Price Competition in Spectrum Markets: How Accurate is the Continuous Prices Approximation?

Aditya MVS, Abhishek Raghuvanshi and Gaurav S. Kasbekar

***Abstract*—Dynamic Spectrum Access technology enables two types of users to operate on a channel– primary users and secondary users, which can use the channel when it is not in use by the primaries. We consider a scenario in which multiple primaries own bandwidth in a large region which is divided into smaller locations. A primary that has unused bandwidth in a time slot would like to lease it out to secondaries in return for a fee. This results in price competition among the primaries. In prior work, this price competition has only been studied under the approximation, made for analytical tractability, that the price of each primary takes values from a continuous set. However, in practice, the set of available prices is discrete. In this paper, we investigate the fundamental question of how the behaviour of the players involved in the price competition changes when this continuity assumption is removed. Our analysis reveals several important differences between the games with continuous and discrete price sets. However, we show that as the number of available prices becomes large, the strategies of the primaries under every symmetric NE converge to the unique NE strategy of the game with continuous price sets.**

I. Introduction

The last decade has seen a tremendous growth in the use of wireless devices, thus increasing the demand for spectrum. Traditionally, a static spectrum allocation policy has been used, where network operators have *exclusive* spectrum rights. This has created an artificial scarcity of spectrum wherein most of the usable radio spectrum is allocated, but under-utilised [1]. *Dynamic spectrum access* (DSA) technology [3] has been proposed as a solution for a more efficient use of spectrum. This technology enables two types of users to operate on a channel– *primary users*, which have prioritised access to the channel and *secondary users*, which can use the channel when it is not in use by the primaries [3].

We consider a scenario in which multiple primaries own bandwidth in a large region (e.g., a state), which is divided into smaller locations (e.g., towns). Time is divided into slots of equal duration. During each slot, a primary can lease its unused (free) channels to secondaries for the duration of that slot. At each location, secondaries lease bandwidth from the primaries who set the lowest prices. This results in *price competition* among the primaries to lease their free channels. This is similar to the classic *Bertrand price competition* [4], wherein a few firms compete among themselves to sell their goods to customers. However, there are several important differences between Bertrand price competition and that in a DSA market. Specifically, a primary may or may not have a free channel in a given slot and hence a primary that *has* a free channel does not know how many other primaries are selling their channels in the slot. The primary will unnecessarily get a low revenue if it sells its bandwidth at a low price when only a few primaries have free channels since its channel would have been bought even if it set a higher price. Conversely, a primary's free channel may remain unsold if it chooses a high price when a large number of primaries have free channels. The other important difference between Bertrand price competition and that in a DSA market is that radio spectrum allows *spatial reuse*: the same band can be simultaneously used at multiple locations provided these locations are far apart; however, transmissions at neighboring locations interfere with each other. Thus, each primary must jointly select a set of mutually non-interfering locations at which to offer bandwidth as well as the price at each location in the set. We formulate the above price competition between the primaries as a game and seek *Nash equilibria* (NE) [5] in it.

Spectrum pricing games have been studied in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. Price competition between secondaries in which a single operator senses and leases the free channels is studied as a Stackelberg game in [12]. Price competition between multiple primaries with uncertain bandwidth availability is studied in [11], [12], [14], [16], [17]. In [16], [17] the quality of bands is taken into account in addition to uncertainty in spectrum availability. NE in the spectrum pricing game are studied in [11], [14], [16], [17]. Our model for the case where there is a single location is similar to the general Bertrand competition with uncertainty model formulated in [22] in which multiple sellers compete through their price selection strategies to sell a homogenous product to a single buyer. However, a seller does not know the number of other sellers with which it is competing. Under this uncertainty the NE of the game is obtained in [22]. However, in all the above papers, it is assumed that each player chooses a price from a *continuous set*, e.g., an interval $[a, b]$, where a and b are real numbers. The continuity of the price set is assumed as an approximation to simplify the mathematical analysis. However, note that in practice, a player can only choose a price from a *discrete set* (e.g., multiples of one cent). This restricts the number of prices a player can choose to a finite number. Also, learning algorithms such as Softmax and Q-learning, which are often used to compute (converge to) NE in games [23], [24], require that a finite number of strategies be available to each player, which is not true when the continuity approximation is used. In this paper we investigate the fundamental question of how

Aditya MVS and G. S. Kasbekar are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai, India. Their email addresses are adityamvs@iitb.ac.in and gskasbekar@ee.iitb.ac.in respectively. Abhishek Raghuvanshi is with Qualcomm India. His email address is abhi.ed754@gmail.com.

A preliminary version of this paper appeared in IEEE VTC 2016 [2].

the behaviour of the players involved in the price competition changes when this continuity assumption is removed. To the best of our knowledge *this work is the first to investigate the effects of the continuity assumption on the NE of a spectrum pricing game*.

In this paper, we study the NE in a scenario where multiple primaries, each of which may or may not have free bandwidth in a given slot in a region, which is divided into multiple locations, sell their free bandwidth to secondaries at individual locations; we assume that the number of prices the players can choose is finite (as in practice). The same model was studied in [11], but under the approximation that players choose prices from a continuous set. Also, this model is important since several generalizations of this model have been studied in prior work [15], [16], [17], [19], [20], [21]. In Section II, we formulate the price competition among the primaries as a game for the case when there is *a single location*. In Section III, the main results obtained for the single location game in [11] are summarized. In Section IV, we analyse the NE of the single location game (with a finite number of prices) for the case when there are two primaries and one secondary. This game, though simple, reveals several important differences between the games with continuous and discrete price sets. For example, no pure strategy NE exists in the game with continuous price sets, whereas a *pure strategy NE may exist* in the game with discrete price sets. Also, in the game with discrete price sets, there exist *multiple symmetric NE*. Moreover, the *expected payoff* that each primary gets under the symmetric NE may be *different under different NE*. This is in sharp contrast to the game with continuous price sets in [11], where there is a unique symmetric NE. However, in Section V, we analyse the price competition between the primaries at a single location when there are an arbitrary number of primaries and secondaries and show that, as the number of available prices becomes large, the *strategies of the primaries under every symmetric NE converge pointwise to the unique symmetric NE strategy of the game with continuous price sets*. In Section VI, we extend the analysis in Section V to the multiple locations case. In Section VII, we provide the proofs of the analytical results stated in Sections IV and V. The results in Sections IV, V and VI show that although, as is consistent with intuition, the equilibrium behaviour of the players in the game with discrete prices is similar to that in the game with continuous prices when the number of prices is large, it is significantly different when the number of prices is small; thus caution must be exercised while using the continuous prices approximation in the context of price competition in spectrum markets. Also, the results in Sections V and VI provide a *formal justification for the continuous prices approximation*; to the best of our knowledge, our work is the first to provide such a justification for *any* spectrum pricing game. In Section VIII, we study the infinitely repeated version of the single location game described in Section II via simulations. We find that when each primary independently uses the well-known *Softmax learning algorithm* [28] to adapt its strategy based on the payoffs it got in the past slots, the long run strategies of the players converge to a NE of the corresponding one-shot game only when there exists a pure strategy NE. However, when only mixed strategy NE exist in the one-shot game, the long run strategies do not converge to a NE. We conclude the paper in Section IX.

Finally, our results for the case where there is a single location more generally apply to any setting where the sellers' supply is uncertain. In particular, microgrids [25] are a newly emerging technology for distributed electricity generation, which consist of a connected network of generators (e.g., solar panels, wind turbines) and loads (e.g., households, factories). There is uncertainty in the power generated by a generator at a given time, e.g, the power produced by a solar panel on a given day depends on the availability of sunlight. Our results provide an analysis of the effects of the continuity assumption on the NE in such electricity markets [19].

II. Network Model

Suppose there are $n \geq 2$ primaries and $k \geq 1$ secondaries in a location. Each primary owns one channel (one unit of bandwidth) and each secondary has a demand of one channel. Time is divided into slots and trade takes place at the beginning of each slot. In every slot, each primary has a free channel with probability (w.p.) $q \in (0, 1)$. Each primary with a free channel selects a price at which to offer its channel to secondaries. Now, a primary that leases a channel to a secondary may incur some cost, *e.g.*, if the secondary uses some of the former's infrastructure. Let this cost be $c \geq 0$ for each primary. A primary does not sell its bandwidth below this price as it would incur a loss. We will also assume that there is a limit, say v, to the maximum price a primary can select. This limit may be due to the following reasons [11]: (i) The spectrum regulator may impose this limit to prevent the primaries from charging excessively when they collude or when the number of primaries with free bandwidth is less than the number of secondaries. (ii) Each secondary may have a valuation of v for a channel and would not buy a channel for a price greater than v. Thus, if we denote primary i's price by p_i, then $c < p_i \leq v$.

Recall that in practice, there is only a finite set of prices a primary can choose from. Let this set be $\{a_1, a_2, \ldots, a_M\}$, where $a_j = c + \left(\frac{v-c}{M}\right) j$. Secondaries buy bandwidth from the primaries that set the lowest prices. Specifically, if Z primaries have free bandwidth, then the bandwidth of $\min(Z, k)$[1] primaries gets sold. We model the above price competition among the primaries as a *game* [5] in which the actions of the primaries (players) are the prices they choose. Note that when primary i has free bandwidth, $p_i \in \{a_1, \ldots, a_M\}$; with a slight abuse of notation, we assume that $p_i = v + 1$ when a primary does not have free bandwidth [2]. Also, let $a_{M+1} = v + 1$. Next, recall that the *utility* or *payoff* represents the level of

[1] $\min(a, b)$, where $a, b \in \mathbb{R}$, denotes the minimum of a and b.

[2] As explained later, the expected payoff of a primary is a function of whether or not each of the other primaries has free bandwidth. This notation simplifies the exposition by eliminating the need to condition on whether they have free bandwidth. Also, the choice $p_i = v+1$ is arbitrary; any price above v can be chosen.

satisfaction of a player [4]. If a primary does not sell its bandwidth, its utility is defined to be 0. Let $u_i(p_1, \ldots, p_n)$ denote the utility of primary i when primary j selects price p_j, $j = 1, \ldots, n$. Consider primary 1 and let X_k denote the k'th lowest price among p_j, $j \in \{2, \ldots, n\}$. Since there are k secondaries, primary 1 sells its bandwidth w.p. 1 if $p_1 < X_k$ and does not sell its bandwidth if $p_1 > X_k$. If $p_1 = X_k$, then note that more than one primary chooses the price X_k. In this case, the tie is broken randomly [3]. Thus:

$$u_1(p_1, p_2, \ldots, p_n) = \begin{cases} p_1 - c, & \text{if} \quad X_k > p_1, \\ \frac{(k-l)(p_1-c)}{m}, & \text{if} \quad X_k = p_1, \\ 0, & \text{if} \quad X_k < p_1, \end{cases}$$

where in the second case, $l < k$ primaries choose a price less than X_k and m primaries (including primary 1) choose X_k. (Note that when primary 1 does not have free bandwidth, its utility is 0 even if $p_1 = X_k$.) The utilities of primaries $j = 2, \ldots, n$ are computed similarly. Each primary i is allowed to randomly choose its price p_i using an arbitrary distribution function (d.f.) $\phi_i(\cdot)$. This d.f. is called the *strategy* [5] of primary i. The vector of strategies of all the players $(\phi_1(\cdot), \ldots, \phi_n(\cdot))$ is called the *strategy profile* [5]. Let $\phi_{-i}(\cdot) = (\phi_1(\cdot), \ldots, \phi_{i-1}(\cdot), \phi_{i+1}(\cdot), \ldots, \phi_n(\cdot))$ denote the strategy profile of all the players except player i. Also, let $E(u_i(\phi_1(\cdot), \ldots, \phi_n(\cdot)))$ denote the expected utility of primary i when the strategy profile selected is $(\phi_1(\cdot), \ldots, \phi_n(\cdot))$. We will use the concept of NE, which is widely used as a solution concept in game theory [5]. A strategy profile $(\phi_1^*(\cdot), \ldots, \phi_n^*(\cdot))$ constitutes a NE [5] if $\forall i \in \{1, 2, \ldots, n\}$:

$$E\left(u_i\left(\phi_i^*(\cdot), \phi_{-i}^*(\cdot)\right)\right) \geq E\left(u_i\left(\phi_i(\cdot), \phi_{-i}^*(\cdot)\right)\right), \quad \forall \phi_i(\cdot).$$

That is, a NE is a strategy profile such that no player can improve its expected utility by unilaterally deviating from its strategy [5]. When $n \leq k$, then clearly $p_i = v \ \forall i = 1, \ldots, n$ is the unique NE of the game since by setting $p_i = v$, primary i can sell its bandwidth regardless of the choices made by the other primaries and also it gets the maximum possible payoff. So henceforth we assume that $n > k$. Note that the game described above is a finite *symmetric game* [5] since all the primaries have the same action sets (available prices), same utility functions and have free bandwidth with equal probabilities. We will seek *symmetric NE*, which are those in which $\phi_1(\cdot) = \phi_2(\cdot) = \ldots = \phi_n(\cdot) = \phi(\cdot)$ (say) [5]. Symmetric NE have been advocated as a solution concept for symmetric games by several game theorists [18], since in practice, it is challenging to implement a NE that is not symmetric. Also, it is shown in [18] that every finite symmetric game has atleast one symmetric NE.

