

Higher dimensional electrical circuits and the matroid dual of a nonplanar graph

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Abstract

In this paper we describe a physical problem, based on electromagnetic fields, whose topological constraints are higher dimensional versions of Kirchhoff's laws, involving 2– simplicial complexes embedded in \mathbb{R}^3 rather than graphs. However, we show that, for the skeleton of this complex, involving only triangles and edges, we can build a matroid dual which is a graph. On this graph we build an 'ordinary' electrical circuit, solving which we obtain the solution to our original problem. Construction of this graph is through a 'sliding' algorithm which simulates sliding on the surfaces of the triangles, moving from one triangle to another which shares an edge with it but which also is adjacent with respect to the embedding of the complex in \mathbb{R}^3 . For this purpose, the only information needed is the order in which we encounter the triangles incident at an edge, when we rotate say clockwise with respect to the orientation of the edge. The dual graph construction is linear time on the size of the 2– complex.

Keywords: Simplicial complex, Kirchhoff's laws, Matroid dual, Nonplanar graph

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1. Introduction

Kirchhoff's laws for electrical networks state that the net current leaving a node is zero (KCL) and the algebraic sum of the voltages around a loop is zero (KVL). This topological model for electrical networks has proved enormously useful both for theoretical studies and for practical computations. Interest in these ideas in other areas of research is growing (see, for instance, [4], [9], [15], [16], [17]). It is therefore pertinent to explore whether there exist variations of this model, which share essential characteristics with it. In our opinion these essential characteristics are

- that the spaces of vectors which satisfy Kirchhoff's voltage law and Kirchhoff's current law are complementary orthogonal (Tellegen's Theorem [1], [11], [18], [14]);
- that the preprocessing, needed for ease of solution, of constraints arising from these laws and from the device characteristic (eg. Ohm's law) can be done far more efficiently than if they are treated as merely linear algebraic constraints.

Early work in the spirit of this paper, but which leads essentially to a graph based model, is available in [8]. The graph based model was studied rigorously for the first time in [14]. Both the general case where the network is regarded as a pair of complementary orthogonal spaces with no connection to complexes and the case where there is an underlying graph have been treated in [11]. This paper is about other situations where both are satisfied but which do not appear to have been considered in the literature with the point of view of efficiency of equation formulation as well as of preprocessing for ease of solution.

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Three dimensional versions of Kirchhoff's laws have already been studied, for instance, in the case of magnetic circuits (for basic ideas see [5]) and in the case of electromagnetic fields (for a general and comprehensive description see [2], [3]). However these problems can be reduced to the case of graphs by a process of 'cell duality'. Suppose we decompose a large tetrahedron which encloses the region of interest into smaller tetrahedra which intersect each other only in mutual faces. The relationship between these smaller tetrahedra can be captured by replacing each of these latter by a node and joining tetrahedra which share a triangle by an edge. This resulting graph could be called the cell dual of the original three dimensional complex.

Two dimensional versions of Kirchhoff's laws, which is the subject of this paper, have features which appear essentially different from the above. In order to describe the work in this paper and also to clearly differentiate it from that available in the literature, we begin by describing a possible generalization. An n -complex (see Section 3) is made up of a series of $(j, j-1)$ skeletons, $0 < j \leq n$, each of which is made up of oriented j -cells and $(j-1)$ -cells. The incidence relationship of this skeleton is captured by a j -coboundary matrix with columns and rows corresponding to j -cells and $(j-1)$ -cells respectively. (A $(1, 0)$ skeleton, for instance, is a graph and its 1-coboundary matrix is the incidence matrix of the graph.) With each $(j, j-1)$ skeleton one can associate a pair of complementary orthogonal spaces. These are simply the row space of the j -coboundary matrix, i.e., the j -coboundaries, and the space of vectors orthogonal to the rows of the matrix, i.e., the j -cycles. If A is the coboundary matrix, these are the space of vectors $y^T = \lambda^T A$ and the space $x^T A^T = 0$. In [20], [21], higher dimensional electrical networks are defined based on such skeletons, in the process unifying physical situations involving heat as well as electromagnetic fields. Here, the first characteristic mentioned above, namely complementary orthogonality, is obviously satisfied, but, in general, not the second characteristic of ease of writing and of preprocessing equations.

In this paper, we consider a new situation where both characteristics are satisfied. This involves 2-complexes embedded in \mathbb{R}^3 , and, more generally, $(n-1)$ -complexes embedded in \mathbb{R}^n . The generalizations of Kirchhoff's laws pertaining to these complexes that we use, are in the form of solution spaces of homogeneous 'physical' equations. The complementary orthogonality is not obvious but has to be derived. Our approach reveals the connection of these ideas to contractibility of the underlying space, complete unimodularity of the relevant vector spaces etc. In addition, one of the main results of this paper is that we can build, in linear time on the size of the problem, an electrical (matroid) dual of the relevant 2-complex that turns out to be graph based. The solution of this graph based electrical network yields the solution to the original 2-complex based electrical network.

To motivate our study, we consider the problem of computing the magnetic intensity H and the magnetic flux density B in a three dimensional region when known current source loops exist in specified physical locations. [Equivalent problems arise in situations involving electric field intensity E and current density J , and static problems involving temperature and heat.]

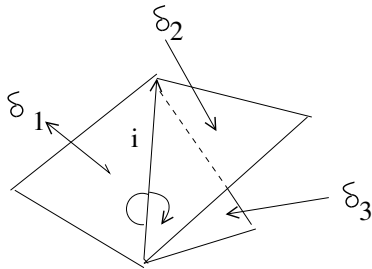
The relevant Maxwell's equations in the integral form are:

$$\oint_C H \cdot dl = \int_S J \cdot ds; \quad \oint B \cdot ds = 0.$$

Here the surface S is bounded by the contour C . A clockwise traversal around it in, say, the plane of the paper, would mean that the ds vector on the right side of the first equation is directed into the plane. The path integral, $\int_P H \cdot dl$ along a directed path P , is called the magnetomotive force (mmf) across the path. The first equation states that the net mmf around the contour is equal to the net current passing through any surface bounded by the contour. The second equation states that the net flux leaving any closed surface is zero. The material property is captured by a relation

$$B = \mu H.$$

This relates the two vectors through the 'permeability' μ which could vary from point to point. Suppose we have a cylinder with a cross sectional area A , length l and uniform permeability μ then the net flux ϕ passing through the cylinder in the direction of the axis is related to the mmf m' along the axis by the



$$m'_{\delta_1} + m'_{\delta_2} + m'_{\delta_3} = i$$

Figure 1: Triangles around an edge

equation

$$m' = \left[\frac{l}{A\mu} \right] \phi;$$

The quantity $\frac{l}{A\mu}$ is called the reluctance of the cylinder in the direction of the axis.

In our problem, the medium is assumed to be composed of high permeability material embedded with thin layers of varying low permeability. Such a situation might arise, for instance, when magnetic material develops cracks due to degradation. This problem can be solved by solving partial differential equations over three dimensional regions but that method provides poor insight while being computationally intensive. We however, choose to model this essentially as a two dimensional complex embedded in 3-space.

We describe here the general discretized version of this problem and later in Section 2, discuss an elementary instance. In the general discretized version, we may imagine a bounded tetrahedral region in \mathbb{R}^3 , being decomposed into smaller tetrahedra. Except for some previously specified facets (triangles) of these tetrahedra, the permeability everywhere may be taken to be infinite. As an idealization, interiors of the tetrahedra have zero reluctance and zero conductance and carry neither mmf nor electric current, the triangular facets may be taken to have zero thickness but can have a positive mmf across (normal) to them and therefore a positive reluctance along the normal, the interior of the triangles may be taken to have no currents while some of the boundaries of triangles may carry loops of currents, and except for the triangles specified, the reluctance everywhere may be taken to be zero.

The constraints are

- the net flux leaving a closed surface (i.e., the boundary of a bounded region) is zero;
- the net mmf, through the set of triangles incident at an edge, taken say clockwise around the edge with respect to its direction, is equal to the current through the edge in that direction (see Figure 1); this constraint arises from the Maxwell equation $\oint_C H \cdot dl = \int_S J \cdot ds$, where there is nonzero contribution to the integral in the left hand side only when the contour crosses a triangle.
- for each triangle, the mmf m' through triangle = flux ϕ through triangle \times reluctance of the triangle.

The first two constraints are topological, analogous to Kirchoff's laws. We show in this paper, that in the case of a 2-complex in the interior of a bounded tetrahedral region, that the two sets of constraints, when the right side is zero, yield a pair of complementary orthogonal spaces. This corresponds, in the case of 1-dimensional electrical circuits on graphs, to voltage and current spaces being complementary orthogonal.

The outline of the paper follows.

Section 2 illustrates the physical problem through a simple example in which the triangles of interest lie on the surface of a cube with one of them carrying a current in its boundary.

Section 3 is on preliminary definitions on simplicial complexes, chains, cycles and coboundaries, boundaries of chains, coboundary matrices associated with the complexes etc. It is shown that the fact that boundary of a boundary is a zero chain is equivalent to product of $(j-1)$ -coboundary matrix and j -coboundary matrix being the zero matrix. It is shown that j -coboundaries and j -cycles of a complex form complementary orthogonal spaces.

Section 4 deals with the special case of a 2-complex embedded in \mathbb{R}^3 . Here, it is shown that the 2-coboundary and 2-cycle spaces are regular (completely unimodular) and, by the use of a basic contractible space theorem, that the row space of the 2-coboundary matrix is complementary orthogonal to the column space of the 3-coboundary matrix of the complex.

