Complexity Analysis, Potential Game Characterization and Algorithms for the Inter Cell Interference Coordination with Fixed Transmit Power Problem

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***Abstract*—We study the Inter Cell Interference Coordination (ICIC) problem in a multi-cell OFDMA based cellular network employing universal frequency reuse. In each cell, only a subset of the available subchannels are allocated to mobile stations (MS) in a given time slot so as to limit the interference to neighboring cells; also, each base station (BS) uses a fixed transmit power on every allocated subchannel. The objective is to allocate the available subchannels in each cell to the MSs in the cell for downlink transmissions taking into account the channel qualities from BSs to MSs as well as traffic requirements of the MSs so as to maximize the weighted sum of throughputs of all the MSs. First, we show that this problem is NP-Complete. Next, we show that when the potential interference levels to each MS on every subchannel are above a threshold (which is a function of the transmit power and the channel gain to the MS from the BS it is associated with), the problem can be optimally solved in polynomial-time via a reduction to the matching problem in bipartite graphs. We also formulate the ICIC problem as a non-cooperative game, with each BS being a player, and prove that although it is an ordinal potential game in two special cases, it is *not* an ordinal potential game in general. Also, we design two heuristic algorithms for the general ICIC problem: a greedy distributed algorithm and a simulated annealing (SA) based algorithm. The distributed algorithm is fast and requires only message exchanges among neighboring BSs. The SA algorithm is centralized and allows a tradeoff between quality of solution and execution time via an appropriate choice of parameters. Our extensive simulations show that the total throughput obtained using the better response (BR) algorithm, which is often used in game theory, is very small compared to those obtained using the SA and greedy algorithms; however, the execution time of the BR algorithm is much smaller than those of the latter two algorithms. Finally, the greedy algorithm outperforms the SA algorithm in dense cellular networks and requires only a small fraction of the number of computations required by the latter algorithm for execution.**

Keywords*—Cellular Networks, Inter Cell Interference Coordination, Complexity, Algorithms, Potential Game

I. Introduction

Modern cellular systems, including those based on the 3GPP Long Term Evolution Advanced (LTE-A) [9] and IEEE 802.16 [13] standards, are based on Orthogonal Frequency Division Multiple Access (OFDMA) technology, and are often deployed with universal frequency reuse, wherein the entire available spectrum is reused in every cell. In addition, dense deployments of a large number of small cells are often used to enhance capacity resulting in high inter cell interference [20]. Thus, avoidance of inter cell interference is an important challenge [1] in these networks.

Interference avoidance techniques can be broadly classified into (i) static schemes, wherein the available frequency band is divided into sub-bands, with different sub-bands being statically assigned to different cells (the frequency reuse factor is greater than 1 and possibly fractional), and (ii) dynamic schemes, wherein frequency resources are assigned to different cells in real-time [16]. An important class of dynamic schemes, which are the focus of this paper, are Inter Cell Interference Coordination (ICIC) techniques, in which different base stations (BS) coordinate among themselves and restrict the transmit power used on a subset of the OFDMA subcarriers in each cell so as to limit interference to neighboring cells [16].

In particular, the available spectrum is divided into groups of consecutive OFDMA subcarriers, each of which is called a *subchannel* and constitutes the basic unit of resource allocation in a time slot [29]. The ICIC problem is to decide what restrictions on transmit power to apply on various subchannels in different cells, taking into account the current traffic requirements of mobile stations (MS) in each cell as well as channel qualities (gains) from BSs to MSs in the current time slot, so as to achieve some objective such as maximization of sum of throughputs of MSs across the network, fairness etc [16], [29]. Two variants of this problem have been widely studied in the literature: (i) a BS may select any transmit power level from a range of power levels on a given subchannel and allocate the subchannel to any MS in its cell [26], [30], and (ii) each BS uses a *fixed* transmit power level on each subchannel that is allocated to a MS and 0 transmit power on the other subchannels, so that the problem becomes that of deciding, for each subchannel, which MS to allocate it to, if any, in each cell [5], [18], [29]. We refer to problem (i) (respectively, (ii)) as the ICIC with variable (respectively, fixed) transmit power problem. Although the ICIC with variable transmit power model allows a more flexible allocation than the ICIC with fixed transmit power model, the latter is simpler, easier to implement, and its performance loss can be negligible relative to the former especially for dense deployments of BSs [14], [16], [31]. Hence, we focus on the ICIC with fixed transmit power problem in this paper.

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[1]Although it is expected that in 5G cellular networks, *mmWave* spectrum will be used, on which communication will take place using highly directional antennas, which reduces the amount of inter-cell interference, it is likely that lower-frequency bands will continue to be used in the future (*e.g.,* to achieve wide coverage, support high mobility users etc), on which a large amount of inter-cell interference can potentially take place [2].

The novel contributions of this paper are as follows:

- We provide a proof of NP-Completeness of the ICIC with fixed transmit power problem with the objective of maximizing the weighted sum of throughputs of all the MSs in the network.
- We show that when the potential interference levels to each MS on every subchannel are above a threshold, the ICIC with fixed transmit power problem can be optimally solved in polynomial time.
- We formulate the ICIC problem as a non-cooperative game with each BS being a player, and show that although two special cases of this game are ordinal potential games, the game is *not* an ordinal potential game in general, which is surprising given the results in prior work [1], [6], [8], [12], [34]–[36].
- We design two low complexity heuristic subchannel allocation algorithms for the ICIC problem– a distributed greedy algorithm and a simulated annealing (SA) based algorithm– and evaluate the performance of these two algorithms and the better response (BR) algorithm, which is often used in game theory, via extensive simulations.

We now describe our contributions in detail. First, from the results in [11], [22], it follows that the ICIC problem with the objective of maximizing the weighted sum of throughputs of MSs is NP-hard under the variable transmit power model. Intuitively, the ICIC with fixed transmit power problem seems to be a simpler problem than the ICIC with variable transmit power problem since the transmit powers that each BS uses on different subchannels are decision variables in the latter problem, but are constants in the former problem. In prior work, the similarity of the ICIC with fixed transmit power problem to various NP-hard problems (*e.g.*, $3D$ matching [29]) has been pointed out, or it has been noted that the problem is a special case of the integer programming problem that is in general NP-hard [5], [18]. However, note that this *does not* imply that the ICIC with fixed transmit power problem is NP-hard. To the best of our knowledge, a formal proof of NP-hardness has not been provided in prior work. In this paper, we provide a rigorous proof of the fact that the *ICIC with fixed transmit power problem with the objective of maximizing the weighted sum of throughputs of all MSs in the network is NP-Complete*. In fact, we show that even the special case of this problem wherein there is only one subchannel and only one MS associated with each BS is NP-Complete (see Section IV-A). This result and its proof provide insight into the structure of the ICIC with fixed transmit power problem, which would be useful to future work on the design of approximation algorithms with a guaranteed approximation ratio for the problem.

Next, we show that when the potential interference levels to each MS on every subchannel are above a threshold (which is a function of the transmit power and the channel gain to the MS from the BS it is associated with), the ICIC with fixed transmit power problem can be *optimally solved in polynomial-time* via a reduction to the matching problem in bipartite graphs (see Section IV-B). The above threshold turns out to be a decreasing function of the transmit power that each BS uses on a subchannel allocated to one of its MSs, and approaches 0 as the transmit power goes to ∞. Equivalently, if the transmit power is above a threshold, which is a function of the channel gains *(which can be arbitrary),* the ICIC with fixed transmit power problem can be optimally solved in polynomial-time. This shows that if sufficiently high transmit powers are used, then the ICIC with fixed transmit power problem is solvable in polynomial-time, which is a surprising result given the fact that this problem is NP-complete in general (see the previous paragraph).

Now, since the general ICIC with fixed transmit power problem is NP-Complete, we seek efficient algorithms to obtain an approximate solution of this problem. Game theory [23] is a promising tool which was used to design distributed algorithms for solving various versions of the ICIC problem in prior literature (see Section II). Specifically, the problem was formulated as a *non-cooperative game* in which each BS is a *player* and an *action* of a BS is an allocation of subchannels to MSs associated with it [23]. Distributed iterative algorithms were devised to obtain a pure strategy Nash equilibrium (PSNE) [23] of the game, which often approximately maximizes the objective function [6], [8], [34]–[36]. The iterative best/ better-response dynamics are frequently used in game theory to reach a PSNE [6], but they are not guaranteed to converge in general. Nonetheless, they are guaranteed to converge for specific classes of games, such as potential games [25]. It has been proved in [1], [6], [8], [12], [34]–[36] that the subchannel allocation game corresponding to the ICIC problem with different objective functions is a potential game and therefore, the simple better response (BR) algorithm can be used to find a PSNE of the game, which is an approximate solution of the ICIC problem (see Section II). In most of the above papers, the objective function (cost) is a measure of the total inter-cell interference in the network; however, note that this objective function does not directly map to the Quality of Service (QoS) experienced by the MSs. Also, frequency selective channel gains are not considered in most of the above papers. On the other hand, in this paper, the objective function is the weighted sum of throughputs of all the MSs in the network, which directly maps to the QoS they experience; also, our model includes frequency selective channel gains (see Section III). In Section V, we study the subchannel allocation game corresponding to the ICIC with fixed transmit power problem with the objective of maximizing the weighted sum of throughputs of all the MSs in the network. To the best of our knowledge, the question of whether this game is a potential game has not been resolved in prior work; we show that it is *not* a potential game in general, which is surprising given the results in [1], [6], [8], [12], [34]–[36]. Nonetheless, we show that some special cases of the above game are potential games. Since the above game is not a potential game in general, the BR algorithm is not guaranteed to converge to a PSNE. We study the convergence probability of the BR algorithm to a PSNE through simulations in Section VII-A. Our simulation results show that the convergence probability of the BR algorithm to a PSNE decreases as the number of MSs, BSs or subchannels is increased.

The fact that the BR algorithm is not guaranteed to converge to a PSNE motivates us to seek other algorithms to solve the ICIC with fixed transmit power problem with the objective of maximizing the weighted sum of throughputs of all the MSs in the network. Accordingly, we design two low complexity heuristic subchannel allocation algorithms for this problem– a distributed greedy algorithm and a simulated annealing based algorithm (see Section VI). The greedy algorithm is fast and requires direct connections among only neighboring BSs (over which messages are exchanged [2] during the execution of the algorithm). Also, it is flexible in that it can be implemented irrespective of how many other BSs a given BS is directly connected to; our simulation results show that the performance of the algorithm improves as the numbers of directly connected BSs of different BSs in the network increase (see Section VII-B). The simulated annealing (SA) based algorithm is centralized and can be implemented if a central entity (*e.g.*, radio network controller [18]) to which all the BSs in the network are directly connected is available. This algorithm allows a tradeoff between quality of the obtained solution and execution time by means of an appropriate choice of parameters. Also, we compare the performance of the BR algorithm, greedy algorithm and SA algorithm via simulations in Section VII-C. Our simulation results show that the total throughput obtained by the BR algorithm is very small compared to those obtained using the SA and greedy algorithms; however, the execution time of the BR algorithm is much smaller than those of the latter two algorithms. Finally, the greedy algorithm outperforms the SA algorithm and uses only a small fraction of the number of computations in dense cellular networks (see Section VII-C).

The rest of the paper is organized as follows. Section II provides a review of related research literature. We describe the system model and problem definition in Section III. The complexity of the problem defined in Section III is analyzed in Section IV. In Section V, the ICIC problem is formulated as a non-cooperative game and the question of whether this game is a potential game is investigated. In Section VI, we propose two heuristic algorithms for the ICIC problem. Section VII provides performance evaluation results obtained via simulations, and we conclude the paper in Section VIII.