III. Background

We now briefly summarise the results obtained in [11], which are for the model described in Section II with the difference that the price a primary selects is allowed to be any *real number* in the interval $(c, v]$. It was proved in [11] that this game has no pure strategy NE [4]. Let:

$$w(q, n) = \sum_{i=k}^{n-1} \binom{n-1}{i} q^i (1-q)^{n-1-i}. \tag{1}$$

Since each primary independently has unused bandwidth w.p. q, $w(q, n)$ is the probability that k or more out of $n-1$ primaries have unused bandwidth. Let:

$$\tilde{p} = c + (v - c)(1 - w(q, n)) \tag{2}$$

and note that $c < \tilde{p} < v$. It was shown in [11] that in the above price competition game, there is a unique symmetric NE. Also, in this NE, each primary selects its price randomly using a d.f. $\phi(\cdot)$, whose *support set* [5] is the interval $[\tilde{p}, v]$. Moreover, $\phi(\cdot)$ is continuous and strictly increasing on $[\tilde{p}, v]$ as illustrated in Fig. 1. The function $\phi(\cdot)$ can be computed as follows [11]. Let X_k be as in Section II and $F(\cdot)$ denote the d.f. of X_k. It was shown in [11] that under the unique symmetric NE:

$$F(x) = \begin{cases} 0, & x \leq \tilde{p}, \\ \frac{x-\tilde{p}}{x-c}, & \tilde{p} < x \leq v. \end{cases} \tag{3}$$

Also, the symmetric NE price selection d.f. $\phi(\cdot)$ is the unique solution of the following equation [11]:

$$F(x) = \sum_{i=k}^{n-1} \binom{n-1}{i} (\phi(x))^i (1 - \phi(x))^{n-1-i}. \tag{4}$$

Note that (4) is consistent with the facts that $F(x) = P(X_k \leq x)$ and $X_k \leq x$ if and only if k or more out of primaries

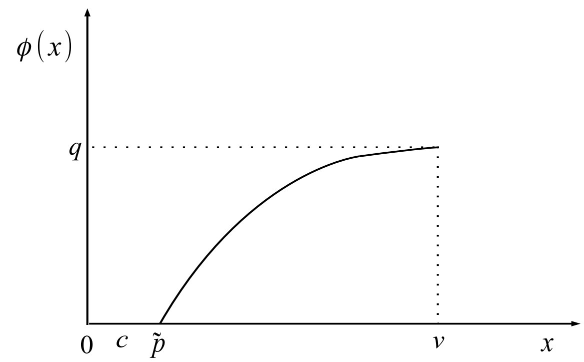

Fig. 1: $\phi(x)$ is continuous and strictly increasing in the interval $[\tilde{p}, v]$.

[3]For example, suppose $k-1$ primaries choose a price less than X_k, 2 primaries (including primary 1) choose the price X_k and the rest choose a price more than X_k. Then, primary 1 sells its bandwidth w.p. $\frac{1}{2}$.

[4]A pure strategy NE is one in which each primary plays a single price w.p. 1 [5].

[5]Recall that the support set of a d.f. is the smallest closed set whose complement has probability 0 under the d.f. [26]. Also, since we defined $p_i = v+1$ if primary i does not have free bandwidth, the price $v+1$ is always in the support set of the price selection strategy of a primary. However, we ignore it throughout the paper since we are concerned with the price selection strategies of primaries that *have* free bandwidth.

$2, \ldots, n$ select a price that is $\leq x$. Also, note that since $\phi(\cdot)$ is strictly increasing on $[\tilde{p}, v]$, prices in every sub-interval of the interval $[\tilde{p}, v]$ are played with positive probability by each primary in this NE; also, prices in $(c, \tilde{p})$ are *not* played. The utility of each primary under the above symmetric NE was shown to be [11]:

$$E\left(u_1\left(\phi(\cdot), \phi_{-1}(\cdot)\right)\right) = \tilde{p} - c = u_{max} \quad \text{(say).} \tag{5}$$

Next, consider the model with discrete price sets described in Section II and let $\phi(\cdot)$ be as defined in that section. We state a lemma [4], which provides necessary and sufficient conditions for $\phi(\cdot)$ to constitute the price selection strategy of each primary under a symmetric NE, and which we will extensively use in the following sections. Let S be the support set of the d.f. $\phi(\cdot)$, *i.e.*, the subset of prices from $\{a_1, \ldots, a_M\}$ that are selected with positive probabilities under $\phi(\cdot)$, and let $S^c = \{a_1, \ldots, a_M\} \setminus S$.

Lemma 1: The d.f. $\phi(\cdot)$ constitutes the price selection strategy of each primary under a symmetric NE iff:

$$E\left(u_1\left(a_l, p_2, \ldots, p_n\right)\right) = E\left(u_1\left(a_m, p_2, \ldots, p_n\right)\right), \quad \forall a_l, a_m \in S,$$

$$E\left(u_1\left(a_l, p_2, \ldots, p_n\right)\right) \geq E\left(u_1\left(a_l', p_2, \ldots, p_n\right)\right), \quad \forall a_l \in S, a_l' \in S^c.$$

Lemma 1 states that the expected payoffs that primary 1 gets at all prices that it plays with positive probability under the symmetric NE are equal, and are $\geq$ the payoff at each price that it does not play under the NE. For ease of terminology, henceforth we refer to the game with continuous price sets as the continuous game and the game with discrete price sets as the discrete game.

IV. Price Competition in the Special Case $n = 2$, $k = 1$

We now analyse the special case of the discrete game with $n = 2$ and $k = 1$. Our analysis reveals the following important differences between the NE in the continuous game and the discrete game:

1) No pure strategy NE exists in the continuous game, whereas a *pure strategy NE may exist* in the discrete game.

2) There exist *multiple symmetric NE* in the discrete game unlike in the continuous game where there exists a unique symmetric NE. Moreover, the *expected payoff that each primary gets under the symmetric NE may be different under different symmetric NE.*

3) Recall from Section III that in the continuous game, for each value of $q \in (0, 1)$, the support set of the symmetric NE price selection strategy $\phi(\cdot)$ of each primary is the set $[\tilde{p}, v]$. However, in the discrete game, this type of symmetric NE, i.e., one in which the support set of each primary's price selection strategy is the set, $\{a_{M-P}, \ldots, a_{M-1}, a_M\}$, of all the available prices above a threshold, *exists only for some values of $q \in (0, 1)$ and not for others, and no matter how large M is, there are certain values of q for which this type of NE does not exist*. Fig. 2 shows, for an example, the set of values of q where this type of NE exists when $n = 2$ and $k = 1$. For ease of terminology, henceforth we refer to a symmetric NE in the discrete game whose support set consists of all the prices above a threshold as a *symmetric NE of type $\mathcal{C}$*.

Fig. 2: The shaded area represents the set of values of q for which a symmetric NE of type $\mathcal{C}$ exists.

Note that the game with $n = 2$, $k = 1$, and a finite number of prices for each player to choose from, is a finite symmetric game and thus has atleast one symmetric NE [18]. Suppose in this NE each primary selects a price from $\{a_1, a_2, \ldots, a_M, a_{M+1}\}$ using the probability mass function (PMF) $R(\cdot)$. Then, $R(\cdot)$ satisfies the following equations:

$$\sum_{i=1}^{M} R(a_i) = q, \tag{6a}$$

$$R(a_{M+1}) = 1 - q. \tag{6b}$$

Now, primary 1 sells its free bandwidth w.p. 1 when $p_1 < p_2$ and w.p. $\frac{1}{2}$ when $p_1 = p_2$. So its expected utility is:

$$E(u_1(p_1, p_2)) = (p_1 - c)\left[P(p_1 < p_2) + \frac{P(p_1 = p_2)}{2}\right], \tag{7}$$

where $P(A)$ denotes the probability of event A. Fix q and suppose the PMF $R(\cdot)$ has support set $\{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$, where $i_j \in \{1, 2, \ldots, M\}$ and $i_1 < i_2 < \ldots < i_m$. Primaries select $a_{i_{m+1}} = v + 1$ if they do not have a free channel. By Lemma 1, we get $\forall\ i_j, i_l \in \{i_1, i_2, \ldots, i_m\}$:

$$E\left(u_1\left(a_{i_j}, p_2\right)\right) = E\left(u_1\left(a_{i_l}, p_2\right)\right), \tag{8a}$$

$$E\left(u_1\left(a_{i_j}, p_2\right)\right) \geq E\left(u_1\left(a_i, p_2\right)\right) \quad \forall i \in \{1, 2, ..., M\}. \tag{8b}$$

By (7) and the fact that $a_i = c + \frac{v-c}{M} i\ \forall i = 1, \ldots, M$, the utility of primary 1 when it chooses the price a_{i_m} is:

$$\begin{aligned} E\left(u_1\left(a_{i_m}, p_2\right)\right) &= \left(a_{i_m} - c\right)\left[P\left(p_2 > a_{i_m}\right) + \frac{P\left(p_2 = a_{i_m}\right)}{2}\right] \\ &= \frac{v-c}{M} i_m \left[R\left(a_{i_{m+1}}\right) + \frac{R\left(a_{i_m}\right)}{2}\right] \\ &= \frac{v-c}{M} i_m \left[1 - q + \frac{R\left(a_{i_m}\right)}{2}\right] \quad \text{(by (6b))} \end{aligned} \tag{9}$$

Similarly we can write for $j = 1, \ldots, m-1$:

$$E\left(u_1\left(a_{i_j}, p_2\right)\right) = \frac{v-c}{M} i_j \left[\sum_{l=j+1}^{m+1} R(a_{i_l}) + \frac{R(a_{i_j})}{2}\right]. \tag{10}$$

TABLE I: Support sets of symmetric NE at different values of q for $M = 4$

Support Set	Valid q
$\{a_4\}$	$(0, 0.5]$
$\{a_3\}$	$[0.4, 0.67]$
$\{a_2\}$	$[0.67, 1]$
$\{a_1\}$	$[0.86, 1]$
$\{a_3, a_4\}$	$[0.4, 0.5]$
$\{a_2, a_4\}$	$[0.67, 0.75]$
$\{a_1, a_4\}$	$[0.86, 0.9]$
$\{a_2, a_3\}$	$[0.57, 0.67]$
$\{a_1, a_3\}$	$[0.84, 0.89]$
$\{a_1, a_2\}$	$[0.8, 1]$
$\{a_2, a_3, a_4\}$	$[0.57, 0.75]$
$\{a_1, a_3, a_4\}$	$[0.84, 0.9]$
$\{a_1, a_2, a_4\}$	$[0.8, 0.875]$
$\{a_1, a_2, a_3\}$	$[0.82, 0.89]$
$\{a_1, a_2, a_3, a_4\}$	$[0.82, 0.875]$

By (8a), we get $E(u_1(a_{i_j}, p_2)) = E(u_1(a_{i_m}, p_2))\ \forall\ j = 1, \ldots, m-1$. This gives us a set of $m-1$ linear equations (one for each $j = 1, \ldots, m-1$) with m unknowns $(R(a_{i_1}), \ldots, R(a_{i_m}))$. These $m-1$ linear equations along with (6a) result in m linear equations with m unknowns. By solving these linear equations, we get the following expressions for $R(a_{i_j})$, $j = 1, 2, \ldots, m$.

