Pricing Games among Interconnected Microgrids

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***Abstract*—We consider a scenario with multiple independent microgrids close to each other in a region that are connected to each other and to the central grid (macrogrid). In each time slot, a given microgrid may produce more than, less than or as much power as it needs, and there is uncertainty on which of these events may occur. The microgrids with excess power, those with deficit power and the macrogrid trade power in an electricity market, in which each microgrid with excess power quotes a price for it and the microgrids with deficit power buy power from the microgrids who quote the lowest prices. This results in price competition among the microgrids with excess power, and this competition has several distinguishing features not normally present in price competition in traditional markets studied in economics. We analyze this price competition using the framework of game theory, explicitly compute a Nash Equilibrium and show its uniqueness.**

I. Introduction

A. Motivation

Traditional power systems often generate power in large power stations using fossil fuel resources, and distribute it over long distances. This results in depletion of fossil fuel resources, environmental pollution and large energy losses during distribution [10]. Microgrids, which are emerging as a promising alternative to traditional centralized power systems, mitigate the above problems through distributed energy generation close to the loads and the use of renewable energy sources [10], [25].

Microgrids are small-scale power supply networks that are designed to supply electric power to small communities such as housing complexes, universities, schools, industrial estates etc. [10]. A microgrid consists of an interconnected network of several energy sources such as solar panels, wind power stations, fuel cells and microturbines, and electrical loads such as households and factories [10]. The energy sources, called microsources, are typically of smaller capacity than the large generators in traditional power systems and are located close to the loads [10].

A microgrid is connected to the central power grid, called macrogrid, and to other microgrids in its vicinity, and can transfer power to or receive power from them. Now, there is often uncertainty in the amount of power generated by a given microsource in a microgrid and also in the amount of power consumed by a given load, *e.g.*, the amount of power generated by a solar panel depends on the weather, and the amount of power consumed by a household depends on the electrical equipment that is being used– both of the above factors are uncertain. So, in a given time slot, the aggregate power generated by all the microsources within a microgrid may be more than, less than or equal to the aggregate power consumed by all the loads in the microgrid. Thus, in each slot, a microgrid generates more than, less than or as much aggregate power as it requires.

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Electricity deficits in a microgrid must be compensated through electricity trades with the macrogrid and also with other microgrids. Note that trades among neighboring microgrids may incur lower costs than trades of the microgrids with the macrogrid due to elimination of electricity wastage during long distance transmission, and thereby improve the profitability of the microgrids. In fact, the above trade may be viewed as a potential opportunity, the smart grid of the future is expected to be an interconnected network of microgrids and the macrogrid [12], and "the bigger promise of microgrids may be in the private sector, not as islands of power unto themselves, but as trading partners, making and sharing electricity with each other and the grid at large [19]." Although significant progress has been made in the microgrid generation technology, electricity trades among microgrids have received limited attention [18]. The existing schemes are primarily centralized [1] and mostly auction-based (requiring a centralized auctioneer) [2], [30]. Note that centralized schemes are not likely to scale as the technology proliferates. The penetration of this nascent technology is therefore contingent on the design of a distributed and scalable electricity trading framework equipped with dynamic pricing strategies that adapt to the load and price fluctuations of the macrogrid and all the microgrids. This is the space where this paper seeks to contribute.

B. Challenges and Contributions

We consider the following mechanism for the trade of power among neighboring microgrids in a region and the macrogrid: at the beginning of every time slot, each microgrid that expects to have excess power in the slot announces a price at which it is willing to sell the power. The microgrids who expect to have deficit power then buy power from the microgrids with excess power who set the lowest prices (and from the macrogrid if the total amount of power required by the microgrids with deficit power is not available with the microgrids with excess power). This results in price competition among the microgrids with excess power. If a microgrid quotes a low price, it will attract buyers, but will earn lower profit per sale. This is a common feature of an *oligopoly* [26], in which multiple firms sell a common good to a pool of buyers. Price competition in an oligopoly is naturally modeled using *game theory* [28], and has been extensively studied in economics using, for example, the classic *Bertrand game* [16], [26] and its variants.

However, a microgrid market has several distinguishing features, which makes the price competition very different from oligopolies encountered in economics. First, in every slot, each microgrid may have excess power, deficit power or neither. Thus, each microgrid who *has* excess power is uncertain about the number of microgrids from whom it will face competition as well as the demand for power. A low price will result in unnecessarily low revenues in the event that very few other microgrids have excess power or several microgrids have deficit power, because even with a higher price the microgrid's power would have been bought, and vice versa. Second, note that the sets of buyers and sellers in a microgrid market are drawn from the same pool of traders (the set of all the microgrids in the vicinity and the macrogrid), whereas in most traditional markets, the sets of buyers and sellers are distinct.

In this paper, we analyze price competition among interconnected microgrids in a region using the framework of game theory [28] and study Nash Equilibria (NE) [28] in the game. We model the system by assuming that in each slot, a microgrid may either have one unit of excess power, one unit of deficit power or may have neither excess nor deficit power, with some probabilities (Section II). First, in Section III, we consider the case where each microgrid has deficit power with the same probability, although the probability that a microgrid has excess power may be different for different microgrids. Since prices can take real values, the *strategy sets of the microgrids are continuous*. In addition, the utilities of the microgrids are not continuous functions of their actions. Thus, classical results, including those for concave and potential games, do not establish the existence and uniqueness of NE in the resulting game, and there is no standard algorithm for finding a NE. Nevertheless, we are able to explicitly compute a NE and show its uniqueness, allowing for player strategies that are arbitrary mixtures of continuous and discrete probability distributions. The structure of the NE reveals several interesting insights, which we discuss in Section III-C. Next, in Section IV, we consider the model analyzed in Section III with arbitrary deficit power probabilities of the microgrids and explicitly compute a NE for the case of three microgrids. Then, in Section V, we generalize the results in Section III to the case where the price at which the macrogrid sells unit power is a random variable. In Section VI, we generalize the results in Section III to the game where the events that different microgrids have excess power, deficit power or neither may be correlated across microgrids. We provide numerical studies in Section VII and conclude in Section VIII.

C. Related Work

Electricity markets, including those with renewable electricity generation components, (e.g., [3], [4], [5], [6], [7], [9], [13], [14], [15], [23], [27], [33]) have largely been analyzed in a macro-economic setting. Such a setting entails a very large number of buyers and sellers, leading to a perfect competition in which the equilibrium price of the commodity to be traded equals the market-clearing price at which the aggregate demand matches the aggregate supply in the market. The individual sellers adopt the market-clearing price, thereby acting as price takers rather than choosing their prices for maximizing individual profits. Some prior works employing the macro-economics viewpoint also allow the individual prices to be determined so as to maximize a social utility *e.g.*, [4], or based on risk neutral pricing (which assumes arbitrary storage capabilities) *e.g.*, [3], [6], [13], [14], [23]. Models for the spot price of electricity in competitive electricity markets have been proposed in [7], [15]. In [5], the long-run gains in efficiency from using retail real-time pricing of electricity (retail pricing that changes hourly to reflect the changing supply and demand balance) are investigated.

Although the microgrid technology is now growing rapidly, it is in the stage where microgrids have only been experimentally deployed so far. Hence, a micro-economic analysis that considers a market with a moderate number of entities and allows the trade dynamics to depend on the choices of and the uncertainties experienced by the individual entities is imperative; the decisions in such a setting would likely be motivated from individual profit considerations rather than global objectives. In the micro-economics literature, the Bertrand game [16], [26] and several of its variants have been used to study price competition. The closest to our work are [17], [24], which analyze price competition where each seller may be inactive with some probability. In our prior work [21], [22], we analyzed price competition among primary users in a Cognitive Radio Network– in that model, each primary may have unused bandwidth with some probability, which it can sell to a secondary user. However, [17], [22], [24], suffer from the limitation that they consider only the *symmetric* model where the good availability probability of each seller is the same [1]. Also, in all of the above papers [17], [21], [22], [24], the set of sellers and the set of buyers are distinct, whereas in case of price competition among interconnected microgrids, the sellers and buyers are drawn from the same set of traders. The facts that in the game we consider, (i) there is uncertainty in whether a given trader (microgrid) has the good (excess power) to sell in a slot and (ii) the sets of sellers and buyers are drawn from the same pool of traders result in significant changes in the structure of the NE in comparison with that in games where one or both of the above features are not present. For example, in the Bertrand price competition game [16], which does not have features (i) and (ii), there is a unique NE, which is of pure strategy type [16]. On the other hand, in the game in this paper, no pure strategy NE exists and there is a unique NE, which is of mixed strategy type, provided there are at least three microgrids in the system. If there are two microgrids, there is a pure strategy NE in the game in this paper, but the NE strategies are different from those in the Bertrand game. Also, the NE in the game studied in [21] (see the preceding paragraph), which has feature (i) but not feature (ii) above, is of mixed-strategy type for the case of two sellers, and hence differs in structure from the game in this paper. Moreover, in the game in this paper, for the case of three microgrids, when the probabilities of the microgrids having deficit power are asymmetric, the expected utilities that the microgrids get are also asymmetric, in contrast to the game in [21], in which the expected utilities are always equal.

[1] In [22], a toy model with 2 sellers and 1 buyer is analyzed in the case with asymmetric good availability probabilities of the sellers.

Proofs of the analytical results in this paper are provided in the Appendix.

II. Model

A. Terminology, Problem Formulation and Solution Concept

We consider a scenario in which there are n microgrids close to each other in a region. Each microgrid consists of an interconnected network of microsources (e.g., solar panels, fuel cells, wind power generators) and loads (e.g., households, factories, shops). Also, the n microgrids are connected to each other and each of the n microgrids is connected to the central grid or macrogrid.

In each microgrid, the microsources are capable of generating electrical power and each load has some demand for power. Time is divided into slots of equal duration. In each slot, there is uncertainty in the amount of power generated by a given microsource as well as the amount of power consumed by a given load. For example, the amount of power generated by a solar panel depends on the weather, and the amount of power consumed by a household depends on the weather, the time of the day, the electrical equipment that is being operated etc.– all of the above factors are uncertain. So in each slot, a given microgrid either generates more, less or as much aggregate power as it requires. To model this, we assume that in every slot, each microgrid $i \in \{1, \dots, n\}$ independently has 1 unit of excess power with probability (w.p.) q_i, 1 unit of deficit power w.p. s_i and neither excess nor deficit w.p. $1 - q_i - s_i$, where $q_i > 0$, $s_i > 0$ and $q_i + s_i < 1$. The independence assumption holds in several important special cases, *e.g.*, when the predominant energy sources and/ or loads in different microgrids are of *different types*, say solar panels, wind power generators, fuel cells etc. and households, factories, shops etc. respectively. In general, however, the power availabilities of different microgrids in a locality may be correlated, particularly when the energy sources are of the same type. In Section VI, we generalize our analysis to allow for such correlations.

Each microgrid is capable of drawing power from or transferring power to the macrogrid or another microgrid. Since microgrids as well as the macrogrid are selfish entities, the transfer of power by any of these entities is done in exchange for a fee. The macrogrid buys power from microgrids at the rate of c per unit and sells power to microgrids at the rate of v per unit, where we assume that $c < v$ since the macrogrid incurs some cost for transmission and distribution of power over long distances and also makes some profit. In practice, the values of c and v would be chosen by the operator that runs the macrogrid based on its transmission and distribution costs and any limits that may have been imposed by the electricity regulator. We assume that c and v are constant and known to all the microgrids, except in Section V, in which we consider the case where v is a random variable.

Now, each microgrid that has 1 unit of excess power announces a price p_i at which it is willing to sell power to a microgrid that has 1 unit of deficit power. Note that a microgrid with excess power always has the option of selling its power to the macrogrid for a price of c; so $p_i \geq c$. Similarly, a microgrid with deficit power has the option of buying power from the macrogrid for a price of v; so when v is constant and known, $p_i \leq v$.

At the beginning of a slot, that is, before a microgrid chooses its price, it knows whether it will have excess power, deficit power or neither in the slot, but does not know the power availabilities of the microgrids other than itself. It however knows the values of $q_1, \dots, q_n$ and $s_1, \dots, s_n$, possibly from historical and/or weather forecast data, and hence knows the *expected* excess, deficit etc. of the microgrids other than itself.

Let N (respectively, K) be the number of microgrids with excess power (respectively, deficit power) in a slot. If $K \leq N$, then the microgrids that have deficit power buy power from the microgrids offering the K lowest prices among those that have excess power, and the remaining $N - K$ microgrids with excess power sell it to the macrogrid (at price c). If $K > N$, then N of the microgrids with deficit power buy power from the microgrids with excess power and the remaining $K - N$ microgrids with deficit power buy power from the macrogrid.

If microgrid i has 1 unit of excess power and microgrid j sets [2] a price p_j, $j = 1, \dots, n$, we define the utility $u_i(p_1, \dots, p_n)$ of microgrid i to be the incremental revenue that it earns over and above its revenue if it were to sell its power to the macrogrid; so $u_i(p_1, \dots, p_n) = p_i - c$ if microgrid i sells its power to another microgrid at price p_i, and $u_i(p_1, \dots, p_n) = 0$ if microgrid i sells its power to the macrogrid at price c.

We allow each microgrid i to choose its price p_i randomly from a set of prices using an arbitrary distribution function [3] (d.f.) $\psi_i(.)$, which is referred to as the *strategy* of microgrid i. The vector $(\psi_1(.), \dots, \psi_n(.))$ of strategies of the microgrids is called a *strategy profile* [26]. Let $\psi_{-i} = (\psi_1(.), \dots, \psi_{i-1}(.), \psi_{i+1}(.), \dots, \psi_n(.))$ denote the vector of strategies of the microgrids other than i. Let $E\{u_i(\psi_i(.), \psi_{-i})\}$ denote the expected utility of microgrid i when it adopts strategy $\psi_i(.)$ and the other microgrids adopt ψ_{-i}. If the strategy $\psi_i(.)$ consists of setting the single price p_i w.p. 1, then we also denote the above expected utility by $E\{u_i(p_i, \psi_{-i})\}$.