II. Related Work

We now briefly review prior literature on the ICIC problem; see [16], [20] for detailed surveys. The ICIC with variable transmit power problem is considered in [26], [30], [33]. However, as explained in Section I, *our model differs from the ICIC with variable transmit power model in [26], [30], [33] since the transmit power on each allocated subchannel is assumed to be fixed.* The ICIC with fixed transmit power problem was studied in [5], [18], [29]. A centralized graph-based algorithm for the downlink allocation problem is presented in [5]. In [18], a two-level scheme is developed, in which the algorithm for radio resource allocation is partly run at a central controller and partly at BSs. In [29], a two-level scheme that seeks to improve the throughput of cell-edge users is presented. *However, in all of the above papers [5], [18], [29], some or all of the allocation decisions are taken at a central controller that is connected to all the base stations; our greedy and BR algorithms do not require the availability of such a controller.*

A distributed ICIC scheme to maximize the critical and overall performance of a multi-cell system was proposed in [17]. The throughput maximization problem using interference avoidance through ICIC was formulated as a complex optimization problem. This optimization problem was decomposed into two sub-problems, *viz.*, user allocation and inter-cell restriction, which are solved at eNodeB (eNB) level and central controller level respectively. A linear programming relaxation by rounding the solution to the nearest binary value was used to solve the inter-cell restriction sub-problem. *Since a network controller is required, the proposed scheme in [17] is not a fully distributed scheme. In contrast, our greedy and BR algorithms are fully distributed because no such network controller is required.* A multi-cell resource allocation problem to maximize the weighted sum rate of user terminals using ICIC was studied in [3]. A relaxed linear programming problem corresponding to this problem was formulated. A primal-decomposition based scheme was proposed to decompose the formulated problem into a master problem and multiple sub-problems. The master problem was solved using projected-subgradient method. The other sub-problems were transformed into an equivalent minimum-cost network flow (MCNF) optimization problem and network flow based algorithms were used to solve them. In the proposed algorithm, each user terminal reports the channel gains from its associated BS and from its first tier interfering BSs to its associated BS. Also, in each iteration of the proposed scheme, each BS needs to exchange the N subgradient values obtained by primal-decomposition and N binary variable values with its neighboring BSs, where N is the number of resource blocks. The ICIC with fixed transmit power problem was considered in [3], [17]. However, *no proof was provided to show that this problem is NP-hard. In contrast, a rigorous proof of the fact that the ICIC with fixed transmit power problem is NP-complete is provided in this paper.* A joint optimization of user BS association and subchannel and transmit power allocation to different users for ICIC in the downlink of heterogeneous networks was studied in [7]. A factor graph model was used to decompose the overall optimization problem into multiple local maximization problems. A belief propagation (BP) algorithm was proposed to solve the local maximization problems by converting them into marginal distribution estimation problems. In each iteration of the BP algorithm, each BS exchanges the mean and variance of the variable to be optimized with its neighboring BSs. *To reduce the computational complexity, a Gaussian approximation was used to calculate the intercell interference. In contrast, we have used exact intercell interference without any approximation.*

A joint user association and downlink interference management problem in two-tier heterogeneous networks was studied in [19]. An optimization problem to maximize the network utility over user association and resource partitioning was formulated and an algorithm was proposed to obtain an integer solution. *The algorithm can be implemented in a centralized or semi-distributed manner. In contrast, our proposed greedy algorithm is completely distributed.* In [24], a scheduler

[2] Note that under our proposed greedy, simulated annealing based and BR algorithms, some channel gain and other information is exchanged among neighboring BSs or between the BSs and a central controller. This is because if no coordination among different BSs were used, then there would be a large amount of inter-cell interference resulting in poor network performance. Hence, most algorithms proposed in prior research literature to solve the ICIC problem (*e.g.,* those in [1], [3], [7], [8], [12], [24], [32], [35]) also require the channel gain information and other information to be shared among different BSs.

that performs dynamic fractional frequency reuse (FFR) while maximizing the sum throughput of the cell users in an LTE-A network was proposed. In each scheduling interval, vulnerable user protection bitmaps are exchanged between an eNodeB and its neighboring eNodeBs to avoid interference. *In contrast to dynamic FFR, we consider universal frequency reuse in this paper.* A centralized and a distributed dynamic ICIC algorithms were proposed for LTE-A networks in [32]. The expected resource usage of a BS needs to be shared for taking muting decisions. In the centralized scheme, BSs exchange this information with the central controller while in the distributed scheme, this information is exchanged among BSs.*In the model in [32], the transmit powers are decision variables, whereas it is fixed in our model.*

Recently, the problem of allocation of subchannels to mitigate inter-cell interference has been extensively studied using the theory of *potential games* [25]. The ICIC problem in the downlink of an OFDMA cellular system was addressed in [1]. The problem was modelled as a non-cooperative game in which BSs are the players, selecting a subset of the available resource blocks (RBs) is the strategy of a player and the cost function of a BS is the total delay experienced by the BS to transmit data over the assigned RBs to its associated MSs. This game was shown to be an exact potential game. Two iterative distributed algorithms were proposed to reach a pure strategy Nash equilibrium (PSNE). *Potential game theory was used in [6], but it was in the context of database-assisted white-space spectrum sharing not ICIC.* The problem was modelled as a non-cooperative game with each access point as a player; selecting a subset of channels from the available set of TV channels was the strategy of a player and downlink throughput was the utility of a player. It was shown that the above non-cooperative channel selection game is a potential game and hence has a PSNE and the best response update algorithm was used to find a PSNE. In [8], a distributed channel allocation algorithm was proposed for the ICIC problem. A channel allocation game was formulated with each BS as a player, the set of orthogonal frequency channels as the action set of a player and the utility function of a player as the negative of the total interference experienced by the player. The game was proved to be an exact potential game. A deterministic best response and a probabilistic better response algorithm were proposed to obtain a PSNE of the game. In [34], downlink interference avoidance by distributed channel selection in heterogeneous small cell networks was studied. The channel selection problem was modelled as a stochastic game with each femtocell BS as a player, the selection of a subset of the set of available channels as the strategy of the player and the negative of the total interference received by a player from other femtocell BSs and the macro BS as the utility function of the player. The game was proved to be an exact potential game. A fully distributed, no-regret learning algorithm was used to obtain a PSNE of the game. To mitigate the inter-cell interference, a novel BS coordination method was proposed in [35]. The problem of network sum-throughput maximization through intercell interference mitigation was formulated as a local cooperation game with each BS as a player. The strategy of a player consists of joint subchannel and power allocation in its own cell. Mean opinion score (MOS), a new metric, was the utility function of a player. The above game was shown to be an exact potential game and a γ-logit based decentralized iterative algorithm was proposed to obtained a PSNE of the game. In each iteration of the proposed algorithm, each BS exchanges information with its neighboring BSs to calculate its current utility. A channel selection problem for interference mitigation in spectrum access networks was formulated as a stochastic game in [36]. Each communication node was a player and the available set of channels was the action set of each player. The expected utility function of a player was defined to be a constant minus the interference experienced by the player. The game was proved to be an exact potential game. A stochastic learning automata (SLA) based dynamic channel selection algorithm was proposed to find a PSNE of the game. *However, it was assumed that only BSs that are within a certain range of a given BS can cause interference to users of that BS. In contrast, in our work interference is not range limited,* i.e.*, a BS at any distance causes interference to another BS that is using the same subchannel.* A semi-static ICIC scheme was proposed in [12], which involves two algorithms: intercell primary subchannel self-configuration and intracell resource allocation. The primary subchannel coordination problem was formulated as a non-cooperative game with each femtocell access point (FAP) as a player; selecting a subset of subchannels from the available set of subchannels was the strategy of a player and the sum of the interference caused to its neighboring FAPs and the received interference was the utility of a player. The game was shown to be an exact potential game. *Frequency selective channel gains were not considered in [1], [6], [8], [35]. In contrast, the model in this paper takes into account frequency selective channel gains. Also, the utility function of a player (BS) in the channel allocation game was considered to be the total time delay experienced by the BS to transmit the data in [1], the negative of the total interference received at the BS in [6], [12], [34], [36] and MOS in [35]. In contrast, we have considered the total throughput of all the MSs associated with a BS to be its utility function.* We show that *the game in this paper is not a potential game in general*, which is surprising, since the games in [1], [6], [8], [12], [34]–[36], which are similar, are potential games [3].

III. Model and Problem Definition

Consider a multicell OFDMA based cellular network and let $\mathcal{B}$ denote the set of all base stations (BS). The set of all OFDMA subcarriers is divided into groups of consecutive subcarriers referred to as *subchannels*, which have equal bandwidth, and are used as the basic units for resource allocation. Let $\mathcal{N} = \{1, \dots, N\}$ be the set of all available subchannels. We assume that universal frequency reuse is used; thus, a given subchannel in $\mathcal{N}$ may potentially be used by any subset of the BSs in $\mathcal{B}$ simultaneously. Let [4] $\mathcal{M}_a$ be the set of mobile stations (MS) associated with BS $a \in \mathcal{B}$, $|\mathcal{M}_a| = M_a$, $\mathcal{M} = \cup_{a \in \mathcal{B}} \mathcal{M}_a$ be the set of all MSs and $M = \sum_{a \in \mathcal{B}} M_a$ be the total number of MSs in the network.

[3]Note that *within the class of subchannel allocation games corresponding to the ICIC problem with different objective functions*, the property of being a potential game is a common characteristic, and hence our proof that the subchannel allocation game corresponding to the ICIC with fixed transmit power problem with the objective of maximizing the weighted sum of throughputs of all the MSs in the network is not a potential game is surprising.

[4]$|A|$ denotes the cardinality of a set A.

For concreteness, we consider the problem of allocating subchannels to MSs for downlink transmissions in a given time slot. We model the traffic requirements of different MSs in the time slot under consideration as follows. Let $w_{a,j}$, $a \in \mathcal{B}, j \in \mathcal{M}_a$ denote the *weight* of MS j associated with BS a. It is a non-negative number and is a measure of the priority of the downlink traffic intended for MS j. Weights may be assigned to MSs so as to achieve various objectives, *e.g.*: (i) MSs to which delay-sensitive traffic (*e.g.*, voice, video) is to be sent may be assigned higher weights than those to which elastic traffic (*e.g.*, file transfer) is to be sent, (ii) weights may be selected to be proportional to the lengths of data packet queues waiting to be transmitted to MSs, (iii) weights may be assigned so as to achieve fairness across MSs by assigning high weights to MSs that have received low bandwidth in the recent time slots and vice-versa.

Let $z^n_{a,j}$ equal 1 if subchannel n is allocated to MS $j \in \mathcal{M}_a$ and 0 otherwise and $\mathbf{Z} = \{z^n_{a,j} : a \in \mathcal{B}, j \in \mathcal{M}_a, n \in \mathcal{N}\}$ denotes the overall allocation. Let

$$y^n_a = \sum_{j \in \mathcal{M}_a} z^n_{a,j}. \tag{1}$$

We assume that within each cell a, a given subchannel n is allocated to at most one MS so as to avoid intra-cell interference; thus, we get the constraint:

$$y^n_a \in \{0, 1\}, \ \forall a \in \mathcal{B}, n \in \mathcal{N}. \tag{2}$$

Note that y^n_a equals 1 if BS a assigns subchannel n to one of its MSs and 0 otherwise.

If $z^n_{a,j} = 1$ for some $a \in \mathcal{B}, j \in \mathcal{M}_a, n \in \mathcal{N}$, then the BS a uses a *fixed* transmit power P on subchannel n; otherwise, the transmitted power is 0. Let N_0 be the noise power per subchannel. We assume that the bandwidth of each subchannel $n \in \mathcal{N}$ is smaller than the channel coherence bandwith, so that fading is approximately flat on each subchannel. Let $H^n_{a,j}$ be the channel gain from BS a to MS j on subchannel n in the time slot under consideration [5]. The channel gains $\{H^n_{i,j} : i \in \mathcal{B}, j \in \mathcal{M}, n \in \mathcal{N}\}$ can be estimated using cell-specific reference signals that are orthogonal across cells [16] and their knowledge can be used in the resource allocation process.

We define a *feasible allocation* to be an allocation $\mathbf{Z} = \{z^n_{a,j} : a \in \mathcal{B}, j \in \mathcal{M}_a, n \in \mathcal{N}\}$ that satisfies (1) and (2). Also, we define the *utility* of the network under a feasible allocation $\mathbf{Z}$ to be the weighted sum of throughputs of MSs:

$$U(\mathbf{Z}) = \sum_{a \in \mathcal{B}} \sum_{j \in \mathcal{M}_a} \sum_{n \in \mathcal{N}} z^n_{a,j} w_{a,j} \log \left(1 + \frac{P H^n_{a,j}}{P \sum_{i \in \mathcal{B}, i \neq a} H^n_{i,j} y^n_i + N_0} \right). \tag{3}$$

Note that as a normalization, the bandwidth of each subchannel is taken to be unity and the throughput of each MS $j \in \mathcal{M}_a$ is assumed to be the Shannon capacity of the channel from a to j. Let $\mathcal{Z}$ denote the set of all feasible allocations. Our objective is to find a utility-maximizing feasible allocation $\mathbf{Z}$:

Problem 1: Find an allocation $\mathbf{Z} \in \mathcal{Z}$ that maximizes the utility in (3).

For later use, let:

$$U_n(\mathbf{Z}) = \sum_{a \in \mathcal{B}} \sum_{j \in \mathcal{M}_a} z^n_{a,j} w_{a,j} \log \left(1 + \frac{P H^n_{a,j}}{P \sum_{i \in \mathcal{B}, i \neq a} H^n_{i,j} y^n_i + N_0} \right). \tag{4}$$

denote the contribution of subchannel n to the utility $U(\mathbf{Z})$ in (3). Note that:

$$U(\mathbf{Z}) = \sum_{n \in \mathcal{N}} U_n(\mathbf{Z}). \tag{5}$$

Remark 1: For ease of exposition, in the above formulation, we have assumed that each MS requires one subchannel per slot. Our formulation and results can be readily generalized to the case where different MSs may require different and arbitrary numbers of subchannels in a slot as follows. The number of subchannels needed by a MS, *i.e.*, the demand, is modelled with the help of "*virtual MSs*". In particular, if a MS needs $m \geq 0$ subchannels, then it is represented by m virtual MSs and one subchannel needs to be assigned to each virtual MS. Note that depending on its demand, different weights may be assigned to the virtual MSs corresponding to a given MS. However, all the virtual MSs corresponding to a given MS $j \in \mathcal{M}_a$ have the same channel gains $\{H^n_{i,j} : i \in \mathcal{B}, n \in \mathcal{N}\}$. All our results apply without change if MSs in the above formulation are replaced with virtual MSs.

IV. Complexity

A. NP-Completeness of the General Problem

The decision version of Problem 1 is as follows: given a number D, does there exist a feasible allocation $\mathbf{Z}$ such that $U(\mathbf{Z}) \geq D$? The following result shows the NP-Completeness of (the decision version of) Problem 1.

Theorem 1: Problem 1 is NP-Complete.

Proof: Given an allocation $\mathbf{Z}$, we can check in polynomial-time using (1) and (2) whether $\mathbf{Z}$ is feasible. Also, in polynomial-time, $U(\mathbf{Z})$ can be found using (3) and it can be checked whether $U(\mathbf{Z}) \geq D$. Thus, Problem 1 is in class NP [15].