Case (i): When m is even:

$$\begin{aligned} R(a_{i_m}) &= -2 + \frac{(2Q-1)}{Q-1}q, \\ R(a_{i_{2l+1}}) &= 2 - \frac{2Q_{i_{2l+1}}}{Q-1}q, \\ R(a_{i_{2l}}) &= -2 + \frac{2Q_{i_{2l}}}{Q-1}q, \end{aligned} \tag{11}$$

where $l = 0, 1, \ldots, \frac{m}{2} - 1$, $Q = \frac{i_m}{i_1} - \frac{i_m}{i_2} + \ldots + \frac{i_m}{i_{m-1}}$, $Q_{i_{2l}} = \frac{i_m}{i_1} - \frac{i_m}{i_2} + \ldots + \frac{i_m}{i_{2l-1}} - \frac{i_m}{2i_{2l}}$ and $Q_{i_{2l+1}} = \frac{i_m}{i_1} - \frac{i_m}{i_2} + \ldots - \frac{i_m}{i_{2l}} + \frac{i_m}{2i_{2l+1}}$.

Case (ii): When m is odd:

$$\begin{aligned} R(a_{i_m}) &= -\frac{2Q}{Q+1} + \frac{(2Q+1)q}{Q+1}, \\ R(a_{i_{2l}}) &= -2 + \frac{4Q_{i_{2l}} - 2Q_{i_{2l}}q}{Q+1}, \\ R(a_{i_{2l+1}}) &= 2 - \frac{4Q_{i_{2l+1}} - 2Q_{i_{2l+1}}q}{Q+1}, \end{aligned} \tag{12}$$

where $l = 0, 1, \ldots, \frac{m-1}{2}$, $Q = \frac{i_m}{i_1} - \frac{i_m}{i_2} + \ldots + \frac{i_m}{i_{m-2}} - \frac{i_m}{i_{m-1}}$, $Q_{i_{2l}} = \frac{i_m}{i_1} - \frac{i_m}{i_2} + \ldots + \frac{i_m}{i_{2l-1}} - \frac{i_m}{2i_{2l}}$, $Q_{i_{2l+1}} = \frac{i_m}{i_1} - \frac{i_m}{i_2} + \cdots - \frac{i_m}{i_{2l}} + \frac{i_m}{2i_{2l+1}}$.

For the special case $n = 2$, $k = 1$ and $M = 4$, Table I provides an *exhaustive* list of all possible support sets, $\{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$, of a symmetric NE price selection PMF $R(\cdot)$ in the first column, and the set of all values of q for which symmetric NE with these support sets exist in the second column. For example, consider the fifth entry: $\{a_3, a_4\}$ constitutes the support set of a symmetric NE price selection PMF $R(\cdot)$ for $q \in [0.4, 0.5]$. The table is obtained by calculating $R(\cdot)$ for every possible combination of prices, $\{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$, as support set and noting that only the combinations which satisfy (8b) constitute a valid support set. Table I along with our analysis of the special case of the game with $n = 2, k = 1$ reveals the following differences between the symmetric NE in the continuous game and the discrete game.

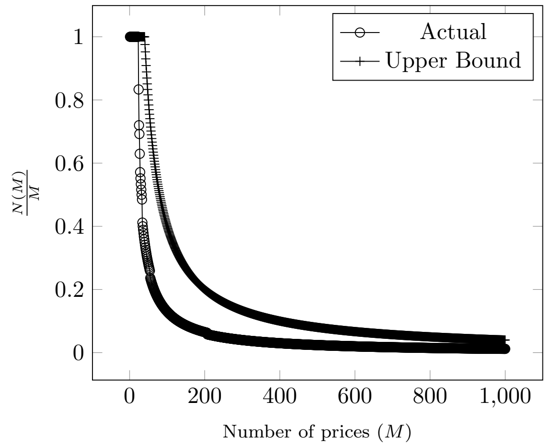

Fig. 3: The plot compares the upper bound on the fraction $\frac{N(M)}{M}$ obtained by Theorem 1 with the actual number of PNE obtained by exhaustive search for the game with $n = 15, k = 10$ as M varies from 1 to 1000.

A. Pure Strategy NE

Recall from Section III that in the continuous game, no pure strategy NE exists. However, the first four entries in Table I show that a *pure strategy NE may exist* in the discrete game. For some values of q, the discrete game has a pure strategy NE. From Table I, in the discrete game with $n = 2, k = 1$ and $M = 4$ there are a total of 4 pure strategy NE. We now state a theorem which gives an upper bound on the total number of pure strategy NE in the discrete game with n primaries and k secondaries.

Theorem 1: For a fixed k, $\forall M > 4k+5$, there exist at most $4k$ pure strategy NE.

The proof of Theorem 1 is provided in Section VII. Let $N(M)$ denote the total number of pure strategy NE (PNE) that exist for all values of $q \in (0, 1)$ when there are a total of M prices. Then for the special case of $n = 2, k = 1$, the above theorem limits $N(M)$ to 4 whenever $M > 9$. Fig. 3 compares the actual fraction of the prices that constitute a PNE in the discrete game with $n = 15, k = 10$ with the upper bound obtained from Theorem 1. Since no pure strategy NE exists in the continuous game, intuitively, Theorem 1 says that *as $M \to \infty$, the discrete game becomes similar to the continuous game in terms of fraction of prices that constitute a PNE.*

B. Multiple NE

Recall from Section III that in the continuous game, there is a unique symmetric NE for a given value of q. In contrast, there are *multiple symmetric NE* in the discrete game. For example Table I shows that at $q = 0.5$, $\{a_4\}$, $\{a_3\}$ and $\{a_3, a_4\}$ all constitute support sets of symmetric NE price selection PMFs $R(\cdot)$. Also, it is easy to check that the expected payoff that each primary gets may also be different under the different symmetric NE for a given value of q. In the above example, the expected payoffs under the three symmetric NE at $q = 0.5$ are $\frac{3}{4}(v-c)$, $\frac{9}{16}(v-c)$ and $\frac{3}{4}(v-c)$ respectively. The above observations show that *the actions (price selection strategies) taken by players as well as the rewards (expected payoffs) they get at equilibrium may differ substantially in the continuous game and the discrete game.*

More importantly, the differences between the NE in the continuous game and the discrete game observed for $M = 4$ in Section IV-A and in the previous paragraph in fact hold *for every value of M, no matter how large.* Specifically, it is easy to verify using the above analysis that selection of the price a_M w.p. 1 by each primary that has a free channel constitutes a symmetric NE when $q \in \left(0, \frac{2}{M}\right]$. Hence, no matter how large the value of M is, there exists a pure strategy NE in the discrete game for certain values of q, in contrast to the continuous game, in which no pure strategy NE exists for any value of q. It can also be checked using the above analysis that $\{a_{M-1}, a_M\}$ constitutes the support set of a symmetric NE price selection PMF $R(\cdot)$ when $q \in \left[\frac{2}{M+1}, \frac{2}{M}\right]$. Hence, for $q \in \left[\frac{2}{M+1}, \frac{2}{M}\right]$, $\{a_M\}$ as well as $\{a_{M-1}, a_M\}$ constitute support sets of symmetric NE price selection PMFs. Thus, no matter how large the value of M is, there exist multiple symmetric NE in the discrete game for certain values of q, in contrast to the continuous game, in which there is a unique symmetric NE for every value of q.

C. Symmetric NE of type $\mathcal{C}$

Next, recall from Section III that for every value of $q \in (0,1)$, the support set of the unique symmetric NE price selection strategy in the continuous game is of the form $[\tilde{p}, v]$. Hence, the support set is the set of all the available prices above a threshold ($\tilde{p}$). We are interested in symmetric NE with a similar support set in the discrete game, i.e., a support set consisting of all the available prices above a threshold; we refer to such an NE as a symmetric NE of type $\mathcal{C}$. Suppose under a symmetric NE of the discrete game with $n = 2, k = 1$, the price selection strategy support set is $\{a_{M-P}, a_{M-P+1}, \ldots, a_{M-1}, a_M\}$; note that there are $P+1$ consecutive prices in the support set. The expressions for the symmetric NE price selection PMF $R(\cdot)$ are obtained by substituting $i_1 = M-P, i_2 = M-P+1, ..., i_P = M-1, i_{P+1} = M$ in (11) when P is odd or in (12) when P is even. Now, since the prices $a_i, i = M-P, M-P+1, ..., M$ are in the support set:

$$R(a_i) > 0, \ i = M-P, M-P+1, ..., M. \tag{13}$$

Also, for $R(\cdot)$ to constitute a symmetric NE price selection strategy, (8b) must be satisfied. Let V^P denote the set of values of q for which inequalities (13) and (8b) are satisfied. The following lemma characterizes the set V^P.

Lemma 2: V^P is an open interval. For P odd, V^P is:

$$\left(\frac{2(Q-1)}{2Q-1}, \frac{Q-1}{Q_{P-1}}\right),$$

where $Q = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots + \frac{M}{M-3} - \frac{M}{M-2} + \frac{M}{M-1}$ and $Q_{P-1} = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots - \frac{M}{M-2} + \frac{M}{2(M-1)}$ and for P even, V^P is:

$$\left(\frac{2Q}{2Q+1}, 2 - \frac{Q+1}{Q_{P-1}}\right),$$

where $Q = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots + \frac{M}{M-2} - \frac{M}{M-1}$ and $Q_{P-1} = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots + \frac{M}{M-2} - \frac{M}{2(M-1)}$.

We provided the proof of above Lemma in Appendix. Let L^P and U^P denote the lower and upper endpoints of the interval V^P respectively. The following Lemma states that for $P \in \{0, 1, 2, \ldots\}$, $U^P < L^{P+1}$.

Lemma 3: For $P \in \{0, 1, 2, \ldots\}$, $U^P < L^{P+1}$.

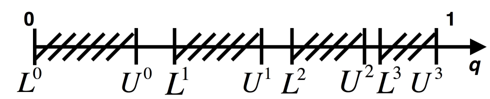

Fig. 4: The figure illustrates L^P and U^P for $P = 0, 1, 2, 3$ on the q line.

We provided the proof of above Lemma in Appendix. Fig. 4 shows an example of the endpoints L^P and U^P. From Lemma 3, it follows that for certain values of q (e.g., for the values in $\left[U^0, L^1\right]$, $\left[U^1, L^2\right]$ and $\left[U^2, L^3\right]$ in Fig. 4, and in general, for the values in $\left[U^P, L^{P+1}\right]$, $P = 0, 1, 2, \ldots$), there *does not exist a symmetric NE whose price selection strategy support set is the set of all the available prices above a certain threshold (a_{M-P})*. Also, surprisingly, this is *true no matter how large M is*. This is in sharp contrast to the NE in the continuous game where for every $q \in (0,1)$, the support set of the unique symmetric NE price selection strategy is the set $[\tilde{p}, v]$, which is the set of all the available prices above $\tilde{p}$ (see Section III). Also, note that as q increases from a value in $\left(L^P, U^P\right)$ to a value in $\left(L^{P+1}, U^{P+1}\right)$, a price ($a_{M-P-1}$) gets added to the lower end of the support set. This is consistent with the intuition that as q, the probability that a primary has free bandwidth, increases, the price competition becomes more intense, and hence each primary chooses lower prices to get its bandwidth sold.