Section 5 is on electrical 2-networks defined on 2-complexes. We prove a generalization of the celebrated Tellegen's theorem of electrical networks for electrical 2-networks. We also explicitly state the equations for such networks and show how to solve them in the case of linear networks. In particular this shows how the physical problem stated in the introduction can be solved.

Section 6 is on the notion of a triangle adjacency graph $tag(\mathcal{C})$ for the 2-complex \mathcal{C} . This graph has two vertices v_+, v_- for each triangle of \mathcal{C} . If the triangle were on the $x-y$ plane in \mathbb{R}^3 , one of these would be above and the other below at a distance ϵ . Two vertices are joined by an edge in $tag(\mathcal{C})$, if the corresponding triangles of \mathcal{C} have a common edge and one can slide from one to another along triangles. For instance, if triangles $\delta_1, \delta_2, \delta_3$ of \mathcal{C} were as in Figure 1, then in $tag(\mathcal{C})$, $v_+(\delta_1)$ would be connected by an edge to $v_-(\delta_2)$. The graph $tag(\mathcal{C})$ can be constructed in time linear in size of \mathcal{C} . It is shown that two vertices of $tag(\mathcal{C})$ can be connected by a path iff physically, the two vertices are in the same connected region of $\mathbb{R}^3 \setminus \mathcal{C}$. The graph $\mathcal{G}_{comptag(\mathcal{C})}$ is built on connected components of vertices of $tag(\mathcal{C})$. There is an edge corresponding to each triangle of \mathcal{C} directed from the component of $tag(\mathcal{C})$ containing v_+ to the one containing v_- of the triangle. Finally, it is shown that the rows of the incidence matrix of $\mathcal{G}_{comptag(\mathcal{C})}$ are 2-cycles of \mathcal{C} .

Section 7 is on the cell dual of a 3-complex $\mathcal{C}_{\mathcal{T}}$ which is the decomposition of a tetrahedron \mathcal{T} into smaller tetrahedra τ_i whose interiors do not intersect. The triangles which are the faces of these tetrahedra are either in the boundary of \mathcal{T} , in which case they belong to only one of the τ_i or are common to exactly two of the τ_i . It is assumed that the given 2-complex \mathcal{C} lies in the interior of \mathcal{T} and has its triangles as a subset of the triangles of $\mathcal{C}_{\mathcal{T}}$. The cell dual is the graph $\mathcal{G}_{\mathcal{T}}$ which is obtained by replacing each τ_i by a node and joining two nodes if they represent tetrahedra which share a common triangle. A region graph $\mathcal{G}_{region(\mathcal{C})}$ corresponding to connected regions of $\mathcal{T} \setminus \mathcal{C}$ is shown to be obtained from $\mathcal{G}_{\mathcal{T}}$ by contracting edges which correspond to triangles which do not belong to \mathcal{C} . The row space of the incidence matrix of this region graph is shown to be the 2-cycle space of \mathcal{C} .

Section 8 is on the proof of the fact that when \mathcal{C} is connected $\mathcal{G}_{region(\mathcal{C})}$ is identical to $\mathcal{G}_{comptag(\mathcal{C})}$. Since the latter can be constructed in time linear on the size of \mathcal{C} , from the results of the previous section we have a convenient and easily constructed representation for the 2-cycle space of \mathcal{C} . This is sufficient for writing a linearly independent set of equations for the linear electrical 2-network.

Section 9 discusses the matroid duality of the complex \mathcal{C} and the graph $\mathcal{G}_{region(\mathcal{C})}$ and also sketches, given a non planar graph, how to enlarge it so that it has a matroid dual which is a 2-complex.

Section 10 describes how to build the dual network to a given 2-network so as to infer the solution of the 2-network from that of the dual, and also indicates how to translate results from graph based networks to 2-networks by considering the case of the Kirchhoff's tree formula.

Section 11 sketches how to generalize the ideas of the paper from 2-complexes embedded in \mathbb{R}^3 to $(n-1)$ -complexes embedded in \mathbb{R}^n .

Section 12 is on conclusions.

2. A simple instance of the problem

We now illustrate the general ideas through a simple example.

In Figure 2, except for the thin layers of the faces, both the inside and outside of the cube are of high permeability (∞ , for simplicity). We divide the faces into triangles. One of the triangles would have a loop carrying current i , in the direction of the orientation of the triangle, on the upper face of the cube. The

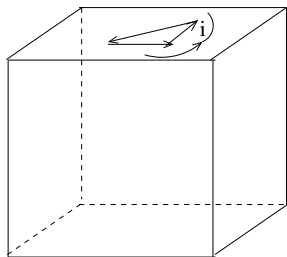


Figure 2: Cube with faces of low permeability

triangles should be small enough that one can take the thickness to be constant and the permeability to be uniform in its interior without causing unacceptable error. After computing the reluctance of the triangles, we will idealize them to have zero thickness.

The solution procedure is as follows.

1. Orient all the triangles consistent with the outward normal of the cube (an orientation of a triangle is an ordering of its vertices - see the beginning of Section 3 and Figure 3).
2. For the k^{th} triangle, compute the reluctance $r_k = \frac{d_k}{\mu_k A_k}$, where μ_k is the permeability in the thin triangular slab, A_k is the area and d_k , the thickness. Let H_k, B_k be the average magnetic field intensity and the average flux density normal to the surface of the triangle. We write

$$H_k d_k = B_k A_k \times \frac{d_k}{\mu_k A_k} \quad \text{i.e.,}$$

$$m'_k = \phi_k \times r_k \quad (\text{mmf} = \text{flux times reluctance}). \quad (1)$$

3. With the orientation specified we have constraints for the fluxes and mmfs associated with triangles (k^{th} - flux, k^{th} - mmf with k^{th} - triangle) as follows. If ϕ denotes flux, we must have net flux leaving every closed surface equal to zero. In the present case there is only one closed surface.

$$\sum_k \phi_k = 0 \quad \text{i.e.,} \quad (2)$$

$$[1 \quad \cdots \quad 1] \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} = 0. \quad (3)$$

The constraint on the mmf vector is that the net mmf around an edge (i.e., the sum of the mmfs associated with the triangles incident at an edge) is the current through the edge in the direction consistent with the direction of rotation. This constraint can be expressed in terms of the rows of the 2-coboundary matrix which has columns corresponding to triangles and rows corresponding to edges. In the row for edge e , the entry for triangle δ is 0 if it is not incident on the edge, is +1 if it is incident and agreeing with the orientation of the edge and is -1 if it is incident but oppositely oriented to the orientation of the edge.

In the present case, every edge is incident on exactly two triangles and if we were to orient the triangles according to the outward normal of the cube, in one it would agree with the orientation and, in the other, it would oppose it. Let $A^{(2)}$ denote the 2-coboundary matrix and m_k , the k^{th} mmf. Further, let the first three rows $A_{1,\cdot}^{(2)}, A_{2,\cdot}^{(2)}, A_{3,\cdot}^{(2)}$ correspond to edges of the triangle carrying the current i , edges

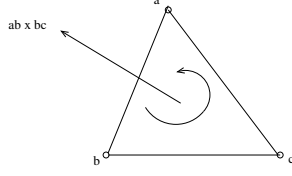


Figure 3: Triangle abc oriented consistent with the vector product $ab \times bc$

oriented in the direction of the current and let the remaining rows be denoted $A_{rem,..}^{(2)}$.

$$\begin{bmatrix} A_{1,..}^{(2)} \\ A_{2,..}^{(2)} \\ A_{3,..}^{(2)} \\ \vdots \\ A_{rem,..}^{(2)} \end{bmatrix} \begin{bmatrix} m'_1 \\ \vdots \\ m'_n \end{bmatrix} = \begin{bmatrix} i \\ i \\ i \\ \vdots \\ \mathbf{0} \end{bmatrix}. \quad (4)$$

4. Solve simultaneously Equations 1, 2 and 4.

The above appears as a straight forward linear algebraic problem but with singular equations. The first step is to build an equivalent nonsingular set of equations. If, however, we use linear algebraic methods at this stage, it would be (relatively speaking) computationally expensive. We will show how to do this combinatorially in linear time and also show that, after this, the problem reduces to solving a conventional electrical circuit based on graphs. This is one of the contributions of this paper. After that we would be left with a set of sparse linear equations for which very efficient practical procedures exist in the literature.

3. Preliminaries

This section follows [23] (more recent references are [7], [10]).

Let a_0, \dots, a_n be vectors in \mathbb{R}^n . An n -simplex $a_0 \cdots a_n$ is the convex hull of the vectors a_0, \dots, a_n , i.e., the set of all points $\sum_0^n \lambda_i a_i$, where $\lambda_i \geq 0$, $\sum_0^n \lambda_i = 1$. A face of the n -simplex $a_0 \cdots a_n$ is the convex hull of any subset of the vectors a_0, \dots, a_n . An oriented n -cell c is a simplex $a_0 \cdots a_n$ with vertices a_0, \dots, a_n and with a specified ordering of vertices $[a_0, \dots, a_n]$. Two orderings $[a_0, \dots, a_n], [b_0, \dots, b_n]$ of the vertices are treated as equivalent if $[b_0, \dots, b_n]$ is an even permutation of $[a_0, \dots, a_n]$ and opposite if an odd permutation. We will usually call a 2-cell, a triangle. If abc is an oriented triangle embedded in \mathbb{R}^3 , we say the orientation $[abc]$ is consistent with the direction of the vector cross product $ab \times bc$ (see Figure 3).