Now, we show that the Maximum Independent Set [6] (MIS) problem, which is known to be NP-Complete [15], is polynomial-time reducible to Problem 1, *i.e.*, MIS $<_p$ Problem 1. Consider the following instance of the MIS problem: given a graph $G = (V, E)$ with vertex set V and edge set E and a positive integer k, does G contain an independent set of size at least k?

From this instance, we construct the following instance of Problem 1: suppose there is only one subchannel [7] ($N = 1$), $w_{a,j} = 1$ for all $a \in \mathcal{B}, j \in \mathcal{M}_a$, and every BS has 1 associated user (*i.e.*, $M_a = 1$ for all $a \in \mathcal{B}$). Corresponding to each node

[5] The duration of a time slot is selected to be some value that is less than the channel coherence time, so that channel gain values are approximately constant in a time slot.

[6] Recall that an *independent set* [15] in a graph is a set of nodes such that there is no edge between any pair of nodes in the set.

[7] Since $N = 1$, we drop the superscript n (subchannel number) in the rest of this proof for simplicity.

$u \in V$, we have a BS u (*i.e.*, $\mathcal{B} = V$). Let j_u denote the MS associated with BS u and (u, v) denote the edge between nodes u and v, $u \neq v$. The channel gains are as follows:

$$H_{u,j_u} = 1 \ \forall u \in V, \tag{6}$$

$$H_{u,j_v} = H_{v,j_u} = \begin{cases} \infty, & \text{if } (u,v) \in E \\ 0, & \text{else,} \end{cases} \quad \forall u, v \in V, v \neq u. \tag{7}$$

Let $\mathbf{Z} = \{z_{u,j_u} \in \{0,1\} : u \in V\}$ be an allocation in the above instance of Problem 1. By (1) and the fact that $M_u = 1$ for all $u \in V$, it follows that $y_u \in \{0,1\}$ for all $u \in V$; so (2) is true. Thus, every allocation $\mathbf{Z} = \{z_{u,j_u} \in \{0,1\} : u \in V\}$ is feasible in the above instance.

Next, in the above instance of Problem 1, we ask: does there exist a (feasible) allocation $\mathbf{Z}$ such that $U(\mathbf{Z}) \geq k \log\left(1 + \frac{P}{N_0}\right)$? We claim that the answer is yes if and only if there exists an independent set of size at least k in G. To show sufficiency, suppose there exists an independent set, I, of size $k' \geq k$ in G. Then the following allocation:

$$z_{u,j_u} = \begin{cases} 1, & \text{if } u \in I, \\ 0, & \text{else,} \end{cases} \tag{8}$$

has utility $k' \log\left(1 + \frac{P}{N_0}\right)$ by (3), (6) and (7), which shows sufficiency. To show necessity, let $\mathbf{Z} = \{z_{u,j_u} \in \{0,1\} : u \in V\}$ be an allocation such that:

$$U(\mathbf{Z}) \geq k \log\left(1 + \frac{P}{N_0}\right), \tag{9}$$

and $I = \{u \in V : z_{u,j_u} = 1\}$. If there is an edge between two nodes $u, v \in I$, then by (6) and (7), it follows [8] that $\log\left(1 + \frac{PH_{u,j_u}}{P\sum_{i \in V, i \neq u} H_{i,j_u} y_i + N_0}\right) = \log\left(1 + \frac{PH_{v,j_v}}{P\sum_{i \in V, i \neq v} H_{i,j_v} y_i + N_0}\right) = 0$. By this fact and by (3), it follows that:

$$U(\mathbf{Z}') = U(\mathbf{Z}), \tag{10}$$

where $\mathbf{Z}'$ is the following allocation:

$$z'_{u,j_u} = \begin{cases} 1, & \text{if } u \in I', \\ 0, & \text{else,} \end{cases} \tag{11}$$

and I' is the independent set obtained by dropping all nodes $u, v \in I$ such that there is an edge between u and v from I. Let $|I'| = k'$. Then:

$$U(\mathbf{Z}') = k' \log\left(1 + \frac{P}{N_0}\right) \tag{12}$$

by (3), (6) and (7). By (9), (10) and (12), we get $k' \geq k$. Thus, an independent set of size at least k exists in G, which shows necessity. The result follows.

[8]Note that $H_{u,j_u} = 1$ by (6). If there is an edge between u and v, *i.e.*, $(u,v) \in E$, then $H_{u,j_v} = H_{v,j_u} = \infty$ by (7). Also, since $v \in I$, $y_v = z_{v,j_v} = 1$. Hence, $\log\left(1 + \frac{PH_{u,j_u}}{P\sum_{i \in V, i \neq u} H_{i,j_u} y_i + N_0}\right) = \log\left(1 + \frac{PH_{u,j_u}}{PH_{v,j_u} y_v + P\sum_{i \in V, i \neq u,v} H_{i,j_u} y_i + N_0}\right) = \log\left(1 + \frac{P(1)}{P(\infty)(1) + P\sum_{i \in V, i \neq u,v} H_{i,j_u} y_i + N_0}\right) = 0$. Similarly, $\log\left(1 + \frac{PH_{v,j_v}}{P\sum_{i \in V, i \neq v} H_{i,j_v} y_i + N_0}\right) = 0$.

■

Our proof of Theorem 1 in fact shows that even *the special case of Problem 1 wherein there is only one subchannel, every BS has only* 1 *associated MS and all weights are unity is NP-complete [10]:*

Proposition 1: Problem 1 with $N = 1$, $M_a = 1$ for all $a \in \mathcal{B}$ and $w_{a,j} = 1$ for all a, j is NP-Complete.

B. Polynomial-Time Solution of the High Interference or Transmit Power Case

Throughout this subsection, we assume that all channel gains are positive, *i.e.*, $H^n_{i,j} > 0$ for all $i \in \mathcal{B}, j \in \mathcal{M}, n \in \mathcal{N}$. However, the gains are allowed to be arbitrarily small, so this is a mild assumption.

For BS $a \in \mathcal{B}$, MS $j \in \mathcal{M}_a$ and subchannel $n \in \mathcal{N}$, let:

$$\tau(a, j, n) = \frac{PH^n_{a,j}}{N_0}, \tag{13}$$

and

$$\beta(a, j, n) = \frac{\min_{b \in \mathcal{B} \setminus \{a\}} H^n_{b,j}}{H^n_{a,j}}. \tag{14}$$

Consider the following condition:

Condition 1:

$$\beta(a, j, n) > \frac{1}{\log(1 + \tau(a, j, n))}, \tag{15}$$

for all $a \in \mathcal{B}$, $j \in \mathcal{M}_a$ and $n \in \mathcal{N}$.

Theorem 2: When Condition 1 is satisfied, Problem 1 can be optimally solved in $O((M+N)^3)$ time using the algorithm in Fig. 1.

We now explain Condition 1 and Theorem 2. Note that for a given MS $j \in \mathcal{M}_a$ and subchannel $n \in \mathcal{N}$, $\beta(a, j, n)$ in (14) is the ratio of the smallest crosstalk coefficient $H^n_{b,j}$ across all BSs b other than the BS a to which MS j is associated to the gain, $H^n_{a,j}$, from BS a to j on channel n. Thus, $\beta(a, j, n)$ is a measure of, and increasing in, the *strength of the potential* [9] *interference to MS j on subchannel n relative to the signal from BS a*. Condition 1 requires that $\beta(a, j, n)$ be above a threshold for all a, j and n, and hence that the potential interference levels across all BSs, MSs and subchannels be sufficiently high. Thus, Theorem 2 states that *when the potential interference levels are sufficiently high, Problem 1 can be solved in polynomial-time.*

Also, note that the threshold interference level, $\frac{1}{\log(1+\tau(a,j,n))}$, in Condition 1 is decreasing in $\tau(a, j, n)$ and hence, by (13), in the transmit power P used by each BS on a subchannel allocated to one of its MSs. In particular, note that for fixed channel gains and noise level N_0, *the threshold approaches* 0 *as* $P \to \infty$. Thus, *Condition 1 becomes more relaxed as the transmit power level increases*. Also, note that *for arbitrary channel gains* $H^n_{i,j}$, $i \in \mathcal{B}$, $j \in \mathcal{M}$, $n \in \mathcal{N}$, there exists a threshold, say P_0, such that if

[9]We say "potential" interference since this interference would be experienced by MS j if subchannel n were allocated to it as well as to an MS of a different cell.

$P \geq P_0$, then Condition 1 is satisfied, and hence Problem 1 can be optimally solved in polynomial-time.

In practice, Condition 1 would hold in several of the time slots in a scenario where there is a dense deployment of BSs close to each other and high transmit powers are used and hence, by Theorem 2, Problem 1 can be optimally solved in polynomial-time using the algorithm in Fig. 1 in this case.

We prove Theorem 2 in the rest of this subsection– we now outline the proof. First, in Section IV-B1, we consider the idealized situation in which all the crosstalk coefficients are ∞, *i.e.*, the following condition holds:

Condition 2: $H_{b,j}^n = \infty$ and $H_{a,j}^n$ are finite for all $a, b \in \mathcal{B}$, $j \in \mathcal{M}_a$, $b \neq a$.

In Section IV-B1, we show that when Condition 2 holds, Problem 1 can be solved in $O((M+N)^3)$ time via a reduction to the bipartite matching problem in graphs. Now, note that Condition 1 holds whenever Condition 2 holds, but the former is a much less stringent requirement. In Section IV-B2, we show that the bipartite matching based algorithm provided in Section IV-B1 in fact optimally solves Problem 1 even when only Condition 1 holds.

1) Infinite Crosstalk Coefficients: Recall that a graph $G = (V, E)$ is said to be *bipartite* if its node set V can be partitioned into two sets A and B such that every edge in E is between a node in A and a node in B [15]. A matching T in a graph $G = (V, E)$ is defined to be a subset of the edges such that no two edges in the subset share a common node [15]. The weight of matching T is defined to be the sum of the weights of the edges it contains, *i.e.*, $W(T) = \sum_{e \in T} w_e$, where w_e is the weight of edge e [15]. The bipartite matching problem is the problem of finding a matching with the maximum weight [15].

Consider Problem 1 under Condition 2. We now reduce this problem to the bipartite matching problem. Let $\mathcal{M}$ (the set of MSs) and $\mathcal{N}$ (the set of subchannels) be the two partitions in a bipartite graph. Consider a MS $j \in \mathcal{M}$ and a subchannel $n \in \mathcal{N}$. Suppose $j \in \mathcal{M}_a$, that is, j is associated with BS a. Then we define the weight of the edge between nodes j and n to be:

$$W(j, n) = w_{a,j} \log\left(1 + \frac{PH_{a,j}^n}{N_0}\right). \tag{16}$$

Let $\mathcal{Z}^1 \subseteq \mathcal{Z}$ denote the set of all feasible allocations in which each subchannel $n \in \mathcal{N}$ is assigned to at most one MS, that is $\sum_{i \in \mathcal{B}} y_i^n \leq 1,\ \forall n \in \mathcal{N}$. Given a matching T in the above bipartite graph, consider a corresponding allocation $\mathbf{Z}(T)$ given by:

$$z_{a,j}^n = \begin{cases} 1, & \text{if } (j, n) \in T, \\ 0, & \text{else.} \end{cases} \tag{17}$$

Since T is a matching, it follows that $\mathbf{Z}(T) \in \mathcal{Z}^1$. Also, given any $\mathbf{Z}^1 \in \mathcal{Z}^1$, there exists a unique matching T such that $\mathbf{Z}(T) = \mathbf{Z}^1$. Thus, $\mathbf{Z}(T)$ is a one-to-one correspondence between the set $\mathcal{Z}^1$ and the set of all matchings in the above bipartite graph. Also, by (16) and (3), the weight of a matching equals the utility of the corresponding allocation, *i.e.*, $W(T) = U(\mathbf{Z}(T))$. Hence, if T^* is a maximum weight matching, then the allocation $\mathbf{Z}(T^*)$ is the allocation with the highest utility in $\mathcal{Z}^1$.

It remains to show that $\mathbf{Z}(T^*)$ is, in fact, the allocation with the highest utility in $\mathcal{Z}$. To this end, note that if in an allocation $\mathbf{Z} \in \mathcal{Z}$, a subchannel n is allocated to two or more MSs, then $U_n(\mathbf{Z}) = 0$ by (4) and Condition 2. Hence, if $\mathbf{Z}^1 \in \mathcal{Z}^1$ is the allocation obtained from $\mathbf{Z}$ by deallocating each subchannel n that was allocated to two or more MSs in $\mathbf{Z}$ from all the MSs to which it was allocated, then $U(\mathbf{Z}^1) = U(\mathbf{Z})$ by (5). So it follows that under Condition 2, there exists a utility-maximizing allocation in $\mathcal{Z}^1$.

Thus, when Condition 2 holds, the allocation $\mathbf{Z}(T^*)$ corresponding to the maximum weight matching T^* in the above bipartite graph optimally solves Problem 1. A maximum weight matching in a bipartite graph with k nodes can be found in $O(k^3)$ time using the Hungarian algorithm [28]. Hence, the matching T^* can be found in $O((M+N)^3)$ time.

Fig. 1 summarizes the algorithm for optimally solving Problem 1 under Condition 2.

1: Consider the complete bipartite graph with partitions $\mathcal{M}$ and $\mathcal{N}$, and for $j \in \mathcal{M}_a \subseteq \mathcal{M}$, $n \in \mathcal{N}$, weight of edge (j, n) given by (16).
2: Find a maximum weight matching T^* in this graph using the Hungarian algorithm.
3: Return the allocation $z_{a,j}^n = \begin{cases} 1, & \text{if } (j, n) \in T^*, \\ 0, & \text{else.} \end{cases}$

Fig. 1: The algorithm for optimally solving Problem 1 under Condition 1.