We use the *Nash Equilibrium* (NE) solution concept, which has been extensively used in game theory as a prediction of the outcome of a game. A NE is a strategy profile such that no player can improve his expected utility by unilaterally deviating from his strategy [26]. Thus, in our context, $(\psi_1^*(.), \dots, \psi_n^*(.))$ is a NE if for each microgrid i: $E\{u_i(\psi_i^*(.), \psi_{-i}^*)\} \geq E\{u_i(\tilde{\psi}_i(.), \psi_{-i}^*)\}, \ \forall \ \tilde{\psi}_i(.)$. When players other than i play ψ_{-i}^*, $\psi_i^*(.)$ maximizes i's expected utility and is thus its *best-response* [26] to ψ_{-i}. Our goal is to find NE in the above price competition game and to investigate its uniqueness.

Now, if $n = 2$, then there are only two microgrids, say 1 and 2. There is no price competition between them, since in no event are they simultaneously prospective sellers to a common

[2] If microgrid j has no excess power, it does not matter what price p_j it sets. Yet, for convenience, we speak of p_j as being its action.

[3] Recall that the distribution function of a random variable (r.v.) X is the function $G(x) = P(X \leq x)$ [11].

set of buyer microgrids, and it is easy to check that the strategy profile in which both microgrids $i = 1, 2$ set the price $p_i = v$ w.p. 1 is the unique NE. So henceforth, we assume that $n \geq 3$.

B. Relation to Existing Electricity Markets and Some Implementation Issues

We now explain how the above framework for electricity trade among microgrids can interface with existing electricity markets. Existing electricity markets (see [20] for a survey) often consist of multiple firms that generate electricity in centralized power stations and sell it to multiple retail electricity providers in a wholesale marketplace that is managed by a market operator who collects asks and bids from the generators and retail providers respectively, and clears the market periodically (e.g., daily, hourly etc). Each retail provider supplies electricity to several customers, e.g., residential and/ or business units in a city or multiple suburbs. Some other electrical markets consist of a single supplier of electricity (monopolist), which generates electricity in large centralized power stations and distributes it over long distances. In markets with multiple retail electricity providers, each retail provider can serve as the macrogrid of the locality. The retail provider now supplies to and receives electricity from the microgrids on a need basis, that is, the microgrids trade power among themselves whenever possible, and with the macrogrid only in the event of necessity arising from a mismatch between the amounts of deficit and excess power with the microgrids. The trading comprises of selling or buying electricity depending on the nature of the mismatch. The retail provider serving as a macrogrid will in general also serve customers other than microgrids, e.g., residential or business units that have not organized themselves as microgrids. These customers will only receive electricity from and not supply electricity to the retail provider. Thus, microgrids have a two-way interaction with the retail provider (the macrogrid) whereas traditional customers have one-way interactions. Our framework can also function in localities with a single electricity provider (the monopolist); the monopolist then serves as the macrogrid and has a two-way interaction with the microgrids. In both the above settings, the trade between the macrogrid and microgrids proceeds at prices (the sale and purchase prices to and from the microgrids) chosen by the macrogrid. The trade among the microgrids, however, proceeds at the prices individual sellers choose.

The above trade among the microgrids and the macrogrid to which they are connected can be carried out in a distributed manner, without the need for a centralized market operator. We describe two possible distributed implementations. In one, microgrids with excess power in a region can announce the availabilities and the price they quote in a web-portal (e.g., like Craigslist for sale of commodities) dedicated for this purpose. Microgrids with deficits can contact their counterparts with excesses through this web-site in increasing order of prices until they locate one who has not yet sold its available power. Alternatively, that is if such a web-site can not be maintained in a specific locality, a microgrid with deficit can learn of the availabilities and prices by directly contacting the rest. It can subsequently contact those with availability in increasing order of the quoted prices, until it locates one who has not yet sold its excess.

To save on interconnection costs, there need not be direct cables between each pair of microgrids. Instead, each microgrid can be connected to a common hub, which contains a system of switches that are capable of interconnecting the microgrids pairwise for power exchange. Also note that microgrids would typically be connected to the Internet, over which data exchanges required for trading can be made.

Due to the intermittent nature of renewable energy sources and fluctuations in demand, a given microgrid may change from having excess power to deficit power or vice versa as time goes on. Each time slot in our model is assumed to be short enough that microgrids can fairly accurately predict at the beginning of a time slot whether they will have excess power, deficit power or neither in the slot. Discrepancies due to sudden changes in generation and/or demand can be handled by microgrids by maintaining a small battery backup.

III. Symmetric Deficit Power Probabilities

For tractability, in the rest of this section, we assume that $s_1 = \ldots = s_n = s$ for some $s \in (0, 1)$. Note that $q_1, \ldots, q_n$ need not be equal. In Section IV, we analyze the generalization where $s_1, \ldots, s_n$ may be unequal (and $q_1, \ldots, q_n$ may also be unequal) for the case $n = 3$.

Without loss of generality, we assume that $q_1, \ldots, q_n$ satisfy:

$$q_1 \geq q_2 \geq \ldots \geq q_n. \tag{1}$$

For convenience, we define the pseudo-price of microgrid $i \in \{1, \ldots, n\}$, p'_i, as the price it selects if it has excess power and $p'_i = v+1$ otherwise [4]. Also, let $\phi_i(.)$ be the d.f. of p'_i. For $c \leq x \leq v$, $p'_i \leq x$ for a microgrid i iff it has excess power and sets a price $p_i \leq x$. So $\phi_i(x) = q_i P(p_i \leq x) = q_i \psi_i(x)$. Thus, $\psi_i(.)$ and $\phi_i(.)$ differ only by a constant factor on $[c, v]$ and we use them interchangeably wherever applicable.

In Section III-A, we state some necessary conditions that any profile of NE strategies must satisfy. In Section III-B, we note that these conditions are sufficient and also explicitly compute the NE and show its uniqueness.

We now briefly discuss the insights that the structure of this NE provides– a more detailed discussion is provided in Section III-C. First, this NE is of mixed-strategy type– every microgrid randomizes over a range of prices– in constrast to the Bertrand price competition game [16], which is somewhat similar to the game we consider, and in which there is a unique NE that is of pure-strategy type. Also, under the NE computed in Section III-B, only the microgrids with a high excess power availability probability (q) play high prices.

A. Necessary Conditions for a NE

Consider a NE under which the d.f. of the price (respectively, pseudo-price) of microgrid i is $\psi_i(.)$ (respectively, $\phi_i(.)$). In Theorem 1 below, we show that the NE strategies

[4] The choice $v + 1$ is arbitrary. Any other choice greater than v also works.

must have a particular structure. Before stating Theorem 1, we describe some basic properties of the NE strategies.

Property 1: $\phi_2(.), \ldots, \phi_n(.)$ are continuous on $[c, v]$. $\phi_1(.)$ is continuous at every $x \in [c, v)$, has a jump [5] of size $q_1 - q_2$ at v if $q_1 > q_2$ and is continuous at v if $q_1 = q_2$.

Thus, there does not exist a pure strategy NE (one in which every microgrid selects a single price with probability (w.p.) 1).

Now, let $u_{i,max}$ be the expected payoff that microgrid i gets in the NE and L_i be the lower endpoint of the support set [6] of $\psi_i(.)$, *i.e.*:

$$L_i = \inf\{x : \psi_i(x) > 0\}. \tag{2}$$

Definition 1: Let N_{-i} (respectively, K_{-i}) be the number of microgrids out of microgrids $\{1, \ldots, n\} \backslash i$ who have 1 unit of excess power (respectively, deficit power). Also, let $w_i = P(N_{-i} \geq K_{-i})$.

Property 2: $L_1 = \ldots L_n = \tilde{p}$, where

$$\tilde{p} = c + (v - c)\frac{1 - w_1}{1 - (1 - s)^{n-1}} \tag{3}$$

Also,

$$u_{i,max} = (\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right], \ i = 1, \ldots, n. \tag{4}$$

Thus, the lower endpoints of the support sets of the d.f.s $\psi_1(.), \ldots, \psi_n(.)$ of all the microgrids are the same and they get the same expected payoff in the NE.

Theorem 1: The following are necessary conditions for strategies $\phi_1(.), \ldots, \phi_n(.)$ to constitute a NE:
1) $\phi_1(.), \ldots, \phi_n(.)$ satisfy Property 1 and Property 2.
2) There exist numbers $R_j, j = 1, \ldots, n + 1$, and a function $\{\phi(x) : x \in [\tilde{p}, v)\}$ such that

$$\tilde{p} = R_{n+1} < R_n \leq R_{n-1} \leq \ldots \leq R_1 \leq v, \tag{5}$$

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x), \ \tilde{p} \leq x < R_j, \ j \in \{1, \ldots, n\}, \tag{6}$$

$$\text{and} \quad \phi_j(R_j) = q_j, \ j = 1, \ldots, n. \tag{7}$$

Also, every point in $[\tilde{p}, R_j)$ is a best response for microgrid j and it plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability. Finally, $R_1 = R_2 = v$.

Theorem 1 says that all n microgrids play prices in the range $[\tilde{p}, R_n)$, the d.f. $\phi_n(.)$ of microgrid n stops increasing at R_n, the remaining microgrids $1, \ldots, n-1$ also play prices in the range $[R_n, R_{n-1})$, the d.f. $\phi_{n-1}(.)$ of microgrid $n-1$ stops increasing at R_{n-1}, and so on. Also, microgrid 1's d.f. $\phi_1(.)$ has a jump of height $q_1 - q_2$ at v if $q_1 > q_2$. Fig. 1 illustrates the structure.

B. Explicit Computation, Uniqueness and Sufficiency

By Theorem 1, for each $i \in \{1, \ldots, n\}$:

$$\phi_i(x) = \begin{cases} \phi(x), & \tilde{p} \leq x < R_i \\ q_i, & x \geq R_i \end{cases} \tag{8}$$

[5]A d.f. $f(x)$ is said to have a *jump* (discontinuity) of size $b > 0$ at $x = a$ if $f(a) - f(a-) = b$, where $f(a-) = \lim_{x \uparrow a} f(x)$ [11].

[6]The support set of a d.f. is the smallest closed set such that its complement has probability zero under the d.f. [11].

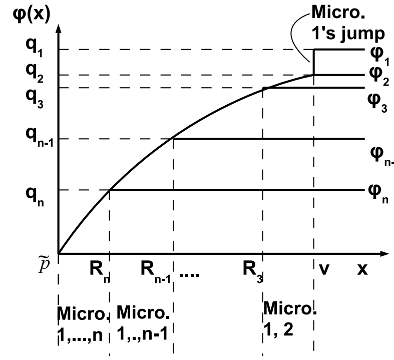

Fig. 1. The figure shows the structure of a NE described in Theorem 1. The horizontal axis shows prices in the range $x \in [\tilde{p}, v]$ and the vertical axis shows the functions $\phi(.)$ and $\phi_1(.), \ldots, \phi_n(.)$.

So the candidate NE strategies $\phi_1(.), \ldots, \phi_n(.)$ are completely determined once $\tilde{p}, R_1, \ldots, R_n$ and the function $\phi(.)$ are specified. Also, Property 2 provides the value of $\tilde{p}$, and $R_1 = R_2 = v$ by Theorem 1. First, we will show that there also exist unique $R_3, \ldots, R_n$ and $\phi(.)$ satisfying (5), (6), and (7) and will compute them. Then, we will show that the resulting strategies given by (8) indeed constitute a NE (sufficiency).

Definition 2: Let p'_{-i} be the K_{-i}'th smallest pseudo-price out of the pseudo-prices, $\{p'_l : l \in \{1, \ldots, n\}, l \neq i\}$, of the microgrids other than i (with $p'_{-i} = 0$ if $K_{-i} = 0$). Also, let $F_{-i}(x)$ denote the d.f. of p'_{-i}.

Since K_{-1} microgrids out of microgrids $2, \ldots, n$ have deficit power, if microgrid 1 has excess power and sets $p_1 = x \in [\tilde{p}, v)$, its power is bought iff [7] $p'_{-1} > x$, which happens w.p. $1 - F_{-1}(x)$. Note that microgrid 1's payoff is $(x - c)$ if its power is bought and 0 otherwise. So, letting $E\{u_i(x, \psi_{-i})\}$ denote the expected payoff of microgrid i if it sets a price x and the other microgrids use the strategy profile ψ_{-i}, we have that for $x \in [\tilde{p}, v)$:

$$E\{u_1(x, \psi_{-1})\} = (x - c)(1 - F_{-1}(x)) \tag{9}$$

$$= (\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right] \tag{10}$$

where the second equality follows from the facts that each $x \in [\tilde{p}, v)$ is a best response for microgrid 1 by Theorem 1, and $u_{1,max} = (\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right]$ by (4). By (10), we get:

$$F_{-1}(x) = g(x), \ x \in [\tilde{p}, v). \tag{11}$$

$$\text{where, } g(x) = \frac{x - c - (\tilde{p} - c)[1 - (1 - s)^{n-1}]}{x - c}, \ x \in [\tilde{p}, v). \tag{12}$$

Next, we calculate $R_i, i = 3, \ldots, n$ and $\phi(.)$ using (11).

1) Computation of $R_i, i = 3, \ldots, n$:

Definition 3: Consider $n - 1$ events, each of which has three possible outcomes– deficit, success and failure. Each event results in deficit w.p. s. Let K_{-1} be the total number of

[7]By Property 1, no microgrid has a jump at any $x \in [\tilde{p}, v)$. So $P(p'_{-1} = x) = 0$.

events that result in deficit [8]. For $0 \le y \le 1$, let $f_i(y)$ be the probability of K_{-1} or more successes out of the $n-1$ events if $i-1$ of them have success probability y and the remaining $n-i$ have success probabilities $q_{i+1}, \ldots, q_n$.

An expression for $f_i(.)$ can be easily computed, using which we prove in the Appendix that:

Lemma 1: $f_i(.)$ is a continuous and strictly increasing function.

