

## **DESIGN OF FIR FILTERS WITH ARBITRARY RESPONSES**

N. V. Murali (Roll No. 04307014)  
Supervisor: Prof. P. C. Pandey

### **Abstract**

This report consists of some of the techniques used to design FIR filters with arbitrary responses. Three techniques used to design linear phase FIR filters are described. In the beginning, the window technique and the frequency sampling techniques are discussed in detail with their relative merits and demerits. Different optimal filter design techniques are also discussed to design linear phase FIR filters. WLS Chebyshev approximation method for designing FIR filters with arbitrary complex frequency response is discussed in detail. An efficient weighted least squares design technique to design FIR filters satisfying prescribed magnitude and phase specifications is discussed. Some MATLAB functions used to design FIR filters are also included.

### **1. Introduction**

FIR filters are particularly useful for applications where exact linear phase response is required. The filter design essentially involves the two stages as follows:

- (i) The approximation stage
- (ii) The realization stage

The approximation stage takes the specifications and gives the transfer function of the FIR filter. The realization stage deals with choosing the specific structure to implement the transfer function obtained from the approximation stage. In this report only approximation stage for FIR filter design is discussed. There are three well-known techniques to design linear phase FIR filters namely:

- (i) The window technique
- (ii) The frequency sampling technique
- (iii) The optimal filter design technique

FIR filters satisfying prescribed phase specifications can also be designed. Weighted Least Squares (WLS) Chebyshev approximation method [5], for designing FIR filters satisfying arbitrary complex frequency response is discussed in detail. Computationally efficient weighted least squares design [6], of FIR filters satisfying prescribed magnitude and phase specifications is also discussed. In this method, the filter coefficients are obtained by solving a system of linear equations that involves Toeplitz matrix.

## 2. The window method

This method is mainly used for designing proto-type filters. In this method [1], [2], from the given desired frequency response specifications  $H_d(\omega)$ , corresponding unit sample response  $h_d(n)$  is determined using the Fourier transform relation

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \quad (1)$$

$$\text{where } H_d(\omega) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad (2)$$

It can be easily observed from (2) that the impulse response of digital filter is infinite in duration. One possible way to obtain an FIR filter that approximates the desired frequency response is to truncate the infinite impulse response at  $n=N-1$ . This truncation of infinite impulse response can be done by multiplying  $h_d(n)$  by the  $N$ -point rectangular window defined as

$$w(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Then the impulse response of the truncated filter becomes

$$\begin{aligned} h(n) &= h_d(n)w(n) \\ &= h_d(n) \quad 0 \leq n \leq N-1 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (4)$$

It is well known fact that multiplication of the window function  $w(n)$  with  $h_d(n)$  in time domain is equivalent to convolution of  $W(\omega)$  with  $H_d(\omega)$  given by the formula

$$H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\lambda) W(\omega - \lambda) d\lambda \quad (5)$$

where  $W(\omega)$  is the Fourier transform of the rectangular window. Since (2) is basically Fourier representation of  $H_d(\omega)$ , the multiplication of  $h_d(n)$  with  $w(n)$  is identical to truncating the Fourier series of the desired filter characteristic. But direct truncation of infinite Fourier series leads to the well known Gibbs phenomenon, which manifests itself as a fixed percentage of overshoot and ripple before and after an approximated discontinuity in the frequency response due to the non-uniform convergence Fourier series at the discontinuity.

The effects of truncating the Fourier coefficients on the frequency response of the designed filter is as follows:

- (i) The discontinuities in the desired frequency response become transition bands between values in either side of the discontinuity.

(ii) The window function produce ripples in the resulting frequency response  $H(\omega)$  for all  $\omega$ .

Some of the windows commonly used are listed below:

1. Barlett triangular window:

$$\begin{aligned} w(n) &= 2n/(N-1) & 0 \leq n \leq N/2 \\ &= 1 - 2n/(N-1) & N/2 \leq n \leq N-1 \end{aligned}$$

2. Blackman window:

$$w(n) = 0.42 - 0.5\cos[2\pi k/(N-1)] + 0.08\cos[4\pi k/(N-1)]$$

3. Hamming window:

$$w(n) = 0.54 - 0.46\cos[2\pi k/(N-1)]$$

4. Hanning window:

$$w(n) = 0.5(1 - \cos[2\pi k/(N-1)])$$

5. Kaiser window:

$$w(n) = I_0[\alpha \sqrt{\{k^2 - (n-k)^2\}}] / I_0[\alpha k] \quad \text{where, } k=(N-1)/2$$

In the Kaiser window,  $I_0[x]$  is the zeroth order Bessel function. The best filter can be designed by using Kaiser window, which has a parameter ' $\alpha$ ' that allows the adjustment of compromise between the overshoot reduction and transition region width spreading. The major advantage of using the window method is the relative ease of using well-defined equations for windows.