2) Finite Crosstalk Coefficients: Recall from Section IV-B1 that the algorithm in Fig. 1 finds an allocation $\mathbf{Z} \in \mathcal{Z}^1$ that maximizes the utility in (3). To prove Theorem 2, consider Problem 1 under Condition 1. Below, we start from an arbitrary given feasible allocation $\mathbf{Z} \in \mathcal{Z}$, and from it, construct an allocation $\mathbf{Z}^1 \in \mathcal{Z}^1$ by deallocating some MSs from one or more subchannels if necessary, such that $U(\mathbf{Z}^1) \geq U(\mathbf{Z})$. From this, it will follow that there exists an allocation in $\mathcal{Z}^1$ that maximizes the utility in (3), which will prove Theorem 2. Note that by (5), to show that $U(\mathbf{Z}^1) \geq U(\mathbf{Z})$, it suffices to show that:

$$U_n(\mathbf{Z}^1) \geq U_n(\mathbf{Z})\ \forall n \in \mathcal{N}. \tag{18}$$

Given $\mathbf{Z} = \{z_{a,j}^n : a \in \mathcal{B}, j \in \mathcal{M}_a, n \in \mathcal{N}\} \in \mathcal{Z}$, fix a subchannel n that is allocated to at least one MS in $\mathbf{Z}$ and let:

$$(a^*, j^*) = \underset{a \in \mathcal{B}, j \in \mathcal{M}_a : z_{a,j}^n = 1}{\operatorname{argmax}} w_{a,j} \times \log\left(1 + \frac{PH_{a,j}^n}{P\sum_{i \in \mathcal{B}\setminus\{a\} : y_i^n = 1} H_{i,j}^n + N_0}\right) \tag{19}$$

Note that (a^*, j^*) is the (BS, MS) pair that has the largest contribution to $U_n(\mathbf{Z})$ in the RHS of (4). Suppose k_n MSs are allocated subchannel n in the allocation $\mathbf{Z}$, *i.e.*:

$$k_n = |\{i \in \mathcal{B} : y_i^n = 1\}|. \tag{20}$$

If $k_n \geq 2$, then in order to get the allocation $\mathbf{Z}^1$ from $\mathbf{Z}$, we deallocate all MSs other than j^* from subchannel n [10]. By (4),

[10] Similar deallocations of MSs from subchannels other than n are performed.

(19) and (20), we get:

$$U_n(\mathbf{Z}) \leq k_n w_{a^*,j^*} \log\left(1 + \frac{PH^n_{a^*,j^*}}{P\sum_{i\in\mathcal{B}\setminus\{a^*\}:y_i^n=1} H^n_{i,j^*} + N_0}\right)$$
$$\leq k_n w_{a^*,j^*} \log\left(1 + \frac{PH^n_{a^*,j^*}}{(k_n-1)PH^n_{b,j^*} + N_0}\right) \quad (21)$$
$$\text{where, } H^n_{b,j^*} = \min_{i\in\mathcal{B}\setminus\{a^*\}} H^n_{i,j^*}. \quad (22)$$

By (21), (22), (13) and (14), we get:

$$U_n(\mathbf{Z}) \leq k_n w_{a^*,j^*} \log\left(1 + \frac{\tau(a^*,j^*,n)}{(k_n-1)\beta(a^*,j^*,n)\tau(a^*,j^*,n)+1}\right). \quad (23)$$

Now, we have the following technical result, whose proof is relegated to our technical report [10] due to space constraints:

Lemma 1: Let τ and β be positive numbers such that $\beta > \frac{1}{\log(1+\tau)}$ and:

$$f(x) = x\log\left(1 + \frac{\tau}{(x-1)\beta\tau+1}\right). \quad (24)$$

Then $f(x) \leq f(1)$ for all real numbers $x \geq 1$.

By (23), Condition 1 and Lemma 1, we get:

$$U_n(\mathbf{Z}) \leq w_{a^*,j^*} \log\left(1+\tau(a^*,j^*,n)\right) = U_n(\mathbf{Z}^1),$$

where the equality follows from (4), (13) and the fact that j^* is the only MS to which subchannel n is allocated in $\mathbf{Z}^1$. This proves (18) and completes the proof of Theorem 2.

V. Potential Game Characterization

Our objective is to design distributed algorithms, in which each BS separately allocates subchannels to its associated MSs, for finding an approximate solution to Problem 1. To this end, we formulate the problem as a non-cooperative game [23] in which each BS is a player. The concept of a Nash equilibrium (NE) [23] is widely used as a solution concept in the analysis of non-cooperative games. Also, in prior work, for several optimization problems that are similar to Problem 1, it has been shown that a NE of the corresponding game approximately maximizes or minimizes the objective function, *e.g.*, the negative sum of interference received by all the BSs is approximately maximized in [8] and the cost function, which is the total time delay experienced by a BS to transmit data, is approximately minimized in [1]. Hence, instead of seeking an allocation that maximizes the utility in (3), we seek a NE of the game corresponding to Problem 1; specifically, we seek a pure strategy Nash equilibrium (PSNE) [23], which is one in which each player (BS) selects a single strategy (allocation of subchannels to its associated MSs) with probability 1. Next, we address the question of how to efficiently find a PSNE: recall that if a game is a potential game [25], then the better response (BR) algorithm [6] can be used to efficiently converge to a PSNE. Also, several related games have been shown to be potential games in [1], [6], [8], [12], [34]–[36]. Hence, we investigate whether the game corresponding to Problem 1 is a potential game. Surprisingly, it turns out that the game is *not* a potential game in general, in contrast to the results in [1], [6], [8], [12], [34]–[36]; we provide a proof of this fact. Also, we show that some special cases of the game are potential games.

A. Problem Formulation

Recall that a *game* is any situation in which multiple individuals called *players* interact with each other, such that each player's welfare depends on the *strategies* of the others [23]. The *utility* of a player in a game is a numerical measure of its satisfaction level [23]. In our context, each BS in the set $\mathcal{B}$ is a player and let $K = |\mathcal{B}|$. A strategy of BS i is an allocation of subchannels to MSs associated with it. Let $z_i = \{z^n_{i,j} : j \in \mathcal{M}_i, n \in \mathcal{N}\}$ denote the strategy of BS i, where $z^n_{i,j}$ is as defined in Section III. We refer to the vector $\mathbf{S} = \{z_1, \ldots, z_K\}$ of strategies of the players as a *strategy profile* [23]. Also, recall that the objective of Problem 1 is to find a feasible allocation that maximizes the weighted sum of throughputs of all the MSs in the network. Hence, $u_a(\mathbf{S})$, the utility of player (BS) a, is defined to be the weighted sum of throughputs of the MSs associated with BS a under the strategy profile $\mathbf{S}$. That is:

$$u_a(\mathbf{S}) = \sum_{j\in\mathcal{M}_a}\sum_{n\in\mathcal{N}} z^n_{a,j} w_{a,j} \log\left(1 + \frac{PH^n_{a,j}}{P\sum_{i\in\mathcal{B},i\neq a} H^n_{i,j} y^n_i + N_0}\right) \quad (25)$$

We impose upper limits $Q_1, \ldots, Q_K$, where Q_a is the maximum number of MSs to which BS a is allowed to allocate subchannels. These upper limits are necessary since in their absence, each BS, being a selfish player in the above game formulation, would allocate subchannels to as many of its associated MSs as possible (specifically, to $\min(N, M_a)$ MSs [11]), resulting in high inter-cell interference and hence low total throughput. In Section VII-A, we investigate as to what values of these upper limits approximately maximize the total throughput achieved under the BR algorithm.

B. Analysis of Game

In this section, we will investigate whether the game formulated in Section V-A is a potential game.

Definition 1: (Ordinal Potential Game [25]) Consider a game in which $\mathcal{B}$ is the set of players, $K = |\mathcal{B}|$, z_i denotes the strategy of player i and $u_i(\cdot)$ is the utility function of player i. The game is an ordinal potential game if there exists a function $\phi(.)$ such that for every player $a \in \mathcal{B}$, for every $\mathbf{S} = \{z_1, \ldots, z_a, \ldots, z_K\}$ and $\mathbf{S}' = \{z_1, \ldots, z'_a, \ldots, z_K\}$,

$$\mathrm{sgn}(\phi(\mathbf{S}') - \phi(\mathbf{S})) = \mathrm{sgn}(u_a(\mathbf{S}') - u_a(\mathbf{S}))$$

where,

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

The function $\phi(.)$ is called a *potential function* of the game.

[11] Note that in the absence of the upper limits, from (25), the utility of BS a can be maximized only if it allocates subchannels to $\min(N, M_a)$ MSs.

Consider the special case of the game formulated in Section V-A in which there is only one MS per BS, *i.e*, $M_a = 1$ $\forall a \in \mathcal{B}$; let $\mathcal{G}_1$ denote this special case game. For each $a \in \mathcal{B}$, let j_a denote the MS associated with BS a. Suppose that under the strategy profile $\mathbf{S}$, for each $a \in \mathcal{B}$, the MS j_a is assigned subchannel $n_a \in \mathcal{N}$. For simplicity of notation, $\forall a \in \mathcal{B}$, let w_a and $H_a^{n_a}$ denote the weight of the MS j_a and the channel gain from BS a to the MS j_a on subchannel n_a respectively. Then, (25) simplifies to the following:

$$u_a(\mathbf{S}) = w_a \log\left(1 + \frac{PH_a^{n_a}}{P \sum_{i \in \mathcal{B}, i \neq a} H_{i,j_a}^{n_a} I_{(n_i = n_a)} + N_0}\right) \quad (26)$$

where,

$$\mathrm{I}_{(n_i=n_a)} = \begin{cases} 1, & \text{if } n_i = n_a, \\ 0, & \text{else.} \end{cases}$$

For convenience, consider the utility function:

$$u'_a(\mathbf{S}) = \frac{PH_a^{n_a}}{P \sum_{i \in \mathcal{B}, i \neq a} H_{i,j_a}^{n_a} I_{(n_i = n_a)} + N_0}, \quad (27)$$

and let $\mathcal{G}'_1$ denote the game $\mathcal{G}_1$, with the change that the utility function of BS a is $u'_a(\cdot)$ instead of $u_a(\cdot)$.

Lemma 2: The game $\mathcal{G}_1$ is an ordinal potential game *if and only if* the game $\mathcal{G}'_1$ is an ordinal potential game. Also, the function $\phi(\cdot)$ is a potential function of the game $\mathcal{G}_1$ *if and only if* it is a potential function of the game $\mathcal{G}'_1$.

Proof: This follows from Definition 1 and the fact that $g(x) = \alpha \log(1+x)$, where α is a positive scalar, is a monotone increasing function, due to which:

$$\operatorname{sgn}(u_a(\mathbf{S}') - u_a(\mathbf{S})) = \operatorname{sgn}(u'_a(\mathbf{S}') - u'_a(\mathbf{S})).$$

■

Now, using Lemma 2, it is proved in Theorems 3 and 4 below that the game $\mathcal{G}_1$ is an ordinal potential game in two different special cases. Let

$$\phi_1(\mathbf{S}) = -\sum_{i \in \mathcal{B}} \sum_{k \in \mathcal{B}, k \neq i} H_{i,j_k}^{n_i} I_{(n_i = n_k)} \quad (28)$$

and

$$\phi_2(\mathbf{S}) = -\sum_{i \in \mathcal{B}} \sum_{k \in \mathcal{B}, k \neq i} \frac{H_{i,j_k}^{n_i}}{H_k^{n_k}} I_{(n_i = n_k)}. \quad (29)$$

Theorem 3: If $H_a^n = H_a, \forall a \in \mathcal{B},\ n \in \mathcal{N}$ and $H_{i,j_a}^n = H_{a,j_i}^n, \forall a, i \in \mathcal{B},\ i \neq a$, and $\forall n \in \mathcal{N}$, then the game $\mathcal{G}_1$ is an ordinal potential game with potential function $\phi_1(\mathbf{S})$, which is as in (28).

Theorem 4: Let $N_0 = 0$. If $H_a^n = H^n$, $\forall a \in \mathcal{B},\ n \in \mathcal{N}$, and $H_{i,j_a}^n = H_{a,j_i}^n, \forall a, i \in \mathcal{B},\ i \neq a$, and $\forall n \in \mathcal{N}$, then the game $\mathcal{G}_1$ is an ordinal potential game with potential function $\phi_2(\mathbf{S})$, which is as in (29).

The proofs of Theorems 3 and 4 are provided in the Appendix. Note that the condition $H_a^n = H_a, \forall a \in \mathcal{B}, \forall n \in \mathcal{N}$, in Theorem 3 would hold when fading is not frequency-selective. Also, the condition $N_0 = 0$ in Theorem 4 would be a good approximation in an interference-limited system. However, the condition $H_{i,j_a}^n = H_{a,j_i}^n, \forall a, i \in \mathcal{B},\ i \neq a$, and $\forall n \in \mathcal{N}$, in Theorems 3 and 4, and the condition $H_a^n = H^n, \forall a \in \mathcal{B}$, $\forall n \in \mathcal{N}$, in Theorem 4 would typically not be satisfied in practice. We now show that in the general case where each BS has more than one MS with frequency selective channel gains, the above game is not an ordinal potential game.

Theorem 5: The game formulated in Section V-A is in general not an ordinal potential game.