Suppose $S := \{c_1, \dots, c_m\}$ is a set of oriented n -cells. Let $c_j, c'_j, j = 1, \dots, m$ be oppositely oriented. Let $S' := \{c'_1, \dots, c'_m\}$. We can think of a new orientation for the cells to be a one to one map $\sigma : S \rightarrow S \cup S'$ with $\sigma(c_j) = c_j$ or c'_j .

Example 1. An oriented 0-cell is a point. An oriented 1-cell is a directed edge. An oriented 2-cell is an oriented triangle. An oriented 3-cell is an oriented tetrahedron (see Figure 4). The 1-cells ab, ba are oppositely oriented. The 2-cells abc, cba are oppositely oriented and similarly the 3-cells $abcd, dcba$. The 3-cells $abcd, badc$ denote the same oriented cell since $\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$ is an even permutation.

Let c_1, \dots, c_m be oriented n -cells. An n -chain is a formal sum $\sum_1^m \alpha_i c_i, \alpha_i \in \mathbb{R}$. When c_i is an n -cell with $n \geq 1$, We take $-c_i$ to be the n -cell with the same vertices but with orientation opposite to that of c_i . The formal sum is the 'zero chain' iff each of the α_i is zero. If $\sum_1^m \alpha_i c_i, \sum_1^m \beta_i c_i$, are two n -chains, we define their sum to be $\sum_1^m (\alpha_i + \beta_i) c_i$. We define $\lambda(\sum_1^m \alpha_i c_i)$ to be $\sum_1^m \lambda \alpha_i c_i$.

Given an oriented n -cell $a_0 \cdots a_n$, the boundary $\partial(a_0 \cdots a_n)$ is defined to be the $(n-1)$ -chain

$$\underline{a}_0 a_1 \cdots a_n + (-1)^1 a_0 \underline{a}_1 \cdots a_n + \cdots + (-1)^j a_0 \cdots \underline{a}_j \cdots a_n + \cdots + (-1)^n a_0 \cdots \underline{a}_n,$$

where $a_0 \cdots \underline{a_j} \cdots a_n$ denotes $a_0 \cdots a_{j-1} a_{j+1} \cdots a_n$. When a_0 is a 0-cell, we take $\partial(a_0) := 0$.

For any n -chain $\sum_1^m \alpha_i c_i$, we take $\partial(\sum_1^m \alpha_i c_i) := \sum_1^m \alpha_i \partial(c_i)$. It follows from the definition that $\partial\partial(a_0 \cdots a_n)$ is the zero $(n-2)$ -chain, for $m \geq 2$. For $m < 2$, we take it to be trivially zero. For any n -chain, $\sum_1^m \alpha_i c_i, n \geq 2$, we therefore have $\partial\partial(\sum_1^m \alpha_i c_i)$ as the zero $(n-2)$ -chain and for $n < 2$, we take it to be trivially zero.

Example 2. $\partial\partial(a_0 a_1 a_2) = \partial(a_1 a_2 - a_0 a_2 + a_0 a_1) = (a_1 - a_2) - (a_0 - a_2) + (a_0 - a_1) = 0$.

Cochains are defined to be linear functionals acting on chains. We will deal with chains and cochains uniformly using the notion of a vector.

Definition 3. A vector is a function $h : S \rightarrow \mathcal{F}$, from a finite set S to a field \mathcal{F} . We say that the vector is on S over \mathcal{F} . In the present paper the field would be \mathbb{R} . The support of h , denoted $\text{supp}(h)$, is the subset of S , where h takes nonzero values. Addition $h_1 + h_2$ of vectors h_1, h_2 on S over \mathcal{F} is defined by $(h_1 + h_2)(e) := h_1(e) + h_2(e)$, $e \in S$; scalar multiplication λh_1 , $\lambda \in \mathcal{F}$, of h_1 is defined by $(\lambda h_1)(e) := \lambda(h_1(e))$, $e \in S$. A vector space on S over \mathcal{F} is a collection of vectors S over \mathcal{F} closed under addition and scalar multiplication.

In the case of chains we can identify $\sum_1^m \alpha_i c_i$ with the vector h such that $h(c_i) = \alpha_i$. If c'_i is oppositely oriented to c_i we take $h(c'_i) = -\alpha_i$. Thus if $S := \{c_1, \dots, c_m\}$, $\alpha c_1 + \beta c_2 + \gamma c_3$ can be identified with

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_m \\ (\alpha & \beta & \gamma & 0 & \cdots & 0) \end{pmatrix}.$$

Definition 4. An n -complex \mathcal{C} , is a collection of n -cells, $(n-1)$ -cells, \dots , 0-cells, such that every face of a cell in \mathcal{C} is also in \mathcal{C} and the intersection of any two cells of \mathcal{C} is a face of each of them. Thus, the cells in the boundary of each j -cell, $0 < j \leq n$, are also present in the complex. Further, a pair of distinct k -cells, $0 < k \leq n$ either has no intersection or intersects in a j -cell, $j \leq k-1$, present in \mathcal{C} . We will denote the collection of oriented k -cells of \mathcal{C} by $S^{(k)}(\mathcal{C})$ or, when clear from the context, by $S^{(k)}$.

We only deal with oriented complexes and will omit the prefix 'oriented' while referring to them.

To define higher dimensional electrical circuits, we need the definition of a $(k, k-1)$ skeleton of a complex.

Definition 5. The $(k, k-1)$ skeleton, $1 \leq k \leq n$, of an n -complex \mathcal{C} is the pair $(S^{(k)}(\mathcal{C}), S^{(k-1)}(\mathcal{C}))$ with the boundary relationship between the k -cells in $S^{(k)}$ and the $(k-1)$ -cells in $S^{(k-1)}$ agreeing with that of the complex \mathcal{C} .

The most natural way of defining new complexes from old is through the notion of a subcomplex.

Definition 6. Let \mathcal{C} be an n -complex and let $k \leq n$. Let $T \subseteq S^{(k)}(\mathcal{C})$. The k -subcomplex \mathcal{C}_k of \mathcal{C} on T has $S^{(k)}(\mathcal{C}_k) = T$ and for $j \leq k$ has $S^{(j-1)}(\mathcal{C}) \supseteq S^{(j-1)}(\mathcal{C}_k)$ with the boundaries of j -cells of \mathcal{C}_k agreeing with the boundaries of these cells in \mathcal{C} .

The 'connectedness' of complexes, defined below, has a role in algorithms developed in the present paper.

Definition 7. An n -complex \mathcal{C} is said to be n -connected (or connected in brief), if given any pair of $(n-1)$ -cells in the complex, c_1, c_{end} , there exists a sequence $c_1, \dots, c_k, c_{\text{end}}$ of $(n-1)$ -cells such that $c_i, c_{i+1}, 1 \leq i < k$, are faces of a common n -cell and so are c_k, c_{end} . A graph is a 1-complex. A 1-subcomplex of a graph is called a subgraph. The connected components of a graph are its maximally connected subgraphs.

Definition 8. A (path) connected region in \mathbb{R}^n is a set of points in \mathbb{R}^n with the property that given any two points in the set, it contains a continuous path between them.

A convenient way of representing an n -complex \mathcal{C} , is through a sequence of coboundary matrices $A^{(1)}(\mathcal{C}), \dots, A^{(n)}(\mathcal{C})$.

Definition 9. The matrix $A^{(k)}(\mathcal{C}), 1 \leq k \leq n$, has its columns indexed by the k -cells of \mathcal{C} , say $c_1^{(k)}, \dots, c_r^{(k)}$, and rows indexed by the $(k-1)$ -cells, say $c_1^{(k-1)}, \dots, c_m^{(k-1)}$. The (i, j) th entry of this matrix would be zero,

if $c_i^{(k-1)}$ does not lie in the boundary of $c_j^{(k)}$, $+1$, if it occurs with a positive sign in the boundary and -1 , if it occurs with a negative sign in the boundary. k -cells can be degenerate in the sense that their boundary is zero or made up of p -chains where $p \leq k-2$. In such a case the corresponding column of $A^{(k)}(\mathcal{C})$ would be the zero column.

When clear from the context we will write $A^{(k)}$ in place of $A^{(k)}(\mathcal{C})$.

It is evident that the column $c_j^{(k)}$ represents the boundary vector $\partial(c_j^{(k)}) := \sum \beta_i c_i^{(k-1)}$ with the (i, j) th entry being β_i .

Example 10. Let \mathcal{C} be a 3-complex and let abc be an oriented 2-cell and let ab, cb, ac be oriented 1-cells in \mathcal{C} . Then, in $A^{(2)}(\mathcal{C})$, the column abc will have $+1$ in the row ab , -1 in the row cb , -1 in the row ac and in all other rows will have 0.