We now state a theorem which establishes the existence of large regions in the interval $q \in (0,1)$ where a symmetric NE of type $\mathcal{C}$ does not exist in the discrete game with $n = 2, k = 1$ for values of M greater than some threshold. Let I_M^P denote the length of the interval V^P when there are a total of M available prices. The following theorem states that *for a fraction of approximately $\frac{3}{8}$ of the interval $q \in (0,1)$, there does not exist a symmetric NE of type $\mathcal{C}$ for values of M greater than some threshold*. This is in contrast to the continuous game, in which there exists a symmetric NE of type $\mathcal{C}$ *for every value of* $q \in (0,1)$.

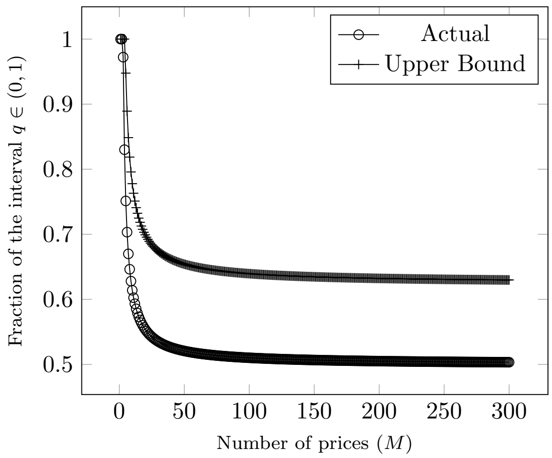

Fig. 5: The plot gives the actual fraction of the interval $q \in (0,1)$ where there exists a symmetric NE of type $\mathcal{C}$ and the upper bound obtained in Theorem 2 for various numbers of prices.

Theorem 2: For every $\epsilon > 0$, there exists an $M(\epsilon)$ such that for all $M > M(\epsilon)$, $\sum_{i=0}^{M-1} I_M^i < \frac{5}{8} + \epsilon$.
The proof of Theorem 2 is provided in Section VII. Fig. 5 compares the upper bound obtained in Theorem 2 to the actual fraction[6] of the interval $q \in (0,1)$ where there exists a symmetric NE of type $\mathcal{C}$.

V. Price Competition for arbitrary n, k and large M

In this section, we show that, as $M \to \infty$, the price selection strategies of the primaries under every symmetric NE of the discrete game converge to the unique symmetric NE strategy of the continuous game.

As in Section IV, let $R(\cdot)$ denote the PMF that each primary adopts over the price set $\{a_1, a_2, \ldots, a_M, a_{M+1}\}$ in a symmetric NE. Then $R(\cdot)$ satisfies (6a) and (6b). Let $\phi_M(\cdot)$ denote the d.f. corresponding to $R(\cdot)$. Then $\phi_M(a_i) = \sum_{l=1}^{i} R(a_l)\ \forall i \in \{1, \ldots, M, M+1\}$. Recall from Section III that in the continuous game, the symmetric NE strategy $\phi(\cdot)$ of a primary is continuous on its support set $(\tilde{p}, v]$. However, $\phi_M(\cdot)$ is a discontinuous function with jumps [7] at the prices in its support set (the prices a_i such that $R(a_i) > 0$). Also, $\phi(\cdot)$ is unique. In contrast, as we have seen in Section IV, there may exist multiple symmetric NE in the discrete game. So $\phi_M(\cdot)$ may not be unique. However, we prove that as $M \to \infty$, *all the possible functions $\phi_M(\cdot)$ that constitute a symmetric NE price selection strategy converge to $\phi(\cdot)$*. Since $\phi(\cdot)$ is a continuous d.f., to prove that a discrete d.f. $\phi_M(\cdot)$ converges pointwise to the former, we first show in Lemma 4 that the sizes of the jumps in $\phi_M(\cdot)$ decrease to 0 as $M \to \infty$.

Lemma 4: Fix $q \in (0,1)$. For every $\epsilon > 0$, $\exists\ M_\epsilon$ such that if $M > M_\epsilon$ then in every symmetric NE strategy $\phi_M(\cdot)$, each price $x \in (c, v]$ is played with probability $\leq \epsilon$.

The proof of Lemma 4 is provided in Section VII. Note that Lemma 4 does not contradict the result stated in Section IV that selection of the price a_M w.p. 1 by each primary that has a free channel constitutes a symmetric NE when $q \in (0, \frac{2}{M}]$. This is because, since each primary has a free channel w.p. q, under this symmetric NE, the effective probability with which a primary selects price a_M is q, which decreases to 0 as $M \to \infty$. Now, let $u_{max}(M)$ be the best response[8] payoff under the symmetric NE when there are M available prices. We state a lemma which says that as $M \to \infty$, $u_{max}(M)$ converges to u_{max} in (5), which is the best response payoff with continuous prices.

Lemma 5: For every $\epsilon' > 0$, $\exists M_{\epsilon'}$ such that if $M \geq M_{\epsilon'}$, then in every symmetric NE:

$$|u_{max}(M) - u_{max}| < \epsilon'.$$

The proof of Lemma 5 is provided in Section VII. Next, we state the main result of this section, which shows that as $M \to \infty$, the price selection d.f. $\phi_M(\cdot)$ under *every* symmetric NE of the discrete game approaches the price selection d.f., $\phi(\cdot)$, in the continuous game.

Theorem 3: As $M \to \infty$, the sequence of functions $\phi_M(x)$ converges pointwise to $\phi(x)\ \forall\ x \in (c, v]$.

The proof of Theorem 3 is provided in Section VII. Theorem 3 shows that as $M \to \infty$, the price selection d.f.s of the primaries under every symmetric NE of the discrete game converge pointwise to the price selection d.f. under the unique symmetric NE of the continuous game. This is a surprising result since as shown in Section IV, important differences exist between the NE in the continuous game and the discrete game for every value of M, no matter how large. Also, Theorem 3 provides a *formal justification for the continuous prices approximation*, which has not been provided in prior work for *any* spectrum pricing game to the best of our knowledge.

VI. Spatial Reuse

In this section, we study a generalization of the model in Section II, in which primaries sell their unused bandwidth at multiple locations. There are n primaries and each primary owns one channel over a large region (e.g., a state) which is divided into several small locations (e.g., towns). There are k secondaries at each location, where $k \in \{1, \ldots, n-1\}$. In every slot, a primary either uses its channel over the entire region or does not use it anywhere in the region. A typical example of this scenario is when a primary uses its channel to broadcast the signal throughout the region (e.g., TV

[6] The actual fraction for each M is obtained by summing the lengths of all intervals V^P, $P = 0, 1, \ldots, M-1$ where V^P is as in Lemma 2.

[7] Recall that the function $\phi_M(\cdot)$ has a jump at x, if $\phi_M(x) - \phi_M(x-) > 0$ [26].

[8] The price selection strategy that gives the highest utility to a player given the price selection strategies of the other players is called the former's *best response* [5].

broadcasting). Like in the single location case, each primary has a free channel w.p. q. A primary that has a free channel in a time slot can lease it out to secondaries at multiple locations. However, simultaneous transmissions on the same channel at two neighbouring locations interfere with each other. So a primary cannot sell its free bandwidth at two locations which are neighbours of each other. The overall region is represented by an undirected graph [27] $G = (V, E)$ where V is the set of nodes (representing locations) and E is the set of edges between the nodes. Two nodes are connected by an edge if transmissions at the corresponding locations interfere with each other. Recall that an *independent set* [27] (I.S.) in a graph is a set of nodes such that there is no edge between any pair of nodes in the set. So a primary can only sell its unused bandwidth at multiple nodes provided they constitute an I.S. Let $\mathscr{I}$ denote the set of all I.S. in G. A primary has to *jointly* select (i) an I.S. from $\mathscr{I}$ at which to offer bandwidth, and (ii) the price at each node in the selected I.S. As in Section II, each price must be from the set $\{a_1, \dots, a_M\}$. A primary incurs an operational cost c at each node at which it sells bandwidth. So if primary i offers bandwidth at the nodes in I.S. I and selects price $p_{i,z}$ at node $z \in I$, its utility is $\sum_{z \in I}(p_{i,z} - c)$. A primary faces the following tradeoff: if it offers bandwidth at nodes of a large I.S., the number of nodes at which it potentially gets revenue is large; however, it is likely to face intense competition from other primaries who would prefer to offer bandwidth at the nodes of the large I.S. to get high revenues. We study symmetric NE in the above game.

Consider a symmetric NE in which each primary selects I.S. $I \in \mathscr{I}$ w.p. $\beta(I)$, where $\sum_{I \in \mathscr{I}} \beta(I) = 1$. The probability, say α_z, with which a primary offers bandwidth at a node $z \in V$ equals the sum of the probabilities associated with all the I.S. that contain the node, *i.e.*, $\alpha_z = \sum\limits_{I \in \mathscr{I}: z \in I} \beta(I)$. A set of probabilities, $\{\alpha_z,\ z \in V\}$, is said to be a *valid distribution* [14] if there exists a PMF $\{\beta(I),\ I \in \mathscr{I}\}$ such that $\alpha_z = \sum_{I \in \mathscr{I}: z \in I} \beta(I)\ \forall z \in V$. We analyse the above price competition for a special class of graphs called *mean valid graphs*, which were introduced in [14] and which model the conflict graphs of several topologies that commonly arise in practice, including line graphs, two and three dimensional grid graphs, the conflict graph of a cellular network with hexagonal cells and a clique of size $e \geq 1$. A graph G is mean valid if it satisfies the following two conditions [14]: (i) Its nodes can be divided into d disjoint maximal [9] I.S. $I_1, \dots, I_d$. Let $|I_j| = M_j$ and $a_{j,l},\ l = 1, \dots, M_j$ be the nodes in I.S. I_j. Assume that $M_1 \geq M_2 \geq \dots \geq M_d$. (ii) For every valid distribution in which a primary offers bandwidth at a node $a_{j,l}$ w.p. $\alpha_{j,l}$, $j = 1, \dots, d$, $l = 1, \dots, M_j$:

$$\sum_{j=1}^{d} \overline{\alpha}_j \leq 1, \quad \text{where} \quad \overline{\alpha}_j = \frac{\sum_{l=1}^{M_j} \alpha_{j,l}}{M_j}, j \in \{1, \dots, d\}.$$

An example of a mean valid graph is the $m \times m$ grid graph, which we denote as $\mathcal{H}_{m,m}$, in part (a) of Fig. 6. In this graph, m^2 nodes (locations) are arranged in a square grid. For example, $\mathcal{H}_{m,m}$ may represent a shopping complex, with the nodes corresponding to the locations of shops with Wi-Fi Access Points (AP) for Internet access. $\mathcal{H}_{m,m}$ is also the conflict graph of a cellular network with square cells as shown in part (b) of Fig. 6.