The issues with the window method are

(i) The window method needs closed form expression of desired frequency response for calculating the filter coefficients. Otherwise the evaluation of filter coefficients becomes more difficult.

(ii) The edge frequencies of the designed filter cannot be specified exactly since the window smears the discontinuities in the frequency response of the desired filter.

(iii) Window method is basically useful for prototype filters only. This makes its use very limited in speech and image processing applications where non-prototype filters are often required.

### 3. The frequency sampling technique

In this technique [1], [3], the desired frequency response is sampled at a set of  $N$  equally spaced frequencies. These  $N$  frequency samples constitute an  $N$ -point DFT  $H(k)$ . Thus by using the inverse DFT, the filter impulse response can be determined using the following relation

$$h(n) = 1/N \sum_{k=0}^{N-1} H(k)e^{j(2\pi nk/N)} \quad (6)$$

By using the impulse response obtained above, the continuous frequency response is calculated as an interpolation of the sampled frequency response. The approximation error then would be zero exactly at the sampled frequencies and be finite between them. The smoother the frequency response is approximated, the smaller the interpolation error between them.

The method to improve the quality of approximation is to make a number of frequency samples as unconstrained variables. The frequency samples in the transition bands as unconstrained variables have been found in [1], to provide good ripple cancellation in adjacent frequency bands. The values of these unconstrained variables are generally optimized by computer by means of linear programming technique to minimize some simple function of the approximation error.

The procedure for designing FIR filters using frequency sampling technique described in [3], is summarized below:

(i) Choose a set of frequencies at which the sampled frequency response is specified. The choice of a set of frequencies is merely the choice of a value for  $N$ , the number of impulse response samples, and an initial frequency.

(ii) For a given value of  $N$ , the designer must determine how fine an interpolation should be used. In [1], they mentioned that for the designs they investigated, where  $N$  varies from 15 to 256, they found 16 to 1 interpolation lead to reliable computations and results.

(iii) Given the set of  $N$  values of  $H(k)$ , compute the inverse DFT to obtain  $h(n)$  [using FFT algorithm.]

(iv) In order to obtain the values of interpolated frequency response two procedures were suggested in [3]. They are

a) Rotate the  $h(n)$  by  $N/2$  samples ( $N$  even) or  $[(N-1)/2]$  samples ( $N$  odd) to remove the sharp edges of the impulse response, and then  $15N$  zero-valued samples are symmetrically placed around the impulse response.

b) Split the  $h(n)$  around the  $N/2$  sample value, and  $15N$  zero-valued samples are placed between the two pieces of the impulse response.

(v) The zero-augmented sequences are transformed using the FFT algorithm to give the interpolated frequency responses.

(vi) The values of these unconstrained variables are optimized by means of linear programming technique to minimize some simple function of the approximation error.

Design example:

The frequency response of the low pass filter with total 64 samples, 16 samples in bandwidth and 3 transition coefficients using frequency sampling method is shown below. The transition coefficients obtained by the linear programming from [3] are 0.030957, 0.275570, 0.744348.

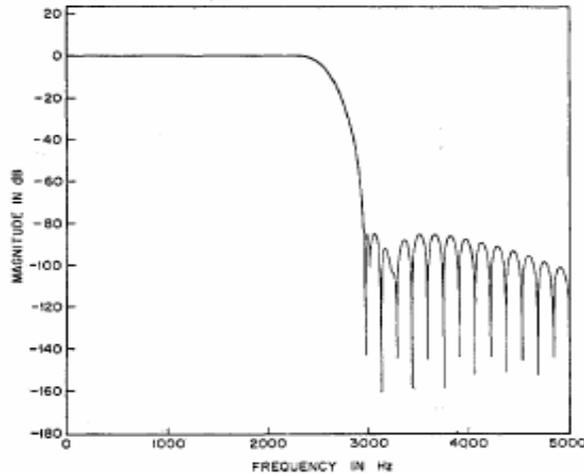


Fig. 1. Magnitude response of the low pass filter with three transition coefficients from [3]

### Merits

(i) The frequency sampling technique can be used to design the non-prototype filters where the desired frequency response can be any irregular shape.