Now, to compute $R_i, i \in \{3, \ldots, n\}$, we note that by (8) and (5), $\phi_j(R_i) = q_i$, $j = 2, \ldots, i$, and $\phi_j(R_i) = q_j$, $j = i+1, \ldots, n$. Also, we define the $n-1$ events in the preceding paragraph as follows: for $j \in \{2, \ldots, n\}$, let the j'th event result in deficit if the j'th microgrid has 1 unit of deficit power, in success if $\{p'_j \le R_i\}$ and in failure otherwise. Then by the definition of $F_{-1}(.)$, we get:

$$F_{-1}(R_i) = f_i(q_i). \tag{13}$$

By (11) and (13):

$$g(R_i) = f_i(q_i). \tag{14}$$

By (12) *and* (14)*,* R_i *is unique and is given by:*

$$R_i = c + \frac{(\tilde{p} - c)\left[1 - (1-s)^{n-1}\right]}{1 - f_i(q_i)}. \tag{15}$$

We now verify that the expression for R_i in (15) is consistent with the necessity condition in (5) in Theorem 1. First, we show that $R_i \ge R_{i+1}$ for $i \in \{3, \ldots, n-1\}$. From Definition 3 and since $q_i \ge q_{i+1}$ by (1), it is easy to check that $f_i(q_i) \ge f_{i+1}(q_{i+1})$. So by (15), $R_i \ge R_{i+1}$. Now, by Definition 3, it follows that $f_n(q_n) > P(K_{-1} = 0) = (1-s)^{n-1}$; so by (15), $R_n > \tilde{p}$. Also, by Definitions 1 and 3 and by (1), it follows that $w_1 \ge f_3(q_3)$. Hence, by (3) and (15), it follows that $R_3 \le v$. Thus, (15) is consistent with (5).

2) Computation of $\phi(.)$*:* Now we compute the function $\{\phi(.) : x \in [\tilde{p}, v)\}$ by separately computing it for each interval $[R_{i+1}, R_i)$, $i \in \{2, \ldots, n\}$. If $R_{i+1} = R_i$, then note that the interval $[R_{i+1}, R_i)$ is empty. Now suppose $R_{i+1} < R_i$. For $x \in [R_{i+1}, R_i)$, by (8) and (5):

$$\phi_j(x) = q_j,\ j = i+1, \ldots, n \tag{16}$$

$$\text{and } \phi_1(x) = \ldots = \phi_i(x) = \phi(x). \tag{17}$$

We define the $n-1$ events in the definition of the function $f_i(.)$ as follows: for $j \in \{2, \ldots, n\}$, let the j'th event result in deficit if the j'th microgrid has 1 unit of deficit power, in success if $\{p'_j \le x\}$ and in failure otherwise. By definition of $F_{-1}(x)$ and using $P\{p'_j \le x\} = \phi_j(x)$, (16) and (17):

$$F_{-1}(x) = f_i(\phi(x)),\ R_{i+1} \le x < R_i. \tag{18}$$

By (11) and (18):

$$f_i(\phi(x)) = g(x),\ R_{i+1} \le x < R_i. \tag{19}$$

Lemma 2: For each x, with $R_{n+1} = \tilde{p}$, $R_n, \ldots, R_3$ given by (15) and $R_2 = R_1 = v$, (19) has a unique solution $\phi(x)$. The function $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, v)$. For $i \in \{2, \ldots, n\}$, $\phi(R_i) = q_i$. Also, $\phi(\tilde{p}) = 0$.

Thus, there is a unique function $\phi(.)$*, and by* (8)*, unique* $\phi_i(.), i = 1, \ldots, n$ *that satisfy the conditions in Theorem 1.*

3) Sufficiency:

Theorem 2: The pseudo-price d.f.s $\phi_i(.), i = 1, \ldots, n$ in (8), with $R_1 = R_2 = v$, R_i, $i = 3, \ldots, n$ given by (15), and $\phi(.)$ being the solution of (19), constitute the unique NE. The corresponding price d.f.s are $\psi_i(x) = \frac{1}{q_i}\phi_i(x)$, $x \in [c, v]$, $i = 1, \ldots, n$.

C. Discussion

The structure of the unique NE identified in Theorems 1 and 2 provides several interesting insights:

1) First, by Property 1, $\psi_1(.)$ has a jump at v iff $q_1 > q_2$ and is continuous everywhere else, whereas $\psi_2(.), \ldots, \psi_n(.)$ are always continuous on $[c, v]$. Thus, each microgrid randomizes over a range of prices. This random selection of prices can be interpreted as follows: each microgrid i that has excess power sets a base price v and randomly holds "sales" to attract the microgrids that have deficit power by lowering the price to some value $p_i < v$ [9].

2) Second, from (1), (5) and the fact that the support set of $\psi_i(.)$ is $[\tilde{p}, R_i]$, it follows that only the microgrids with a high excess power availability probability (q) play high prices (see Fig. 1). Intuitively this is because all the microgrids play low prices (near $\tilde{p}$), so if a microgrid sets a high price, it is undercut by all the other microgrids. But a microgrid with a high q runs a lower risk of being undercut than one with a low q because of the lower excess power availability probabilities of the set of microgrids *other than itself.*

3) Third, note that there does not exist a pure strategy NE, and the unique NE is of mixed-strategy type. We contrast this with the Bertrand price competition game [16], in which (i) there are n sellers, each of whom owns 1 unit of a good w.p. 1, (ii) there are $k \in \{1, \ldots, n-1\}$ buyers (distinct from the sellers), each of whom needs 1 unit of the good w.p. 1, (iii) each seller i sets a price $p_i \in [c, v]$, where $c < v$ and (iv) the utility $u_i(p_1, \ldots, p_n)$ of seller i if seller j sets a price p_j, $j = 1, \ldots, n$, is $p_i - c$ if seller i's good is bought and 0 otherwise. Note that the game in our paper differs from the Bertrand game in that there is uncertainty in the availability of the goods with the sellers, and the sets of buyers and sellers are drawn from a common pool of traders. In the Bertrand game, the pure strategy profile under which each seller deterministically selects c as his price is the unique NE [16]. This strategy profile is not a NE in our context as it provides 0 utility for each microgrid, whereas by quoting any price above c (and below v), each microgrid with excess power can attain a positive expected utility since it will sell its power at least in the event that it is the only microgrid with excess power and at least one microgrid has deficit power, which happens with positive probability. Thus, uncertainty in the availability of goods with the sellers and the drawing of the sets of buyers and sellers from a common pool of traders

[8]Note that we have defined K_{-1} twice since we had previously defined it to be the number of microgrids out of microgrids $2, \ldots, n$ who have 1 unit of deficit power. However, when we use the function $f_i(.)$, there will be no conflict between the two definitions, since we will define the $n-1$ events so that the two definitions match.

[9]This interpretation has been suggested in [31] for random selection of prices in a different context.

fundamentally alters the structure of the NE.
4) Finally, we compare the NE found in this section with that in the game studied in [21], which is like the Bertrand game described in 3) above, with the difference that each seller $i \in \{1, \ldots, n\}$ owns 1 unit of the good w.p. $q_i \in (0,1)$ (instead of w.p. 1) and 0 units w.p. $1 - q_i$. Note that in the game studied in this section as well as in the game in [21], there is uncertainty in the availability of the goods with the sellers. However, the difference between the two games is that in the former, the sets of buyers and sellers are drawn from the same pool of traders, whereas in the latter, the sets of buyers and sellers are distinct. Recall that in the former game, (i) for the case $n = 2$, as noted at the end of Section II, the strategy profile in which both microgrids $i = 1, 2$ set the price $p_i = v$ w.p. 1 is the unique NE and (ii) for the case $n \geq 3$, the structure of the unique NE is as in Theorems 1 and 2. However, in the latter game, the structure of the NE for $n \geq 3$ *as well as for* $n = 2$ is similar to the structure in Theorems 1 and 2 [21].

IV. Asymmetric Deficit Power Probabilities

In Section III, we assumed that $s_1 = \ldots = s_n = s$ for some $s \in (0,1)$. In this section, we relax that assumption and allow $s_1, \ldots, s_n$ to be unequal. This makes the analysis significantly harder. So for tractability, we consider only the case $n = 3$ and find a NE.

In this section, we do not assume that (1) holds, but instead, consider a generalization of that condition. Consider the quantity $\frac{1}{q_i}\left(1 - \frac{s_i}{2}\right), i \in \{1,2,3\}$. Without loss of generality, assume that $i = 3$ maximizes it, *i.e.*:

$$\frac{1}{q_3}\left(1 - \frac{s_3}{2}\right) = \max_{i=1,2,3} \frac{1}{q_i}\left(1 - \frac{s_i}{2}\right). \tag{20}$$

Also, suppose:

$$q_3(s_2 - s_1) + s_3(q_2 - q_1) \leq (1 - s_3)(q_1 s_2 - q_2 s_1). \tag{21}$$

Note that the conditions in (20) and (21) together generalize condition (1), and they reduce to (1) when $s_1 = s_2 = s_3 = s$. In the sequel, we will present a strategy profile that is a NE when (20) and (21) hold. When (21) does not hold, then the strategy profile obtained by swapping the roles of microgrids 1 and 2 everywhere in the above strategy profile is a NE.

A. The NE

Let p_i be the price selected by microgrid i and let the corresponding pseudo-price p'_i be as defined in Section III. Also, as before, let $\psi_i(.)$ and $\phi_i(.)$ be the d.f. of p_i and p'_i respectively. Let L_i (respectively, R_i) be the left (respectively, right) endpoint of the support set of $\psi_i(.)$.

We will now describe the NE strategies. Let

$$\tilde{p} = c + \left\{1 - \frac{(s_3 q_2 + s_2 q_3)}{(s_3 + s_2 - s_3 s_2)}\right\}(v - c) \tag{22}$$

It is easy to check that $c < \tilde{p} < v$. We will later see that in the NE, $L_1 = L_2 = L_3 = \tilde{p}$. Also, $\tilde{p} < R_3 \leq R_2 = R_1 = v$, where:

$$R_3 = c + \frac{(\tilde{p} - c)}{1 - \frac{q_3}{1 - \frac{s_3}{2}}} \tag{23}$$

Let:

$$F(x) = \frac{x - \tilde{p}}{x - c}, \tag{24}$$

$$\psi_1(x) = \begin{cases} \frac{1}{q_1}\left(1 - \frac{s_1}{2}\right) F(x), & \tilde{p} \leq x < R_3 \\ \frac{1}{s_3 q_1}\left\{(s_1 + s_3 - s_1 s_3)F(x) - s_1 q_3\right\}, & R_3 \leq x < v \\ 1, & x \geq v \end{cases} \tag{25}$$

$$\psi_2(x) = \begin{cases} \frac{1}{q_2}\left(1 - \frac{s_2}{2}\right) F(x), & \tilde{p} \leq x < R_3 \\ \frac{1}{s_3 q_2}\left\{(s_2 + s_3 - s_2 s_3)F(x) - s_2 q_3\right\}, & R_3 \leq x < v \\ 1, & x \geq v \end{cases} \tag{26}$$

$$\psi_3(x) = \begin{cases} \frac{1}{q_3}\left(1 - \frac{s_3}{2}\right) F(x), & \tilde{p} \leq x < R_3 \\ 1 & x \geq R_3 \end{cases} \tag{27}$$

Theorem 3: The strategies $\psi_1(.)$, $\psi_2(.)$ and $\psi_3(.)$ in (25), (26) and (27) constitute a NE.

B. Discussion

It can be checked that when $s_1 = s_2 = s_3 = s$, the NE strategies in (25), (26) and (27) reduce to those computed in Section III.

We now compare the structure of the NE given by (25), (26) and (27) with that in Theorem 1. First, note that by (25), (26) and (27), for x in the range $[\tilde{p}, R_3)$, each of $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$ equals $F(x)$ times a constant factor (*i.e.*, a factor that does not depend on x), and hence they differ only by a constant multiplicative factor. This is similar to the NE strategies in Theorem 1 (with $n = 3$), for which, by (6) and the fact that $\phi_i(x) = q_i \psi_i(x)$, each of $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$ equals $\phi(x)$ times a constant factor on $x \in [\tilde{p}, R_3)$. However, a difference is that for the NE strategies in Theorem 1, for x in the range $[R_3, v)$, $\psi_1(.)$ and $\psi_2(.)$ differ only by a constant multiplicative factor, whereas this is not the case in general for the NE in (25), (26) and (27). Thus, a structure similar to that in (6) does not hold in general for the NE in (25), (26) and (27).

Property 1 generalizes to the NE in (25), (26) and (27) as follows. $\phi_2(.)$ and $\phi_3(.)$ are continuous on $[c, v]$. $\phi_1(.)$ is continuous at every $[c, v)$; it is continuous at v if (21) holds with equality and has a jump at v otherwise, whose size can be obtained from (25). Property 2 generalizes to give the following. $L_1 = \ldots = L_n = \tilde{p}$, where $\tilde{p}$ is given by (22). Also, the expected payoffs of the microgrids in the NE are given by:

$$u_{1,max} = (\tilde{p} - c)(s_2 + s_3 - s_2 s_3), \tag{28}$$

$$u_{2,max} = (\tilde{p} - c)(s_1 + s_3 - s_1 s_3) \tag{29}$$

and

$$u_{3,max} = (\tilde{p} - c)(s_1 + s_2 - s_1 s_2). \tag{30}$$

Note that when $s_1 = s_2 = s_3$, by Property 2, the expected payoffs of the three microgrids in the NE are equal. Also, recall that in point 4 in the discussion in Section III-C, we noted that the structure of the unique NE in the game studied in [21] is similar to that in Theorems 1 and 2; it also turns out that the expected payoffs of all the sellers in that NE are equal [21]. However, for the NE in (25), (26) and (27), *the expected payoffs of the three microgrids are not equal in general*, as can be seen from (28), (29) and (30). This is an interesting idiosyncrasy brought about by the inequity among s_1, s_2 and s_3.

V. Random v

In Sections III and IV, we assumed that v, the price at which the macrogrid sells unit power, is constant and known to the microgrids. However, this need not always be the case in practice. So in this section, we analyze the scenario where v is a random variable whose value is unknown to the microgrids. Suppose there are n microgrids, where $n \geq 3$. For simplicity, as in Section III, we assume that (i) $s_1 = \ldots = s_n = s$ for some $s \in (0,1)$ and (ii) c is a constant that is known to the microgrids. Also, as in Section III, assume without loss of generality that (1) holds.

Suppose v takes values in the interval $[\underline{v}, \overline{v}]$ w.p. 1, where $c < \underline{v} < \overline{v}$. Note that the price v at which the macrogrid sells unit power is always upper bounded in practice by some finite constant $\overline{v}$. Also, as we mentioned in Section III, the macrogrid always sells power at a higher price than the price c at which it buys power, due to redistribution costs and since it makes a profit; so $\underline{v} > c$.