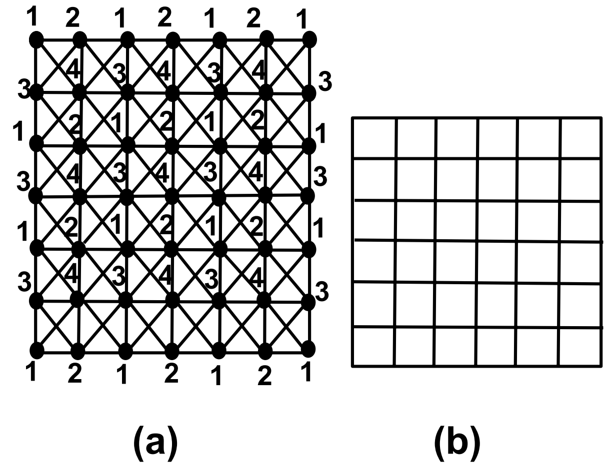

Fig. 6: Part (a) shows a grid graph $\mathcal{H}_{m,m}$ with $m = 7$. It is mean valid with $d = 4$ and the disjoint maximal I.S. $I_1, \dots, I_4$ (in the notation of the definition of a mean valid graph in Section VI), where the nodes labelled j, $j \in \{1, 2, 3, 4\}$, constitute I.S. I_j. Part (b) shows a tiling of a plane with squares, *e.g.* cells in a cellular network. Transmissions at neighboring cells interfere with each other. The corresponding conflict graph is $\mathcal{H}_{6,6}$.

We now state a separation lemma due to which once the PMF, $\{\beta(I),\ I \in \mathscr{I}\}$, that each primary uses in selecting I.S. under a symmetric NE is known, its price selection strategy at each node follows.

Lemma 6: Suppose a primary selects each node $z \in V$ w.p. α_z under a symmetric NE. Then the price selection d.f. that each primary uses at node z in the symmetric NE will be $\phi_M(\cdot)$, which is as in Section V but with q replaced with $q\alpha_z$ throughout.

The proof of Lemma 6 is similar to that of Lemma 2 in [14] and is omitted. Next, let $W(\alpha) = (v-c)(1-w(q\alpha, n))$, where $w(q, n)$ is as in (1). For the continuous game, *i.e.*, the above game with the change that the price of a primary at a node can be any *real number* from the interval $(c, v]$, it was shown in [14] that there exists a unique symmetric NE. The following result from [14] characterizes that NE.

Theorem 4: In a mean valid graph, for every $q \in (0, 1)$, there is a unique symmetric NE in the continuous game. In this NE, each primary offers bandwidth at every node in I_j, $j \in \{1, \dots, d\}$, w.p. t_j, i.e., $\alpha_{j,l} = t_j$, $l = 1, \dots, M_j$, where $(t_1, \dots, t_d)$ is the unique distribution satisfying the following conditions:

1. There exists $d' \in \{1, \dots, d\}$ such that $t_j = 0$ if $j > d'$.
2. $M_1 W(t_1) = \dots = M_{d'} W(t_{d'}) > M_{d'+1}(v - c)$.

[9] An I.S. I is maximal if for every $z \in V \setminus I$, $I \cup z$ is not an I.S [27].

It was shown in [14] that when primaries $2, \dots, n$ play their symmetric NE strategies, primary 1 gets an expected payoff of $M_j W(t_j)$ if it selects I_j. Thus, condition 2 in Theorem 4 states that a primary gets equal expected payoffs by choosing I.S. in $I_1, I_2, \dots, I_{d'}$ and this payoff exceeds the maximum payoff it could have got by selecting an I.S. in $I_{d'+1}, \dots I_d$; hence, it never opts for the latter choice, *i.e.*, condition 1 holds.

Next, we state a theorem that characterizes the symmetric NE in the discrete game in which there are M available prices at each location, for the case where M is large.

Theorem 5: Let $\{\alpha_z^M : z \in V\}$ be node selection probabilities that constitute a symmetric NE in the discrete game with M available prices at each location. Let $\{\alpha_z : z \in V\}$ be the node selection probabilities that constitute the unique symmetric NE in the continuous game described in Theorem 4, i.e., $\alpha_z = t_j$ if $z \in I_j$. Given $\epsilon > 0$, there exists M_ϵ such that for all $M \geq M_\epsilon$, $|\alpha_z^M - \alpha_z| < \epsilon$ for all $z \in V$.

Theorem 5 says that *as $M \to \infty$, the strategies of the primaries under all symmetric NE in the discrete game converge to those in the unique symmetric NE in the continuous game*. Thus, Theorem 5 provides a *formal justification for the continuous prices approximation* for the above price competition game with spatial reuse. Theorem 5 can be proved using Theorem 3 and techniques similar to the proof of Theorem 4 in [14]. We omit the proof for brevity.

VII. Proof of Analytical Results

In this section we prove the analytical results stated in Sections IV and V.

A. Proofs of Analytical Results in Section IV

Proof of Theorem 1: Consider a price a_{M-P}, where $P \in \{1, \dots, M-2\}$. Let the strategy profile $(a_{M-P}, a_{M-P}, \dots, a_{M-P})$ constitute a pure strategy NE, i.e., each primary with available bandwidth selects the price a_{M-P} w.p. 1. The utility of primary 1 under this pure strategy NE is:

$$E\left[u_1(a_{M-P}, X_k)\right] = \quad (14)$$

$$\frac{(v-c)}{M}(M-P)\left[\sum_{i=0}^{k-1}\binom{n-1}{i}q^i(1-q)^{n-1-i} + \sum_{i=k}^{n-1}\frac{k}{i+1}\binom{n-1}{i}q^i(1-q)^{n-1-i}\right]. \quad (15)$$

The first term, $\sum_{i=0}^{k-1}\binom{n-1}{i}q^i(1-q)^{n-1-i}$, in the RHS of the above equation represents the probability of the event where the number of primaries with free bandwidth is less than or equal to k, in which case, primary 1's bandwidth gets sold w.p. 1 and the second term represents the probability of the event where $p_1 = X_k$ (atleast k primaries other than primary 1 have free bandwidth and choose the price a_{M-P}) and primary 1's bandwidth gets sold. For the strategy profile $(a_{M-P}, a_{M-P}, \dots, a_{M-P})$ to constitute a pure strategy NE, the following inequalities should hold:

$$E\left[u_1\left(a_{M-P}, X_k\right)\right] \geq \begin{cases} E\left[u_1\left(a_M, X_k\right)\right], \\ E\left[u_1\left(a_{M-P-1}, X_k\right)\right]. \end{cases} \quad (16)$$

where $E\left[u_1\left(a_M, X_k\right)\right] = \frac{(v-c)}{M}M[\sum_{i=0}^{k-1}\binom{n-1}{i}q^i\left(1-q\right)^{n-1-i}]$ and $E[u_1(a_{M-P-1}, X_k)] = \frac{v-c}{M}(M-P-1)$. We substitute these expressions along with (14) in the above two inequalities in (16). By some rearrangement of terms we get:

$$\sum_{i=k}^{n-1}\left(P\left(1-\frac{k}{i+1}\right)+\frac{Mk}{i+1}\right)\binom{n-1}{i}q^i(1-q)^{n-1-i} \geq P, \quad (17)$$

$$\sum_{i=k}^{n-1}\left(1-\frac{k}{i+1}\right)\binom{n-1}{i}q^i(1-q)^{n-1-i} \leq \frac{1}{M-P}. \quad (18)$$

Substituting the above result in (18) into the LHS of (17), we get an upper bound on the LHS of (17). The upper bound on the LHS of (17) is $\frac{P}{M-P} + \frac{Mk}{M-P}$. Clearly, if this upper bound is less than P, then the strategy profile $(a_{M-P}, \dots, a_{M-P})$ does not constitute a pure strategy NE. Thus a sufficient condition for the strategy profile $(a_{M-P}, \dots, a_{M-P})$ ***not*** to constitute a pure strategy NE is:

$$P + Mk < P\left(M-P\right). \quad (19)$$

Note that the LHS of (19) increases with P and the RHS increases initially and later decreases with P (with a maximum at $\frac{M}{2}$). Also, the RHS has the same value for $P = i$ and $P = M - i$. So, if we prove that the RHS is greater than the LHS for $P = M - i$, where $M - i > \frac{M}{2}$, then it can be said that the RHS is greater than the LHS for all $P \in \{i, i+1, \dots, M-i\}$. For $k = 1$, it can be easily shown that the inequality in (19) is not satisfied for only $P = 0, 1, M-2, M-1$ for all $M > 6$. Consider the case when $k > 1$. We now show that if we choose $M > 4k + 5$, (19) is satisfied for all $P \in \{2k, \dots, M-2k\}$. Substituting $P = M - 2k$ in (19), we get $M - 2k + Mk < 2k(M-2k)$. This is equivalent to:

$$\frac{2k\left(2k-1\right)}{k-1} < M \quad (20)$$

It can be checked that (20) is satisfied when $M > 4k + 5$ and $k > 1$. Hence (19) holds for all $P \in \{2k, \dots, M-2k\}$. Thus, the strategy profile $(a_{M-P}, \dots, a_{M-P})$ does not constitute a pure strategy NE for $P \in \{2k, \dots, M-2k\}$. The result follows. ∎

We now prove some lemmas (Lemmas 7, 8 and 9) which will be used to prove Theorem 2. From Lemma 2 and the fact that I_M^P is the length of interval V^P, we have

$$I_M^P = \frac{Q+1}{2\left(M-1\right)\left(Q+\frac{1}{2}\right)\left(Q+\frac{M}{2(M-1)}\right)} \quad \text{for } P \text{ even and} \tag{21}$$

$$I_M^P = \frac{Q-1}{2\left(M-1\right)\left(Q-\frac{1}{2}\right)\left(Q-\frac{M}{2(M-1)}\right)} \quad \text{for } P \text{ odd} \tag{22}$$

where $Q = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots + \frac{M}{M-2} - \frac{M}{M-1}$ when P is even and $Q = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots + \frac{M}{M-3} - \frac{M}{M-2} + \frac{M}{M-1}$ when P is odd. We assume that M is odd (the proof is similar when M is even). Consider the case when P is odd. We have the following lemma.

Lemma 7: $\sum_{i=0}^{\frac{M-3}{2}} I_M^{2i+1} < \frac{1}{2\left(1+\sqrt{\frac{M-2}{M-1}}\right)^2}$.

Proof: I_M^P, the length of the interval V^P, is given by (22) when P is odd. It can be checked that $Q > 1$ when P is odd. From this fact and (22), it follows that $I_M^P > 0$ for all $P \in \{1, 3, \ldots, M-2\}$. Taking the derivative of (22) w.r.t. Q, we get:

$$\frac{-Q^2 + 2Q - \frac{M}{4(M-1)} - \frac{1}{2}}{2(M-1)\left(Q-\frac{1}{2}\right)^2\left(Q-\frac{M}{2(M-1)}\right)^2}.$$

From the above equation, I_M^P has a stationary point[10] at $Q = 1 + \frac{1}{2}\sqrt{\frac{M-2}{M-1}}$ at which it is maximum[11]. Substituting $Q = 1 + \frac{1}{2}\sqrt{\frac{M-2}{M-1}}$ into the RHS of (22), we get that the maximum value of I_M^P is less than or equal to $\frac{1}{(M-1)\left(1+\sqrt{\frac{M-2}{M-1}}\right)^2}$. This implies that $\sum_{i=0}^{\frac{M-3}{2}} I_M^{2i+1} < \frac{M-1}{2}\frac{1}{(M-1)\left(1+\sqrt{\frac{M-2}{M-1}}\right)^2}$. ∎

We now consider the case when P is even. Let $\alpha = I_M^{M-1}$.

Lemma 8: $\frac{2}{M}\left(1-\frac{P}{M-1}\right) + \frac{P\alpha}{M-1} - I_M^P \geq 0 \ \forall P \in \{0, 2, \ldots, M-1\}$.

The above Lemma can be proved by simple algebraic comparisons. The proof can be found in Appendix. We now upper bound the sum $\sum_{i=0}^{\frac{M-1}{2}} I_M^{2i}$ using Lemma 8.