(ii) Unlike the window method, once the transition bandwidth has been chosen, the filter can be designed that will have such a transition bandwidth. Whereas in the window method there is a trade-off between overshoot and transition bandwidth.

### Demerits

The interpolated frequency response is equal to the desired frequency response only at sampled frequencies. There will be a finite error present at all the other frequencies.

## 4. Optimal filter design methods

Here optimal filter in the sense that a filter that is the best that can be achieved for the given number of impulse response coefficients. Many methods are available for designing linear phase optimal filters. The basic idea in all these methods is to design the filter coefficients so that a particular error is being minimized. Three of them are listed below:

- (i) Weighted Chebyshev approximation
- (ii) Nonlinear equation solution for maximal ripple FIR filter
- (iii) Polynomial interpolation solution for maximal ripple FIR filters

### 4.1. Weighted Chebyshev approximation

The design of optimal linear phase FIR filters can be achieved by considering the filter design problem as Chebyshev approximation problem. Thus it is possible to derive a set of conditions for which the designed filter is optimal and unique.

In this method [1], following terms are defined to formulate the FIR filter design problem as Chebyshev approximation problem.

$H_d(\omega)$  = The desired frequency response of the filter

$H(\omega)$  = The frequency response of the designed filter

$W(\omega)$  = The frequency response of the weighing function

The weighing function enables the designer to choose the relative size of the error in different frequency bands. The general form of the frequency response for four different types of linear phase FIR filters (i.e. whether length  $N$  of the filter is odd or even and filter is symmetric or anti-symmetric) can be written as

$$H(\omega) = e^{-j\omega(N-1)/2} e^{j\pi L/2} |H(\omega)| \quad (7)$$

**TABLE – I**

The values of  $L$  and  $|H(\omega)|$  for four different types of FIR filters from [1]

	$L$	$ H(\omega) $
Case1: $N$ odd Symmetrical impulse response	0	$\sum_{n=0}^{(N-1)/2} a(n)\cos(\omega n)$
Case2: $N$ even Symmetrical impulse response	0	$\sum_{n=1}^{N/2} b(n)\cos[\omega(n-0.5)]$
Case3: $N$ odd Anti-Symmetrical impulse response	1	$\sum_{n=1}^{(N-1)/2} c(n)\sin(\omega n)$
Case4: $N$ even Anti-Symmetrical impulse response	1	$\sum_{n=1}^{N/2} d(n)\sin[\omega(n-0.5)]$

Each of the expressions for  $|H(\omega)|$  in the above table can be written as a product of fixed function of  $\omega$  [call this as  $Q(\omega)$ ] and a term that is a sum of cosines [call this as  $P(\omega)$ ]. Thus the expressions for  $|H(\omega)|$  in TABLE-I become as

**TABLE – II**

Expressions for  $P(\omega)$  and  $Q(\omega)$  for different types of filters from [1]

	$Q(\omega)$	$P(\omega)$
Case 1	1	$\sum_{n=0}^{(N-1)/2} p(n)\cos(\omega n)$
Case 2	$\cos(\omega/2)$	$\sum_{n=0}^{N/2-1} q(n)\cos(\omega n)$
Case 3	$\sin(\omega)$	$\sum_{n=0}^{(N-3)/2} r(n)\cos(\omega n)$
Case 4	$\sin(\omega/2)$	$\sum_{n=0}^{N/2-1} s(n)\cos(\omega n)$

The weighted error of approximation  $E(\omega)$  is, by definition

$$\begin{aligned} E(\omega) &= W(\omega)[H_d(\omega) - |H(\omega)|] \\ &= W(\omega)[H_d(\omega) - P(\omega)Q(\omega)] \end{aligned} \quad (8)$$

Since  $Q(\omega)$  is a fixed function of frequency it can be factored out from (8), gives

$$E(\omega) = \hat{W}(\omega)[\hat{H}_d(\omega) - P(\omega)] \quad (9)$$

where  $\hat{W}(\omega) = W(\omega)Q(\omega)$

$$\hat{H}_d(\omega) = H_d(\omega) / Q(\omega)$$

The Chebyshev approximation problem may now be stated as finding the set of coefficients  $[p(n), q(n), r(n), \text{ or } s(n)]$  to minimize the maximum absolute value of  $E(\omega)$  over the frequency bands in which the approximation is being performed. The Chebyshev approximation problem may be stated mathematically as follows:

$$\text{Min}_{\text{coefficients}} [\max |E(\omega)|] \quad (10)$$

The solution to this problem is given by Parks and McClellan [2], who applied a theorem in theory of Chebyshev approximation called the alternation theorem.