We provide a proof of Theorem 5 in the rest of this sub-section. The proof is based on the *better response (BR)* algorithm, which is a distributed iterative algorithm and is often used in game theory [25]. This algorithm, when applied to the game formulated in Section V-A, operates as follows. The initial allocation of subchannels to MSs in each BS, *i.e.*, the initial strategy profile, is selected randomly. Next, at each iteration $t \in \{1, 2, 3, \ldots\}$, any one of the K BSs, say BS a_t, finds a new candidate strategy (*i.e.*, allocation of subchannels to its own MSs) at random, and changes its strategy to the candidate strategy only if it results in an increase in its utility; otherwise the strategy of BS a_t is not changed [25]. Also, the strategies of the BSs other than BS a_t do not change at iteration t. Formally, let $\mathbf{S}_0$ be the initial strategy profile, and for $t \in \{1, 2, 3 \ldots\}$, let $\mathbf{S}_t = \{z_{1,t}, \ldots, z_{K,t}\}$ denote the strategy profile after t iterations. Let $z'_{a_t,t}$ be the candidate strategy selected by BS a_t at iteration t and let $\mathbf{S}'_t = \{z_{1,t-1}, \ldots, z_{a_t-1,t-1}, z'_{a_t,t}, z_{a_t+1,t-1}, \ldots, z_{K,t-1}\}$. Then:

$$\mathbf{S}_t = \begin{cases} \mathbf{S}'_t, & \text{if } u_{a_t}(\mathbf{S}'_t) > u_{a_t}(\mathbf{S}_{t-1}), \\ \mathbf{S}_{t-1}, & \text{else.} \end{cases}$$

Note that the BR algorithm can be implemented in a distributed manner as follows: whenever a BS, a_t, changes its strategy at some iteration t, it sends its new strategy, $z_{a_t,t}$, to all the other BSs in the network, which thereby have the required information to execute iteration $t+1$.

Definition 2: (Cycle [25]) If the present strategy profile is equal to one of the previous strategy profiles at some iteration during an execution of the BR algorithm, then a cycle is said to exist in the game. That is, there exists a cycle if $\mathbf{S}_{t_1} = \mathbf{S}_{t_2}$ for some $t_1 < t_2$.

Lemma 3: If there exists a cycle, then the game is not an ordinal potential game.

Proof: Since there exists a cycle, $\mathbf{S}_t = \mathbf{S}_{t_r}$ for some non-negative integers t and $t_r > t$ during an execution of the BR algorithm. Suppose the strategy profile changes from $\mathbf{S}_t$ to $\mathbf{S}_{t_1}$ at iteration t_1, from $\mathbf{S}_{t_1}$ to $\mathbf{S}_{t_2}$ at iteration t_2,..., from $\mathbf{S}_{t_{r-1}}$ to $\mathbf{S}_{t_r}$ at iteration t_r, where $t < t_1 < t_2 < \ldots < t_{r-1} < t_r$. Let $p_k \in \mathcal{B}$ be the BS that changes its strategy at iteration t_k for each $k = 1, \ldots, r$. Then $u_{p_1}(\mathbf{S}_{t_1}) > u_{p_1}(\mathbf{S}_t)$ and $u_{p_k}(\mathbf{S}_{t_k}) > u_{p_k}(\mathbf{S}_{t_{k-1}})$ for $k = 2, \ldots, r$. Now, suppose the game is an ordinal potential game with potential function $\phi(.)$. Then by Definition 1:
$u_{p_1}(\mathbf{S}_{t_1}) - u_{p_1}(\mathbf{S}_t) > 0 \Rightarrow \phi(\mathbf{S}_{t_1}) - \phi(\mathbf{S}_t) > 0.$
Similarly for each $k = 2, \ldots, r$:
$u_{p_k}(\mathbf{S}_{t_k}) - u_{p_k}(\mathbf{S}_{t_{k-1}}) > 0 \Rightarrow \phi(\mathbf{S}_{t_k}) - \phi(\mathbf{S}_{t_{k-1}}) > 0.$

By adding the above r equations, we get:

$$\phi(\mathbf{S}_{t_1})-\phi(\mathbf{S}_t)+\phi(\mathbf{S}_{t_2})-\phi(\mathbf{S}_{t_1})+\ldots+\phi(\mathbf{S}_{t_r})-\phi(\mathbf{S}_{t_{r-1}}) > 0$$
$$\Rightarrow \phi(\mathbf{S}_{t_r}) - \phi(\mathbf{S}_t) > 0. \quad (30)$$

However, $\mathbf{S}_t = \mathbf{S}_{t_r} \Rightarrow \phi(\mathbf{S}_{t_r}) = \phi(\mathbf{S}_t)$, which is a contradiction. The result follows. ∎

Next, we provide an example in which a cycle exists. By this example and Lemma 3, Theorem 5 follows.

In this example, there are 3 BSs, $\{a, b, c\}$, and a total of 9 MSs. Two MSs with indices $\{3, 6\}$, four MSs with indices $\{5, 7, 8, 9\}$ and three MSs with indices $\{1, 2, 4\}$ are associated with BS a, b and c respectively. There are a total of 5 subchannels numbered $1, 2, 3, 4, 5$. The channel gain values are provided in Table I. Specifically, in each 5-tuple vector $H_{i,j}$, where $i \in \{a, b, c\}$ and $j \in \{1, 2 \ldots, 9\}$, in Table I, the n^{th} element represents the channel gain [12] from BS i to the j^{th} MS over subchannel $n \in \{1, 2, 3, 4, 5\}$. The power P, N_0 and each weight, $w_{a,j}$, are assumed to be 8, 7.224 $\times 10^{-15}$ and 1 respectively. The BR algorithm was executed in the above example network. Fig. 2 shows the existence of a cycle formed by the strategy profiles $\mathbf{S}_{t_0+1}, \ldots, \mathbf{S}_{t_0+38} = \mathbf{S}_{t_0+1}$ at 38 consecutive iterations, where t_0 is a constant positive integer. The first row in each box denotes a strategy profile $\mathbf{S}_{t_0+i}$ $\forall i = 1, \ldots, 37$. In the box of the strategy profile $\mathbf{S}_{t_0+i}$, the entries in the three parentheses in the second row and the third row represent [13] the subchannel allocations to the MSs of BSs a, b and c and the corresponding utilities of BSs a, b and c respectively under the strategy profile $\mathbf{S}_{t_0+i}$. For example, in the first box, the first row $\mathbf{S}_{t_0+1}$ denotes a strategy profile, the second row $(4, 5), (4, 2, 1, 3), (1, 5, 4)$ represents the subchannel allocations and third row $(36.24), (114.86), (16.78)$ represents the utilities of BSs a, b and c respectively under the strategy profile $\mathbf{S}_{t_0+1}$. Also, on each arrow in the figure, the BS that changes its subchannel allocation during the corresponding strategy profile update is indicated. For example, in Fig. 2, the strategy profile changes from $\mathbf{S}_{t_0+1}$ to $\mathbf{S}_{t_0+2}$ due to change in the allocation of BS 'c' only from $(1, 5, 4)$ to $(2, 3, 5)$ and the utility of BS 'c' increases from 16.78 to 17.24. From Fig. 2, we can see that $\mathbf{S}_{t_0+1} = \mathbf{S}_{t_0+38}$, i.e., *the strategy profile at iteration* $t_0 + 38$ *is equal to that at iteration* $t_0 + 1$. Hence, by Definition 2, there exists a cycle in the above game. Therefore, the above game is not an ordinal potential game by Lemma 3. This completes the proof of Theorem 5.

Although the above game is not an ordinal potential game in general, this does not imply that the BR algorithm never converges to a PSNE. We study the convergence probability of the BR algorithm, that is, the fraction of times it converges to a PSNE, via simulations in Section VII-A.

Remark 2: Recall that the BR algorithm requires some information to be exchanged among BSs. However, to be able to apply the BR algorithm to solve the ICIC problem, it needs to be formulated as a non-cooperative game since in each iteration t of the BR algorithm, the BS a_t changes its strategy from $z_{a_t,t}$ to $z'_{a_t,t}$ only if strategy $z'_{a_t,t}$ constitutes a better response of BS a_t to the strategies of the other BSs, *i.e.*, provides a higher utility, than strategy $z_{a_t,t}$. Note that the players, strategies and utilities need to be identified to be able to apply the BR algorithm.

TABLE I: 5-tuple Channel gain vectors from each BS to each MS corresponding to 5 subchannels

$H_{\mathbf{a},\mathbf{1}} = [\,4.04, 3.22, 1.56, 3.65, 1.45] \times 10^{-6}$	$H_{\mathbf{b},\mathbf{1}} = [\,23.5, 6.6, 44.0, 10.0, 9.2] \times 10^{-9}$	$H_{\mathbf{c},\mathbf{1}} = [\,1.5, 3.32, 0.95, 0.3, 0.43] \times 10^{-9}$
$H_{\mathbf{a},\mathbf{2}} = [\,0.02, 0.04, 0.04, 0.06, 0.09]$	$H_{\mathbf{b},\mathbf{2}} = [\,71.5, 42.2, 36.1, 36.3, 16.2]$	$H_{\mathbf{c},\mathbf{2}} = [\,2.2, 4.1, 1.1, 1.8, 6.2] \times 10^{-4}$
$H_{\mathbf{a},\mathbf{3}} = [\,1.03, 1.84, 0.36, 3.97, 1.23] \times 10^{-6}$	$H_{\mathbf{b},\mathbf{3}} = [\,10.3, 7.82, 10, 7.1, 4.23] \times 10^{-10}$	$H_{\mathbf{c},\mathbf{3}} = [\,3.3, 5.8, 6.7, 2.06, 3.17] \times 10^{-10}$
$H_{\mathbf{a},\mathbf{4}} = [\,2.08, 0.93, 1.20, 0.18, 0.46] \times 10^{-6}$	$H_{\mathbf{b},\mathbf{4}} = [\,5.33, 1.9, 1.14, 2.3, 2.4] \times 10^{-8}$	$H_{\mathbf{c},\mathbf{4}} = [\,0.03, 0.016, 0.007, 0.02, 0.05]$
$H_{\mathbf{a},\mathbf{5}} = [\,2.03, 1.16, 1.31, 2.24, 2.67] \times 10^{-11}$	$H_{\mathbf{b},\mathbf{5}} = [\,2.5, 0.36, 3.8, 1.1, 2.5] \times 10^{-4}$	$H_{\mathbf{c},\mathbf{5}} = [\,6.84, 3.22, 5.83, 5.6, 6.63] \times 10^{-18}$
$H_{\mathbf{a},\mathbf{6}} = [\,20, 30, 1.5, 17, 54] \times 10^{-5}$	$H_{\mathbf{b},\mathbf{6}} = [\,4.72, 2.53, 9.66, 16, 8.84] \times 10^{-5}$	$H_{\mathbf{c},\mathbf{6}} = [\,5.4, 2.86, 1.65, 1.2, 2.9] \times 10^{-11}$
$H_{\mathbf{a},\mathbf{7}} = [\,4.14, 1.93, 8.56, 10, 1.54] \times 10^{-6}$	$H_{\mathbf{b},\mathbf{7}} = [\,0.01, 0.01, 0.01, 0.02, 0.02]$	$H_{\mathbf{c},\mathbf{7}} = [\,2, 2, 1.76, 1.6, 3.2] \times 10^{-5}$
$H_{\mathbf{a},\mathbf{8}} = [\,2.81, 2.87, 0.84, 0.3, 2.2] \times 10^{-11}$	$H_{\mathbf{b},\mathbf{8}} = [\,10, 6, 4.76, 10.2, 2.14] \times 10^{-5}$	$H_{\mathbf{c},\mathbf{8}} = [\,0.64, 3.83, 3.9, 2.6, 4.5] \times 10^{-11}$
$H_{\mathbf{a},\mathbf{9}} = [\,8.1, 3.34, 0.72, 6.4, 3.5] \times 10^{-6}$	$H_{\mathbf{b},\mathbf{9}} = [\,4.6, 4.25, 3.82, 3.8, .91] \times 10^{-8}$	$H_{\mathbf{c},\mathbf{9}} = [\,2.1, 0.82, 1.27, 1.8, 2.6] \times 10^{-8}$

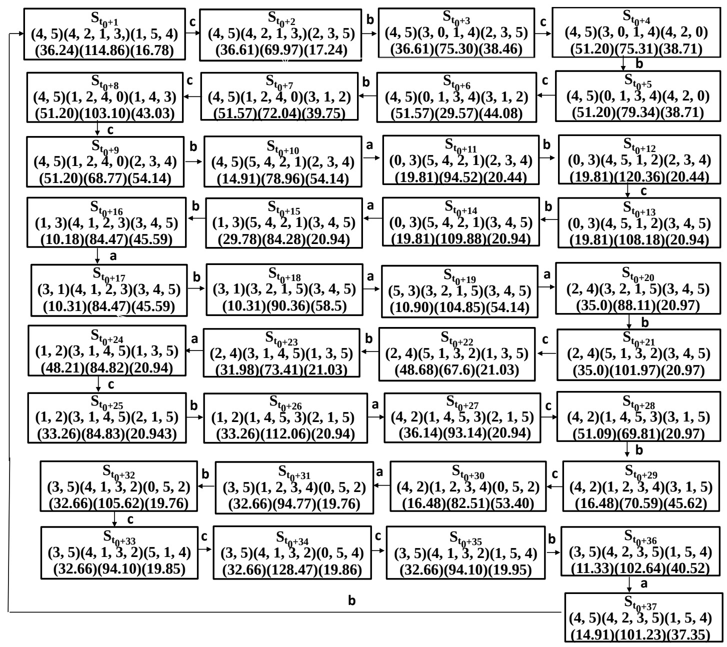


Fig. 2: The figure shows the existence of a cycle formed by the strategy profiles $\mathbf{S}_{t_0+1}, \ldots, \mathbf{S}_{t_0+37}$. Each box represents the strategy profile along with the subchannels allocated to MSs in the second row and corresponding utilities of the BSs in the third row. In each strategy profile update, the BS indicated on the arrow changes its allocation such that its utility increases.