Lemma 9: $\sum_{i=0}^{\frac{M-1}{2}} I_M^{2i} < \frac{1}{2} + \frac{1}{2M} + \frac{1}{M} + \frac{1}{M^2}$.

Proof: We have from (21) that:

$$I_M^{M-1} = \frac{1}{2(M-1)\left(Q+\frac{M}{2(M-1)}\right)} \tag{23}$$

$$+ \frac{1}{4(M-1)\left(Q+\frac{1}{2}\right)\left(Q+\frac{M}{2(M-1)}\right)} \tag{24}$$

where $Q = \frac{M}{1} - \frac{M}{2} + \ldots + \frac{M}{M-2} - \frac{M}{M-1}$. It can be easily verified that $Q + \frac{1}{2} > \frac{M}{2}$ and $Q + \frac{M}{2(M-1)} > \frac{M}{2}$. Substituting this result in (23), we can write $I_M^{M-1} < \frac{1}{M(M-1)} + \frac{1}{M^2(M-1)}$. From the above inequalities and Lemma 8, we get:

$$\begin{aligned}\sum_{i=0}^{\frac{M-1}{2}} I_M^{2i} &< \sum_{i=0}^{\frac{M-1}{2}} \left(\frac{2}{M} + \left(\alpha - \frac{2}{M}\right)\frac{2i}{M-1}\right) \\ &= \frac{M+1}{2M} + \frac{M+1}{4}\alpha \\ &< \frac{1}{2} + \frac{1}{2M} + \frac{1}{M} + \frac{1}{M^2}\end{aligned}$$

The last inequality is due to the fact that $I_M^{M-1} = \alpha < \frac{1}{M(M-1)} + \frac{1}{M^2(M-1)}$ and $\frac{M+1}{4(M-1)} < 1$ for all $M \geq 2$. ∎

Proof of Theorem 2: The proof follows from Lemmas 7 and 9. ∎

B. Proofs of Analytical Results in Section V

Proof of Lemma 4: Choose $\epsilon' < (v-c)P(X_k > v)$. When a primary chooses a price $x < c + \epsilon'$, then its utility is at most ϵ', which is less than the utility it gets by selecting v in which case the primary gets a utility of atleast $(v-c)P(X_k > v)$. So a primary does not choose a price less than $c + \epsilon'$. Let $x \in (c+\epsilon', v]$ be the smallest price where a jump of more than ϵ occurs. Let $I(x)$ be the probability that $X_k = x$ and primary 1's bandwidth is sold when $p_1 = x$. The expected utility of primary 1 at x is:

$$E(u_1(x, X_k)) = (x-c)P(X_k > x) + (x-c)I(x).$$

As x is the lowest price of jump greater than ϵ, let us consider the price $x - \delta$, where $\delta = \frac{v-c}{M}$. Then the expected utility of primary 1 at $x - \delta$ is:

$$E(u_1(x-\delta, X_k)) = (x-\delta-c)P(X_k > x-\delta) + (x-\delta-c)I(x-\delta).$$

Taking the difference of the above two equations, we get:

$$\begin{aligned}&E(u_1(x-\delta, X_k)) - E(u_1(x, X_k)) = \\ &(x-c)(P(X_k = x) + I(x-\delta) - I(x)) \\ &\quad -\delta(P(X_k \geq x) + I(x-\delta)).\end{aligned}$$

The RHS is greater than 0 if

$$\begin{aligned}&(x-c)(P(X_k = x) + I(x-\delta) - I(x)) > \\ &\quad \delta(P(X_k \geq x) + I(x-\delta)),\end{aligned}$$

which, since $\delta = \frac{v-c}{M}$, is equivalent to

$$M > \frac{(v-c)(P(X_k \geq x) + I(x-\delta))}{(x-c)(P(X_k = x) - I(x) + I(x-\delta))}. \tag{25}$$

The RHS of the above inequality is bounded above by $\frac{v-c}{\epsilon'(P(X_k=x)-I(x))}$. Also, the term $P(X_k = x) - I(x)$ is the probability of the event, say E_1, that primary 1 does not sell its bandwidth when it chooses x and $X_k = x$. The probability of this event is greater than the probability of the event, say E_2, that all $n-1$ primaries $2, \ldots, n$ choose price x and primary 1's bandwidth is not sold. It is easy to check that the probability of event E_2 is atleast $\frac{\epsilon^{n-1}}{n}$. So by (25) and the above arguments,

[10]Note that out of the two stationary points $1 \pm \frac{1}{2}\sqrt{\frac{M-2}{M-1}}$, we select the one that is greater than 1 as our maximizer.

[11]It can be easily verified that the second derivative is negative at this stationary point.

if we choose $M > \frac{n(v-c)}{\epsilon^{n-1}\epsilon'}$, then primary 1's expected utility is greater when it chooses the price $x-\delta$ than when it chooses the price x. This contradicts the assumption that price x is selected w.p. more than ϵ. By the above facts the proof of Lemma 4 follows.

■

We first prove some Lemmas which will be used to prove Lemma 5 and Theorem 3. Let $z_l(M)$ (respectively, $z_r(M)$) be the lowest (respectively, highest) price from $\{a_1, ..., a_M\}$ in the support set of $\phi_M(\cdot)$. There may be prices from the set $\{a_1, .., a_M\}$ within the interval $(z_l(M), z_r(M))$ that are not in the support set; however, we prove in the following lemma that the distance between neighbouring prices which are in the support set decreases to 0 as $M \to \infty$. Consider a price $y \in (z_l(M), z_r(M))$ which is not in the support set. Let $y_l(M)$ (respectively, $y_r(M)$) be the highest (respectively, lowest) price such that $y_l(M) < y$ (respectively, $y_r(M) > y$) which is in the support set of $\phi_M(\cdot)$. Note that $y, y_l(M), y_r(M) \in \{a_1, \ldots, a_M\}$. For ease of presentation we omit M from the variables $z_l(M), z_r(M), y_l(M)$ and $y_r(M)$.

Lemma 10: For every price $y \in (z_l, z_r)$ which is not in the support set, $y_r - y_l \to 0$ as $M \to \infty$.

Proof: Since no primary chooses a price in the interval (y_l, y_r) and also $y_l < y$, we have $P(X_k > y) = P(X_k > y_l)$. Also y_l, y_r are in the support set and by Lemma 1 we can write:

$$E(u_1(y_l, X_k)) = E(u_1(y_r, X_k)).$$

As y_l, y_r are in the support set and y is not, by Lemma 1 we can write:

$$E(u_1(y_l, X_k)), E(u_1(y_r, X_k)) \geq E(u_1(y, X_k)). \tag{26}$$

We also have:

$$E(u_1(y_l, X_k)) = (y_l - c)(P(X_k > y_l) + I(y_l)), \tag{27}$$

$$E(u_1(y, X_k)) = (y-c)P(X_k > y) = (y-c)P(X_k > y_l). \tag{28}$$

By (26), (27) and (28):

$$(y_l - c)P(X_k > y_l) + (y_l - c)I(y_l) \geq (y-c)P(X_k > y_l).$$

Hence:

$$(y_l - c)I(y_l) \geq (y - y_l)P(X_k > y_l).$$

So:

$$(y - y_l) \leq \frac{(y_l - c)I(y_l)}{P(X_k > y_l)}. \tag{29}$$

We can write for any price x in the support set:

$$I(x) < P(X_k = x) \leq (n-1)\epsilon, \tag{30}$$

by Lemma 4 and since $P(X_k = x) < P($atleast one of the primaries chooses price $x)$. By (29) and (30):

$$(y - y_l) < \frac{(v-c)}{P(X_k > v)}(n-1)\epsilon, \tag{31}$$

where $P(X_k > v) = \sum_{i=0}^{k-1} \binom{n-1}{i} q^i (1-q)^{n-1-i}$. (31) is valid for all $y \in (y_l, y_r)$. We complete the proof by choosing a y just below y_r and we have:

$$y_r - y = \frac{v-c}{M} \to 0 \quad \text{as} \quad M \to \infty. \tag{32}$$

Combining (31), (32) we can say $y_r - y_l \to 0$ as $M \to \infty$.

■

Lemma 10 shows that the support set of $\phi_M(\cdot)$ contains most of the available prices from the set $\{a_1, \ldots, a_M\}$ that lie in the interval (z_l, z_r). Recall from Section III that the support set of $\phi(\cdot)$ is $(\tilde{p}, v]$. Next, we prove that z_l and z_r, which are the lowest and highest prices from the set $\{a_1, \ldots, a_M\}$ in the support set of $\phi_M(\cdot)$, converge to $\tilde{p}$ and v respectively.

Lemma 11: For every $\epsilon' > 0$, $\exists M_{\epsilon'}$ such that if $M \geq M_{\epsilon'}$, then in every symmmetric NE:

1)
$$z_r > v - \epsilon'.$$

2)
$$|z_l - \tilde{p}| < \epsilon'.$$

Proof: Since z_r is in the support set and v may or may not be in the support set, by Lemma 1, we can write:

$$E(u_1(z_r, X_k)) \geq E(u_1(v, X_k)).$$

So:

$$(z_r - c)(P(X_k > z_r) + I(z_r)) \geq (v-c)(P(X_k > v) + I(v)). \tag{33}$$

Since z_l (respectively, z_r) is the lowest (highest) price in the set $\{a_1, \ldots, a_M\}$ which is in the support set, we have:

$$P(X_k > z_l) = 1 - P(X_k = z_l), \tag{34a}$$

$$P(X_k > z_r) = P(X_k > v). \tag{34b}$$

Substituting (34b) in (33) we get:

$$(z_r - c)(P(X_k > v) + I(z_r)) \geq (v-c)(P(X_k > v) + I(v))$$

From the above inequality, (30) and the fact that $I(v) \geq 0$:

$$(v - z_r) \leq \frac{(z_r - c)I(z_r)}{P(X_k > v)} \leq \frac{(v-c)}{P(X_k > v)}(n-1)\epsilon \tag{35}$$

Thus, part 1) of Lemma 11 follows. Since z_l, z_r are in the support set, we can write:

$$E(u_1(z_l, X_k)) = E(u_1(z_r, X_k)), \tag{36a}$$

$$E(u_1(z_l, X_k)) = (z_l - c)(P(X_k > z_l) + I(z_l)), \tag{36b}$$

$$E(u_1(z_r, X_k)) = (z_r - c)(P(X_k > z_r) + I(z_r)). \tag{36c}$$

Using (34a), (36b):

$$E(u_1(z_l, X_k)) = (z_l - c) - (z_l - c)(P(X_k = z_l) - I(z_l)), \tag{37}$$

and using (34b), (36c), (2) with the fact that $P(X_k > v) = 1 - w(q,n)$:

$$E(u_1(z_r, X_k)) = (z_r - v + v - c)(P(X_k > v) + I(z_r)) = \tilde{p} - c - (v - z_r)P(X_k > v) + (z_r - c)I(z_r). \quad (38)$$

Substituting (37) and (38) in (36a) and simplifying we get:

$$z_l - \tilde{p} = (z_l - c)(P(X_k = z_l) - I(z_l)) - \quad (39)$$
$$(v - z_r)P(X_k > v) + (z_r - c)I(z_r). \quad (40)$$