#### 4.2. Nonlinear equation solution for maximal ripple FIR filters

The filters designed by this method [1], have maximum number of ripples in their frequency responses. The maximum number of ripples i.e. the number of frequencies at which  $H(\omega)$  could attain an extremum is strictly a function of the type of filter. At each extremum the value of  $|H(\omega)|$  is pre-determined by a combination of the weighing function  $|H(\omega)|$ , the desired frequency response  $H_d(\omega)$ , and the quantity  $\delta$  that represents the peak error of approximation. By distributing the frequencies at which  $|H(\omega)|$  attained an extremal value among the different frequency bands over which a desired response was being approximated a unique optimal filter can be obtained.

The procedure for obtaining a set of nonlinear equations is as follows:

At each of the unknown extremal frequencies,  $E(\omega)$  attains the maximal value of either  $\delta$  or  $-\delta$ , the derivative of  $|H(\omega)|$  becomes zero. Thus equations of the form

$$|H(\omega_i)| = \pm\delta/W(\omega_i)+H_d(\omega_i) \quad i=1, 2 \dots k \quad (11)$$

$$d/d\omega[|H(\omega)|]_{\omega=\omega_i} = 0 \quad i=1, 2 \dots k \quad (12)$$

are obtained. Where,  $k = \text{Number of extrema of } |H(\omega)|$

These equations represent a set of  $2k$  nonlinear equations in  $2k$  unknowns [ $k$  impulse response coefficients and  $k$  frequencies at which  $|H(\omega)|$  obtains the extremal value]. The set of  $2k$  equations may be solved iteratively using a nonlinear optimization procedure.

It should be noted that the quantity,  $\delta$  is a fixed quantity and is not minimized by the optimization scheme. Thus the shape of  $|H(\omega)|$  is postulated a priori and only the frequencies at which  $|H(\omega)|$  is attains the extremal values are unknown.

The main disadvantage of this technique is that the design procedure has no way of specifying band edges for the different frequency bands of the filter. Thus the optimization algorithm is free to select exactly where the bands will lie.

### 4.3. Polynomial interpolation solution for maximal ripple FIR filters

This technique [1], is basically an iterative technique for producing a polynomial  $H(\omega)$  that has extrema of desired values. The algorithm for this technique is as follows:

- (i) Make an initial estimate of frequencies at which the extrema in  $|H(\omega)|$  will occur.
- (ii) Use the Lagrangian interpolation formula to obtain a polynomial that alternatively goes through the maximal ripple values at these frequencies.
- (iii) Check whether the polynomial associated with the frequencies have extrema that achieve the maximum allowable ripple.
- (iv) If not locate the frequencies at which the extrema of previous Lagrangian interpolation occur and then update the extremal frequency values by these frequencies.
- (v) Use the updated frequencies to construct another Lagrangian polynomial and then go to step (iii)

This procedure is repeated until the polynomial satisfies the required specifications. The disadvantage of this method is the band edge frequencies cannot be specified a priori. They must be calculated from the final solution. The low pass filter designed by polynomial interpolation method is shown below:

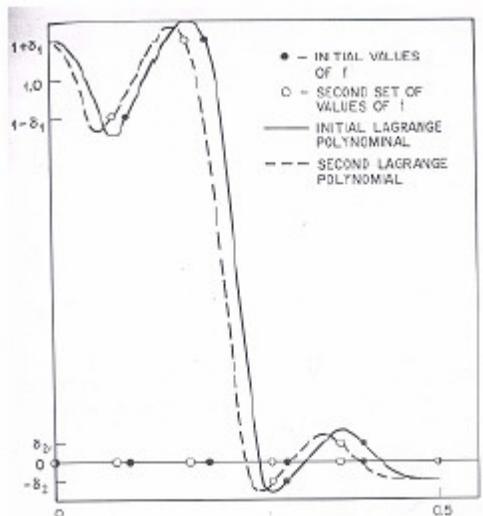


Fig. 2. Iterative solution for a maximal ripple lowpass filter from [1]

## 5. Design of FIR filters with arbitrary complex frequency responses

In this section, two methods for design of FIR filters with arbitrary complex frequency response are described. Firstly, weighted least squares Chebyshev approximation method proposed in [5], is discussed. The computational complexity involved in this method is reduced in [6], which is also discussed.