Let $G(.)$ be the d.f. of v. We assume that $G(.)$ is known to all the microgrids. Also, let $h(x) = (x-c)(1-G(x))$. For tractability, we make the following technical assumption on $G(.)$:

Assumption 1: $G(.)$ is continuous. Also, the function $h(.)$ has a unique maximizer, say v_T, and $h(.)$ is strictly increasing on the interval $[c, v_T]$.

Note that a large class of distribution functions $G(.)$, including the uniform distribution on $[\underline{v}, \overline{v}]$, satisfy the above assumption.

Similar to the constant v case, we define the pseudo-price of microgrid i, p'_i, as the price p_i it selects if it has 1 unit of excess power and $p'_i = \overline{v} + 1$ otherwise. Also, let $\psi_i(.)$ (respectively, $\phi_i(.)$) be the d.f. of p_i (respectively, p'_i). As before, $\phi_i(x) = q_i \psi_i(x)$.

As before, let K_{-i} be the number of microgrids out of $\{1, \ldots, n\} \backslash i$ who have 1 unit of deficit power and let p'_{-i} be as in Definition 2. Now, if a microgrid i sets a price p_i, then its power is sold iff (i) $v \geq p_i$ (if $v < p_i$, then a microgrid with deficit power would prefer to buy power from the macrogrid instead of from microgrid i) and (ii) microgrid i is one of the microgrids with the K_{-i} lowest pseudo-prices (and among the randomly selected ones in case of ties in pseudo-prices); let the probability of the event in (ii) be $B(p_i)$. So microgrid i's expected revenue is:

$$\begin{aligned} E\{u_i(p_i, \psi_{-i})\} &= (p_i - c)P(p_i \leq v)B(p_i) \\ &= (p_i - c)(1 - G(p_i))B(p_i) \\ &= h(p_i)B(p_i) \end{aligned} \tag{31}$$

Now, by Assumption 1, $h(.)$ has a unique maximizer at v_T. Also, by definition, the function $B(p_i)$ is a nonincreasing function of p_i. So by (31), $E\{u_i(p_i, \psi_{-i})\} < E\{u_i(v_T, \psi_{-i})\}$ for all $p_i > v_T$ and hence *no microgrid sets a price greater than* v_T. In fact, as the analysis below shows, v_T *plays the role that* v *plays in the constant* v *case in Section III*.

Now, since $\underline{v} \leq v \leq \overline{v}$, $G(p_i) = 1$ for $p_i \geq \overline{v}$. So

$$h(p_i) = 0, \; p_i \geq \overline{v}. \tag{32}$$

Also, $G(p_i) = 0$ for $p_i \leq \underline{v}$. So $h(p_i) = p_i - c$ for $c \leq p_i \leq \underline{v}$, which is a strictly increasing function of p_i and $h(\underline{v}) = \underline{v} - c > 0$. This, combined with (32), gives:

$$\underline{v} \leq v_T < \overline{v}. \tag{33}$$

Next, we explicitly compute a NE and show that it is unique. The results are similar to those in the constant v case in Section III, and hence we only state the differences.

First, we state some necessary conditions that the NE strategies must satisfy. Property 1 in Section III-A holds in the present context with v_T in place of v. Let w_i be as in Definition 1.

Lemma 3: The equation

$$h(x) = \frac{h(v_T)(1-w_1)}{[1-(1-s)^{n-1}]} \tag{34}$$

has a unique solution $x \in (c, v_T)$. Let $\tilde{p}$ be this solution. Let

$$u_{i,max} = h(\tilde{p}) \left[1 - (1-s)^{n-1}\right]. \tag{35}$$

Property 2 in Section III-A holds except that now, $\tilde{p}$ and $u_{i,max}$ are as defined above. Also, Theorem 1 holds with Properties 1 and 2 and $\tilde{p}$ as described above, and v_T in place of v.

Equation (8) holds in the present context, and to completely determine the NE strategies $\phi_1(.), \ldots, \phi_n(.)$, it remains to specify $R_3, \ldots, R_n$ and the function $\phi(.)$. Let $F_{-i}(.)$ be as in Section III. Similar to the derivation of (10) and using (31) and (35), we get that for $x \in [\tilde{p}, v_T)$:

$$E\{u_1(x, \psi_{-1})\} = h(x)(1 - F_{-1}(x)) = h(\tilde{p}) \left[1 - (1-s)^{n-1}\right] \tag{36}$$

By (36), we get

$$F_{-1}(x) = g(x), \; x \in [\tilde{p}, v), \tag{37}$$

where,

$$g(x) = \frac{h(x) - h(\tilde{p})[1-(1-s)^{n-1}]}{h(x)}. \tag{38}$$

Next, we compute $R_i, i = 3, \ldots, n$ and the function $\phi(.)$ using (37) and (38). We define the function $f_i(.)$ as in Section III and the $n-1$ events in the definition of $f_i(.)$ as in the derivation of (15). Similar to the derivation of (14), we get:

$$g(R_i) = f_i(q_i),$$

which, using (38), becomes:

$$h(R_i) = \frac{h(\tilde{p}) \left[1 - (1-s)^{n-1}\right]}{1 - f_i(q_i)}. \tag{39}$$

Lemma 4: Equation (39) has a unique solution $R_i \in [\tilde{p}, v_T]$.

Also, similar to the derivation of (19), for each $i \in \{2, \ldots, n\}$ such that $R_{i+1} < R_i$:

$$f_i(\phi(x)) = g(x), \; R_{i+1} \leq x < R_i. \tag{40}$$

Finally, Lemma 2 holds in the present context with (40) in place of (19) and v_T in place of v in the statement of the lemma and Theorem 2 holds with $R_i, i \in \{3, \dots, n\}$ being the solutions of (39) instead of the values in (15), (40) in place of (19) and v_T in place of v in the statement of the theorem.

Thus, a unique NE exists and the above discussion identifies a procedure for computing it. Also, the NE has a structure similar to that in Section III; in particular, the discussion in Section III-C applies to it.

VI. Correlated Power Availability Events

A. Model

So far, we have assumed that the events that the microgrids have excess power, deficit power or neither in a given slot are independent across microgrids. Although this is a good approximation in several important special cases, in general, these events could be correlated across microgrids. For example, if solar panels are the pre-dominant energy sources in two neighboring microgrids, then when one of these microgrids has excess power, the other is likely to as well and vice versa. This is because the power output of solar panels depends on the weather conditions, which are likely to be similar at the two microgrids. In this section, we analyze the scenario where the above events may be correlated.

Specifically, consider the model described in Section II with the following differences. For $i \in \{1, \dots, n\}$, let E_i, D_i and B_i be the events that microgrid i has excess power, deficit power and neither respectively in a given slot. These events are allowed to be correlated across different i. For a triple of disjoint subsets E, D and B of the set $\{1, \dots, n\}$ such that $E \cup D \cup B = \{1, \dots, n\}$, let $P(E, D, B)$ be the probability that microgrids in the sets E, D and B have excess power, deficit power and neither respectively. The distribution $P(E, D, B)$ is assumed to be known to the microgrids, possibly from historical and/ or weather forecast data.

We make the following technical assumption:

Assumption 2: For every triple of disjoint subsets E, D and B of the set $\{1, \dots, n\}$ such that $E \cup D \cup B = \{1, \dots, n\}$, $P(E, D, B) > 0$.

Note that although $P(E, D, B)$ is required to be positive, it may be arbitrarily small; hence, the above assumption would typically be satisfied in practice.

For tractability, we analyze the case where the distribution $P(E, D, B)$ is *symmetric across different microgrids*; in particular, note that $P(E, D, B)$ is completely determined by [10] $|E|$, $|D|$ and $|B|$, *i.e.*, $P(E_1, D_1, B_1) = P(E_2, D_2, B_2)$ if $|E_1| = |E_2|$, $|D_1| = |D_2|$ and $|B_1| = |B_2|$.

B. Symmetric NE

Since the distribution $P(E, D, B)$ is symmetric across different microgrids, the above game is a *symmetric game*, which is one in which all players have the same parameters, action sets and utility functions. We focus on a specific class of NE, known as *symmetric NE*. A NE $(\psi_1^*(\cdot), \dots, \psi_n^*(\cdot))$ is a symmetric NE if all players play identical strategies under it, i.e., $\psi_1^*(\cdot) = \psi_2^*(\cdot) = \dots = \psi_n^*(\cdot)$. In practice it is challenging to implement any other NE in a symmetric game – see Section 3.3 in [22] for a detailed discussion. Symmetric NE has indeed been advocated for symmetric games by several game theorists [8]. The natural question now is whether there exists at least one symmetric NE, and also whether there is a unique symmetric NE (note that some symmetric games are known to have multiple symmetric NE, *e.g.*, the simple "Meeting in New York game" [26]). In Section VI-C, we explicitly compute a symmetric NE and show its uniqueness in our context.

C. Symmetric NE Computation and Uniqueness

The computation and proof of uniqueness of the symmetric NE can be done using techniques similar to those in Section III, as we now explain.

Consider a symmetric NE in which every microgrid that has excess power uses the d.f. $\psi(\cdot)$ to set its price. Let $u_{i,max}$ be the maximum expected payoff that microgrid i can get, *conditional on the event that microgrid i has excess power*, when all other microgrids use the strategy $\psi(\cdot)$. Also, let N_{-i} and K_{-i} be as in Definition 1, and let $w_i = P(N_{-i} \geq K_{-i} | E_i)$.

The following two lemmas provide some necessary conditions that $\psi(\cdot)$ and $u_{i,max}$ must satisfy.

Lemma 5: $\psi(\cdot)$ is continuous and its support set is $[\tilde{p}, v]$, where:

$$\tilde{p} = c + \frac{(v - c)(1 - w_1)}{P(K_{-1} \geq 1 | E_1)}. \tag{41}$$

Also,

$$u_{1,max} = (\tilde{p} - c) P(K_{-1} \geq 1 | E_1). \tag{42}$$

Proof: First, we show by contradiction that $\psi(\cdot)$ is continuous. Note that the price c is dominated for every microgrid that has excess power by every price $x > c$. Now suppose microgrids other than i select a price $x \in (c, v]$ with positive probability. Then for microgrid i, a price just below x fetches a strictly higher payoff than the price x. So x is not a best response for microgrid i, which contradicts the fact that $\psi(\cdot)$ is a symmetric NE strategy. Thus, $\psi(\cdot)$ is continuous.

Now, the proof of (42) is as in the last paragraph of Lemma 10, except that $P(K_{-1} \geq 1)$ is replaced with $P(K_{-1} \geq 1 | E_1)$.

Now, Lemma 11 continues to hold in the present context and its proof is the same as before. By Lemma 11 and the continuity of $\psi(\cdot)$, v is a best response for each microgrid. So:

$$u_{1,max} = (v - c)(1 - w_1) \tag{43}$$

Expression (41) follows from (42) and (43). ■

Now, Lemma 15 continues to hold in the present context and its proof is the same as before. Hence, we get:

Lemma 6: $\psi(\cdot)$ is strictly increasing in $[\tilde{p}, v]$ and in particular, every price $x \in [\tilde{p}, v]$ is a best response for microgrid 1 under the symmetric NE.

Now, let the pseudo-price, p_i', of microgrid i be equal to p_i if it has excess power, and $v + 1$ otherwise. Let p_{-i}' be as in Definition 2, and let $F_{-1}(x | E_1) = P(p_{-1}' \leq x | E_1)$. Similar

[10] $|A|$ denotes the cardinality of set A.

to the derivation of (9), if microgrid 1 has excess power and sets price $x \in [\tilde{p}, v]$, its expected payoff is given by:

$$E\{u_1(x, \psi_{-1})\} = (x - c)(1 - F_{-1}(x|E_1)). \tag{44}$$

By Lemma 6, x is a best response. So by (42) and (44), we get:

$$F_{-1}(x|E_1) = g(x), \tag{45}$$

where,

$$g(x) = \frac{x - c - (\tilde{p} - c)P(K_{-1} \geq 1|E_1)}{x - c}. \tag{46}$$

Now, by the definition of $F_{-1}(x|E_1)$, when microgrids $2, \ldots, n$ use the strategy $\psi(\cdot)$:

$$F_{-1}(x|E_1) = \sum_{k,j:j \geq k} P(K_{-1} = k, N_{-1} = j|E_1) \times$$

$$\left[\sum_{l=k}^{j} \binom{j}{l} (\psi(x))^l (1 - \psi(x))^{j-l}\right] = \tilde{f}(\psi(x)) \text{ (say)}. \tag{47}$$

We get the above expression by conditioning on the event $\{K_{-1} = k, N_{-1} = j\}$ and using the fact that given $N_{-1} = j$, each of these j microgrids independently sets a price $\leq x$ w.p. $\psi(x)$. By (45) and (47):

$$\tilde{f}(\psi(x)) = g(x). \tag{48}$$

Lemma 7: For each $x \in [\tilde{p}, v]$, (48) has a unique solution $\psi(x)$. The function $\psi(\cdot)$ is strictly increasing and continuous on $[\tilde{p}, v]$. Also, $\psi(\tilde{p}) = 0$ and $\psi(v) = 1$.

The proof of Lemma 7 is similar to that of Lemma 2.

Note that by Lemma 7, the solution $\psi(\cdot)$ of (48) satisfies the necessary conditions on $\psi(\cdot)$ in Lemmas 5 and 6 and hence is a candidate for the symmetric NE strategy. Finally, similar to Theorem 2, we get:

Theorem 4: The solution $\psi(\cdot)$ of (48) constitutes the unique symmetric NE strategy.

Thus, a unique symmetric NE exists, and its structure is similar to that of the NE in Section III (with $q_1 = \ldots = q_n$).

VII. Numerical Studies

In this section, using numerical experiments, we compare the trade of power among interconnected microgrids proposed in this paper with a scheme in which microgrids only trade power with the macrogrid, and also further study the NE studied in Sections III and IV. Throughout, we use the parameter values $c = 0$ and $v = 1$.