By (35), we can upper bound the negative term in the RHS of the above equation by $(z_r - c)I(z_r)$ which results in:

$$z_l - \tilde{p} \geq (z_l - c)(P(X_k = z_l) - I(z_l)) \geq 0 \quad (41)$$

The second inequality is the result of the fact that z_l is in the support set. The above inequality shows that the lowest price of the support set of a symmetric NE in the discrete game is $\geq \tilde{p}$. By applying triangle inequality to (39) and by using the fact that $P(X_k = z_l) - I(z_l) < P(X_k = z_l) < (n-1)\epsilon$ and $z_r - c, z_l - c < v - c$ and by (30), (35) we can write:

$$|\tilde{p} - z_l| \leq 3(v - c)(n - 1)\epsilon. \quad (42)$$

Thus, part 2) of Lemma 11 follows. ■

Proof of Lemma 5: This can be shown using part 1 of Lemma 11. Since z_r is in the support set of $\phi_M(\cdot)$:

$$u_{max}(M) = (z_r - c)(P(X_k > v) + I(z_r))$$

By (2) and (5), $u_{max} = (v-c)(1-w(q,n)) = (v-c)P(X_k > v)$ as $P(X_k > v) = \sum_{i=0}^{k-1} q^i(1-q)^{n-1-i} = 1 - w(q,n)$. So:

$$u_{max}(M) - u_{max} = (z_r - c)I(z_r) - (v - z_r)P(X_k > v).$$

By triangle inequality:

$$|u_{max}(M) - u_{max}| \leq (z_r - c)I(z_r) + (v - z_r)P(X_k > v) \leq (v - c)I(z_r) + (v - z_r)P(X_k > v). \quad (43)$$

By using (30) and (35) in the above inequality, we get:

$$|u_{max}(M) - u_{max}| \leq 2(v - c)(n - 1)\epsilon. \quad (44)$$

The result follows. ■

Proof of Theorem 3: Fix $x \in (z_l, z_r)$. If x is in the support set of $\phi_M(\cdot)$ then

$$E(u_1(x, X_k)) = E(u_1(z_l, X_k)). \quad (45)$$

We have:

$$E(u_1(x, X_k)) = (x - c)(P(X_k > x) + I(x)) \quad (46)$$

Also from (43) we have:

$$|u_{max}(M) - u_{max}| \leq (v - c)I(z_r) + (v - z_r)P(X_k > v). \quad (47)$$

By substituting the result from (46) and from the equation $u_{max} = (x - c)(1 - F(x))$ (which follows from (3) and (5)) into (47) we get:

$$(x - c)|F(x) - P(X_k \leq x) + I(x)| \leq (v - c)I(z_r) + (v - z_r)P(X_k > v)$$

Since x is in the set (z_l, z_r) and by (41) we have $x - c \geq z_l - c \geq \tilde{p} - c = (v - c)P(X_k > v)$. Also from (35) and (30), we have the following inequalities

$$|F(x) - P(X_k \leq x) + I(x)| \leq \frac{(v - c)I(z_r)}{\tilde{p} - c} + \frac{(v - z_r)P(X_k > v)}{\tilde{p} - c}. \leq \frac{2I(z_r)}{P(X_k > v)} \leq \frac{2(n-1)\epsilon}{P(X_k > v)}.$$

Since $I(x) \geq 0$ and from (30) it can be shown that

$$|F(x) - P(X_k \leq x)| \leq \frac{3(n-1)\epsilon}{P(X_k > v)} \quad (48)$$

For any point $z \in \{a_1, ..., a_M\}$ which is not in the support set of $\phi_M(\cdot)$, we can say by Lemma 10 that z is bounded by prices which are in the support set and the gap reduces to 0 as $M \to \infty$. Since d.f.'s $P(X_k \leq x)$ converge pointwise to $F(x)$ for all x in the support set and since $F(.)$ is a continuous and increasing function we can say that $P(X_k \leq z)$ converges pointwise to $F(z)$. Next, consider the function $\mathcal{F}(y) = \sum_{i=k}^{n-1} y^i(1-y)^{n-1-i}$. Note that by (4), $\mathcal{F}(\phi(x)) = F(x)$; also, $\mathcal{F}(y)$ is a continuous and strictly increasing function of $y \in [0,1]$. Thus $\mathcal{F}(\cdot)$ is invertible and $\mathcal{F}^{-1}$ itself is continuous. The d.f. $P(X_k \leq x) = \mathcal{F}(\phi_M(x)) = \sum_{i=k}^{n-1} \phi_M(x)^i(1 - \phi_M(x))^{n-1-i}$ converges pointwise to $\mathcal{F}(\phi(x)) = F(x)$. So it follows that $\phi_M(\cdot) \to \phi(\cdot)$ in pointwise convergence. ■

VIII. SIMULATIONS

So far, we have studied NE in the price competition game in a single time slot. However, in practice, primaries in a region would repeatedly interact with each other in different time slots. To model this situation, in this section, we consider a scenario in which there are an infinite number of time slots, and in each slot, n primaries sell bandwidth to k secondaries as in the model in Section II. Also, in practice, the players (primaries) would not know all the parameters of the game (e.g., n, k, q) and hence would use *learning algorithms* to adapt their price selection strategies based on the prices they selected and the payoffs they got in previous slots. We assume that each primary independently adapts its price selection strategy using the *Softmax learning algorithm*, which was proposed to solve the multi-armed bandit problem in [28], and investigate under what conditions the strategies of the primaries converge to the NE of the one-shot game. The algorithm is initiated by each player i by playing all the available prices $\{a_1, ..., a_M\}$ randomly atleast once. Then, the utilities obtained by primary i in time slots $1, ..., t-1$ are used to compute the PMF that is used by primary i to select the price in time slot t. Specifically, in slot t, primary i selects price a_j, $j \in \{1, ..., M\}$ with the following probability:

$$R_{i,t}(a_j) = \frac{\exp\left(\frac{u_{i,t-1}(a_j)}{\tau N_{i,t-1}(a_j)}\right)}{\sum_{l=1}^{M} \exp\left(\frac{u_{i,t-1}(a_l)}{\tau N_{i,t-1}(a_l)}\right)}, \tag{49}$$

where $N_{i,t-1}(a_j)$ is the number of time slots in which primary i played the price a_j so far, $u_{i,t-1}(a_j)$ is the total utility that primary i got in the time slots in which it played the price a_j so far, and τ is the temperature constant [28]. Note that the algorithm assigns a probability to each price which is an increasing function of the payoffs that player i got by playing that price in the time slots elapsed so far.

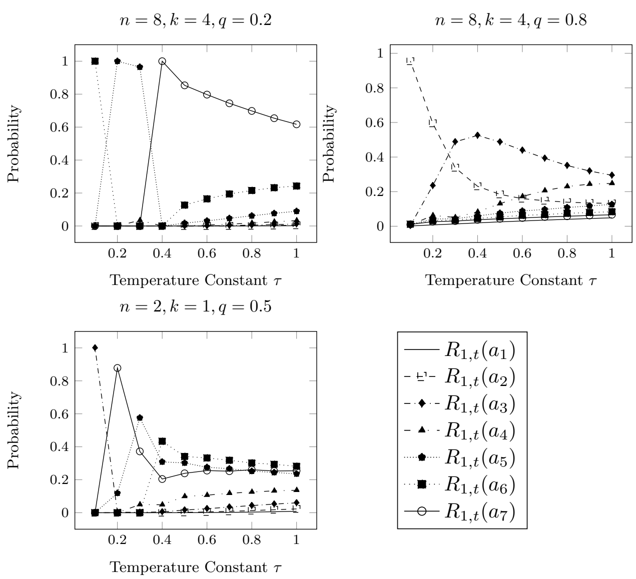

Fig. 7: The figure shows the steady state probability distributions versus the temperature constant τ for different values of n, k and q.

We simulated a scenario in which primaries adapt their price selection strategies using the Softmax algorithm and the *steady state probability distributions* to which the price selection PMFs, $R_{i,t}(\cdot)$, of the primaries converge after the simulation has run for a large number of time slots were obtained for different values of n, k, M, q and τ. Throughout, we observed that whenever a *pure strategy* symmetric NE exists in the one-shot game for given values of n, k, M and q (*i.e.*, the NE price selection strategy support set contains a single price), the price selection PMFs of the primaries under the Softmax algorithm converge to their price selection PMFs under at least one of the pure strategy NE for some values of τ. However, when only mixed strategy NE exist in the one-shot game, the price selection PMFs of the primaries under the Softmax algorithm do not converge to the NE price selection PMFs *for any value of* τ. For example, with $M = 7$ throughout, (i) with $n = 8$, $k = 4$ and $q = 0.2$, one pure strategy NE exists and has support set $\{a_7\}$; the top-left plot in Fig. 7 shows that the Softmax algorithm converges to this NE at $\tau = 0.4$, (ii) with $n = 8$, $k = 4$ and $q = 0.8$, there exist two pure strategy NE with support sets $\{a_2\}$ and $\{a_3\}$, and the top-right plot shows that the Softmax algorithm converges to $\{a_2\}$ at $\tau = 0.1$, (iii) with $n = 2$, $k = 1$ and $q = 0.5$, only a mixed strategy NE exists, and the bottom-left plot shows the steady state probability distributions under Softmax. It is easy to check that the probabilities in the plot do not equal the price selection strategy probabilities under the mixed NE for any value of τ. The design of learning algorithms that converge to the NE even when only mixed strategy NE exist is a direction for future work.

IX. Conclusions and Future Work

In this paper, we investigated the fundamental question of how the behavior of the players involved in price competition in a DSA market changes when the widely used continuous prices approximation is removed. Our analysis reveals several important differences between the discrete game and the continuous game. Although our results show that for the games at a single location as well as at multiple locations, as the number of available prices becomes large in the discrete game, the strategies of the primaries under every symmetric NE converge to the unique NE strategy of the continuous game, they are significantly different when the number of prices is small. Hence caution must be exercised while using the continuous prices approximation in the context of price competition in spectrum markets. For simplicity, we assumed in this paper that q, the probability with which a primary has unused bandwidth, is the same for each primary. A direction for future work is to generalize our results to the case where these probabilities are different for different primaries.

X. Appendix

Proof of Lemma 2: Let P is odd. For each price in the support set, the corresponding probability $R(a_{M-P+i})$ for $i = 0, ..., P$ must be strictly positive. Since P is odd, and the number of prices in the support set is $P + 1$, we use (12) to obtain the probabilities. For $R(a_{M-P+i}) > 0$ for $i = 0, .., P$, we need

$$q > \frac{2(Q-1)}{2Q-1} \tag{50}$$

Similarly,

$$q > \frac{Q-1}{Q_{2k+1}} \quad \text{for } R(a_{M-P+2k+1}),\ k = 0, 1, ..., \frac{P-3}{2} \tag{51a}$$

$$q < \frac{Q-1}{Q_{2k}} \quad \text{for } R(a_{M-P+2k}),\ k = 0, 1, ... \frac{P-1}{2} \tag{51b}$$

Let V_j denote the interval where $R(a_{M-P+j})$, $j = 0, 1, ..P$ is valid. The interval $\cap_{j=0}^{P} V_j$ satisfies all the inequalities in (50), (51a), (51b). To get $\cap_{j=0}^{P} V_j$ we take the highest lower bound and least upper bound of all the inequalities. Out of all the upper bounds for $\frac{Q-1}{Q_{2k}}$, $k = 0, 1, ..., \frac{P-1}{2}$, numerator term is the same. Consider the denominator term Q_{2k}, $Q_{2k} = \frac{M}{M-P} - \frac{M}{M-P+1} + ... - \frac{M}{M-P+2k-1} + \frac{M}{2(M-P+2k)}$. The least upper bound correspond to the term which has the highest denominator. For an even number i

$$Q_{i+2} - Q_i = \frac{M}{2(M-P+i)} - \frac{M}{M-P+i+1} + \frac{M}{2(M-P+i+2)}$$
$$= \frac{M}{2(M-P+i)(M-P+i+1)} - \frac{M}{2(M-P+i+1)(M-P+i+2)} > 0$$