### 5.1. Weighted least squares Chebyshev approximation method

The basic idea in this technique [5], is to obtain the suitable frequency dependent weighting function by an iterative procedure that yields an equi-ripple design. Let the desired complex frequency response  $H_d(\omega)$ , is conjugate symmetric and includes  $p$  disjoint nontransition bands as follows:

$$B_{NT} = B_1 \cup B_2 \dots \cup B_p$$

where  $B_m = \{\omega \mid \omega_{m1} \leq \omega \leq \omega_{m2}\}$ ,  $m = 1, 2, \dots, p$

$\omega_{m1}$  and  $\omega_{m2}$  denote the specified cutoff frequencies in the  $m$ th frequency band. Then the union of the transition bands, denoted  $B_{TS}$ , is given by

$$B_{TS} = \{\omega \mid 0 \leq \omega \leq \pi, \omega \notin B_{NT}\}.$$

The frequency response of the  $(N-1)$ th order FIR filter can be written as

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{M-1} h(n)e^{-j\omega n} \\ &= \sum_{n=0}^{M-1} h(n)\cos(\omega n) - j \sum_{n=0}^{M-1} h(n)\sin(\omega n) \end{aligned} \quad (13)$$

Then the complex approximation error between  $H_d(\omega)$  and  $H(\omega)$  can be expressed as

$$\begin{aligned} E(\omega) &= H_d(\omega) - H(\omega) \\ &= E_r(\omega) + jE_i(\omega), \quad \omega \in B_{NT} \end{aligned} \quad (14)$$

where  $E_r(\omega)$  and  $E_i(\omega)$  are the real part and the imaginary part of  $E(\omega)$ , respectively. Let  $W_e(\omega)$ ,  $\omega \in B_{NT}$ , be a piece-wise constant function associated with the desired relative approximation error ratio among the  $p$  frequency bands, defined as

$$W_e(\omega) = \rho_m, \quad \text{if } \omega \in B_m, \quad (15)$$

where  $\rho_m > 0, \rho_2 > 0, \dots, \rho_p > 0$ ,  $\max\{\rho_1, \rho_2, \dots, \rho_p\} = 1$ , and the ratio  $(1/\rho_1) : (1/\rho_2) : \dots : (1/\rho_p)$  denotes the desired relative approximation error ratio among  $B_1, B_2, \dots, B_p$ . Now the object is to find the impulse response  $h(n)$ , such that  $H(\omega)$  is equiripple with  $\delta_1 : \delta_2 : \dots : \delta_p = (1/\rho_1) : (1/\rho_2) : \dots : (1/\rho_p)$ . where  $\delta_m$  is the maximum approximation error in  $B_m$ .

For notational simplicity, let  $H_d(k)$ ,  $W_c(k)$ ,  $E(k)$ ,  $E_r(k)$  and  $E_i(k)$  also denote  $H_d(\omega=k/2N)$ ,  $W_c(\omega=k/2N)$ ,  $E(\omega=k/2N)$ ,  $E_r(\omega=k/2N)$  and  $E_i(\omega=k/2N)$  at  $N$  total number of uniform samples. Thus the error can be expressed as in the following linear vector form:

$$\begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_I \end{bmatrix} = \begin{bmatrix} \text{Re}(\mathbf{H}_d) \\ \text{Im}(\mathbf{H}_d) \end{bmatrix} - \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix} \mathbf{h} \quad (16)$$

where  $\mathbf{E}_r$  and  $\mathbf{E}_i$  denote the real part and imaginary part of  $\mathbf{E}$ , respectively,  $\text{Re}(\mathbf{H}_d)$  and  $\text{Im}(\mathbf{H}_d)$  denote the real part and imaginary part of  $\mathbf{H}_d$ , respectively,

$$\begin{aligned} \mathbf{h} &= [h(0) \ h(1) \ h(2) \ \dots \ h(M-1)]^T, \\ \mathbf{E} &= [E(0) \ E(1) \ E(2) \ \dots \ E(N-1)]^T, \\ \mathbf{H}_d &= [H_d(0) \ H_d(1) \ H_d(2) \ \dots \ H_d(N-1)]^T, \end{aligned}$$

and  $\mathbf{D}_1$  and  $\mathbf{D}_2$  are  $N \times M$  matrices with the  $(i, j)$ th element  $[\mathbf{D}_1]_{ij} = \cos((i-1)(j-1)\pi/N)$  and  $[\mathbf{D}_2]_{ij} = -\sin((i-1)(j-1)\pi/N)$ , respectively. Then the sum of weighted error squares can be written as