First, we consider $q_1, \ldots, q_n$ that are uniformly spaced in $[q_L, q_H]$ for some parameters q_L and q_H, and $s_1 = \ldots = s_n = s$. Let $q = \frac{q_L + q_H}{2}$ be the mean probability of having excess power of the microgrids. We consider two schemes: (i) the scheme considered in this paper; and (ii) a centralized scheme in which a microgrid who has deficit power (respectively, excess power) in a slot buys power from (respectively, sells power to) the macrogrid alone. Recall that in scheme (i), a microgrid with excess power sells power to a microgrid with deficit power for a price in $[c, v]$, whereas the macrogrid buys power at price c and sells power at price v; thus, the microgrids trade power among themselves whenever possible, and with the macrogrid only in the event of necessity arising from a mismatch between the amounts of deficit and excess power with the microgrids. Let T_D and T_C be the total expected power traded (bought or sold) by all the microgrids with the macrogrid in one slot in schemes (i) and (ii) respectively. Note that since the microgrids are close to each other, power that is traded by the microgrids with the macrogrid is typically transmitted over longer distances than the power that is traded among the microgrids, resulting in larger transmission losses. The left plot in Fig 2 plots T_D, T_C and the ratio $\eta = \frac{T_D}{T_C}$ versus q, and shows that scheme (i) results in considerable savings in the total expected power exchanged with the macrogrid over scheme (ii) (the savings are between 43.4% and 68.4% in the current example). Also, note that T_D achieves its lowest value around $q = 0.25$, which is close to the value of s (0.3). This is consistent with the intuition that when the mean excess power and deficit power probabilities of the microgrids are close to each other, the average total excess power available with the microgrids with excess power roughly matches the average total deficit power required by the microgrids with deficit power, and hence only a small amount of power needs to be exchanged with the macrogrid. We also considered two extreme cases: (a) $s = 0.8$ and $q = 0.1$, and (b) $s = 0.1$ and $q = 0.8$ [11], which correspond, for example, to peak demand hours and off peak hours respectively. We found that in both cases, T_D is lower than T_C by about 22.2%, which is a significant amount. Finally, the right plot in Fig. 2 plots T_D, T_C and η versus n and shows, as well, a substantial reduction in the total expected power exchanged with the macrogrid in scheme (i) over scheme (ii) (the reduction is between 27% and 61% in the current example). The plot also shows that η decreases in n, which is consistent with intuition– in scheme (i), the amount of power traded among the microgrids increases in the number of participants in the trade, and hence in n. In summary, trade of power among interconnected microgrids can result in substantial savings in the amount of power that needs to be transmitted over large distances (with a corresponding reduction in transmission losses) especially when the average excess power and deficit power probabilities of the microgrids are close to each other. Also, the savings increase in the number of microgrids.

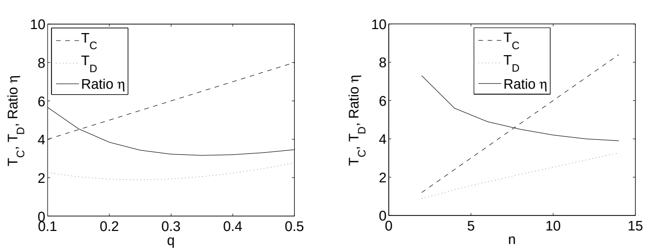

Fig. 2. The left plot shows T_C, T_D and $\eta = \frac{T_D}{T_C}$ versus q. The parameter values used are $n = 10$, $q_H - q_L = 0.2$ and $s = 0.3$. The right plot shows T_C, T_D and η versus n. The parameter values used are $q = 0.2$, $q_H - q_L = 0.2$ and $s = 0.4$. In both plots, η is scaled by a factor of 10 in order to show it on the same plot as the other curves.

Now, we consider the model analyzed in Section III with the

[11] The other parameters are the same as in Fig. 2.

parameter values $n = 4$, $q_1 = 0.6, q_2 = 0.5, q_3 = 0.4, q_4 = 0.3$ and $s = 0.3$. For these parameters, the left plot in Fig. 3 shows the price selection d.f.s of the microgrids in the NE in Theorems 1 and 2. The structure of the functions in the plot is as in Theorem 1 with $\tilde{p} = 0.45$, $R_1 = R_2 = 1$, $R_3 = 0.93$ and $R_4 = 0.82$. In particular, microgrid 1's price selection d.f. has a jump at v. Next, we consider the model analyzed in Section IV with the parameter values $n = 3$, $q_1 = 0.6$, $q_2 = 0.5$, $q_3 = 0.4$, $s_1 = 0.3$, $s_2 = 0.2$ and $s_3 = 0.1$. The right plot in Fig. 3 shows the price selection d.f.s of the microgrids in the NE for these parameter values. Their structure is as found in Section IV (see (25), (26) and (27)) with $\tilde{p} = 0.54$, $R_1 = R_2 = 1$ and $R_3 = 0.92$. In particular, microgrid 1's price selection d.f. has a jump at v. The two plots in Fig. 3 shows that the price selection d.f.s in the NE with symmetric and asymmetric deficit power probabilities are qualitatively similar.

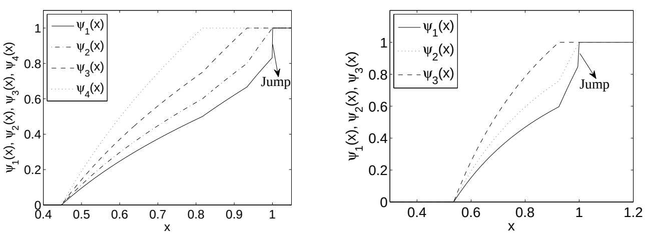

Fig. 3. The left plot shows the functions $\psi_1(.), \psi_2(.), \psi_3(.)$ and $\psi_4(.)$ for the model in Section III with the parameter values in the text. The right plot shows the functions $\psi_1(.), \psi_2(.)$ and $\psi_3(.)$ for the model in Section IV with the parameter values in the text.

Next, we again consider the model in Section III with $q_1, \ldots, q_n$ that are uniformly spaced in $[q_L, q_H]$ for some parameters q_L and q_H. Let $q = \frac{q_L + q_H}{2}$ be the mean probability of having excess power of the microgrids. We first define the *efficiency*, γ, of the NE as $\gamma = \frac{R_{\text{NE}}}{R_{\text{OPT}}}$, where R_{NE} is the expected sum of payoffs of the players (microgrids with excess power) at the NE and R_{OPT} is the maximum possible (optimal) expected sum of payoffs, attained under the scheme OPT in which all players cooperate and set the maximum price v. Clearly, $\gamma \leq 1$ quantifies the loss in aggregate revenue incurred owing to lack of cooperation among the players. The left plot in Fig. 4 reveals, as expected, that price competition significantly reduces the aggregate revenue of the players under the NE relative to OPT. Also, the efficiency (γ) decreases as q increases since the competition increases.

Finally, the right plot in Fig. 4 plots the mean price of excess power quoted by microgrid 1 in the NE found in Section III versus q. The figure shows that the mean price is decreasing in q; this is because, since s is constant, as q increases, the expected supply of power in the market increases relative to the expected demand for power and the price competition among the microgrids with excess power becomes more intense, driving down the prices.

VIII. Conclusions and Future Work

We analyzed price competition among interconnected microgrids and found NE in the corresponding game. The analysis provides several insights– for example, there is randomization in the selection of prices by the microgrids who have excess power and, when the probabilities of having deficit power are symmetric, only microgrids with a high excess power availability probability set high prices. Numerical experiments showed that trade of power among interconnected microgrids results in significant savings in the total expected power transmitted over long distances and hence the transmission losses.

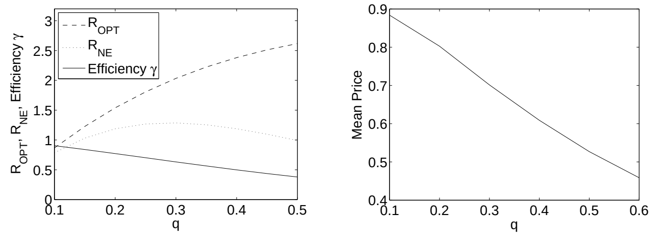

Fig. 4. The left plot shows the aggregate revenues of the microgrids with excess power, R_{NE} and R_{OPT}, under the NE and OPT respectively, and the efficiency of the NE, $\gamma = \frac{R_{NE}}{R_{OPT}}$, versus q. The parameter values $n = 10$, $q_H - q_L = 0.2$ and $s = 0.3$ are used. The right plot shows the mean price of excess power quoted by microgrid 1 versus q for the parameter values $n = 8$, $q_H - q_L = 0.2$ and $s = 0.25$.

We noted that explicit computation and an investigation of the uniqueness of the NE is complicated when the deficit power probabilities of the microgrids are asymmetric; in this paper, we have computed a NE for the case $n = 3$. A direction for future work is to compute the NE and to investigate its uniqueness for arbitrary n. In addition, in each slot, the power corresponding to the outcome of the trade among the microgrids needs to be physically exchanged among the microgrids and in practice, several technical issues such as the connectivity among the microgrids, capacities of the transmission lines, meshed-grid loop-flow effects etc. would constrain this exchange. Another direction for future work is to extend our model by incorporating these technical issues as well as the measures that would ensure system protection during the exchange of this power.

In Section V, we analyzed the scenario where v, the price at which the macrogrid sells unit power, is random. Investigation of the scenario where c, the price at which the macrogrid *buys* unit power, is random is an open problem. Finally, in this paper, we have only characterized the NE strategies in a one-shot game. In practice, microgrids may play this game repeatedly and may use their experience from previous slots and a learning algorithm to choose their strategy in the current slot. An investigation into whether the NE for the one-shot game constitutes a steady-state outcome of some natural learning algorithms in such a setting is an interesting direction for future research. Also note that the probabilities q_i and s_i, $i = 1, \ldots, n$ may not be constant across time slots; a study of the multiple slot game with time-varying probabilities is another open problem.

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APPENDIX

A. Proofs of results in Section III-A

We first prove a lemma (Lemma 8) that we use throughout. Next we prove Property 2 and then Property 1 and Theorem 1.

Lemma 8: For $i = 1, \ldots, n$, $\psi_i(.)$ is continuous, except possibly at v. Also, at most one microgrid has a jump at v.

Proof: Suppose $\psi_i(.)$ has a jump at a point x_0, $c < x_0 < v$. Then for some $\epsilon > 0$, no microgrid $j \neq i$ chooses a price in $[x_0, x_0 + \epsilon]$ because it can get a strictly higher payoff by choosing a price just below x_0 instead. This in turn implies that microgrid i gets a strictly higher payoff at the price $x_0 + \epsilon$ than at x_0. So x_0 is not a best response for microgrid i, which contradicts the assumption that $\psi_i(.)$ has a jump at x_0. Thus, $\psi_i(.)$ is continuous at all $x < v$.

Now, suppose microgrid i has a jump at v. Then a microgrid $j \neq i$ gets a higher payoff at a price just below v than at v. So v is not a best response for microgrid j and it plays it with 0 probability. Thus, at most one microgrid has a jump at v. ■

1) Proof of Property 2: We prove Property 2 in Lemmas 10 and 12. We first prove Lemma 9, which will be used to prove Lemma 10. Let $u_{i,max}$ and L_i be as defined in Section III-A.

Lemma 9: For $i = 1, \ldots, n$, L_i is a best response for microgrid i.

Proof: By (2), either microgrid i has a jump at L_i or plays prices arbitrarily close to L_i and above it with positive probability.

Case (i): If microgrid i has a jump at L_i, then L_i is a best response for i because in a NE, no microgrid plays a price other than a best response with positive probability.

Case (ii): If microgrid i does not have a jump at L_i, then by (2), $\psi_i(L_i) = 0$. Since every microgrid selects a price in $[c, v]$, $\psi_i(v) = 1$. So $L_i < v$. So by Lemma 8, no microgrid among $\{1, \ldots, n\} \backslash i$ has a jump at L_i. Hence, microgrid i's payoff at a price above L_i and close enough to it is arbitrarily close to its payoff *at* L_i. But since microgrid i does not have a jump at L_i, by (2), it plays prices just above L_i with positive probability and they are best responses for it. So L_i is also a best response for microgrid i. ■

Lemma 10: For some $c < \tilde{p} < v$, $L_1 = \ldots L_n = \tilde{p}$. Also, $u_{i,max} = (\tilde{p} - c)\left[1 - (1-s)^{n-1}\right]$, $i = 1, \ldots, n$.
That is, the lower endpoint of the support set of the price distribution of every microgrid is the same.

Proof: Let K_{-i} be as in Definition 1. Note that the expected payoff that a microgrid i gets at a given price p_i depends on the pseudo-price distribution functions of the microgrids other than i and the distribution of K_{-i}. Also, since $s_1 = \ldots = s_n = s$, the distribution of the random variable K_{-i} for $i = 1, \ldots, n$ is the same.

Now, suppose $L_i < L_j$ for some i, j. By Lemma 9, L_j is a best response for microgrid j. Now, the expected payoff that microgrid j gets for $p_j = L_j$ is strictly less than the expected payoff that microgrid i would get if it set p_i to be just below L_j. This is because, if microgrids i or j set a price of approximately L_j, then they see the same pseudo-price distribution functions of the microgrids other than i and j. But microgrid j may be undercut by microgrid i, since $L_i < L_j$, whereas microgrid i may not be undercut by microgrid j. Also, microgrid j's expected payoff is strictly lowered due to this undercutting by microgrid i because microgrid j's excess power is not sold in the event that microgrid i has excess power and sets a price below L_j, and exactly one microgrid has deficit power, which happens with positive probability. Hence, $u_{i,max} > u_{j,max}$.

Now, by Lemma 9, L_i is a best response for microgrid i. If microgrid j were to play price L_i, then it would get a payoff of $u_{i,max}$. This is because, when microgrid i plays price L_i, it gets payoff $u_{i,max}$. Since $L_j > L_i$, microgrid i is, w.p. 1, not undercut by microgrid j. If microgrid j were to set the price L_i, then w.p. 1, it would not be undercut by microgrid i. Also, the pseudo-price distributions of the microgrids other than i and j are exactly the same from the viewpoints of microgrids i and j. Thus, microgrid j can strictly increase its payoff from $u_{j,max}$ to $u_{i,max}$ by playing price L_i, which contradicts the fact that L_j is a best response for it.

Thus, $L_i < L_j$ is not possible. By symmetry, $L_i > L_j$ is not possible. So $L_i = L_j$. Let $L_1 = \ldots = L_n = \tilde{p}$.