Q_{2k} increases as k increases. So, the highest of all Q_{2k}, $k = 0, 1, ..., \frac{P-1}{2}$ is Q_{P-1} and thus the least upper bound is $\frac{Q-1}{Q_{P-1}}$. The lower bounds are $\frac{Q-1}{Q-\frac{1}{2}}$ for $R(a_M)$ and $q > \frac{Q-1}{Q_{2k+1}}$ for $k = 0, 1, .., \frac{P-3}{2}$. Again, the numerators are the same in all the bounds. So, we take the term with least denominator as the overall lower bound. Consider the denominator term $Q_{2k+1} = \frac{M}{M-P} - \frac{M}{M-P+1} + ... + \frac{M}{M-P+2k} - \frac{M}{2(M-P+2k+1)}$. Let i be an odd number. Then

$$Q_{i+2} - Q_i = -\frac{M}{2(M-P+i+2)} + \frac{M}{M-P+i+1} - \frac{M}{2(M-P+i)} < 0$$

Thus Q_{2k+1} decreases as k increases. So the lowest term is $Q_{P-2} = \frac{M}{M-P} - \frac{M}{M-P+1} + ... + \frac{M}{M-3} - \frac{M}{2(M-2)}$ which is greater than $Q - \frac{1}{2} = \frac{M}{M-P} - \frac{M}{M-P+1} + ... + \frac{M}{M-3} - \frac{M}{M-2} + \frac{M}{M-1} - \frac{1}{2}$. So, the highest lower bound is $\frac{2(Q-1)}{2Q-1}$. Thus P odd, we have

$$\frac{2(Q-1)}{2Q-1} < q < \frac{Q-1}{Q_{P-1}}$$

We can obtain the bounds for even P on similar comparison. ■

Proof of Lemma 3: Let P is odd. Then the upper bound of the interval V^P, is $U^P = \frac{Q-1}{Q-\frac{M}{2(M-1)}}$ and the lower bound of the interval V^{P+1} is $L^{P+1} = \frac{\frac{M}{M-P-1}-Q}{\frac{M}{M-P-1}-Q+1}$ where $Q = \frac{M}{M-P} - \frac{M}{M-P+1} + ... + \frac{M}{M-3} - \frac{M}{M-2} + \frac{M}{M-1}$. The difference $U^P - L^{P+1}$ is

$$\frac{Q-1}{Q-\frac{M}{2(M-1)}} - \frac{\frac{M}{M-P-1}-Q}{\frac{M}{M-P-1}-Q+\frac{1}{2}}$$

The numerator part (after neglecting the positive denominator part) of the difference is

$$Q - 1 + \frac{1}{M-1}[\frac{M}{M-P-1} - Q] - [\frac{M}{M-P-1} - Q]$$

Since,

$$Q - 1 = \frac{M}{(M-P)(M-P+1)} + ... + \frac{M}{M(M-1)}$$

and

$$\frac{M}{M-P-1} - Q = \frac{M}{(M-P-1)(M-P)} + ... + \frac{M}{(M-2)(M-1)}$$

so,

$$Q - 1 - \left[\frac{M}{M-P-1} - Q\right] = \frac{-2M}{(M-P-1)(M-P)(M-P+1)} + ... + \frac{-2M}{(M-2)(M-1)M}$$

and we have,

$$\frac{1}{M-1}\left[\frac{M}{M-P-1} - Q\right] = \frac{M}{(M-P-1)(M-P)(M-1)} + ... + \frac{M}{(M-2)(M-2)(M-1)}$$

finally,

$$Q - 1 - \left[\frac{M}{M-P-1} - Q\right] + \frac{1}{M-1}\left[\frac{M}{M-P-1} - Q\right] = \frac{M}{(M-P-1)(M-P)}(\frac{-2}{M-P-1} + \frac{1}{M-1}) + ... + \frac{M}{(M-2)(M-1)}(\frac{-2}{M} + \frac{1}{M-1}) < 0$$

So, L_{P+1} is greater than U_P. This proves that the intervals V^P and V^{P+1} are not contiguous when P is odd. We can prove on similar lines if we take P as an even number. ■

Proof of Lemma 8: Clearly for $P = 0$ and $M-1$, $\frac{2}{M}(1 - \frac{P}{M-1}) + \frac{P\alpha}{M-1} - I_M^P = 0$. We consider the cases for $P \in \{2, 4, \ldots, M-3\}$. Consider the term $\frac{2}{M}(1 - \frac{P}{M-1}) - I_M^P$, where I_M^P is as in (21). Considering only the numerator part of the difference, we get

$$= 4(1 - \frac{P}{M-1})(M-1)(Q_P + \frac{1}{2})(Q_P + \frac{M}{2(M-1)})$$
$$- M(Q_P + 1)$$
$$= 4(M-P-1)(Q_P+1)(Q_P + \frac{M}{2(M-1)})$$
$$- M(Q_P+1) - 2(M-P-1)(Q_P + \frac{M}{2(M-1)})$$
$$= (M-P-1)(Q_P+1)(4(Q_P + \frac{M}{2(M-1)}) - \frac{M}{M-P-1})$$
$$- 2(M-P-1)(Q_P + \frac{M}{2(M-1)}) \qquad (52)$$

We can easily show that $3(Q_P + \frac{M}{2(M-1)}) - \frac{M}{M-P-1} \geq 0$ where $Q_P = \frac{M}{M-P} - \frac{M}{M-P+1} + \ldots + \frac{M}{M-2} - \frac{M}{M-1}$. Substituting this result in (52), the difference is

$$= (M-P-1)(Q_P+1)(X + \frac{M}{2(M-1)} + Q_P + \frac{M}{2(M-1)})$$
$$-2(M-P-1)(Q_P + \frac{M}{2(M-1)})$$
$$= (M-P-1)((Q_P+1)(Q_P + X + \frac{M}{M-1}) - 2Q_P - \frac{M}{M-1})$$
$$= (M-P-1)(Q_P^2 + Q_P X + \frac{Q_P}{M-1} + X) > 0$$
$$\forall P \in \{2, 4, \ldots, M-3\}.$$

■

References

[1] FCC Spectrum Policy Task Force, "Report of the spectrum efficiency working group" 2002.

[2] Aditya MVS, A. Raghuvanshi and G. S. Kasbekar "Price Competition in Spectrum Markets: How Accurate is the Continuous Prices Approximation?", In *Proc. of IEEE VTC 2016*, Montreal, Canada.

[3] Q. Zhao and B.M. Sadler, "A Survey of Dynamic Spectrum Access", In *IEEE Signal Processing Magazine*, Vol. 24, No. 3, pp 79-89, May 2007.

[4] A. Mas-Colell, M.D. Whinston and J.R. Green, "*Microeconomic Theory*", Oxford University Press, 1995.

[5] R. Myerson, "*Game Theory: Analysis of Conflict*", Harvard University Press, 1997.

[6] D. Niyato and E. Hossain, "Optimal Price Competition for Spectrum Sharing in Cognitive Radio: A Dynamic Game-Theoretic Approach", In *IEEE GLOBECOM*, pp. 4625-4629, Washington, DC, USA, Nov. 2007.

[7] D. Niyato and E. Hossain, "Competitive Pricing for Spectrum Sharing in Cognitive Radio Networks: Dynamic Game, Inefficiency of Nash Equilibrium, and Collusion", In *IEEE JSAC*, Vol. 26, No. 1, pp. 192-202, Jan. 2008.

[8] S. Sengupta and M. Chatterjee, "An Economic Framework for Dynamic Spectrum Access and Service Pricing", In *IEEE/ACM Tran. on Networking*, Vol. 17, No. 4, pp. 1200-1213, Aug. 2009.

[9] S.K. Jayaveera and T. Li, "Dynamic Spectrum Leasing in Cognitive Radio Networks via Primary-Secondary User Power Control Games", In *IEEE TWC*, Vol. 8, No. 6, pp. 3300-3310, June 2009.

[10] S.K. Jayaveera, G. Vazquez-Vilar and C. Mosquera, "Dynamic Spectrum Leasing: A New Paradigm for Spectrum Sharing in Cognitive Radio Networks", In *IEEE TVT*, Vol. 59, No. 5, pp. 2328-2339, June 2010.

[11] G.S. Kasbekar and S. Sarkar, "Spectrum Pricing Games with Bandwidth Uncertainty and Spatial Reuse in Cognitive Radio Networks." In *Proc. of MobiHoc*, pp. 251-260, Chicago, Illinois, USA, 2010

[12] L. Duan and B. Shou, "Investment and Pricing with Spectrum Uncertainty: A Cognitive Operator's Perspective", In *IEEE TMC*, Vol. 10, No. 11, pp. 1590-1604, Nov. 2011.

[13] Y. Xiao, G. Bi, D. Niyato and L.A. DaSilva, "A Hierarchical Game Theoretic Framework for Cognitive Radio Networks", In *IEEE JSAC*, Vol. 30, No. 10, pp. 2053-2069, Nov. 2012.

[14] G.S. Kasbekar and S. Sarkar, "Spectrum Pricing Games with Spatial Reuse in Cognitive Radio Networks", In *IEEE JSAC*, Vol. 30, No. 1, pp. 153-164, Jan. 2012.

[15] M. H. Lotfi and S. Sarkar, "Uncertain Price Competition in a Duopoly With Heterogeneous Availability", In *IEEE Transactions in Automatic Control*, Vol. 61, no. 4, pp. 1010-1025, April 2016.

[16] A. Ghosh and S. Sarkar, "Quality- Sensitive Price Competition in Secondary Market Spectrum Oligopoly- Single Location Game", In *IEEE/ACM Transactions on Networking*, Vol. 24, no. 3, pp. 1894-1907, June 2016.

[17] A. Ghosh, S. Sarkar, "Quality Sensitive Price Competition in Spectrum Oligopoly over Multiple Locations", In *Proc. of 48th Annual IEEE CISS*, pp. 1-6, 2014.

[18] S.-F. Cheng, D.M. Reeves, Y. Vorobeychik and M.P. Wellman, "Notes on equilibria in symmetric games", In *AAMAS-04 Workshop on GTDT Agents*, 2004

[19] G. S. Kasbekar and S. Sarkar, "Pricing Games among Interconnected Microgrids", In *Proc. of 2012 IEEE Power and Energy Society General Meeting*, San Diego, CA, pp. 1-8.

[20] G. S. Kasbekar and S. Sarkar, "Spectrum White Space Trade in Cognitive Radio Networks", In *IEEE Transactions on Automatic Control*, Vol. 61, no. 3, pp. 585-600, March 2016.

[21] A. Ghosh, S. Sarkar and R. Berry, "Secondary spectrum market: To acquire or not to acquire side information?", In *2016 IEEE International Symposium on Information Theory (ISIT)*, Barcelona, pp. 1636-1640, 2016.

[22] Janssen, Maarten, and Eric Rasmusen, "Bertrand Competition under Uncertainty", In *The Journal of Industrial Economics*, Vol. 50, no.1, 2002, pp. 11-21.

[23] Junling Hu and M. P. Wellman, "Nash q-learning for general-sum stochastic games", In *The Journal of Machine Learning Research*, Vol. 4, pp. 1039-1069, 2003.

[24] X. Wang and T. Sandholm, "Reinforcement learning to play an optimal Nash equilibrium in team Markov games", In *Proceedings of the 15th International Conference on Neural Information Processing Systems*, pp. 1603-1610, 2002.

[25] R. H. Lasseter, "Microgrids and Distributed Generation", In *Journal of Energy Engineering*, Vol. 133, pp. 144-149, Sept. 2007.

[26] B.S. Everitt, "*The Cambridge Dictionary of Statistics*", Cambridge University Press, 2006.

[27] D. B. West, "*Introduction to Graph Theory*", 2nd ed. Princeton Hall, 2000.

[28] J. Vermorel and M. Mohri, "Multi-armed Bandit Algorithms and Empirical Evaluation", In *Machine Learning: ECML 2005*, Springer Lecture Notes in Computer Science, Vol. 3720, pp. 437-448, 2005.