$$\begin{aligned} \mathbf{J}(\mathbf{h}) &= \sum_{k=0}^{N-1} w(k) |E(k)|^2 \\ &= \mathbf{E}_R^T \mathbf{W} \mathbf{E}_R + \mathbf{E}_I^T \mathbf{W} \mathbf{E}_I \\ &= \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_I \end{bmatrix}^T \begin{bmatrix} \mathbf{W} & 0 \\ 0 & \mathbf{W} \end{bmatrix} \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_I \end{bmatrix}, \end{aligned} \quad (17)$$

where  $\mathbf{W} = \text{diag} [w(0) \ w(1) \ w(2) \ \dots \ w(N-1)]$  with  $w(k) \geq 0$  for all  $0 \leq k \leq N-1$ . The WLS estimate,  $\hat{\mathbf{h}}$ , of  $\mathbf{h}$  which minimizes  $\mathbf{J}(\mathbf{h})$  is given by

$$\hat{\mathbf{h}} = [\mathbf{D}^T \mathbf{W}_1 \mathbf{D}]^{-1} \mathbf{D}^T \mathbf{W}_1 \begin{bmatrix} \text{Re}(\mathbf{H}_d) \\ \text{Im}(\mathbf{H}_d) \end{bmatrix} \quad (18)$$

where  $\mathbf{D}^T = [\mathbf{D}_1^T \ \mathbf{D}_2^T]$  and  $\mathbf{W}^T = \text{diag} [\mathbf{W}_1^T \ \mathbf{W}_2^T]$ .

The following terms are defined in [5], to explain this method

Error ripple  $E_m^i(k)$ :

$$E_m^i(k) = |E(k)|, \quad k/2N \in B_m^i, \quad i=1, 2, \dots, q. \quad (19)$$

where  $q$  is the total number of error ripples in  $B_m$ .

Amplitude  $e_m^i$  of error ripple  $E_m^i(k)$ :

$$e_m^i = \max \{E_m^i(k), k/2N \in B_m^i\} \quad (20)$$

Piecewise-constant function  $R(k)$ ,  $k/2N \in B_{NT}$  :

$$\begin{aligned} R(k) &= W_e(k)e_m^i \\ &= \rho_m e_m^i, \quad \text{if } k/2N \in B_m^i. \end{aligned} \quad (21)$$

Now the problem is to find the  $w(k)$  such that  $|E(k)|$  is equiripple with the desired approximation error ratio among the nontransition bands or, equivalently  $|W_e(k)E(k)|$  for  $k/2N \in B_{NT}$  is equiripple. The design method for this is shown below:

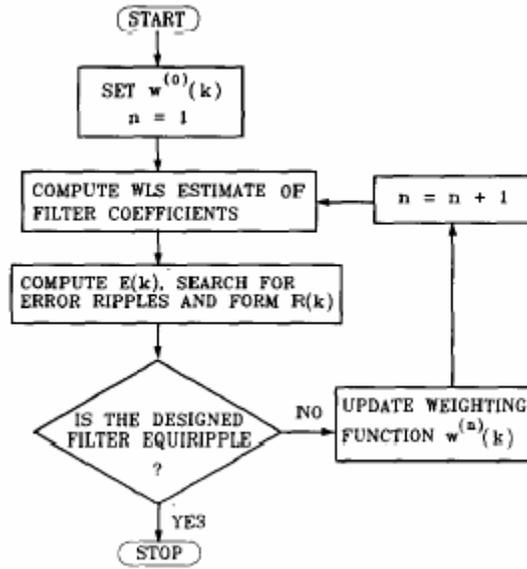


Fig. 1. The WLS Chebyshev approximation method from [6]

The method begins with the initial weighing function

$$\begin{aligned} w^{(0)}(k) &= W_e(k), \quad k/2N \in B_{NT} \\ &= 0, \quad k/2N \in B_{TS} \end{aligned} \quad (22)$$

Assume that the above method is ended up with the weighing function  $w(k) = w^{n-1}(k)$  at the  $(n-1)$  th iteration. Then the sequence of steps to be followed is stated below:

- (i) For the  $n$ th iteration, compute the WLS estimate,  $\hat{h}$  by (18), in which  $w(k) = w^{n-1}(k)$ .
- (ii) Then compute the associated  $E(k)$  by (16).
- (iii) Search for  $E_m^i(k)$  and  $e_m^i$ , for all  $i$  and  $m$ .
- (iv) Check whether  $|W_e(k)E(k)|$  for  $k/2N \in B_{NT}$  is equiripple by

$$\frac{R_{\max} - R_{\min}}{R_{\max}} \leq \sigma \quad (23)$$

where  $R_{\max} = \max\{R(k), k/2N \in B_{NT}\}$ ,

$R_{\min} = \min\{R(k), k/2N \in B_{NT}\}$ ,

$\sigma =$  preassigned small positive constant.