Consider the k -chain $\sum_1^n x_j c_j^{(k)}$. The boundary $\partial(\sum_1^n x_j c_j^{(k)}) = \sum_1^n x_j \partial(c_j^{(k)})$. Therefore this vector is represented in terms of $(k-1)$ -cells of the complex by $[A^{(k)}(\mathcal{C})] [\mathbf{x}]$. Since the boundary of any boundary vector is the zero chain, we must have $[A^{(k-1)}(\mathcal{C})] [A^{(k)}(\mathcal{C})] [\mathbf{x}] = [\mathbf{0}]$, for any vector \mathbf{x} . It follows that $[A^{(k-1)}(\mathcal{C})] [A^{(k)}(\mathcal{C})] = [\mathbf{0}]$, $k \geq 2$, i.e., every row vector of $[A^{(k-1)}(\mathcal{C})]$ is orthogonal to every column vector of $[A^{(k)}(\mathcal{C})]$. On the other hand, if $[A^{(k-1)}(\mathcal{C})] [A^{(k)}(\mathcal{C})] = [\mathbf{0}]$, it is clear that $[A^{(k-1)}(\mathcal{C})] [A^{(k)}(\mathcal{C})] [\mathbf{x}] = [\mathbf{0}]$, which implies that boundary of any boundary vector is zero. Thus the statement that the boundary of the boundary of a k -chain is the zero $(k-2)$ -vector is equivalent to $[A^{(k-1)}(\mathcal{C})] [A^{(k)}(\mathcal{C})] = [\mathbf{0}]$. Note that this only implies that the row space of $[A^{(k-1)}(\mathcal{C})]$ and the column space of $[A^{(k)}(\mathcal{C})]$ are orthogonal and not that they are complementary orthogonal. The situation relevant to this paper is where they are actually complementary orthogonal (i.e., where the ' $(k-1)$ th homology group' is zero).

Let $S^{(k)}$, $0 \leq k \leq n$ denote the set of k -cells of the n -complex \mathcal{C} . Given a vector $y : S^{(k)} \rightarrow \mathbb{R}$, we define the coboundary z of y by $\mathbf{z}^T := \mathbf{y}^T [A^{(k)}]$ (treating z, y as row vectors $\mathbf{z}^T, \mathbf{y}^T$).

Since $\mathbf{y}^T ([A^{(k)}] [\mathbf{x}]) = (\mathbf{y}^T [A^{(k)}]) [\mathbf{x}]$, it is clear that the 'action' of y on the $(k-1)$ -chain represented by the boundary vector $[A^{(k)}] [\mathbf{x}]$ is the same as the 'action' of the coboundary of y on the k -chain $\sum_1^n x_j c_j^{(k)}$ represented by the vector \mathbf{x} . We call a vector belonging to the row space of $[A^{(k)}(\mathcal{C})]$ a k -coboundary of \mathcal{C} and a k -chain $\sum_1^n x_j c_j^{(k)}$, a k -cycle of \mathcal{C} , if its boundary is zero, i.e., satisfies $[A^{(k)}(\mathcal{C})] [\mathbf{x}] = [\mathbf{0}]$.

We summarize the above discussion in the following theorem.

Theorem 11. For any n -complex \mathcal{C} , $0 < k \leq n$,

1. the boundary of a k -chain x is $[A^{(k)}] [\mathbf{x}]$;
2. every boundary of a k -chain is a $(k-1)$ -cycle;
3. $[A^{(k-1)}(\mathcal{C})] [A^{(k)}(\mathcal{C})] = [\mathbf{0}]$;
4. the space of k -coboundaries and the space of k -cycles are complementary orthogonal.

4. 2-complexes in \mathbb{R}^3

In this paper our interest is in $(2, 1)$ skeletons of 3-complexes which are embedded in \mathbb{R}^3 since we build our higher dimensional electrical networks on such skeletons. We show in this section that the 2-coboundaries and 2-cycles associated with these skeletons have certain properties, for instance complete unimodularity, which are useful for our purpose.

We are given a 2-complex \mathcal{C} embedded in \mathbb{R}^3 , i.e., the vertices of triangles (2-cells) of \mathcal{C} are vectors in \mathbb{R}^3 . We consider the region of interest to be a tetrahedron \mathcal{T} which contains \mathcal{C} in its interior. The tetrahedron \mathcal{T} is decomposed into a set of tetrahedra whose interiors do not intersect and which together with their faces constitute the 3-complex $\mathcal{C}_{\mathcal{T}}$. Additionally, the decomposition is such that the set of triangles $S^{(2)}(\mathcal{C})$ of \mathcal{C} is contained in the set of triangles $S^{(2)}(\mathcal{C}_{\mathcal{T}})$ of $\mathcal{C}_{\mathcal{T}}$. We begin by assigning an orientation to \mathcal{T} . Using the orientation of \mathcal{T} , as the reference, we can orient all the tetrahedra of $\mathcal{C}_{\mathcal{T}}$ consistently so that if two of them

$$\partial(\text{abcd}) = \text{bcd} - \text{acd} + \text{abd} - \text{abc}$$

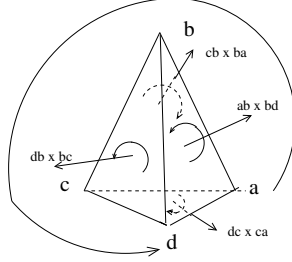


Figure 4: Regular orientation of a tetrahedron

intersect in a triangle, the boundaries of the two tetrahedra assign opposite orientations to the triangle. We could, without loss of generality, assume this orientation of the triangle to be consistent with the outward normal with respect to the tetrahedron. Let us call this orientation, a regular orientation for the tetrahedra of $\mathcal{C}_{\mathcal{T}}$ (see Figure 4).

We will work with regions which are unions of tetrahedra of $\mathcal{C}_{\mathcal{T}}$, which we will call regions of $\mathcal{C}_{\mathcal{T}}$. Let P be a subset of regularly oriented tetrahedra of $\mathcal{C}_{\mathcal{T}}$. The chain $\sum_{\tau_i \in P} \tau_i$ can be identified with the region $\mathcal{R}_P := \bigcup_{\tau_i \in P} \tau_i$. We then have

Theorem 12. *The boundary of \mathcal{R}_P , i.e., the formal sum of the oriented boundary triangles of \mathcal{R}_P , is equal to the 2-cycle $\sum_{\tau_i \in P} \partial(\tau_i)$.*

Proof. When tetrahedra $\tau_i, \tau_j, i \neq j$ intersect in a triangle, because of the regular orientation of tetrahedra, this common triangle is oppositely oriented in τ_i, τ_j , and so cancels out in $\sum_{\tau_i \in P} \partial(\tau_i)$. The only terms that remain are those triangles which occur only once as the boundary of a tetrahedron in P . These are precisely the triangles at the boundary of \mathcal{R}_P and their orientation would be consistent with the outward normal of the tetrahedron in question and therefore also with the outward normal with respect to the region \mathcal{R}_P . Since we have $\partial(\sum_{\tau_i \in P} \partial(\tau_i)) = \sum_{\tau_i \in P} \partial\partial(\tau_i) = 0$, it follows that $\sum_{\tau_i \in P} \partial(\tau_i)$ is a 2-cycle of $\mathcal{C}_{\mathcal{T}}$. Thus the formal sum of the oriented boundary triangles of the region \mathcal{R}_P is equal to the 2-cycle $\sum_{\tau_i \in P} \partial(\tau_i)$. \square

We now state a basic result in algebraic topology (Theorem 19.5 of [10]) after a preliminary definition.

Definition 13. *Let \mathbf{u} denote the closed interval $[0, 1]$. Let \mathcal{C} be an n -complex in $\mathbb{R}^m, m \geq n$, and let $\hat{\mathcal{C}}$ be the union of all n -cells of \mathcal{C} . We say $\hat{\mathcal{C}}$ is contractible if there exists a continuous map $h : \hat{\mathcal{C}} \times \mathbf{u} \rightarrow \hat{\mathcal{C}}$, such that*

*for all $y \in \hat{\mathcal{C}}, h(y, 0) = y$,
for all $y \in \hat{\mathcal{C}}$ and for some $x \in \hat{\mathcal{C}}, h(y, 1) = x$ and, further,
for this $x, h(x, z) = x$, for all $z \in I$.*

Theorem 14. *If the union of n -cells of an n -complex \mathcal{C} is contractible, then for $0 < j \leq n$, every $(j - 1)$ -cycle of the complex is the boundary of a j -chain.*

If an n -complex has the property that for $0 < j \leq n$, every $(j - 1)$ -cycle of the complex is the boundary of a j -chain, then it is said to be acyclic.

Since every convex set in \mathbb{R}^n is contractible it follows that the tetrahedron \mathcal{T} in \mathbb{R}^3 is contractible. We thus have the following result about 2-cycles and boundaries of 3-chains.

Theorem 15. *A 2-chain x of $\mathcal{C}_{\mathcal{T}}$ is a 2-cycle of $\mathcal{C}_{\mathcal{T}}$ iff it is the boundary of a 3-chain of $\mathcal{C}_{\mathcal{T}}$. Equivalently, the row space of the 2-coboundary matrix $A^{(2)}(\mathcal{C}_{\mathcal{T}})$ and the column space of the 3-coboundary matrix $A^{(3)}(\mathcal{C}_{\mathcal{T}})$ are complementary orthogonal. Hence the column space of the 3-coboundary matrix $A^{(3)}(\mathcal{C}_{\mathcal{T}})$ is the 2-cycle space of $\mathcal{C}_{\mathcal{T}}$.*

Remark 1. A direct proof of Theorem 15 through induction on the number of regions into which the tetrahedron \mathcal{T} is divided is routine. However the link to Theorem 14 appears insightful. See Section 11 for an enlargement of the ideas involved.

Definition 16. Let \mathcal{V} be a vector space on S over \mathbb{R} . We say that a vector $x \in \mathcal{V}$ has minimal support (see Definition 3) iff the support is nonempty and no other $z \in \mathcal{V}$ has its support properly contained in that of x .

We now have the following useful result about $\mathcal{C}_{\mathcal{T}}$.