[12]The locations of the BSs and MSs and the channel gains were generated using the procedures described in the second and third paragraphs of Section VII. In particular, each MS is associated with the BS closest to it. However, note that due to shadowing and fast fading, the channel gain to a MS from the BS it is associated with may be less than that from a BS with which it is not associated.

[13]In the second row, the element (l, m, n) for BS c can be interpreted as: the first MS of BS c is assigned subchannel l, second MS is assigned subchannel m and third MS is assigned subchannel n. Entry 0 in the place of l, m or n represents that the corresponding MS(s) is/ are not allocated any subchannel. For example in Fig. 2, the allocation of BS c in strategy profile $\mathbf{S}_{t_0+4}$ is $(4, 2, 0)$, which implies that the first MS of BS c is allocated subchannel 4, the second MS of BS c is allocated subchannel 2 and the third MS of BS c is not allocated any subchannel. The interpretations of the elements for BSs a and b are similar. The BS utilities in the third row have been calculated using (25).

VI. Algorithms

Since Problem 1 is NP-Complete, in this section, we design two heuristic algorithms– a greedy and a simulated annealing based algorithm– to solve it; these algorithms are presented in Sections VI-A and VI-B respectively.

A. Greedy Algorithm

For each BS $a \in \mathcal{B}$, let the neighbor set $\mathcal{B}_a \subseteq \mathcal{B}$ denote the set of BSs that are directly connected to BS a via high-speed links over which messages can be exchanged in real-time during the algorithm operation. For example, in LTE networks, adjacent BSs are typically connected to each other by means of X2 interfaces [9], which can be used for this purpose.

This algorithm has two phases, Initialization and Main Operation, which are explained below.

1) Initialization: Recall from Section III that all channel gains $\{H^n_{i,j} : i \in \mathcal{B}, j \in \mathcal{M}, n \in \mathcal{N}\}$ can be estimated using reference signals. In the initialization phase, each BS a exchanges messages with its associated MSs and with the BSs in $\mathcal{B}_a$ through which each BS a obtains the channel gains $\{H^n_{a,j} : j \in \mathcal{M}_a, n \in \mathcal{N}\}$ between itself and its associated MSs as well as the channel gains $\{H^n_{b,j} : j \in \mathcal{M}_a, b \in \mathcal{B}_a, n \in \mathcal{N}\}$ between its neighbouring BSs and its associated MSs [14]. Note that BS a *does not* obtain the channel gains $\{H^n_{b,j} : j \in \mathcal{M}_a, b \in \mathcal{B}\backslash\mathcal{B}_a, n \in \mathcal{N}\}$ from the BSs that are not directly connected to itself to its associated MSs. Typically in practice, only a small number of BSs would be directly connected to a BS a and hence the amount of channel gain information that would need to be shared would be small.

2) Main Operation: This phase proceeds in rounds and the variables $\{\tilde{z}^n_{a,j} : j \in \mathcal{M}_a, n \in \mathcal{N}\}$, $\tilde{y}^n_a$ and $\tilde{y}^n_b$, $b \in \mathcal{B}_a$, are maintained by each BS a during these rounds. Before the first round, each BS a sets $\tilde{z}^n_{a,j} = 0$, $\tilde{y}^n_a = 0$ and $\tilde{y}^n_b = 0$ for all $j \in \mathcal{M}_a, n \in \mathcal{N}$, $b \in \mathcal{B}_a$ and at each round: $\tilde{z}^n_{a,j}$ equals 1 if subchannel n has so far been assigned to MS $j \in \mathcal{M}_a$ and 0 otherwise and $\tilde{y}^n_a = \sum_{j \in \mathcal{M}_a} \tilde{z}^n_{a,j}$. Note that these variables contain the current values of the variables $z^n_{a,j}$ and y^n_a defined in Section III. The following actions are performed in each round $t = 1, 2, 3, \ldots$:

(1) For a BS a, let $p_a =$

$$\max_{j \in \mathcal{M}_a} \max_{n \in \mathcal{N} : \tilde{y}^n_a = 0} \left\{ w_{a,j} \log \left(1 + \frac{P H^n_{a,j}}{P \sum_{b \in \mathcal{B}_a} H^n_{b,j} \tilde{y}^n_b + N_0} \right) \right\}. \tag{31}$$

At the beginning of a round t, each BS a calculates and sends p_a to all its neighboring BSs $b \in \mathcal{B}_a$.

(2) Each BS a compares p_a with p_b for its neighbors $b \in \mathcal{B}_a$. If $p_a \geq p_b \ \forall b \in \mathcal{B}_a$, then BS a assigns subchannel n to MS j, where j and n are the maximizers in (31), and sets $\tilde{z}^n_{a,j} = 1$ and $\tilde{y}^n_a = 1$. Note that *multiple BSs may make assignments in parallel in a single round.*

(3) Each BS that has assigned a subchannel, say n, to one of its MSs in step 2 sends the value of n to all its neighboring BSs. Using this information, each BS a updates the variables $\tilde{y}^n_b$ for each $b \in \mathcal{B}_a$.

[14]Note that if each channel gain is quantized and represented using c bits, then $|\mathcal{N}||\mathcal{M}_a|c$ bits need to be sent to BS a by each BS $b \in \mathcal{B}_a$.

The above three steps are repeatedly performed by BS a until at least one of the following three conditions is satisfied:

(i) Every MS $j \in \mathcal{M}_a$ is assigned a subchannel.
(ii) Each subchannel in $\mathcal{N}$ is assigned to a MS in $\mathcal{M}_a$.
(iii) p_a becomes less than p_0, where $p_0 \geq 0$ is a parameter of the algorithm (see Remark 2 below).

Once the algorithm stops at BS a, the allocation found (stored in the variables $\tilde{z}^n_{a,j}$) is used by BS a in the current time slot.

Remarks:
1) From (31) and step (2), it can be seen that the algorithm uses a *greedy* approach, wherein during each round, *(MS, subchannel) pairs with high weights and throughputs are selected.*
2) It follows from the rule for updation of the variables $\{\tilde{y}^n_i : i \in \mathcal{B}, n \in \mathcal{N}\}$ and (31) that for each BS a, p_a decreases or remains constant at each round of the algorithm. Setting the threshold p_0 in condition (iii) above to too low a value may result in allocations with high inter cell interference; conversely, setting it to too high a value may result in too many subchannels being unallocated. The impact of p_0 on the algorithm performance is investigated via simulations in Section VII.
3) Note that the larger the sets $\mathcal{B}_a, a \in \mathcal{B}$, the more centralized the algorithm becomes; in particular, it is a completely centralized algorithm if $\mathcal{B}_a = \mathcal{B}$ for each a. Our simulation results (see Section VII) show that the performance of the algorithm improves as the sets $\mathcal{B}_a, a \in \mathcal{B}$ become larger, *i.e.*, as the numbers of directly connected neighbors of BSs increase.

B. Simulated Annealing Based Algorithm

Simulated annealing is a widely used randomized heuristic for approximating the global optimum in combinatorial optimization problems [15]. The general algorithm for approximately solving the problem of finding a utility-maximizing solution from a set, $\mathcal{Z}$, of feasible solutions is described in Fig. 3. The algorithm starts from a random solution and at each iteration, selects a random neighbor, $\mathbf{Z}'$, of the current solution $\mathbf{Z}$. If the utility of $\mathbf{Z}'$ is higher than that of $\mathbf{Z}$, then it accepts, *i.e.*, switches to $\mathbf{Z}'$. If the utility of $\mathbf{Z}'$ is lower than that of $\mathbf{Z}$, then it is still accepted with some probability (see line 8) to *avoid getting stuck at a local maximum [15].* The probability, $\exp\left(-[U(\mathbf{Z}) - U(\mathbf{Z}')]/T\right)$, with which the algorithm switches to a solution worse than the current solution depends on a parameter $T > 0$ called the *current temperature* [15]. T is initialized to a large value T_0 and decreased at each iteration of the while loop (see line 15) according to some rule called the cooling schedule [15]– thus, the probability of accepting a solution worse than the current solution is high towards the beginning of the algorithm and decreases as the algorithm proceeds.

Finally, the entire algorithm in Fig. 3 is run κ times and the best solution found across the κ runs is returned.

We adapt this algorithm to solve Problem 1 as follows. To find the random initial allocation $\mathbf{Z}_0 \in \mathcal{Z}$ (see line 1), a random integer $n_a \in \{0, 1, \ldots, \min(N, M_a)\}$ is selected for each BS $a \in \mathcal{B}$, independently across BSs, and for each BS a, Definitions:

- Let $\mathcal{Z}$ be the set of feasible solutions.
- For $\mathbf{Z} \in \mathcal{Z}$, let $U(\mathbf{Z})$ be the utility of solution $\mathbf{Z}$.
- At any time during the execution of the algorithm, let $\mathbf{Z}$ be the current solution and T be the current temperature.

Begin

```
1: Start with a random initial solution Z_0 ∈ 𝒵 and an initial temperature T = T_0.
2: Z_best ← Z_0 and Z ← Z_0.
3: while T ≥ T_f do
4:     Select a solution Z' ∈ 𝒵 randomly from the neighbors of Z.
5:     if U(Z') ≥ U(Z) then
6:         Z ← Z'
7:     else
8:         With probability exp(−[U(Z) − U(Z')]/T), let Z ← Z'; otherwise
           leave Z unchanged.
9:     end if
10:    if U(Z) ≥ U(Z_best) then
11:        Z_best ← Z.
12:    else
13:        Leave Z_best unchanged.
14:    end if
15:    Decrease T according to the cooling schedule.
16: end while
17: Return Z_best and its utility U(Z_best).
```

End

Fig. 3: The general simulated annealing algorithm [15].

n_a subchannels from $\mathcal{N}$ are randomly assigned to n_a of the MSs in $\mathcal{M}_a$.

We find a neighboring allocation $\mathbf{Z}'$ from $\mathbf{Z}$ (see line 4) as follows. A BS $a \in \mathcal{B}$ is selected at random, and one of the following actions is performed at random: (i) the assignments of two of the subchannels in $\mathcal{N}$ to MSs in $\mathcal{M}_a$ are swapped, (ii) one of the subchannels in $\mathcal{N}$ is deallocated from the MS to which it was allocated in $\mathbf{Z}$, (iii) a subchannel in $\mathcal{N}$ that was not allocated to any MS in $\mathcal{M}_a$ under $\mathbf{Z}$ is allocated to an MS. The utility $U(\mathbf{Z}')$ is required to perform the comparison in step 5 and is deduced from $U(\mathbf{Z})$. For example, suppose in $\mathbf{Z}$, subchannels n_1 and n_2 were allocated to MSs j_1 and j_2 respectively associated with BS a, and were swapped as in (i) above to get $\mathbf{Z}'$. Then by (3):

$$U(\mathbf{Z}') = U(\mathbf{Z}) - w_{a,j_1} \log\left(1 + \frac{PH^{n_1}_{a,j_1}}{P\sum_{i \in \mathcal{B}\setminus a} H^{n_1}_{i,j_1} y_i^{n_1} + N_0}\right) - w_{a,j_2} \log\left(1 + \frac{PH^{n_2}_{a,j_2}}{P\sum_{i \in \mathcal{B}\setminus a} H^{n_2}_{i,j_2} y_i^{n_2} + N_0}\right) + w_{a,j_1} \log\left(1 + \frac{PH^{n_2}_{a,j_1}}{P\sum_{i \in \mathcal{B}\setminus a} H^{n_2}_{i,j_1} y_i^{n_2} + N_0}\right) + w_{a,j_2} \log\left(1 + \frac{PH^{n_1}_{a,j_2}}{P\sum_{i \in \mathcal{B}\setminus a} H^{n_1}_{i,j_2} y_i^{n_1} + N_0}\right)$$

Note that $U(\mathbf{Z}')$ is not found using (3), but is found by updating $U(\mathbf{Z})$ as in the above example since the former is computationally expensive.

The following cooling schedule (see line 15) is used. Let $\alpha \in (0,1)$ and η, a positive integer, be parameters. Every η iterations of the while loop (lines 3 to 16), T is set to αT, *i.e.*, T is multiplicatively decreased by the factor α. The temperature remains unchanged for the iterations between consecutive multiplicative decreases.

Remark 3: It can be seen from the cooling schedule described above that the performance of each run of the algorithm improves as η and α are increased since the amount of time the algorithm spends exploring the search space increases [15]. Also, clearly the performance of the overall algorithm improves as the number of runs, κ, is increased. Thus, as η, α and κ increase, the algorithm approaches the optimal solution with a high probability (though at the expense of increased execution time).

VII. Simulations

We present three sets of simulation results in this section. In Sections VII-A and VII-B, we present performance evaluations of the BR algorithm described in Section V-B and the greedy algorithm described in Section VI-A respectively. In Section VII-C, we provide a performance comparison of the BR, greedy and simulated annealing based (SA) (see Section VI-B) algorithms.