(v) If  $|W_e(k)E(k)|$  is not equiripple, update the weighing function by

$$\begin{aligned} w^{(n)}(k) &= w^{n-1}(k)R(k)/w_{\max}, & k/2N \in B_{NT} \\ &= 0, & k/2N \in B_{TS} \end{aligned} \quad (24)$$

where  $w_{\max} = \max\{w^{n-1}(k)R(k), k/2N \in B_{NT}\}$  (25)

This procedure is repeated until the equiripple filter satisfying prescribed magnitude and phase specifications are obtained.

Design example:

The desired frequency response for linear phase low pass filter is given as follows:

$$H_d(f) = \begin{cases} e^{-j2\pi f(M-1)/2}, & \text{if } f \in B_1 = [0, 0.2] \text{ (passband)}, \\ 0, & \text{if } f \in B_2 = [0.3, 0.5] \text{ (stopband)}. \end{cases}$$

$$W_c(f) = \begin{cases} \rho_1 = 0.1, & f \in B_1, \\ \rho_2 = 1, & f \in B_2. \end{cases}$$

The Magnitude response of the designed filter for above specifications using this method is shown below:

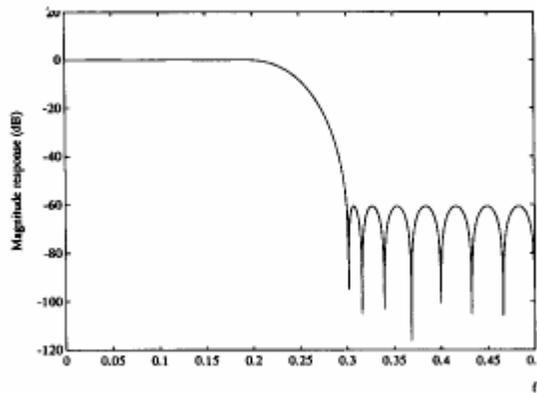


Fig. 3. Magnitude response of the linear phase low pass filter from [5]

## 5.2. Computationally efficient weighted least squares method

The computational complexity involved in the WLS Chebyshev approximation method to obtain the FIR filter coefficients with arbitrary complex frequency response is reduced in this method [6]. In this method, the filter coefficients are obtained by solving a system of linear equations that involves Toeplitz matrix.

The frequency response of an  $N$ -point FIR filter with real valued impulse response coefficients can be written as

$$\begin{aligned}
 H(\omega) &= \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \\
 &= \sum_{n=0}^{N-1} h(n)\cos(\omega n) - j \sum_{n=0}^{N-1} h(n)\sin(\omega n) \\
 &= \mathbf{H}^T \mathbf{C}(\omega) - j\mathbf{H}^T \mathbf{S}(\omega)
 \end{aligned} \tag{26}$$

where  $\mathbf{H} = [h(0) \ h(1) \ h(2) \ \dots \ h(N-1)]^T$

$$\mathbf{C}(\omega) = [1 \ \cos(\omega) \ \cos(2\omega) \ \dots \ \cos((N-1)\omega)]^T$$

$$\mathbf{S}(\omega) = [0 \ \sin(\omega) \ \sin(2\omega) \ \dots \ \sin((N-1)\omega)]^T$$

The desired frequency response  $H_d(\omega)$  having an amplitude response  $H(\omega)$  and phase response  $\Phi(\omega)$  is given by

$$\begin{aligned}
 H_d(\omega) &= H(\omega)e^{j\Phi(\omega)} \\
 &= H_R(\omega) + jH_I(\omega)
 \end{aligned} \tag{27}$$

Then the mean squared error between  $H_d(\omega)$  and  $M(\omega)$  can be expressed as

$$\begin{aligned}
 E &= \sum_{k=1}^M W(\omega_k) |H_d(\omega_k) - H(\omega_k)|^2 \\
 &= \sum_{k=1}^M W(\omega_k) [(H_R(\omega_k) - H^T C(\omega_k))^2 + (H_I(\omega_k) - H^T S(\omega_k))^2]
 \end{aligned} \tag{28}$$

where  $W(\omega)$  nonnegative frequency dependent weighing function and  $M$  is the number of points at which  $H_d(\omega)$  is sampled. Then a set of linear equations can be obtained by setting  $\partial E/\partial H=0$  to minimize the error  $E$ . These equations in the matrix form can be written as

$$\mathbf{QH} = \mathbf{D} \tag{29}$$

where  $\mathbf{Q}(n,m) = \sum_{k=1}^M W(\omega_k)\cos((n-m)\omega_k) \quad 0 \leq n,m \leq N-1$  (30)

$$\mathbf{D}(n) = \sum_{k=1}^M W(\omega_k)H(\omega_k)\cos(n\omega_k + \Phi(\omega_k)) \tag{31}$$