By Lemma 9, a price of $\tilde{p}$ is a best response for every microgrid i. Since no microgrid sets a price lower than $\tilde{p}$, a price of $\tilde{p}$ fetches a payoff of $\tilde{p} - c$ for microgrid i if $K_{-i} \geq 1$ and a payoff of 0 if $K_{-i} = 0$. So $u_{i,max} = (\tilde{p} - c)P(K_{-i} \geq 1) = (\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right]$, $i = 1, \ldots, n$. ■

Let w_i be as in Definition 1. Using (1), it can be easily shown that:

$$w_1 \leq w_2 \leq \ldots \leq w_n. \tag{49}$$

We now prove Lemma 11, which will be used to prove Lemma 12.

Lemma 11: For every $\epsilon > 0$, there exist microgrids m and j, $m \neq j$, such that $\psi_m(v - \epsilon) < 1$ and $\psi_j(v - \epsilon) < 1$. That is, at least two microgrids play prices just below v with positive probability.

Proof: Suppose not. Fix i and let:

$$y = \inf\{x : \psi_l(x) = 1 \ \forall l \neq i\}. \tag{50}$$

By definition of y, $\psi_l(x) = 1 \ \forall l \neq i$ and $x > y$. Also, since $\psi_l(.)$ is a distribution function, it is right continuous [11]. So

$$\psi_l(y) = 1 \ \forall l \neq i. \tag{51}$$

Suppose $y < v$. By (51):

$$P\{p_l \in (y, v]\} = 0, \ \forall l \neq i. \tag{52}$$

So every price $p_i \in (y, v)$ is dominated by $p_i = v$. Hence:

$$P\{p_i \in (y, v)\} = 0 \tag{53}$$

By (52) and (53):

$$P\{p_j \in (y, v)\} = 0, \ j = 1, \ldots, n. \tag{54}$$

By (50), $\forall \epsilon > 0$, $\psi_l(y - \epsilon) < 1$ for at least one microgrid $l \neq i$; otherwise the infimum in the RHS of (50) would be less than y. So this microgrid l plays prices just below y with positive probability. Now, if microgrid l sets a price $p_l < v$, it gets a payoff equal to the revenue, $(p_l - c)$, if power is sold, times the probability that power is sold. Also, by Lemma 8, $\psi_j(.), j = 1, \ldots, n$ are continuous at all prices below v. So by (54), a price p_l just below v yields a higher payoff than a price just below y. This is because, $p_l - c$ is lower by approximately $v - y$ for p_l just below y than for p_l just below v, but by (54) and continuity of $\psi_j(.), j = 1, \ldots, n$, the probability that power is sold for a price p_l just below y can be made arbitrarily close to the probability that power is sold for a price p_l just below v. This contradicts the assumption that microgrid l plays prices just below y with positive probability.

Thus, y in (50) equals v and hence at least one microgrid $j \neq i$ plays prices just below v with positive probability. The above arguments with j in place of i imply that at least one microgrid other than j plays prices just below v with positive probability. Thus, at least two microgrids in $\{1, \ldots, n\}$ play prices just below v with positive probability. ■

Lemma 12: $\tilde{p} = c + (v - c)\frac{1 - w_1}{1 - (1 - s)^{n-1}}$.

Proof: If microgrid 1 sets the price $p_1 = v$, then it gets an expected payoff of at least $(v - c)(1 - w_1)$ because its power is sold at least in the event that $K_{-1} - 1$ or fewer microgrids out of $2, \ldots, n$ have excess power. So $u_{1,max} \geq (v - c)(1 - w_1)$. Since $u_{1,max} = (\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right]$ by Lemma 10, we get:

$$\tilde{p} \geq c + (v - c)\frac{1 - w_1}{1 - (1 - s)^{n-1}}. \tag{55}$$

Now, by Lemma 11, at least two microgrids, say m and j, play prices just below v with positive probability. By Lemma 8, at most one of them has a jump at v. So assume, WLOG, that no microgrid other than j has a jump at v. Then a price of $p_j = v$ is a best response for microgrid j and fetches a payoff of $u_{j,max} = (v - c)(1 - w_j) \leq (v - c)(1 - w_1)$, where the inequality follows from (49). Since $u_{j,max} = (\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right]$ by Lemma 10, we get:

$$\tilde{p} \leq c + (v - c)\frac{1 - w_1}{1 - (1 - s)^{n-1}}. \tag{56}$$

The result follows from (55) and (56). ■

Property 2 follows from Lemmas 10 and 12.

2) Proof of Property 1 and Theorem 1: We start by proving Lemma 13, which proves most of Property 1.

Lemma 13: (i) $\phi_2(.), \ldots, \phi_n(.)$ are continuous at v. (ii) $\phi_1(.)$ is continuous at v if $q_1 = q_2$ and has a jump of size *at most* $q_1 - q_2$ at v if $q_1 > q_2$. Also,

$$\phi_1(v-) \geq q_2. \tag{57}$$

Proof: If no microgrid $i > 1$ has a jump at v, then microgrid 1 gets a payoff of $(v - c)(1 - w_1)$, which equals $(\tilde{p} - c)\left[1 - (1 - s)^{n-1}\right]$ by Lemma 12, for a price p_1 just below v in the limit as $p_1 \to v-$. So if a microgrid $i \geq 2$ *has* a jump at v, microgrid 1 can get a payoff strictly greater than $(\tilde{p} - c)\left[1-(1-s)^{n-1}\right]$ by playing a price close enough to v. This contradicts the fact that $u_{1,max} = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$ (see Lemma 10). Thus, no microgrid $i \geq 2$ has a jump at v and $\phi_2(.), \ldots, \phi_n(.)$ are continuous.

First, suppose $q_1 = q_2$. If microgrid 1 has a jump at v, then similar to the preceding paragraph, microgrid 2 can get a payoff strictly greater than $(\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$ by playing a price just below v, which contradicts the fact that $u_{2,max} = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$. So $\psi_1(.)$ is continuous.

Now suppose $q_1 > q_2$. First, suppose microgrid 1 has a jump of size *exactly* $q_1 - q_2$ at v. Then if microgrid 2 sets a price just below v, then the probability of being undercut by microgrid $j \in \{3, \ldots, n\}$ is approximately q_j. Also, since microgrid 1 has a jump of size $q_1 - q_2$ at v, the probability of being undercut by microgrid 1 is approximately $q_1 - (q_1 - q_2) = q_2$. So at a price just below v, microgrid 2 sees the same set of probabilities of being undercut by microgrids other than itself as microgrid 1 would see if it set a price just below v. Hence, by the first paragraph of this proof, microgrid 2 gets a payoff of approximately $(\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$ at a price just below v.

Hence, if microgrid 1 has a jump of size, not equal to, but greater than $q_1 - q_2$ at v, microgrid 2 gets a payoff of strictly greater than $(\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$ at a price just below v. This contradicts the fact that $u_{2,max} = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$.

Thus, microgrid 1 has a jump of at most size $q_1 - q_2$ at v. So $\phi_1(v) - \phi_1(v-) \leq q_1 - q_2$. This, along with $\phi_1(v) = q_1$, gives (57). ■

Given Lemmas 8 and 13, Property 1 follows once we show that the jump of $\phi_1(.)$ at v is *exactly* $q_1 - q_2$. We will prove this in Lemma 18, which we will prove after proving Part 2 of Theorem 1 in Lemma 17.

Let $F_{-i}(x)$ be as in Definition 2. The following lemma will be used later.

Lemma 14: For a fixed $x \in (\tilde{p}, v]$, and microgrids i and j, (i) $F_{-i}(x) = F_{-j}(x)$ iff $\phi_i(x) = \phi_j(x)$, (ii) $F_{-i}(x) < F_{-j}(x)$ iff $\phi_i(x) > \phi_j(x)$.

Proof: Let $K_{-(i,j)}$ be the number of microgrids out of $\{1, \ldots, n\} \backslash \{i, j\}$ that have 1 unit of deficit power. Let $p'_{(l)}$ be the l'th smallest pseudo-price out of the pseudo-prices of microgrids $\{1, \ldots, n\} \backslash \{i, j\}$ (with $p'_{(l)}$ defined to be 0 if $l \leq 0$ and $v+1$ if $l > n-2$). Now, microgrid j has (i) 1 unit of deficit power w.p. s, (ii) neither excess nor deficit power w.p. $1 - q_j - s$, (iii) 1 unit of excess power and $p_j \leq x$ w.p. $q_j \psi_j(x)$ and (iv) 1 unit of excess power and $p_j > x$ w.p. $q_j(1-\psi_j(x))$. Conditioning on the preceding four events, we get:

$$\begin{aligned} &F_{-i}(x) \\ &= P\{p'_{-i} \leq x\} \\ &= sP\{p'_{(K_{-(i,j)}+1)} \leq x\} + (1-q_j-s)P\{p'_{(K_{-(i,j)})} \leq x\} \\ &\quad +q_j\psi_j(x)P\{p'_{(K_{-(i,j)}-1)} \leq x\} \\ &\quad +q_j(1-\psi_j(x))P\{p'_{(K_{-(i,j)})} \leq x\} \\ &= sP\{p'_{(K_{-(i,j)}+1)} \leq x\} + (1-s)P\{p'_{(K_{-(i,j)})} \leq x\} \\ &\quad +\phi_j(x)\left[P\{p'_{(K_{-(i,j)}-1)} \leq x\} - P\{p'_{(K_{-(i,j)})} \leq x\}\right] \end{aligned} \tag{58}$$

Similarly:

$$\begin{aligned} &F_{-j}(x) \\ &= sP\{p'_{(K_{-(i,j)}+1)} \leq x\} + (1-s)P\{p'_{(K_{-(i,j)})} \leq x\} \\ &+ \phi_i(x)\left[P\{p'_{(K_{-(i,j)}-1)} \leq x\} - P\{p'_{(K_{-(i,j)})} \leq x\}\right] \end{aligned} \tag{59}$$

Subtracting (59) from (58), we get:

$$\begin{aligned} &F_{-i}(x) - F_{-j}(x) = (\phi_j(x) - \phi_i(x)) \times \\ &\left[P\{p'_{(K_{-(i,j)}-1)} \leq x\} - P\{p'_{(K_{-(i,j)})} \leq x\}\right] \end{aligned} \tag{60}$$

Now, since $x > \tilde{p}$, all microgrids play prices in $(\tilde{p}, x)$ with positive probability by Lemma 10. So:

$$\phi_l(x) = P\{p'_l \leq x\} > 0, \ l = 1, \ldots, n. \tag{61}$$

Also,

$$\phi_l(x) \leq \phi_l(v) = q_l < 1, \ l = 1, \ldots, n. \tag{62}$$

By (61) and (62):

$$0 < \phi_l(x) < 1, \ l = 1, \ldots, n. \tag{63}$$

Also, $P\{p'_{(K_{-(i,j)}-1)} \leq x\} - P\{p'_{(K_{-(i,j)})} \leq x\}$ is the probability of the event that exactly $K_{-(i,j)} - 1$ pseudo-prices out of the pseudo-prices of the microgrids $\{1, \ldots, n\} \backslash \{i, j\}$ are $\leq x$, which happens in particular when $K_{-(i,j)} = 1$ and no pseudo-price out of $\{1, \ldots, n\} \backslash \{i, j\}$ is $\leq x$. By (63), the probability of the latter event is positive and hence $P\{p'_{(K_{-(i,j)}-1)} \leq x\} - P\{p'_{(K_{-(i,j)})} \leq x\} > 0$. The result now follows from (60). ■

Now, in a sequence of two lemmas, we prove that each microgrid plays prices in every sub-interval of its support set with positive probability– a result that will be used to prove part 2 of Theorem 1. The following lemma generalizes Lemma 11.

Lemma 15: Let $\tilde{p} \leq a < b \leq v$. Then at least two microgrids play prices in (a, b) with positive probability.

Proof: If $b = v$, then the claim is true by Lemma 11. If $a = \tilde{p}$, then the claim is true by Lemma 8 and Lemma 10, since $\tilde{p} < v$ is the lower endpoint of the support set of all microgrids and no microgrid has a jump at $\tilde{p}$; hence all microgrids play prices just above $\tilde{p}$ with positive probability.

Now, fix any a, b such that $\tilde{p} < a < b < v$. Let:

$$\underline{a} = \inf\{x \leq a : \psi_j(x) = \psi_j(a) \ \forall j = 1, \ldots, n\} \tag{64}$$

By Lemma 10, $\underline{a} > \tilde{p}$. Also, by definition of $\underline{a}$, $P\{p_j \in [\underline{a}, a]\} = 0 \ \forall j = 1, \ldots, n$.

By definition of $\underline{a}$, at least one microgrid, say microgrid i, plays prices just below $\underline{a}$ with positive probability. (If not, then the infimum in (64) would be less than $\underline{a}$.) This implies that at least one microgrid $j \neq i$ plays prices in $(\underline{a}, b)$ with positive probability. (If not, then $p_i = b$ would yield a strictly higher payoff to microgrid i than prices just below $\underline{a}$.) Now, if microgrid j is the only microgrid among microgrids $\{1, \ldots, n\}$ who play prices in $(\underline{a}, b)$ with positive probability, then $p_j = b$ yields a strictly higher payoff than $p_j \in (\underline{a}, b)$, which is a contradiction. So at least two microgrids play prices in $(\underline{a}, b)$ with positive probability. But $P\{p_l \in [\underline{a}, a]\} = 0 \ \forall l = 1, \ldots, n$ by definition of $\underline{a}$. Hence, at least two microgrids play prices in (a,b) with positive probability. ∎

Lemma 16: If $\tilde{p} \leq x < y < v$ and $\psi_i(x) = \psi_i(y)$ for some microgrid i, then $\psi_i(v-) = \psi_i(x)$.
Thus, if $x \geq \tilde{p}$ is the left endpoint of an interval of constancy of $\psi_i(.)$ for some i, then to the right of x, the interval of constancy extends at least until v (there may be a jump at v).

Proof: Suppose not, *i.e.*:

$$\psi_i(v-) > \psi_i(x). \tag{65}$$

Let:

$$\overline{y} = \sup\{z \geq x : \psi_i(z) = \psi_i(x)\} \tag{66}$$

By (65) and (66), we get $\overline{y} < v$. So by Lemma 8, no microgrid among $\{1,\ldots,n\}\backslash i$ has a jump at $\overline{y}$. Also, microgrid i uses prices just above $\overline{y}$ with positive probability (if not, the supremum in the RHS of (66) would be $> \overline{y}$). So $\overline{y}$ is a best response for microgrid i and hence:

$$\begin{aligned} E\{u_i(\overline{y},\psi_{-i})\} &= (\overline{y}-c)(1-F_{-i}(\overline{y})) \\ &= u_{i,max} = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right] \end{aligned} \tag{67}$$

where the last equality follows from Lemma 10.

