dy}{dx} = \frac{u - y}{\tau}$

EULER'S method: $y_{n+1} = y_n + hf(x_n, y_n)$

The solution is convergent if the Eigen values of the state equation are less than 1.

Taking $\frac{h}{\tau} = c$

$$y_{n+1} = (1-c)y_n + cu \tag{35}$$

This is stable for $|1-c| < 1 \Rightarrow c < 2$.

This method is stable for $h < 2\ \tau$.

HEUN'S method

$$y^p{}_{n+1} = y_n + hf(x_n, y_n);\ y_{n+1} = y_n + \frac{h}{2}\left(f(x_{n+1}, y^p{}_{n+1}) + f(x_n, y_n)\right)$$

$$y_{n+1} = (1-c+c^2/2)y_n + (c-c^2/2)u \tag{36}$$

This is stable if $|1-c+c^2/2| < 1 \Rightarrow c < 2$

This method is stable for $h < 2\ \tau$.

RALSTON'S method:

$$k_1 = \frac{(u-y_n)}{\tau} \qquad k_2 = \frac{(u-y_n - \frac{1}{2}k_1 h)}{\tau}$$

$$\Rightarrow \quad y_{n+1} = y_n(1-c+c^2/3) + u(1-c^2/3) \tag{37}$$

This is stability if $|1-c+c^2/3| < 1 \Rightarrow c < 3$

This method is stable for $h < 3\ \tau$. This method has more stability than other methods.

Classical RUNGE-KUTTA method

$$k_1 = \frac{(u-y_n)}{\tau} \quad k_2 = \frac{(u(1-c/2) - y_n(1-c/2)}{\tau}$$

$$k_3 = \frac{(u(1-c/2+c^2/4) - y_n(1-c/2+c^2/4)}{\tau}$$

$$k_4 = \frac{(u(1-c+c^2/2-c^3/4) - y_n(1-c+c^2/2-c^3/4)}{\tau}$$

$$\Rightarrow y_{n+1} = y_n(1-c+c^2/2-c^3/6+c^4/24) + u(1-c+c^2/2-c^3/6+c^4/24) \tag{38}$$

For this to be stable $|1-c+c^2/2-c^3/6+c^4/24| < 1 \Rightarrow c < 2.8$

This method is stable for $h < 2.8\ \tau$.

Conditions for the existence and uniqueness of the solution, to a differential equation.

If the equation satisfies certain conditions, then it is guaranteed to have a solution and only then all these method give required solution. These conditions are [1]

1) $f(x,y)$ should be defined and continuous for $a < x < b$ and $-\infty < y < \infty$, where a and b are finite.

2) There should exist a constant "L" such that for any two real numbers y_1 and y_2

$$\left|\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}\right| \le L .$$

This means the function $f(x,y)$ should have a continuous derivative with respect to y in the bounded interval. This is also called LIPSCHITZ condition.

The equation $\frac{dy}{dx} = |y|$ shows that the existence of $\frac{\partial f(x,y)}{\partial y}$ is not necessary for second condition.

The equation $\frac{dy}{dx} = \sqrt{y}$ and $\frac{dy}{dx} = \sqrt[3]{y}$ show that continuity of $f(x,y)$ is not sufficient condition.

b) $\frac{dy}{dx} = \sqrt{y}$ and $y_0 = 0$.

This equation does not satisfy the Lipschitz condition. It has two solutions.

I) $y = 0$

II) $y = \frac{x^2}{4}$

EULER'S method: $y_{n+1} = y_n + hf(x_n, y_n)$

Only first solution is obtained.

If another solution is required the initial condition should not be equal to zero. Even then the error in solution obtained is very large.

HEUN'S method and **RALSTON method** etc cannot be applied as the intermediate value of y obtained will be zero, and they also behave in the same for same initial conditions.

ADAMS-MOULTON method: In such cases ADAMS-MOULTON method is better. By giving the starter values corresponding to the solution that is desired, the required result can be obtained. This method also provides results with less error as compared with EULER'S explicit or implicit methods. There is an increase in the computational effort. The specific number of substitutions is equal to two if corrector is used only once and more than two if corrector is used more than once.

If the desired result is $y = \frac{x^2}{4}$ then by obtaining the starter values with $h = 0.1$, $y_0 = 0 \quad y_1 = 0.0025 \quad y_2 = 0.01 \quad y_3 = 0.0225$ obtained analytically, and substitution in

$$y_4^p = y_3 + \frac{0.1}{24}\left[55\sqrt{y_3} - 59\sqrt{y_2} + 37\sqrt{y_1} - 9\sqrt{y_0}\right] \tag{32}$$

$$y_4^c = y_3 + \frac{0.1}{24}\left[9\sqrt{y_4^p} + 19\sqrt{y_3} - 5\sqrt{y_3} - 9\sqrt{y_3}\right] \tag{33}$$

The value of y_4 is obtained as 0.03988, 0.04 being the exact value.

If greater accuracy is needed, corrector equation must be used once more. The computational effort can be decreased by considering up to three terms in this method (as a consequence error increases). During implementing in real time we have to store the previous values of $\sqrt{y_n}$ once calculated for further three cycles. Thereby computing the square root values decreases. Then two square root values must be calculated per cycle. If the corrector is used for one more time, one more square root calculation is required in a cycle [2]. The corrector modifier effectively increases the order of the method. The modifier reduces the number of iterations required for convergence.

Application of these methods to System of Ordinary Differential Equations

The above methods can be extended to a system of simultaneous ordinary differential equations. Then for n such simultaneous equations n initial conditions have to be known at the starting value of independent variable. For solving initial value problems special one step methods are available for systems of ordinary differential equations. In this slope for each function at a value of independent variable is computed. This slope value is used to extrapolate the value of dependent variable at next step of independent variable. These special one step methods are methods of RUNGE-KUTTA type, methods based on Quadrature etc. The existence of solution, the convergence etc can be observed by representing the equations in vector notation.

V. CONCLUSION.

Of many methods present to obtain the solution of Differential Equations, there are mainly two trade-off marks for their use in real time. These trade-off quantities are the time taken for evaluation and the error of the result. As error decreases for a method, the time required to evaluate increases. If solution is required within tolerance the computational complexity increases. As the complexity of the method increases initially the time taken to obtain the result decreases and then increases without that great increase in the accuracy of the result. So, a compromise has to be made between the time and accuracy, for a method that can be programmed on a computer or DSP system. The method has to be selected appropriately that has greater stability, easy to calculate and even accurate for that Differential Equation. For most of the practical situations Predictor-Corrector method (ADAMS-MOULTON) is used and for obtaining the starter values, Classical RUNGE-KUTTA method is used. This combination is found to be successful for many of the applications.

Thus, depending on the Differential Equations governing the real time process, the method for obtaining the solution must be chosen that gives solution at a faster rate within the tolerance and with minimum computational effort.

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