Theorem 17. Let the tetrahedra of $\mathcal{C}_{\mathcal{T}}$ be regularly oriented. Let x be a 2-cycle of $\mathcal{C}_{\mathcal{T}}$ with minimal support. Then,

1. x is the boundary of a 3-chain $\tilde{y} := \sum \alpha_i \tau_i$, where all the nonzero α_i are equal to a constant;
2. x is the multiple of a $0, +1, -1$ vector and the support of x is the set of boundary triangles of the union of tetrahedra in the support of \tilde{y} .

Proof. 1. By Theorem 15, we know that $x = \partial y$ for some 3-chain of $\mathcal{C}_{\mathcal{T}}$. Let the support of y be P and, without loss of generality, let the maximum entry of y be positive and equal to, say, k . Consider the vector \tilde{y} defined by $\tilde{y}(\tau) = y(\tau)$, if $y(\tau) = k$, and $\tilde{y}(\tau) = 0$, otherwise.

We claim that the support of the 2-cycle $\partial\tilde{y}$ is contained in the support of ∂y . To see this, we first observe that the support \tilde{P} of \tilde{y} is a set of regularly oriented tetrahedra in $\mathcal{C}_{\mathcal{T}}$, whose union is the region $\mathcal{R}_{\tilde{P}}$. Suppose a triangle δ occurs in the support of $\partial\tilde{y}$. Then by Theorem 12, it is a boundary triangle of $\mathcal{R}_{\tilde{P}}$. Therefore it occurs as a facet of only one tetrahedron, say τ_{in} in \tilde{P} . It can meet at most one other tetrahedron in $P \setminus \tilde{P}$. If it meets none, δ will continue to occur in the support of $x = \partial y$. Suppose it meets one other tetrahedron say τ_{out} in $P \setminus \tilde{P}$. Now $k = y(\tau_{in}) > |y(\tau_{out})|$. So $\partial y(\delta) = k \pm y(\tau_{out}) \neq 0$. In other words, δ occurs in the support of $x = \partial y$. But we know that x is a 2-cycle of $\mathcal{C}_{\mathcal{T}}$ with minimal support. We conclude that $y = \tilde{y}$. Now $\tilde{y} = k\hat{y}$, where \hat{y} is a $0, +1$ vector and $x = k\partial\hat{y}$.

2. Since \hat{y} is a $0, +1$ vector, in $\partial\hat{y}$, the boundary triangles of $\mathcal{R}_{\tilde{P}}$ will acquire value $0, \pm 1$, and the other triangles which are common to two tetrahedra in \tilde{P} will occur with opposite orientations in those two and will cancel out. Thus $\partial\hat{y}$ is a $0, \pm 1$ vector which makes x a multiple of a $0, \pm 1$ vector and it is clear that the support of x is the set of boundary triangles of $\mathcal{R}_{\tilde{P}}$. \square

Definition 18. A vector space \mathcal{V} on S over \mathbb{R} is said to be regular or completely unimodular iff every minimal support vector is a multiple of a $0, \pm 1$ vector.

In the following theorem, we present a few well known and useful facts about regular vector spaces. But first we need a few definitions.

Definition 19. A matrix is said to be totally unimodular iff all its subdeterminants are $0, \pm 1$.

Definition 20. A matrix whose rows form a basis for a vector space \mathcal{V} is called a representative matrix for \mathcal{V} . If, after column permutation, it can acquire the form $[I \mid K]$, where I denotes an identity matrix of the appropriate order, we say it is a standard representative matrix for \mathcal{V} . The standard representative matrix is said to be with respect to the column subset B , if this latter is the set of columns of an identity matrix.

Definition 21. Let S be a finite set and \mathcal{I} be a family of subsets of S such that maximal members of \mathcal{I} contained in any subset of S have the same size. Then $\mathcal{M} := (S, \mathcal{I})$ is called a matroid on S . Members of \mathcal{I} are called independent sets of \mathcal{M} . A maximal independent set of \mathcal{M} contained in S is called a base of \mathcal{M} . Complements of bases of \mathcal{M} are called cobases of \mathcal{M} .

The most natural example of a matroid is the pair (S, \mathcal{I}) , where S is the set of columns of a matrix A , and \mathcal{I} is the family of independent subsets of columns of A . We denote this matroid by $\mathcal{M}(A)$.

It is clear that a set of columns of a representative matrix of \mathcal{V} is independent iff the corresponding set of columns is independent in any other representative matrix of \mathcal{V} .

Definition 22. Let \mathcal{V} be a vector space on S over \mathbb{R} . Let columns of a representative matrix A of \mathcal{V} be identified with elements of S . The matroid $\mathcal{M}(A)$ (which is independent of the representative matrix A chosen for \mathcal{V}) is called the matroid $\mathcal{M}(\mathcal{V})$ associated with \mathcal{V} .

The following elementary result can be proved using ideas from matroid theory or by translating these ideas to vector spaces. We omit the proof.

Lemma 23. If x is a minimal support vector in \mathcal{V} , then there exists a base of $\mathcal{M}(\mathcal{V})$, which intersects $\text{supp}(x)$ in a single element.

Definition 24. Let x, y be vectors on S over a field \mathcal{F} . Then the dot product $\langle x, y \rangle := \sum_{e \in S} x(e)y(e)$. We say x, y are orthogonal iff $\langle x, y \rangle = 0$. Let \mathcal{K} be a collection of vectors on S . $\mathcal{K}^\perp := \{y : \langle x, y \rangle = 0, x \in \mathcal{K}\}$. Let $T \subseteq S$. Then $\mathcal{K} \circ T := \{x_T : x_T = y|_T, y \in \mathcal{K}\}$ is called the restriction of \mathcal{K} to T and $\mathcal{K} \times T := \{x_T : x_T = y|_T, y \in \mathcal{K}, y(e) = 0, e \notin T\}$ is called the contraction of \mathcal{K} to T .

Theorem 25. Let \mathcal{V} be a vector space on S over \mathbb{R} . Let Q_B be a standard representative matrix with respect to a base B of $\mathcal{M}(\mathcal{V})$. Let $e \in B$ and let x_e be the row of Q_B with $x_e(e) = 1$ and $x_e(e') = 0, e' \in B \setminus e$. Then,

1. x_e has minimal support in \mathcal{V} ;
2. the collection of minimal support vectors in \mathcal{V} spans \mathcal{V} . Therefore if a vector is orthogonal to all minimal support vectors of \mathcal{V} , then it is orthogonal to all vectors in \mathcal{V} ;
3. if \mathcal{V} is regular, then Q_B has $0, \pm 1$ entries;
4. if \mathcal{V} is regular, then Q_B is totally unimodular;
5. if \mathcal{V} is regular, so is \mathcal{V}^\perp .

Proof. 1. Let Q_B be an $r \times n$ matrix. Let y be a vector in \mathcal{V} such that $\text{supp}(y)$ is a proper subset of $\text{supp}(x_e)$. Now $\mathbf{y} = \mathbf{z}^T \mathbf{Q}_B$ for some vector z . Since B is the set of columns of an identity matrix, we must have $y(e') = z(e'), e' \in B$. We have $\text{supp}(x_e) \cap B = \{e\} \supseteq \text{supp}(y) \cap B$. So $z(e') = 0, e' \neq e, e' \in B$. It follows that if $y(e) \neq 0$, then $y = z(e)x_e$ so that $\text{supp}(y) = \text{supp}(x)$ and if $y(e) = 0$, then y must be the zero vector. Hence x_e has minimal support in \mathcal{V} .

2. This follows since the rows of Q_B form a basis for \mathcal{V} and have minimal support.

3. Since \mathcal{V} is regular and x_e has minimal support in \mathcal{V} , we must have x_e as a multiple of a $0, \pm 1$ vector. But $x_e(e) = 1$. So x_e must be a $0, \pm 1$ vector.

4. We will first show that every $r \times r$ submatrix has determinant $0, \pm 1$. If the matrix is singular the determinant is zero. So let us suppose that a submatrix M is nonsingular with columns B' . Consider the standard representative matrix $Q_{B'}$. Let N be the submatrix of $Q_{B'}$ with columns B . It is clear that $Q_B = MQ_{B'}$ and $NQ_B = Q_{B'}$, so that $MN = I$, and $\det(M) \times \det(N) = 1$. Since $Q_B, Q_{B'}$ have $0, \pm 1$ entries, $\det(M), \det(N)$ are integers. It follows that $\det(M)$ is ± 1 .

Next consider a $k \times k$ submatrix P of Q_B where $k \leq r$. We can extend the column set of Q_B corresponding to columns of P using only columns in B so that we get a nonsingular $r \times r$ submatrix M' . Since the columns in B are columns of the identity matrix, we must have $\pm 1 = \det(M') = \pm \det(P)$. We conclude that $\det(P)$ is ± 1 .

5. It is clear that $[I \mid K]$ is a representative matrix of \mathcal{V} iff $[-K^T \mid I]$ is a representative matrix for \mathcal{V}^\perp . Since, by Lemma 23, every minimal support vector is a multiple of a row of some standard representative matrix, and, we have shown above that, standard representative matrices of \mathcal{V} have $0, \pm 1$ entries, the result follows.

Remark 2. If the j -coboundary space is regular, it may not in general be true that the j -coboundary matrix is totally unimodular. For instance, the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is a submatrix of the 2-coboundary matrix of a 2-complex embedded in \mathbb{R}^3 . But, as can be verified, it has determinant 2. For a discussion of these ideas, relating total unimodularity to orientability, see [6].

□

We have the following well known and fundamental results whose proofs, however, are simple.