Now, by Lemma 15, there exists a microgrid $j \neq i$ who plays prices just below $\overline{y}$ with positive probability. Since no microgrid among $\{1,\ldots,n\}\backslash j$ has a jump at $\overline{y}$, $\overline{y}$ is a best response for microgrid j. Hence:

$$\begin{aligned} E\{u_j(\overline{y},\psi_{-j})\} &= (\overline{y}-c)(1-F_j(\overline{y})) \\ &= u_{j,max} = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right] \end{aligned} \tag{68}$$

By (67) and (68), $F_{-i}(\overline{y}) = F_{-j}(\overline{y})$. So by Lemma 14:

$$\phi_i(\overline{y}) = \phi_j(\overline{y}). \tag{69}$$

But since microgrid j plays prices just below $\overline{y}$ with positive probability, there exists $\epsilon > 0$ such that $x < \overline{y}-\epsilon$ and $\overline{y}-\epsilon$ is a best response for microgrid j. So

$$\phi_j(\overline{y}-\epsilon) < \phi_j(\overline{y}). \tag{70}$$

But by (66) and the continuity of $\phi_i(.)$ at $\overline{y}$:

$$\phi_i(\overline{y}) = \phi_i(\overline{y}-\epsilon). \tag{71}$$

By (69), (70) and (71), $\phi_i(\overline{y}-\epsilon) > \phi_j(\overline{y}-\epsilon)$. So by Lemma 14:

$$F_{-j}(\overline{y}-\epsilon) > F_{-i}(\overline{y}-\epsilon)$$

This implies:

$$\begin{aligned} (\tilde{p}-c)\left[1-(1-s)^{n-1}\right] &= E\{u_j(\overline{y}-\epsilon,\psi_{-j})\} \\ &= (\overline{y}-\epsilon-c)(1-F_{-j}(\overline{y}-\epsilon)) \\ &< (\overline{y}-\epsilon-c)(1-F_{-i}(\overline{y}-\epsilon)) \\ &= E\{u_i(\overline{y}-\epsilon,\psi_{-i})\} \end{aligned}$$

which contradicts the fact that every microgrid gets a payoff of $(\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$ at a best response in the NE. ∎

Lemma 17: Part 2 of Theorem 1 holds.

Proof: We prove the result by induction. Let:

$$R_n = \inf\{x \geq \tilde{p} : \exists\ y > x \text{ and } i \text{ s.t. } \phi_i(y) = \phi_i(x)\} \tag{72}$$

Note that R_n is the smallest value $\geq \tilde{p}$ that is the left endpoint of an interval of constancy for some $\phi_i(.)$. For this i, $\phi_i(R_n) = \phi_i(y)$ for some $y > R_n$ [12]. We must have $R_n > \tilde{p}$. This is because, if $R_n = \tilde{p}$, then $\phi_i(y) = \phi_i(\tilde{p})$. But $\phi_i(\tilde{p}) = 0$, since $\tilde{p}$ is the lower endpoint of the support set of $\phi_i(.)$ by Lemma 10. So $\phi_i(y) = 0$, which implies that the lower endpoint of the support set of $\phi_i(.)$ is $\geq y > \tilde{p}$. This contradicts Lemma 10. Thus, $R_n > \tilde{p}$.

Now, by definition of R_n, all microgrids play every subinterval in $[\tilde{p}, R_n)$ with positive probability and hence every price $x \in [\tilde{p}, R_n)$ is a best response for every microgrid. So similar to the derivation of (10), for $j \in \{1,\ldots,n\}$ and $x \in [\tilde{p}, R_n)$, $E\{u_j(x,\psi_{-j})\} = (x-c)(1-F_{-j}(x)) = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right]$. Hence, $F_{-1}(x) = \ldots = F_{-n}(x)$ and by Lemma 14,

$$\phi_1(x) = \ldots = \phi_n(x) = \phi(x) \text{ (say)},\ \tilde{p} \leq x < R_n. \tag{73}$$

which proves (6) for $j = n$.

Case (i): Suppose $R_n = v$. Then $\phi_l(R_n) = q_l$, $l = 1,\ldots,n$ (since $\psi_l(v) = 1$), which proves (7).

Case (ii): Now suppose $R_n < v$. Then $\phi_j(.), j = 1,\ldots,n$ are continuous at R_n by Lemma 8. So by (73):

$$\phi_1(R_n) = \phi_2(R_n) = \ldots = \phi_n(R_n). \tag{74}$$

Since R_n is the left endpoint of an interval of constancy of $\phi_i(.)$, by Lemma 16:

$$\phi_i(R_n) = \phi_i(v-) = \phi_n(R_n) \leq q_n \tag{75}$$

where the second equality follows from (74).

Now, suppose $i = 1$. Then by (57) and (75):

$$\phi_i(R_n) \geq q_2. \tag{76}$$

By (75), (76) and (1), $q_2 = q_3 = \ldots = q_n = \phi_i(R_n)$. Also, by (74), $\phi_j(R_n) = q_j$, $j = 2,\ldots,n$. So $\psi_j(R_n) = 1$, $j = 2,\ldots,n$. This implies, since $R_n < v$ by assumption, that at most one microgrid (microgrid 1) plays prices in the interval (R_n, v) with positive probability, which contradicts Lemma 11. Thus, $i \neq 1$.

So by Lemma 13, $\phi_i(.)$ is continuous at v and $\phi_i(v-) = \phi_i(v) = q_i$. So by (75):

$$\phi_i(R_n) = q_i. \tag{77}$$

By (74) and (77), $\phi_n(R_n) = q_i$. If $q_i > q_n$, then $\phi_n(R_n) > q_n$, which is a contradiction because $\phi_n(R_n) = q_n\psi_n(R_n) \leq q_n$. So $q_i \leq q_n$. Also, since $q_i \geq q_n$ by (1), $q_i = q_n$. So:

$$\phi_n(R_n) = q_n. \tag{78}$$

which proves (7) for $j = n$.

Now, as induction hypothesis, suppose there exist thresholds:

$$\tilde{p} < R_n \leq R_{n-1} \leq \ldots \leq R_{i+1} \leq v$$

such that for each $j \in \{i+1,\ldots,n\}$, $\phi_j(R_j) = q_j$,

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x),\ \tilde{p} \leq x < R_j, \tag{79}$$

[12]Note that $\phi_i(.)$ is a distribution function and hence is right continuous [11]. So $\phi_i(R_n+) = \phi_i(R_n)$.

and each of microgrids $1,\dots,j$ plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability.

First, suppose $R_{i+1} < v$. Let:

$$R_i = \inf\{x \geq R_{i+1} : \exists\ y > x \text{ and } j \in \{1,\dots,i\} \text{ s.t. } \phi_j(y) = \phi_j(x)\}.$$

If $R_i = R_{i+1}$, then clearly by (79):

$$\phi_1(x) = \dots = \phi_i(x) = \phi(x),\ \tilde{p} \leq x < R_i \tag{80}$$

which proves (6) for $j = i$. Also, similar to (78), it can be shown that $\phi_i(R_i) = q_i$, which proves (7) for $j = i$ and completes the inductive step. Now suppose $R_i > R_{i+1}$. Then similar to the proof of (73), it can be shown that:

$$\phi_1(x) = \dots = \phi_i(x) = \phi(x),\ R_{i+1} \leq x < R_i. \tag{81}$$

By (79) and (81):

$$\phi_1(x) = \dots = \phi_i(x) = \phi(x),\ \tilde{p} \leq x < R_i.$$

which proves (6) for $j = i$. Also, similar to the proof of (78), it can be shown that $\phi_i(R_i) = q_i$, which proves (7) for $j = i$. This completes the induction.

If $R_{i+1} = v$, then the induction is completed by simply setting $R_1 = \dots = R_i = v$.

It remains to show that $R_1 = R_2 = v$. If $R_1 < v$, then no microgrid plays a price in (R_1, v), which contradicts Lemma 11. So $R_1 = v$. If $R_2 < v$, then only microgrid 1 plays prices in (R_2, v) with positive probability, which again contradicts Lemma 11. So $R_2 = v$. ■

Now, Lemma 13 showed that if $q_1 > q_2$, then $\phi_1(.)$ has a jump of size *at most* $q_1 - q_2$ at v. The following lemma shows that the size of the jump is in fact *exactly* $q_1 - q_2$.

Lemma 18: If $q_1 > q_2$, then $\phi_1(.)$ has a jump of size $q_1 - q_2$ at v.

Proof: By Lemma 17, $\phi_1(x) = \phi_2(x)$ for all $x < R_2 = v$. So:

$$\begin{aligned}\phi_1(v-) &= \phi_2(v-)\\ &= \phi_2(v) \quad \text{(since } \phi_2(.) \text{ is continuous by Lemma 13)}\\ &= q_2\end{aligned}$$

Also, $\phi_1(v) = q_1\psi_1(v) = q_1$. So $\phi_1(v) - \phi_1(v-) = q_1 - q_2$. ■

Finally, (i) Property 1 follows from Lemmas 8, 13 and 18; and (ii) Theorem 1 follows from Properties 1 and 2 and Lemma 17.

B. Proofs of results in Section III-B

Proof of Lemma 1: Since each of the $n-1$ events in the definition of $f_i(y)$ results in deficit w.p. s, we get:

$$P(K_{-1} = k) = \binom{n-1}{k} s^k (1-s)^{n-1-k}. \tag{82}$$

Let $v_{k,l_1,l_2}(q_{i+1},\dots,q_n,s)$ be the probability that out of the $n-i$ events with success probabilities $q_{i+1},\dots,q_n$ in the definition of $f_i(.)$, exactly l_1 result in deficit and l_2 result in success given that $K_{-1} = k$. Also, let $h_{k,l_1,l_2}(y)$ be the probability that out the $i-1$ events with success probability y each in the definition of $f_i(.)$, $k - l_2$ or more result in success given that exactly $k - l_1$ result in deficit. Now, given that exactly $k - l_1$ events result in deficit, the remaining $(i-1)-(k-l_1)$ events do not result in deficit, and hence the probability that each of these results in success is $\frac{y}{1-s}$. So:

$$h_{k,l_1,l_2}(y) = \sum_{l_3=k-l_2}^{i-1-k+l_1} \binom{i-1-k+l_1}{l_3} \left(\frac{y}{1-s}\right)^{l_3} \times \left(1 - \frac{y}{1-s}\right)^{i-1-k+l_1-l_3} \tag{83}$$

Also:

$$\begin{aligned} f_i(y) &= \sum_{k,l_1,l_2} P(K_{-1} = k) v_{k,l_1,l_2}(q_{i+1},\dots,q_n,s) h_{k,l_1,l_2}(y)\\ &= \sum_{k,l_1,l_2} \binom{n-1}{k} s^k (1-s)^{n-1-k} \times\\ &\quad v_{k,l_1,l_2}(q_{i+1},\dots,q_n,s) h_{k,l_1,l_2}(y). \end{aligned} \tag{84}$$

where the second step follows from (82). Now, $h_{k,l_1,l_2}(.)$ in (83) is a polynomial function of y and hence continuous in y. Also, it is a strictly increasing function of y [32]. So by (84), $f_i(y)$ is a strictly increasing and continuous function of y. ■

Proof of Lemma 2: It can be checked from the definition of the function $f_i(.)$ (see Definition 3) that:

$$f_i(q_{i+1}) = f_{i+1}(q_{i+1}). \tag{85}$$

Also, replacing i with $i+1$ in (14), we get:

$$f_{i+1}(q_{i+1}) = g(R_{i+1}). \tag{86}$$

By (85) and (86), we get:

$$f_i(q_{i+1}) = g(R_{i+1}). \tag{87}$$

Now, by Lemma 1, $f_i(.)$ is invertible. By (19), $\phi(.)$ is unique and is given by:

$$\phi(x) = f_i^{-1}(g(x)),\ R_{i+1} \leq x < R_i. \tag{88}$$

Also, by (87) and (14), $f_i(q_{i+1}) = g(R_{i+1})$ and $f_i(q_i) = g(R_i)$. So $f_i(.)$ is a continuous one-to-one map from the compact set $[q_{i+1}, q_i]$ onto $[g(R_{i+1}), g(R_i)]$, and hence $f_i^{-1}(.)$ is continuous (see Theorem 4.17 in [29]). Also, $g(x)$ in (12) is continuous for all $x \in [\tilde{p}, v)$ since $x \geq \tilde{p} > c$. So from (88), $\phi(.)$ is a continuous function on $[R_{i+1}, R_i]$, since it is the composition of continuous functions f_i^{-1} and g (see Theorem 4.7 in [29]). Also, by Lemma 1, $f_i(.)$ is strictly increasing; so $f_i^{-1}(.)$ is strictly increasing. Also, it follows from (12) that $g(.)$ is strictly increasing. By (88), $\phi(.)$ is the composition of the strictly increasing functions $f_i^{-1}(.)$ and $g(.)$ and hence is strictly increasing on $[R_{i+1}, R_i]$. Also, by (87) and (88), $\phi(R_i) = f_i^{-1}(g(R_i)) = q_i$.

Thus, the function $\phi(.)$ is strictly increasing and continuous within each individual interval $[R_{i+1}, R_i]$; also, $\phi(R_i) = q_i$, $i = 2,\dots,n$, and hence $\phi(.)$ is continuous at the endpoints $R_i, i = 2,\dots,n$ of these intervals. So $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, v)$.

It remains to show that $\phi(\tilde{p}) = 0$. By definition of the function $f_i(.)$, $f_n(0) = (1-s)^{n-1}$. As shown above, $f_n(.)$ is one-to-one. So $f_n^{-1}((1-s)^{n-1}) = 0$. Also, by (12), $g(\tilde{p}) = (1-s)^{n-1}$; also, recall that $R_{n+1} = \tilde{p}$. Putting $i = n$ and $x = R_{n+1} = \tilde{p}$ in (88), we get $\phi(\tilde{p}) = f_n^{-1}(g(\tilde{p})) = f_n^{-1}((1-s)^{n-1}) = 0$. ■

Proof of Theorem 2: By Lemma 2 and equation (8), the functions $\phi_i(.)$, $i = 1, \ldots, n$ computed in Section III-B are continuous and non-decreasing on $[\tilde{p}, v]$; also, $\phi_i(\tilde{p}) = 0$ and $\phi_i(v) = q_i$. This is consistent with the fact that $\phi_i(.)$ is the d.f. of the pseudo-price p'_i and hence should be non-decreasing and right continuous [11], and $\phi_i(v) = q_i\psi_i(v) = q_i$ (see the beginning of Section III).