From (30) it can be seen that  $Q$  is real, Toeplitz-symmetric, and positive definite matrix. And also note that the entries are independent of  $H_d(\omega)$ . Hence only first row of elements has to be evaluated. Consequently, the system of equations can be solved by Levinson method.

## 6. Some MATLAB functions to design FIR filters

MATLAB is software that is used in a number of applications including signal processing. The Signal Processing Toolbox provides several functions to design FIR filters. Some of them are listed below:

- fir1: This function implements the classical window method for FIR digital filter design. It designs filters in standard low pass, high pass, band pass, and band stop configurations. By default the filter is normalized so that the magnitude response of the filter at the center frequency of the pass band is 0 dB.
- fir2: This function is used for designing of frequency sampling based FIR filters with arbitrary shaped frequency response.
- firls: This function designs a linear-phase FIR filter that minimizes the weighted, integrated squared error between an ideal piecewise linear function and the magnitude response of the filter over a set of desired frequency bands.
- firpm: This function designs a linear-phase FIR filter using the Parks-McClellan algorithm. The Parks-McClellan algorithm uses the Chebyshev approximation theory to design filters with an optimal fit between the desired and actual frequency responses. The filters are optimal in the sense that the maximum error between the desired frequency response and the actual frequency response is minimized.
- firls: This function uses an iterative least-squares algorithm to obtain an equiripple response. The algorithm is a multiple exchange algorithm that uses Lagrange multipliers method.
- cfirpm: This function designs FIR filters with arbitrary complex responses. This design technique may be used to produce nonlinear-phase FIR filters, asymmetric frequency-response filters. The design algorithm optimizes the Chebyshev error using an extended Remez-exchange algorithm for an initial estimate.

## 7. Conclusions

There are several techniques available for designing linear phase FIR filters. Every method has its own advantages and disadvantages. The choice of technique depends heavily on the decision of designer whether to compromise accuracy of approximation or ease of design. The window technique is most suitable for prototype filters like the low-pass, high-pass, band-pass etc. The frequency sampling technique is suitable for designing of filters with arbitrary frequency response. Optimal filter design techniques give best filters for given length of the FIR filter, but these techniques are more complex.

Weighted least squares Chebyshev approximation method for the design of FIR filters with complex frequency response provides the flexibility of meeting phase specifications also. Weighted least squares design technique is an efficient and less complex technique for the design of FIR filters with prescribed magnitude and phase specifications.

## References

- [1] L.R.Rabiner and B.Gold, *Theory and Applications of Digital Signal Processing*. New Jersey: Prentice-Hall, 1975.
- [2] J.G.Proakis and D.G.Manolakis, *Digital Signal Processing-Principles, Algorithms and Applications*. New Delhi: Prentice-Hall, 2000.
- [3] L.R.Rabiner, B.Gold and C.A.McGonegal, "An approach to the approximation problem for nonrecursive digital filters," *IEEE Trans. Audio and Electroacoust.*, vol. AU-18, pp. 83-105, June 1970.
- [4] L.R.Rabiner, "Techniques for designing finite-duration impulse-response digital filters," *IEEE Trans. Communication Technology*, vol. COM-91, pp.188-195, April 1971.
- [5] C. Y. Chi and S. L. Chiou, "A new WLS Chebyshev approximation method for the design of FIR digital filters with arbitrary complex frequency response, " *Signal Processing*, vol. 29, pp. 335-347, Dec. 1992.
- [6] S. S. Kidambi, "Computationally efficient weighted least-squares design of FIR filters satisfying prescribed magnitude and phase specifications." *Signal Processing*, vol. 60, pp. 127-130, July 1997.
- [7] <http://www.mathworks.com/access/helpdesk/help/toolbox/signal/signal.html>  
Signal Processing Toolbox, The Mathworks, Inc., accessed October 20, 2004.

This document was created with Win2PDF available at <http://www.daneprairie.com>.  
The unregistered version of Win2PDF is for evaluation or non-commercial use only.