Theorem 26. *Let \mathcal{V} be a vector space on a finite set S and let $T \subseteq S$. Then*

1. $\mathcal{V}^{\perp\perp} = \mathcal{V}$;
2. $(\mathcal{V} \circ T)^\perp = \mathcal{V}^\perp \times T$;
3. $(\mathcal{V} \times T)^\perp = \mathcal{V}^\perp \circ T$.

Let \mathcal{C} be a 2-complex embedded in \mathbb{R}^3 , \mathcal{T} be a tetrahedron whose interior contains \mathcal{C} , and let $\mathcal{C}_\mathcal{T}$ be defined as earlier with the set of triangles $S := S^{(2)}(\mathcal{C})$ contained in the set of triangles $S^{(2)}(\mathcal{C}_\mathcal{T})$. We have

Lemma 27. *Let $\mathcal{V}_\mathcal{T}$ be the space of 2-coboundaries of $\mathcal{C}_\mathcal{T}$. Then*

1. $\mathcal{V}_\mathcal{T} \circ S$ is the space of 2-coboundaries of \mathcal{C} ;
2. $\mathcal{V}_\mathcal{T}^\perp$ is the space of 2-cycles of $\mathcal{C}_\mathcal{T}$ and $\mathcal{V}_\mathcal{T}^\perp \times S$ is the space of 2-cycles of \mathcal{C} .

Proof. 1. We have, $\mathcal{V}_\mathcal{T}$ as the row space of $A^{(2)}(\mathcal{C}_\mathcal{T})$, and $\mathcal{V}_\mathcal{T} \circ S$ as the row space of the matrix obtained by deleting the columns of $A^{(2)}(\mathcal{C}_\mathcal{T})$ that are not in S . But, except for zero rows, this matrix is the same as the coboundary matrix $A^{(2)}(\mathcal{C})$, of \mathcal{C} . The result follows.

2. The spaces of 2-coboundaries and of 2-cycles of $\mathcal{C}_\mathcal{T}$ are complementary orthogonal by Theorem 11. Therefore $\mathcal{V}_\mathcal{T}^\perp$ is the space of 2-cycles of $\mathcal{C}_\mathcal{T}$ and $(\mathcal{V}_\mathcal{T} \circ S)^\perp$ is the space of 2-cycles of \mathcal{C} . By Theorem 26, we therefore have that $\mathcal{V}_\mathcal{T}^\perp \times S$ is the space of 2-cycles of \mathcal{C} . □

Let $\mathcal{V}_\mathcal{T}$ be the space of 2-coboundaries of $\mathcal{C}_\mathcal{T}$. By Theorem 17, we know that every minimal support 2-cycle of $\mathcal{C}_\mathcal{T}$ is a multiple of a $0, \pm 1$ vector which latter is the boundary vector of the union of a subset of regularly oriented tetrahedra of $\mathcal{C}_\mathcal{T}$. Let $\mathcal{K}_\mathcal{T}$ be the collection of boundary vectors of regions in \mathbb{R}^3 which are unions of tetrahedra of $\mathcal{C}_\mathcal{T}$. By Theorem 25, such vectors span the space $\mathcal{V}_\mathcal{T}^\perp$ of 2-cycles of $\mathcal{C}_\mathcal{T}$. Let $\mathcal{K} \subseteq \mathcal{K}_\mathcal{T}$, be the collection of minimal support vectors with support contained in $S = S^{(2)}(\mathcal{C})$. Clearly \mathcal{K} spans the subspace of vectors of $\mathcal{V}_\mathcal{T}^\perp$ whose support is contained in S , and therefore $\mathcal{K} \times S$ spans the space $\mathcal{V}_\mathcal{T}^\perp \times S$. Suppose a vector on S is orthogonal to $\mathcal{K} \times S$. Then it must also be orthogonal to $\mathcal{V}_\mathcal{T}^\perp \times S$, i.e., (by Lemma 27) to all 2-cycles of \mathcal{C} , i.e., must be a 2-coboundary of \mathcal{C} . Thus we have

Lemma 28. *Let the tetrahedra of $\mathcal{C}_\mathcal{T}$ be regularly oriented. Let \mathcal{K} be the collection of boundary vectors of regions in \mathbb{R}^3 which are unions of tetrahedra of $\mathcal{C}_\mathcal{T}$, and whose support is contained in the set of triangles $S = S^{(2)}(\mathcal{C})$. If a vector on S is orthogonal to all vectors of $\mathcal{K} \times S$, then it is a 2-coboundary of \mathcal{C} .*

5. Electrical 2-networks

In this section we define electrical 2-networks, prove a generalized Tellegen's Theorem and describe procedures of solution for them.

5.1. Flux and current adjusted mmf

Let B and H be the magnetic flux density and magnetic intensity vectors defined on \mathbb{R}^3 . Let δ be an oriented triangle in \mathbb{R}^3 . Then the flux ϕ_δ through the triangle, in the direction of the normal consistent with the orientation of the triangle (see Figure 3), is equal to $\int_\delta B \cdot ds$. The mmf m'_δ is the mmf associated with the triangle in the direction of the normal. This is the average value of $H \Delta l$ over the triangle, i.e., the value of $\frac{\int_\delta H \Delta l \cdot ds}{\text{Area of } \delta}$. The triangles we work with are zero thickness idealizations of physical triangles with nonzero thickness Δl . From Maxwell's equations, we have the relation $\oint_C H \cdot dl = i$, where i is the current through an edge e and C is a contour around the edge traversed in a direction consistent with the orientation of the edge. (If O is a point in e , A, B , are points encountered successively as we go around the contour,

the vector product $OA \times OB$ is in the direction of e .) Suppose as in Figure 1, we have triangles $\delta_1, \delta_2, \delta_3$ incident at and around an edge e . Let the triangle orientation be such that in each case it agrees with the orientation of e and let a current i_{δ_j} circulate around each triangle δ_j in the direction of orientation of the triangle. Let us go around e , along a contour C which intersects all the triangles incident at e , in a direction consistent with the orientation of e . The only nonzero contribution, to the integral $\oint_C H \cdot dl$, will be where the contour intersects the triangles. This is because we are assuming that except for the triangles, the medium has infinite permeability and flux everywhere is finite. So the equation $\oint_C H \cdot dl = i$, reduces to $m'_{\delta_1} + m'_{\delta_2} + m'_{\delta_3} = i_{\delta_1} + i_{\delta_2} + i_{\delta_3} = i$. This equation can be rewritten as $m_{\delta_1} + m_{\delta_2} + m_{\delta_3} = 0$, where $m_{\delta_j} := m'_{\delta_j} - i_{\delta_j}$. We call the quantity $m_{\delta_j} := m'_{\delta_j} - i_{\delta_j}$, the current adjusted mmf associated with δ_j .

We state the following constraints in relation to the 3-complex $\mathcal{C}_{\mathcal{T}}$ and the 2-complex \mathcal{C} , both embedded in \mathbb{R}^3 .

Constraint K_1 (Generalized KVL)

The net outward flux ϕ_S through 2-cells of \mathcal{C} which are boundaries of regions of tetrahedra of $\mathcal{C}_{\mathcal{T}}$ is zero. Equivalently, the flux vector ϕ_S is orthogonal to 2-cycles of \mathcal{C} which are boundaries of regions of tetrahedra of $\mathcal{C}_{\mathcal{T}}$.

Constraint K_2 (Generalized KCL)

If a set of triangles S_e of \mathcal{C} are incident at an edge e and oriented to agree with the orientation of e , then $\sum_{\delta_i \in S_e} m_{\delta_i} = 0$, where m_{δ_i} is the ‘current adjusted mmf’ defined above associated with δ_i . Equivalently, the current adjusted mmf vector m_S is orthogonal to the rows of the coboundary matrix $A^{(2)}(\mathcal{C})$ and is therefore a 2-cycle of \mathcal{C} .

5.2. Generalized Tellegen’s Theorem

We are now in a position to define an electrical 2-network \mathcal{N} .

Definition 29. *An electrical 2-network \mathcal{N} is a pair $(\mathcal{C}_S, \mathcal{D}_S)$, where \mathcal{C}_S is a 2-complex embedded in \mathbb{R}^3 with S as its 2-cells,*

\mathcal{D}_S , the device characteristic, is a collection of ordered pairs (ϕ_S, m_S) of vectors on S over \mathbb{R} . A solution of \mathcal{N} is an ordered pair (ϕ_S, m_S) satisfying the following constraints

1. ϕ_S satisfies Constraint K_1 ;
2. m_S satisfies Constraint K_2 ;
3. $(\phi_S, m_S) \in \mathcal{D}_S$.

We now have the following ‘generalized Tellegen’s Theorem’ for an electrical 2-network \mathcal{N} .

Theorem 30. *Let $\mathcal{N} := (\mathcal{C}_S, \mathcal{D}_S)$, be an electrical 2-network. Let $\mathcal{V}_S^\phi, \mathcal{V}_S^m$ be the spaces of vectors which satisfy constraints K_1, K_2 respectively for \mathcal{N} . Then $\mathcal{V}_S^\phi, \mathcal{V}_S^m$ are complementary orthogonal, being respectively the 2-coboundary and 2-cycle spaces of \mathcal{C} .*

Proof. Constraint K_1 states that the flux vector ϕ_S is orthogonal to 2-cycles of \mathcal{C} which are boundaries of regions of tetrahedra of $\mathcal{C}_{\mathcal{T}}$. This, by Lemma 28, is equivalent to saying that it is orthogonal to all 2-cycles of \mathcal{C} and therefore is a 2-coboundary of \mathcal{C} . Thus \mathcal{V}_S^ϕ is the space of 2-coboundaries of \mathcal{C} .