Now, we have shown in Sections III-A and III-B that (8) is a necessary condition for the functions $\phi_i(.), i = 1, \ldots, n$ to constitute a NE. We now show sufficiency. Suppose for each $i \in \{1, \ldots, n\}$, microgrid i uses the strategy $\phi_i(.)$ in (8). Similar to the derivation of (10), the expected payoff that microgrid i gets at a price $x \in [\tilde{p}, v)$ is:

$$E\{u_i(x, \psi_{-i})\} = (x - c)(1 - F_{-i}(x)). \quad (89)$$

Now, for $x \in [\tilde{p}, R_i)$, by (8), $\phi_i(x) = \phi_1(x) = \phi(x)$, and hence by Lemma 14, $F_{-i}(x) = F_{-1}(x)$. By (10), (89) and the fact that $F_{-i}(x) = F_{-1}(x)$, for microgrid i, prices $x \in [\tilde{p}, R_i)$ fetch an expected payoff of $(\tilde{p} - c)\left[1 - (1-s)^{n-1}\right]$.

Now let $x \in [R_i, v)$. Note that $R_i \leq x < v = R_1$. So by (8), $\phi_i(x) = q_i$ and $\phi_1(x) = \phi(x) \geq \phi(R_i) = q_i$ by Lemma 2. So $\phi_1(x) \geq \phi_i(x)$. Hence, by Lemma 14, $F_{-1}(x) \leq F_{-i}(x)$, which by (10) and (89) implies $E\{u_i(x, \psi_{-i})\} \leq (\tilde{p} - c)\left[1 - (1-s)^{n-1}\right]$.

Finally, note that a price below $\tilde{p}$ fetches a payoff of less than $(\tilde{p} - c)\left[1 - (1-s)^{n-1}\right]$ for microgrid i. So each price in $[\tilde{p}, R_i)$ is a best response for microgrid i; also, by (8), it randomizes over prices only in this range under $\phi_i(.)$. So $\phi_i(.)$ is a best response. Thus, the functions $\phi_i(.), i = 1, \ldots, n$ constitute a NE. ■

C. Proofs of results in Section IV

Proof of Theorem 3: First, we show that the functions $\psi_1(.)$, $\psi_2(.)$ and $\psi_3(.)$ in (25), (26) and (27) are valid d.f.s. It can be easily checked that $\psi_2(.)$ and $\psi_3(.)$ are continuous everywhere and $\psi_1(.)$ is continuous everywhere except possibly at v. Also, $\psi_1(v-) \leq 1$ iff (21) holds, which is true by assumption. If $\psi_1(v-) < 1$, then $\psi_1(.)$ has a jump at v. Since $\psi_1(x) = 1$ for $x \geq v$, $\psi_1(.)$ is right continuous at v. Also, with $F(.)$ as in (24), $F'(x) = \frac{\tilde{p}-c}{(x-c)^2} > 0$ for $x \in [\tilde{p}, v]$ and hence $F(.)$ is strictly increasing on $[\tilde{p}, v]$. So by (25), (26) and (27), $\psi_1(.)$, $\psi_2(.)$ and $\psi_3(.)$ are non-decreasing. Thus, $\psi_1(.)$, $\psi_2(.)$ and $\psi_3(.)$ are non-decreasing and right continuous, and hence are valid d.f.s [11].

Now, note that under the strategies $\psi_1(.)$, $\psi_2(.)$ and $\psi_3(.)$ in (25), (26) and (27), microgrid 3 (respectively, microgrids 1 and 2) play every sub-interval in the range $[\tilde{p}, R_3)$ (respectively, $[\tilde{p}, v)$) with positive probability and microgrid 1 can have a jump at v. The microgrids do not set prices other than these. In the rest of the proof, we will show that microgrid 1 (respectively, 2, 3) gets an expected payoff of $u_{1,max}$ (respectively, $u_{2,max}$, $u_{3,max}$) at a price $x \in [\tilde{p}, v]$ (respectively, $x \in [\tilde{p}, v)$, $x \in [\tilde{p}, R_3)$) and a payoff less than or equal to $u_{1,max}$ (respectively, $u_{2,max}$, $u_{3,max}$) at every other price. It will follow that in the strategy profile in (25), (26) and (27), every microgrid randomizes only over best responses and hence it is a NE.

Now, if no microgrid out of microgrids 2 and 3 has a jump at price x and microgrid 1 sets the price x, then its power is sold (i) if both of microgrids 2 and 3 have deficit power, (ii) one of them has deficit power and the other has neither excess nor deficit or (iii) one of them has deficit power, the other has excess power and sets a price greater than x [13]. So:

$$\begin{aligned} &E\{u_1(x, \psi_{-1})\} \\ &= (x - c)\{s_2 s_3 + s_2(1 - s_3 - q_3) + s_3(1 - s_2 - q_2) \\ &\quad + s_2 q_3(1 - \psi_3(x)) + s_3 q_2(1 - \psi_2(x))\} \\ &= (x - c)(s_2 + s_3 - s_2 s_3 - s_2\phi_3(x) - s_3\phi_2(x)) \quad (90) \end{aligned}$$

Similarly,

$$\begin{aligned} &E\{u_2(x, \psi_{-2})\} \\ &= (x - c)(s_1 + s_3 - s_1 s_3 - s_1\phi_3(x) - s_3\phi_1(x)) \quad (91) \end{aligned}$$

and

$$\begin{aligned} &E\{u_3(x, \psi_{-3})\} \\ &= (x - c)(s_1 + s_2 - s_1 s_2 - s_1\phi_2(x) - s_2\phi_1(x)) \quad (92) \end{aligned}$$

Using (90), (91) and (92), $\psi_1(.)$, $\psi_2(.)$ and $\psi_3(.)$ from (25), (26) and (27) and the fact that $\phi_i(x) = q_i\psi_i(x)$, $i = 1, 2, 3$, we get $E\{u_1(x, \psi_{-1})\} = u_{1,max}$ for $x \in [\tilde{p}, v]$, $E\{u_2(x, \psi_{-2})\} = u_{2,max}$ for $x \in [\tilde{p}, v)$ and $E\{u_3(x, \psi_{-3})\} = u_{3,max}$ for $x \in [\tilde{p}, R_3)$, where $u_{1,max}$, $u_{2,max}$ and $u_{3,max}$ are as in (28), (29) and (30).

Next, we show that microgrid 3's expected payoff at a price $x \in (R_3, v)$ is $\leq u_{3,max}$. The value of $\phi_1(x)$ is given by (25) along with the fact that $\phi_1(x) = q_1\psi_1(x)$. So:

$$\phi_1(x) = \frac{1}{s_3}\{(s_1 + s_3 - s_1 s_3)F(x) - s_1 q_3\}, \ x \in (R_3, v) \quad (93)$$

Let:

$$\tilde{\phi}_1(x) = \left(1 - \frac{s_1}{2}\right)F(x), \ x \in (R_3, v) \quad (94)$$

By (93) and (94), on $x \in (R_3, v)$:

$$\phi_1(x) - \tilde{\phi}_1(x) = s_1\left\{\left(\frac{1}{s_3} - \frac{1}{2}\right)F(x) - \frac{q_3}{s_3}\right\}. \quad (95)$$

Now, $\frac{1}{q_3}\left(1 - \frac{s_3}{2}\right)F(x) \geq 1$ because $\frac{1}{q_3}\left(1 - \frac{s_3}{2}\right)F(R_3) = \psi_3(R_3) = 1$ (by (27) and the continuity of $\psi_3(.)$), $x \geq R_3$ and $F(.)$ in (24) is an increasing function of x. So by (95):

$$\phi_1(x) \geq \tilde{\phi}_1(x), \ x \in (R_3, v). \quad (96)$$

Similarly:

$$\phi_2(x) \geq \tilde{\phi}_2(x), \ x \in (R_3, v). \quad (97)$$

where:

$$\tilde{\phi}_2(x) = \left(1 - \frac{s_2}{2}\right)F(x), \ x \in (R_3, v) \quad (98)$$

[13] Note that microgrid 1's power can also be sold if one or both of microgrids 2 and 3 set the price x, but the probability of this event is 0 by assumption.

Now, for $x \in (R_3, v)$, by (92):

$$\begin{aligned} & E\{u_3(x, \psi_{-3})\} \\ & = (x-c)[s_1 + s_2 - s_1 s_2 - s_2\phi_1(x) - s_1\phi_2(x)] \\ & \leq (x-c)[s_1 + s_2 - s_1 s_2 - s_2\tilde{\phi}_1(x) - s_1\tilde{\phi}_2(x)] \\ & \quad \text{(by (96) and (97))} \\ & = (\tilde{p}-c)(s_1 + s_2 - s_1 s_2) \text{ (by (94) and (98))} \\ & = u_{3,max} \text{ (by (30))} \end{aligned}$$

Thus, the expected payoff of microgrid 3 is $\leq u_{3,max}$ for prices in (R_3, v). The expected payoff at v is also $\leq u_{3,max}$ because the payoff at every $x < v$ is $\leq u_{3,max}$ and microgrid 1 possibly has a jump at v.

As shown above, for microgrid 2, the expected payoff at every price in $(\tilde{p}, v)$ is $u_{2,max}$ and the expected payoff at v is $\leq u_{2,max}$ since microgrid 1 possibly has a jump at v. Also, for microgrid 1, the expected payoff at every price in $[\tilde{p}, v]$ equals $u_{1,max}$.

Finally, by (90), (26), (27) and the fact that $\phi_i(x) = q_i\psi_i(x)$, $i = 2, 3$, at a price $x < \tilde{p}$, $E\{u_1(x, \psi_{-1})\} = (x-c)(s_2+s_3-s_2s_3) < (\tilde{p}-c)(s_2+s_3-s_2s_3) = u_{1,max}$ and similarly $E\{u_2(x, \psi_{-2})\} < u_{2,max}$ and $E\{u_3(x, \psi_{-3})\} < u_{3,max}$.

The result follows. ■

D. Proofs of results in Section V

The proofs are similar to the proofs (in Appendices A and B) of the results in Section III, as we now explain. Throughout, v in Appendices A and B is replaced with v_T.

First, we show that Properties 1 and 2 and Theorem 1 hold with the changes stated in Section V. Lemmas 8, 11, 9 and their proofs hold in the present context without change.

Lemma 10 holds in the present context with the change that $u_{i,max}$ is as in (35). The proof of Lemma 10 is as in Appendix A, except that now, for microgrid i, a price of $\tilde{p}$ fetches a payoff of $(\tilde{p}-c)$ if $K_{-i} \geq 1$ and $v \geq \tilde{p}$ and a payoff of 0 otherwise. So $u_{i,max} = (\tilde{p}-c)P(K_{-i} \geq 1)P(v \geq \tilde{p}) = (\tilde{p}-c)\left[1-(1-s)^{n-1}\right](1-G(\tilde{p})$. Thus, $u_{i,max}$ is given by (35).

Now, in the present context, $\tilde{p}$ is given by Lemma 3 instead of the value in Lemma 12. To prove Lemma 3, we first proceed as in the proof of Lemma 12 described in Appendix A, except that everywhere, we use the fact that if microgrid i sets a price p_i, then its power is sold only if $v \geq p_i$, which results in an additional factor of $(1-G(p_i))$ in the expression for microgrid i's expected payoff. Thus proceeding, we conclude that $\tilde{p}$ must satisfy (34). It remains to show that (34) has a unique solution in (c, v_T). Note that

$$h(c) = 0 < \frac{h(v_T)(1-w_1)}{1-(1-s)^{n-1}}. \tag{99}$$

Let N_{-i} and K_{-i} be as defined in Section III. Clearly, the event $\{K_{-1} = 0\}$ is a strict subset of the event $\{N_{-1} \geq K_{-1}\}$. So $w_1 = P(N_{-1} \geq K_{-1}) > P(K_{-1} = 0) = (1-s)^{n-1}$. Hence:

$$h(v_T) > \frac{h(v_T)(1-w_1)}{1-(1-s)^{n-1}}. \tag{100}$$

By Assumption 1, $h(.)$ is continuous. So by (99), (100), and the intermediate value theorem [29], (34) has a solution in (c, v_T). Also, since $h(.)$ is strictly increasing in (c, v_T) by Assumption 1, this solution is unique, which completes the proof of Lemma 3.

Finally, Lemmas 15, 14, 13, 16, 17 and 18 go through unchanged and their proofs are similar to those in Appendix A. This completes the proof of the fact that Properties 1 and 2 and Theorem 1 hold with the changes stated in Section V.

Next, we prove Lemma 4. It can be easily checked using the definition of the function $f_i(.)$ that $f_i(q_i) \geq (1-s)^{n-1}$. Hence:

$$h(\tilde{p}) \leq \frac{h(\tilde{p})\left[1-(1-s)^{n-1}\right]}{1-f_i(q_i)} \tag{101}$$

Also, using the definitions of $f_i(.)$ and w_1 and equation (1), it can be checked that $f_i(q_i) \leq w_1$. So:

$$\begin{aligned} h(v_T) & \geq \frac{h(v_T)(1-w_1)}{1-f_i(q_i)} && (102) \\ & = \frac{h(\tilde{p})\left[1-(1-s)^{n-1}\right]}{1-f_i(q_i)} && (103) \end{aligned}$$

where (103) follows from (102) by Lemma 3. By Assumption 1, $h(.)$ is continuous. So by (101), (103), and the intermediate value theorem [29], (39) has a solution in $[\tilde{p}, v_T]$. Also, since $h(.)$ is strictly increasing in $[\tilde{p}, v_T]$ by Assumption 1, this solution is unique, which completes the proof of Lemma 4.

Finally, the proof of the fact that Lemma 2 holds in the present context with the changes stated in Section V is similar to the proof of Lemma 2 in Appendix B. Also, the proof of the fact that Theorem 2 goes through in the present context with the changes stated in Section V is similar to the proof of Theorem 2.