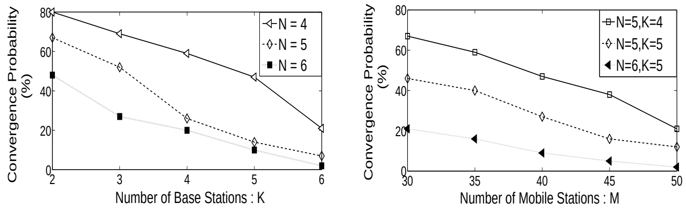
The simulation model is as follows for all the simulation results. Consider a square of dimensions 1 unit $\times$ 1 unit. We place K BSs and M MSs in the square as follows. The locations of the K BSs are selected uniformly at random in the square, while ensuring that the distance between any two BSs is at least d_{min}. Here, d_{min} is a parameter that ensures adequate separation between pairs of BSs. The locations of the M MSs are generated uniformly at random in the square. Each MS is associated with the BS closest to it. The available spectrum is divided into N subchannels.

The channel gains $H^n_{i,j}$ are generated as $H^n_{i,j} = \frac{kS_{ij}X^n_{ij}}{d^\gamma_{ij}}$, where k is a constant, d_{ij} is the distance from BS i to MS j, γ is the path loss exponent, which may be taken to be some value between 2 and 4, and S_{ij} is a log-normal random variable that models shadow fading. The random variables S_{ij} are independent and identically distributed across different pairs (i,j). X^n_{ij} is a Rayleigh distributed random variable. The random variables X^n_{ij} for different triples (i,j,n) are independent and identically distributed. Note that the channel gains take into account path loss, shadowing (slow fading) and fast fading.

A. Performance Evaluation of the BR Algorithm

1) Convergence Probability of the BR algorithm: Recall from Section V that for the game described in Section V-A, the BR algorithm is not guaranteed to converge to a PSNE. So, we are interested in finding the probability that the BR algorithm converges to a PSNE, when a large number of iterations of the algorithm are executed. Assuming $Q_1 = \ldots = Q_K = 3$, we selected the initial allocation (strategy profile) randomly and then ran the BR algorithm for 10^6 iterations; 50 such runs were executed. We calculated the convergence probability of the BR algorithm as the fraction of runs in which a PSNE was achieved. Fig. 4(a) represents the convergence probability in percentage with varying number of BSs for different values of N and Fig. 4(b) represents the convergence probability in percentage with varying number of MSs for different values of K and N. The plots show that the convergence probability is low when the number of MSs, BSs or subchannels is large, as would typically be the case in practice. That is, *with a high probability, the BR algorithm does not converge to a PSNE.*

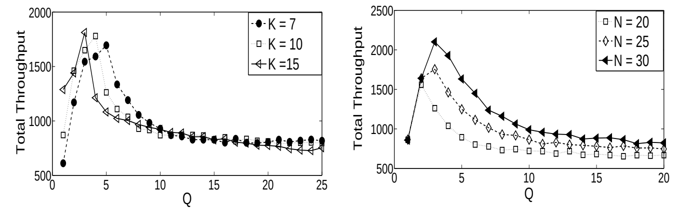
2) Throughput Performance of the BR Algorithm: Note that although with a high probability the BR algorithm does not converge to a PSNE, it finds a *feasible* allocation of



(a) (b)

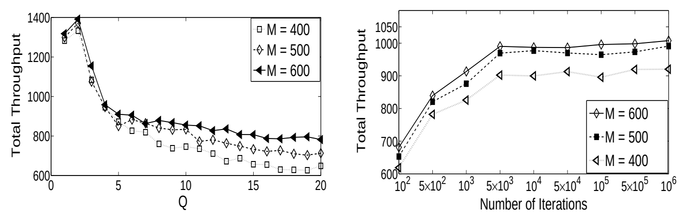
Fig. 4: The plot on the left (respectively, right) shows the variation of probability of convergence of the BR algorithm to PSNE in percentage with number of BSs K (respectively, number of MSs M).

subchannels to MSs for the model in Section III. So next, we evaluate the performance of the BR algorithm in terms of the total throughput achieved under the allocation that it finds after being executed for a large number of iterations. Let $Q_1 = \ldots = Q_K = Q$. First, we investigate as to what values of the parameter Q approximately maximize the total throughput achieved under the BR algorithm. The variation of the total throughput obtained using the BR algorithm, executed for 10^5 iterations, with parameter Q is plotted [15] in Fig. 5(a) (respectively, Fig. 5(b), Fig. 6(a)) for different values of K (respectively, N, M). In Fig. 5(a), we have considered $N = 25$ and $M = 500$. The value of parameter Q for which the total throughput is large, decreases with K because if the values of K as well as Q are large, then subchannels are allocated to a large number of MSs, which results in high inter-cell interference and hence low total throughput. In Fig. 5(b), we have considered $K = 15$ and $M = 400$. It can be seen that the value of Q at which the total throughput is large, increases in N; this is because more resources (subchannels) are available. Also, for a fixed value of Q, the total throughput increases in N; this is because the higher number of available subchannels allow an allocation of subchanels to the MSs in a manner that results in less inter-cell interference. In Fig. 6(a), we have considered $N = 20$, $K = 15$. It can be seen that the values of parameter Q for which the total throughput is close to maximum, are roughly invariant with total number of MSs in the system. Also, the total throughput increases with number of MSs; this is because of the increase in diversity, *i.e.*, the Q MSs to which subchannels are allocated can be selected from a larger set of available MSs, which results in a higher total throughput on average. After a large number of simulations for different values of K, M and N, it was found that *the total throughput of the system is close to maximum for those values of the parameter Q for which the total number of MSs to which subchannels are allocated in the whole system is between N and $2N$* [16]. This observation can be used to select the value of Q so as to achieve a close to maximum total throughput in an implementation of the BR algorithm to solve the ICIC with fixed transmit power problem.



(a) (b)

Fig. 5: The plot on the left (respectively, right) shows the variation of the total throughput obtained using the BR algorithm with Q *i.e.* number of MSs allowed to be allocated per BS for different values of number of BSs K (respectively, number of subchannels N).



(a) (b)

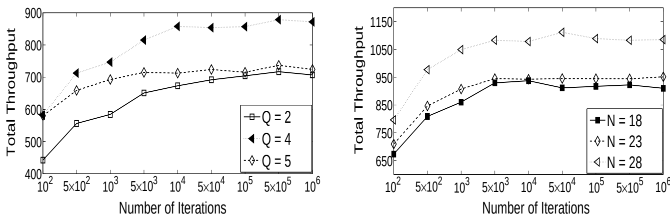
Fig. 6: The plot on the left (respectively, right) shows the variation of the total throughput obtained using the BR algorithm with Q (respectively, number of iterations) for different values of number of MSs M.

Next, we investigate as to how many iterations the BR algorithm should be run for, in order to achieve a high total throughput. Figs. 6(b), 7(a) and 7(b) depict the variation of total throughput with number of iterations the BR algorithm runs for, for different parameter values. Fig. 6(b) shows the total throughput for three different values of M with $K = 15$, $N = 20$ and $Q = 3$. The total throughput increases in M similar to the trend in Fig. 6(a). The total throughput for three different values of Q has been plotted in Fig. 7(a) for $K = 15, N = 20$ and $M = 400$. The throughput is maximized for the middle value of Q ($Q = 4$); intuitively this is because, for the values of K, M and N considered, the channel resources are under utilized for the lowest value of Q ($Q = 2$) and for the highest Q value ($Q = 5$),

[15] For all the plots in Figs. 5 to 11, each data point was obtained by averaging across 25 runs with different random seeds.

[16] For example, in the plot on the left in Fig. 5, $N = 25$ is used and the plot shows that the sets of values of Q for which the total throughput of the system is close to maximum are $\{3, 4, 5, 6\}$, $\{2, 3, 4, 5\}$ and $\{1, 2, 3\}$ for $K = 7, 10$ and 15 respectively. Since there are K BSs, the total number of MSs to which subchannels are allocated in the whole system is $Q \times K$; hence, the corresponding total numbers of MSs to which subchannels are allocated in the whole system are $\{21, 28, 35, 42\}$, $\{20, 30, 40, 50\}$ and $\{15, 30, 45\}$ for $K = 7, 10$ and 15 respectively.

the interference is high, which reduces the total throughput. Fig. 7(b) shows the total throughput for three different values of N with $K = 15$, $M = 400$ and $Q = 4$. The total throughput increases in N similar to the trend in Fig. 5(b). It can be seen from Figs. 6(b), 7(a) and 7(b) that *for a wide range of parameter values, the average total throughput achieved under the BR algorithm increases in the number of iterations and saturates around* 10^5 *iterations*. This suggests that in an implementation of the BR algorithm, the algorithm should be run for approximately 10^5 iterations.



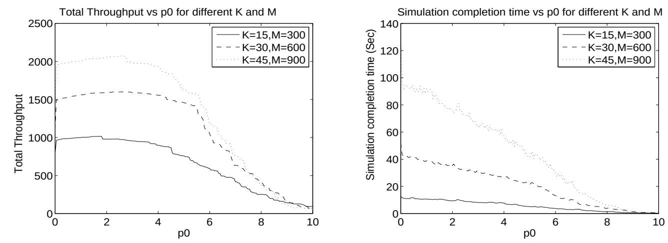
(a) (b)

Fig. 7: The plot on the left (respectively, right) shows the variation of the total throughput obtained using the BR algorithm with number of iterations for different values of Q (respectively, number of subchannels N).

B. Performance Evaluation of the Greedy Algorithm

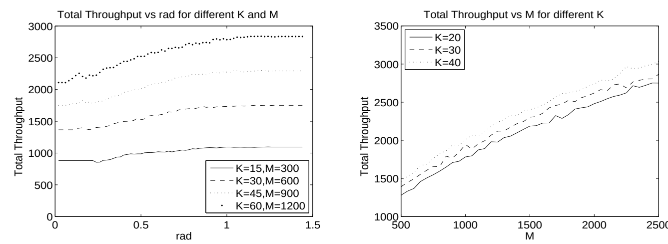
For the greedy algorithm, the set, $\mathcal{B}_a$, of neighbors of BS a (see Section VI-A) is assumed to consist of all BSs that are at distance less than rad from BS a, where rad is a parameter. The greedy algorithm was run for this topology for different parameter values, and the total throughput of the allocation returned by the algorithm and the time that the simulation took to complete [17] were noted. The latter is used as a measure of the total number of computations required by the greedy algorithm.

We first study the impact of the parameter p_0 used by the greedy algorithm (see Section VI-A, in particular, Remark 2) on its performance. Fig. 8 plots the total throughput and the simulation completion time of the greedy algorithm versus p_0 for different values of K and M. The total throughput plot shows that the algorithm performs poorly for values of p_0 that are very low or very high, consistent with the trend explained in Remark 2 in Section VI-A. On the other hand, the algorithm performs quite well for all values of p_0 in $[0.1, 4]$, which is a *wide range*; thus, *it is easy to choose a value of* p_0 *that provides good performance in a practical implementation* of the algorithm. The simulation completion time in Fig 8 decreases in p_0; this is consistent with the fact that by the termination condition (iii) in Section VI-A, the smaller the value of p_0, the longer the algorithm is allowed to run. Fig. 9(a) shows that *the performance of the algorithm improves as more direct connections between pairs of BSs become available*. Next, Fig. 9(b) illustrates that the total throughput of the greedy algorithm increases in M for fixed K, consistent with the intuition that as M increases, the greedy algorithm can choose to allocate subchannels to MSs from a larger set.



(a) (b)

Fig. 8: The figure plots the total throughput and simulation completion time (in seconds) for the greedy algorithm versus p_0. $N = 50$ is used.



(a) (b)

Fig. 9: The plot on the left (respectively, right) shows the total throughput under the greedy algorithm versus rad (respectively, M) for different values of K and M (respectively, K).

Next, recall that under the greedy algorithm, for a given BS a, the channel gains $\{H^n_{b,j} : j \in \mathcal{M}_a, b \in \mathcal{B}_a, n \in \mathcal{N}\}$ between its neighboring BSs and its associated MSs need to be sent to BS a. These channel gains can be quantized and represented using a small number of bits (*e.g.*, 4 or 6 bits). We have evaluated the performance of the proposed greedy algorithm in the cases where 4 bits and 6 bits are used to represent each channel gain. The total throughput achieved under the greedy algorithm with exact channel gain values and with quantized ones for 15 randomly generated network topologies, each with the parameter values $K = 15, M = 300$ and $N = 50$, are provided in Fig. 10. The figure shows that the performance of the greedy algorithm with quantized gains is close to that with exact gains. Thus, the amount of channel gain information that needs to be exchanged under the greedy algorithm can be significantly reduced using quantized channel gains with only a small decrease in its performance.

C. Performance Comparison of the BR, Greedy and SA Algorithms

We now compare the performance of the BR, greedy and SA algorithms in terms of the total throughput obtained and

[17] All simulations was done in MATLAB and run on a computer with a 2.20 GHz Intel i7-2670QM CPU, 6 GB RAM and Windows 7.

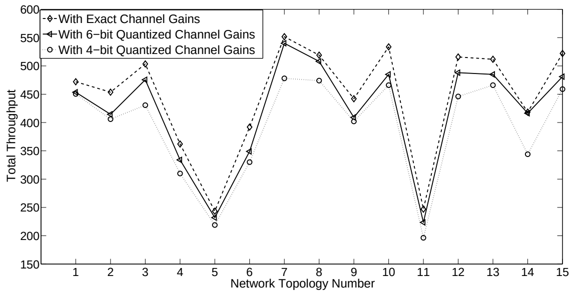
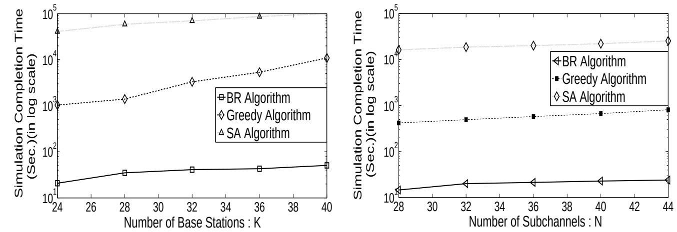


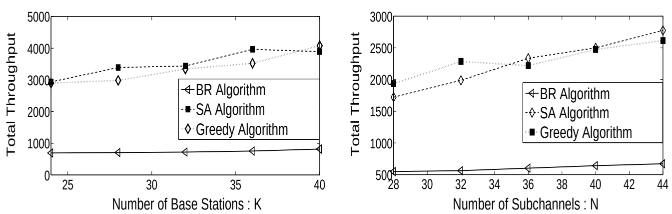
Fig. 10: The plot shows the total throughput achieved under the greedy algorithm with exact, 6-bit quantized and 4-bit quantized channel gain values.