Constraint K_2 states that the current adjusted mms vector m_S is a 2-cycle of \mathcal{C} . Thus \mathcal{V}_S^m is the space of 2-cycles of \mathcal{C} .

Since by Theorem 11, the spaces of 2-cycles and of 2-coboundaries of \mathcal{C} are complementary orthogonal, the result follows. □

Let $A := A^{(2)}(\mathcal{C})$, be the 2-coboundary matrix of \mathcal{C} . Using Theorem 30, we can restate the constraints of the network \mathcal{N} as follows.

$$\begin{aligned} A \mathbf{m}_S &= \mathbf{0}. \\ A^T \mathbf{y} &= \phi_S \\ (\phi_S, m_S) &\in \mathcal{D}_S. \end{aligned} \tag{5}$$

In the special case where the flux is linearly related to mmf, we may write

$$G \phi_S = [\mathbf{m}_S + \mathbf{I}_S], \tag{6}$$

where G is a positive diagonal matrix with its (j, j) entry being the reluctance of the triangle δ_j , m_S is the current adjusted mmf vector and ϕ_S is the flux vector associated with $S = S^{(2)}(\mathcal{C})$.

5.3. Solution of an electrical 2-network

We now discuss the procedure for solving Equation 5, when the device characteristic is linear.

Let \hat{A} be composed of a maximal linearly independent set of rows of the 2-coboundary matrix A of \mathcal{C} . We have, writing $\hat{A} \mathbf{m}_S = \mathbf{0}$ as $\hat{A}[\mathbf{m}_S + \mathbf{I}_S] = \hat{A} \mathbf{I}_S$ and $A^T \mathbf{y} = \phi_S$ as $\hat{A}^T \mathbf{z}$,

$$\begin{aligned} \hat{A} [\mathbf{m}_S + \mathbf{I}_S] &= \hat{A} \mathbf{I}_S, \text{ i.e.,} \\ \hat{A} G \phi_S &= \hat{A} \mathbf{I}_S, \text{ i.e.,} \end{aligned} \tag{7}$$

$$\hat{A} G \hat{A}^T \mathbf{z} = \hat{A} \mathbf{I}_S. \tag{8}$$

When G is positive diagonal, since the rows of \hat{A} are linearly independent, the matrix $\hat{A} G \hat{A}^T$ is positive definite and so invertible so that a unique solution for z and therefore also for the variables ϕ_S, m_S , is guaranteed. If \mathcal{C} is large the matrix $\hat{A} G \hat{A}^T$ would be sparse so that existing efficient techniques for solution of such equations can be adopted. As opposed to conventional graph based electrical networks, the main extra computational effort is in computing \hat{A} from A . In the case of graph based circuits this computation is trivial since we simply have to drop one row per connected component of the graph. Here it appears as though we require linear algebraic computations which though efficient are not trivial. There are other advantages to dealing with graph based circuits - the equations can be ‘preprocessed’ in near linear time to make them more suitable for, say, parallel processing. Such convenient structures are obtained, for instance, through ‘multiport decomposition’ and ‘topological transformation’ ([11]). The algorithms required are graph based. It is not immediately clear if anything similar can be done in the present situation.

We will show, in subsequent pages, that the present network can actually be reduced to a graph based ‘dual network’. To appreciate what would be the procedure after such conversion let us discuss a cycle-based rather than coboundary-based (as above) equations for the network.

5.4. Cycle based equations for the network \mathcal{N}

Let P be a representative matrix of the 2-cycle space of \mathcal{C} . Using Theorem 30, we can restate the constraints of the network \mathcal{N} as follows.

$$\begin{aligned} P \phi_S &= \mathbf{0}. \\ P^T \mathbf{q} &= \mathbf{m}_S \\ (\phi_S, m_S) &\in \mathcal{D}_S. \end{aligned} \tag{9}$$

In the special case where the flux is linearly related to mmf, we may write

$$\phi_S = R [\mathbf{m}_S + \mathbf{I}_S]. \tag{10}$$

Let us, for handling more general but analogous situations, replace this equation by

$$\phi_S + \mathbf{E}_S = R [\mathbf{m}_S + \mathbf{I}_S], \tag{11}$$

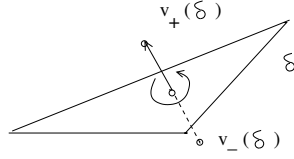


Figure 5: $v_+(\delta)$ and $v_-(\delta)$ for triangle δ

where \mathbf{E}_S is an additional source term. We have,

$$\begin{aligned}
 P \phi_S &= \mathbf{0}, \text{ i.e.,} \\
 P [\phi_S + \mathbf{E}_S] &= P \mathbf{E}_S, \text{ i.e.,} \\
 P R [\mathbf{m}_S + \mathbf{I}_S] &= P \mathbf{E}_S, \text{ i.e.,} \\
 P R \mathbf{m}_S &= -P R \mathbf{I}_S + P \mathbf{E}_S, \text{ i.e.,}
 \end{aligned} \tag{12}$$

$$P R P^T \mathbf{q} = -P R \mathbf{I}_S + P \mathbf{E}_S. \tag{13}$$

When R is positive diagonal, since the rows of P are linearly independent, the matrix PRP^T is positive definite and so invertible so that a unique solution is guaranteed. We will show that the matrix P can be chosen as the reduced incidence matrix (dropping one row per connected component from the incidence matrix) of a graph which is the ‘matroid dual’ of \mathcal{C} . Thus Equation 9 will look exactly like the constraints of a graph based network. We will build this latter graph in linear time from \mathcal{C} .

6. Triangle adjacency graph $tag(\mathcal{C})$ and the graph $\mathcal{G}_{tag(\mathcal{C})}$

The triangle adjacency graph $tag(\mathcal{C})$ of \mathcal{C} is constructed from the embedding information about \mathcal{C} in \mathbb{R}^3 as to how triangles lie around an edge they are incident on. We show later, that for the purpose of writing minimal equations for the network, we only have to find the connected components of $tag(\mathcal{C})$.

Let \mathcal{C} be a 2-complex embedded in \mathbb{R}^3 .

We assign each oriented triangle $\delta := abc$, a ‘positive’ node $v_+(\delta)$ and a ‘negative’ node $v_-(\delta)$. Physically, $v_+(\delta)$ is placed at a distance ϵ from the centroid of the triangle moving in the direction of $ab \times bc$, (i.e., normal to the triangle consistent with the orientation of the triangle) and $v_-(\delta)$ is placed at a distance ϵ from the centroid of the triangle in the opposite direction (see Figure 5). The value of ϵ is sufficiently small so that the line segment $(v_+(\delta), v_-(\delta))$ does not intersect any other triangle of \mathcal{C} . Note that if δ is given the opposite orientation and denoted δ' , then $v_+(\delta') = v_-(\delta)$ and $v_-(\delta') = v_+(\delta)$.

Definition 31. To build $tag(\mathcal{C})$, we start with the vertex set

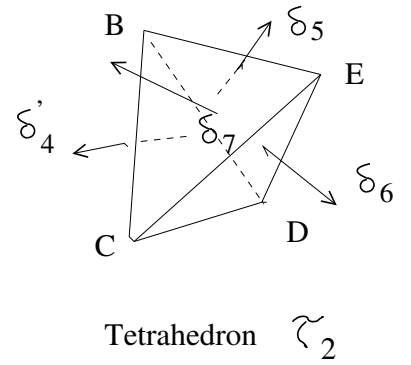
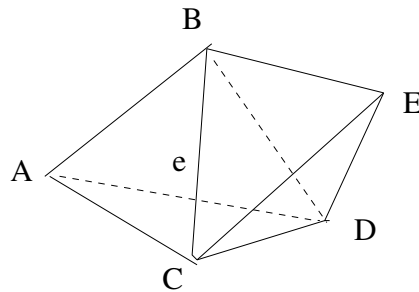
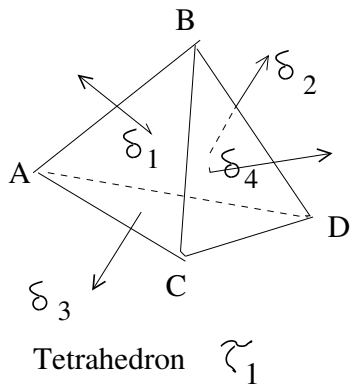
$$V(tag(\mathcal{C})) := \{v_+(\delta_i), \delta_i \text{ a triangle in } \mathcal{C}\} \cup \{v_-(\delta_i), \delta_i \text{ a triangle in } \mathcal{C}\}.$$

Suppose triangles $\delta_1, \dots, \delta_k$ are incident at edge e and are oriented to agree with the orientation of e . Further, suppose when we rotate around e in a direction consistent with the orientation of e , we encounter them successively as $\delta_1, \delta_2, \dots, \delta_k, \delta_1$ (see Figure 1 for the case $k = 3$). Then in $tag(\mathcal{C})$, we construct the undirected edges

$$(v_+(\delta_1), v_-(\delta_2)), (v_+(\delta_2), v_-(\delta_3)), \dots, (v_+(\delta_{k-1}), v_-(\delta_k)), (v_+(\delta_k), v_-(\delta_1)).$$

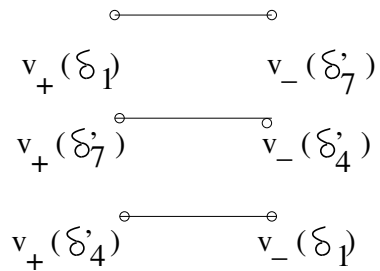
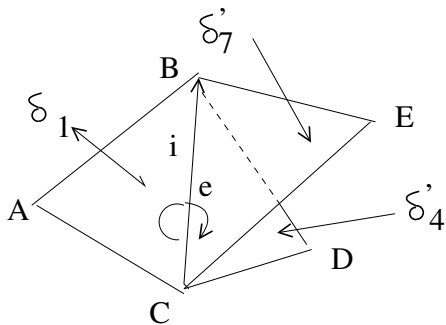
This is done for every edge e of \mathcal{C} (see Figure 6). The resulting set of edges is the edge set of $tag(\mathcal{C})$.