(a) (b)

Fig. 12: The plot on the left (respectively, right) shows the comparison of the greedy, SA and BR algorithms in terms of simulation completion time for different values of the number of BSs K (respectively, number of subchannels N).

simulation completion time. For the greedy algorithm, the parameter values $rad = 0.4$ and $p_0 = 0.1$ are used, for the SA algorithm, $\alpha = 0.998$, $\eta = 100$ and $\kappa = 5$ are used and for the BR algorithm, $Q = 3$ and 10^5 iterations are used. The variation of the total throughput (respectively, simulation completion time) versus K and N have been plotted for the three algorithms in Fig. 11 (respectively, Fig. 12). From Fig. 11, it can be observed that the total throughput obtained using the BR algorithm is very small compared to those obtained using the SA and greedy algorithms. The total throughput obtained using the BR algorithm is 0.33 times or less as compared to that obtained using the greedy algorithm for all the parameter values considered.



(a) (b)

Fig. 11: The plot on the left (respectively, right) shows the comparison of the greedy, SA and BR algorithm in terms of the total throughput obtained for different values of the number of BSs K (respectively, number of subchannels N).

However, the simulation completion time for the BR algorithm is very small compared to those for the SA and greedy algorithms. Also, the simulation completion time for the SA algorithm is very high as compared to that of the greedy algorithm. Finally, the performance of the greedy algorithm is close to that of the SA algorithm in terms of total throughput for all the parameter values considered; also, the former algorithm outperforms the latter algorithm in dense networks, *i.e.*, those in which K is large or N is small.

VIII. Conclusions, discussion and Future Work

We showed that the general ICIC with fixed transmit power problem is NP-complete and that when the potential interference levels are above a threshold, the problem can be optimally solved in polynomial-time via reduction to the bipartite matching problem. We also formulated the ICIC problem as a non-cooperative game and proved that although it is an ordinal potential game in two special cases, it is *not* an ordinal potential game in general. Also, we designed two heuristics for the general ICIC problem– a fast and distributed greedy algorithm, which only requires direct connections between pairs of neighboring BSs, and a centralized simulated annealing (SA) based algorithm. Our simulations show that the total throughput obtained using the better response (BR) algorithm is very small compared to those obtained using the SA and greedy algorithms; however, the execution time of the BR algorithm is much smaller than those of the latter two algorithms. Finally, the greedy algorithm outperforms the SA algorithm in dense cellular networks and requires only a small fraction of the number of computations required by the latter algorithm for execution.

Our analytical results provide insight into the structure of the ICIC with fixed transmit power problem, which would be useful to future work on the design of approximation algorithms with a provable approximation ratio for the problem. Also, since the subchannel allocation game corresponding to the ICIC with fixed transmit power problem is not a potential game in general, the BR algorithm is not guaranteed to converge to a Nash equilibrium (NE). The problem of designing iterative algorithms that *always* converge to a NE of this game is a direction for future research.

Appendix

A. Proof of Lemma 1: First, we will show that the function $f(x)$ is quasiconvex [18] on the domain $x \geq 1$. Let

$$y = (x-1)\beta + \frac{1}{\tau}. \tag{32}$$

Then

$$f(x) = g(y), \tag{33}$$

[18] Recall that a function $f : R \to R$ is *quasiconvex* if its domain is convex and for every $\alpha \in R$, the sublevel set $S = \{x | f(x) \leq \alpha\}$ is convex [4].

where:

$$g(y)=\left(\frac{y}{\beta}+1-\frac{1}{\tau\beta}\right)\log\left(1+\frac{1}{y}\right). \tag{34}$$

Now, a sufficient condition for a function $h(\cdot)$ to be quasi-convex is provided by the following property [4]:

Property 1: If $h''(y)>0$ whenever $h'(y)=0$, then $h(\cdot)$ is quasi-convex.

We will now show using Property 1 that $g(\cdot)$ is quasi-convex, and deduce quasi-convexity of $f(\cdot)$ from it.

Differentiating (34), we get:

$$g'(y)=\frac{1}{\beta}\left[\log\left(1+\frac{1}{y}\right)-\frac{1}{y(y+1)}\left[y+\beta-\frac{1}{\tau}\right]\right] \tag{35}$$

and:

$$g''(y)=\frac{1}{\beta}\left[\frac{y\left(2\left(\beta-\frac{1}{\tau}\right)-1\right)+\left(\beta-\frac{1}{\tau}\right)}{y^2(y+1)^2}.\right] \tag{36}$$

Suppose $g'(y)=0$. Then by (35):

$$\log\left(1+\frac{1}{y}\right)=\frac{1}{y}\left[\frac{y+\left(\beta-\frac{1}{\tau}\right)}{y+1}\right]. \tag{37}$$

Now, we have the following inequality [21]:

$$\log(1+u)>\frac{u}{1+\frac{u}{2}}, \forall u>0. \tag{38}$$

By (37), (38) with $u=\frac{1}{y}$, and some algebraic simplification:

$$y\left(2\left(\beta-\frac{1}{\tau}\right)-1\right)+\left(\beta-\frac{1}{\tau}\right)>0 \tag{39}$$

By (36) and (39), it follows that $g''(y)>0$. Hence, by Property 1, it follows that $g(\cdot)$ is quasi-convex.

Now, by (32) and (33), we get $f'(x)=\beta g'(y)$ and $f''(x)=\beta^2 g''(y)$. Since $g(\cdot)$ satisfies the sufficient condition in Property 1, so does $f(\cdot)$, and hence $f(\cdot)$ is quasi-convex.

Next, it is easy to show using L'Hôpital's rule that:

$$\lim_{x\to\infty} f(x)=\frac{1}{\beta}<\log(1+\tau)=f(1). \tag{40}$$

Now, consider the sublevel set:

$$S=\{x\geq 1 | f(x)\leq f(1)\}.$$

By (40), there exists $x'>1$ such that $x\in S$ for all $x\geq x'$. Also, clearly $1\in S$. Since $f(\cdot)$ is quasi-convex, the set S is convex [4]; so $x\in S$ for all $x\geq 1$. That is, $f(x)\leq f(1)$ for all $x\geq 1$ and the result follows.

■

B. Proof of Theorem 3: Let $\mathbf{S}$ and $\mathbf{S}'$ be as in Definition 1, $u'_a(.)$ be as in (27) and $\phi_1(.)$ be as in (28). We will show that:

$$\text{sgn}(\phi_1(\mathbf{S}')-\phi_1(\mathbf{S}))=\text{sgn}(u'_a(\mathbf{S}')-u'_a(\mathbf{S})). \tag{41}$$

The proof of Theorem 3 will then follow from Definition 1 and Lemma 2.

Consider: $u'_a(\mathbf{S}')-u'_a(\mathbf{S})$

$$=\frac{PH_a^{n'_a}}{N_0+P\sum_{i\neq a,i\in B}H_{i,j_a}^{n'_a}I_{(n_i=n'_a)}}-\frac{PH_a^{n_a}}{N_0+P\sum_{i\neq a,i\in B}H_{i,j_a}^{n_a}I_{(n_i=n_a)}}$$

$$\begin{aligned}=\quad&\alpha\Bigg[N_0\left(H_a^{n'_a}-H_a^{n_a}\right)+P\Bigg(\sum_{i\neq a}H_a^{n'_a}H_{i,j_a}^{n_a}I_{(n_i=n_a)}\\&-\sum_{i\neq a}H_a^{n_a}H_{i,j_a}^{n'_a}I_{(n_i=n'_a)}\Bigg)\Bigg],\end{aligned} \tag{42}$$

where,
$\alpha=\frac{P}{\left(N_0+P\sum_{i\neq a,i\in B}H_{i,j_a}^{n'_a}I_{(n_i=n'_a)}\right)\left(N_0+P\sum_{i\neq a,i\in B}H_{i,j_a}^{n_a}I_{(n_i=n_a)}\right)}$.

Note that $\alpha>0$. Since $H_a^n=H_a\ \forall\, n,a$, by (42):

$$\begin{aligned}&u'_a(\mathbf{S}')-u'_a(\mathbf{S})=\\&P\alpha\left[\sum_{i\neq a}H_aH_{i,j_a}^{n_a}I_{(n_i=n_a)}-\sum_{i\neq a}H_aH_{i,j_a}^{n'_a}I_{(n_i=n'_a)}\right].\end{aligned} \tag{43}$$

Let $\phi_1(\mathbf{S})$ be as in (28). Then,

$$\begin{aligned}\phi_1(\mathbf{S}')\quad=\quad&-\sum_{i\neq a}\sum_{k\neq i,k\neq a}H_{i,j_k}^{n_i}I_{(n_i=n_k)}-\sum_{i\neq a}H_{i,j_a}^{n'_a}I_{(n_i=n'_a)}\\&-\sum_{k\neq a}H_{a,j_k}^{n'_a}I_{(n'_a=n_k)}.\end{aligned}$$

Using a change of variables and $H_{i,j_a}^n=H_{a,j_i}^n, \forall n,a,i$, and $i\neq a$, we get:

$$\begin{aligned}\phi_1(\mathbf{S}')-\phi_1(\mathbf{S})\quad&=\quad2\left(\sum_{i\neq a}H_{i,j_a}^{n_a}I_{(n_i=n_a)}-\sum_{i\neq a}H_{i,j_a}^{n'_a}I_{(n_i=n'_a)}\right)\\&=\quad2\left[\frac{u'_a(\mathbf{S}')-u'_a(\mathbf{S})}{P\alpha H_a}\right]\quad\text{(by (43))}.\end{aligned} \tag{44}$$

Equation (41) follows from (44), which completes the proof.

■

C. Proof of Theorem 4: Let $\mathbf{S}$ and $\mathbf{S}'$ be as in Definition 1, $u'_a(.)$ be as in (27) and $\phi_2(.)$ be as in (29). We will show that:

$$\text{sgn}(\phi_2(\mathbf{S}')-\phi_2(\mathbf{S}))=\text{sgn}(u'_a(\mathbf{S}')-u'_a(\mathbf{S})). \tag{45}$$

The proof of Theorem 4 will then follow from Definition 1 and Lemma 2.

Now, using the facts that $H_a^n=H^n\ \forall a,n$ and $N_0=0$ in (42) we get:

$$\begin{aligned}&u'_a(\mathbf{S}')-u'_a(\mathbf{S})\\&=PH^{n'_a}H^{n_a}\alpha\left[\left(\sum_{i\neq a}\frac{H_{i,j_a}^{n_a}I_{(n_i=n_a)}}{H^{n_a}}-\sum_{i\neq a}\frac{H_{i,j_a}^{n'_a}I_{(n_i=n'_a)}}{H^{n'_a}}\right)\right].\end{aligned} \tag{46}$$

Let $\phi_2(\mathbf{S})$ be as in (29). Then,

$$\begin{aligned}\phi_2(\mathbf{S}')\quad=\quad&-\sum_{i\neq a}\sum_{k\neq i,k\neq a}\frac{H_{i,j_k}^{n_i}}{H_k^{n_k}}I_{(n_i=n_k)}\\&-\sum_{i\neq a}\frac{H_{i,j_a}^{n'_a}}{H_a^{n'_a}}I_{(n_i=n'_a)}-\sum_{k\neq a}\frac{H_{a,j_k}^{n'_a}}{H_k^{n_k}}I_{(n'_a=n_k)}.\end{aligned}$$

Hence, $\phi_2(\mathbf{S}') - \phi_2(\mathbf{S})$

$$= \sum_{i \neq a} \left[\frac{H^{n_a}_{i,j_a}}{H^{n_a}_a} I_{(n_i = n_a)} - \frac{H^{n'_a}_{i,j_a}}{H^{n'_a}_a} I_{(n_i = n'_a)} \right] + \sum_{k \neq a} \left[\frac{H^{n_a}_{a,j_k}}{H^{n_a}_k} I_{(n_k = n_a)} - \frac{H^{n'_a}_{a,j_k}}{H^{n'_a}_k} I_{(n_k = n'_a)} \right]. \quad (47)$$

Using a change of variables and $H^n_a = H^n \; \forall a, n$, $H^n_{i,j_a} = H^n_{a,j_i}, \forall a, i$ and $i \neq a$, in (47):

$$\begin{aligned} \phi_2(\mathbf{S}') - \phi_2(\mathbf{S}) &= 2 \sum_{i \neq a} \left[\frac{H^{n_a}_{i,j_a}}{H^{n_a}} I_{(n_i = n_a)} - \frac{H^{n'_a}_{i,j_a}}{H^{n'_a}} I_{(n_i = n'_a)} \right] \\ &= 2 \left[\frac{u'_a(\mathbf{S}') - u'_a(\mathbf{S})}{P \alpha H^{n'_a} H^{n_a}} \right] \quad \text{(by (46))}. \quad (48) \end{aligned}$$

Equation (45) follows from (48), which completes the proof. ∎

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