Remark 3. We note that in the above definition, ‘ $v_+(\delta_i), v_-(\delta_i)$ ’ are used to denote both points in \mathbb{R}^3 and the corresponding vertices in $tag(\mathcal{C})$. It would be clear from the context which entity is being referred to.



The 3-complex \mathcal{Q}^3 obtained by pasting ζ_1, ζ_2 across δ_4

The complex \mathcal{Q} is the 2-subcomplex on triangles of \mathcal{Q}^3



Triangles incident at edge e

Corresponding edges in $\text{tag}(\mathcal{Q})$

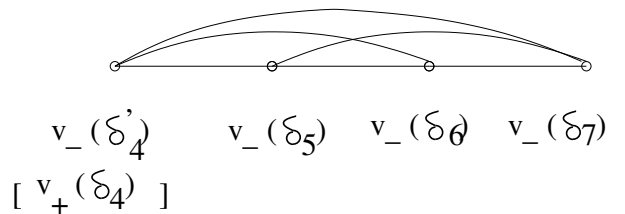
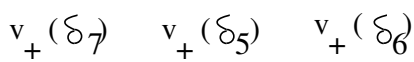
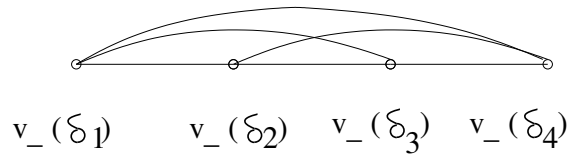
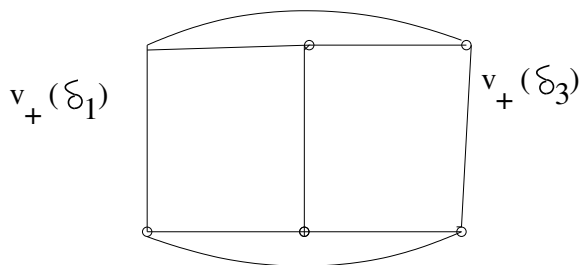


Figure 6: The graph $\text{tag}(\mathcal{C})$ for a complex \mathcal{C}

In Figure 6, we have constructed the graph $\text{tag}(\mathcal{C})$ for a 2–complex \mathcal{C} which is a subcomplex of a 3–complex $\mathcal{C}_{\mathcal{T}}$ obtained by pasting two tetrahedra τ_1, τ_2 across the face δ_4 . The 2–complex \mathcal{C} has 2–cells $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7$ and 1–cells and 0–cells, the edges and vertices of these triangles. The tetrahedra τ_1, τ_2 are regularly oriented (i.e., so that in the boundary the triangles appear with an orientation consistent with the outward normal as in Figure 4). The triangle δ_4 in tetrahedron τ_1 appears as δ'_4 with opposite orientation in tetrahedron τ_2 . Observe that this graph has three connected components corresponding to the ‘outer surface’ of the complex composed of the triangles $\delta_1, \delta_2, \delta_3, \delta_5, \delta_6, \delta_7$ and the ‘inner surfaces’ of the tetrahedra τ_1, τ_2 composed respectively of the triangles $\delta_1, \delta_2, \delta_3, \delta_4$ and $\delta'_4, \delta_5, \delta_6, \delta_7$. Our regular orientation of the tetrahedra has led to these surfaces being represented by v_+ vertices for the outer surface and v_- vertices for the inner surfaces (noting that $\delta_4 = -\delta'_4$).

To build the graph $\mathcal{G}_{\text{comptag}(\mathcal{C})}$, we first find the connected components of the graph $\text{tag}(\mathcal{C})$. Let V_1, \dots, V_k be the vertex sets of these components. We take the vertex set $V(\mathcal{G}_{\text{comptag}(\mathcal{C})})$ to be the set $\{V_1, \dots, V_k\}$. Let δ be a triangle of \mathcal{C} and let $v_+(\delta) \in V_i, v_-(\delta) \in V_j$. Then in $\mathcal{G}_{\text{comptag}(\mathcal{C})}$ we introduce a directed edge $e(\delta)$ from V_i to V_j . (If $V_i = V_j$ for δ then $e(\delta)$ would be a self loop.) This is done for each triangle δ . Figure 7 shows the graph $\mathcal{G}_{\text{comptag}(\mathcal{C})}$ for the complex in Figure 6. This has been constructed by building the three supernodes

$$V_1 := \{v_+(\delta_1), v_+(\delta_2), v_+(\delta_3), v_+(\delta_5), v_+(\delta_6), v_+(\delta_7)\}, V_2 := \{v_-(\delta_1), v_-(\delta_2), v_-(\delta_3), v_-(\delta_4)\},$$

$$V_3 := \{v_-(\delta'_4), v_-(\delta_5), v_-(\delta_6), v_-(\delta_7)\} \equiv \{v_+(\delta_4), v_-(\delta_5), v_-(\delta_6), v_-(\delta_7)\}$$

and introducing an edge for each δ from the super node in which its v_+ vertex lies to the supernode in which its v_- vertex lies.

It is clear that when \mathcal{C} is removed from \mathcal{T} , $\mathcal{T} \setminus \mathcal{C}$ could split into connected regions (see beginning of Section 4 for a description of $\mathcal{C}, \mathcal{T}, \mathcal{C}_{\mathcal{T}}$ etc.). We now have the following intuitively obvious result.

Lemma 32. *If $v_{\pm}(\delta_1)$ and $v_{\pm}(\delta_2)$ are path connected in $\text{tag}(\mathcal{C})$, then they are also path connected in $\mathcal{T} \setminus \mathcal{C}$.*

Proof. It is sufficient to consider the case where the path length in $\text{tag}(\mathcal{C})$ is of length 1, i.e., when δ_1, δ_2 share a common edge e in \mathcal{C} . We may assume that e appears with a positive sign in the boundary of both δ_1 and δ_2 . This means in \mathbb{R}^3 , as we rotate about e consistent with its orientation, we encounter δ_2 immediately after δ_1 (see Figure 1). In $\mathcal{C}_{\mathcal{T}}$, there would be tetrahedra τ_1, τ_2 such that $v_+(\delta_1)$ belongs to tetrahedron τ_1 and $v_-(\delta_2)$ belongs to tetrahedron τ_2 . In $\mathcal{C}_{\mathcal{T}}$, there would be tetrahedra τ'_1, \dots, τ'_k all having e as an edge, such that τ_1, τ'_1 share a common triangle, $\tau'_i, \tau'_{i+1}, i = 1, \dots, k-1$, share a common triangle, τ'_k, τ_2 share a common triangle with none of these common triangles belonging to \mathcal{C} . Draw an arc of radius ϵ from a point x_1 in τ_1 to a point x_2 in τ_2 . Choose ϵ to be sufficiently small so that it lies entirely in the union of the tetrahedra $\tau_1, \tau'_1, \dots, \tau'_k, \tau_2$. The path $v_+(\delta_1), x_1$ lies in τ_1 , the path $x_2, v_-(\delta_2)$ lies in τ_2 , and the ϵ –arc from x_1 to x_2 lies in the union of the tetrahedra $\tau_1, \tau'_1, \dots, \tau'_k, \tau_2$. This path will continue to exist even if \mathcal{C} is deleted from \mathcal{T} . Observe that this argument is valid in the simpler case where τ_1, τ_2 share a common triangle. The result follows. \square

We will call the algorithm for finding the connected component of $\text{tag}(\mathcal{C})$, the ‘sliding algorithm’, because if we translate the traversal of nodes of $\text{tag}(\mathcal{C})$ to a ‘physical’ traversal of the triangles of \mathcal{C} , it appears as though we are sliding on the surface of triangles of \mathcal{C} .

It is clear that the construction of $\text{tag}(\mathcal{C})$ from \mathcal{C} is linear time on the size of \mathcal{C} . Since finding connected component of a graph is linear time on the size of the graph, it is clear that the construction of $\mathcal{G}_{\text{comptag}(\mathcal{C})}$ from \mathcal{C} is also linear time on the size of \mathcal{C} .

Definition 33. *Let \mathcal{G} be a directed graph. The incidence matrix $A(\mathcal{G})$ of \mathcal{G} is defined as follows. $A(\mathcal{G})$ has rows corresponding to nodes of \mathcal{G} and columns corresponding to edges of \mathcal{G} . The entry $A(\mathcal{G})_{i,j}$ is zero if edge j is not incident on node i , is +1 if the edge j is incident on node i but directed away and is –1 if the edge j is incident on node i but directed inward. If an edge has only one end point, then the corresponding column is a zero column.*

We have the following well known lemma whose proof is routine (see [11], for instance).

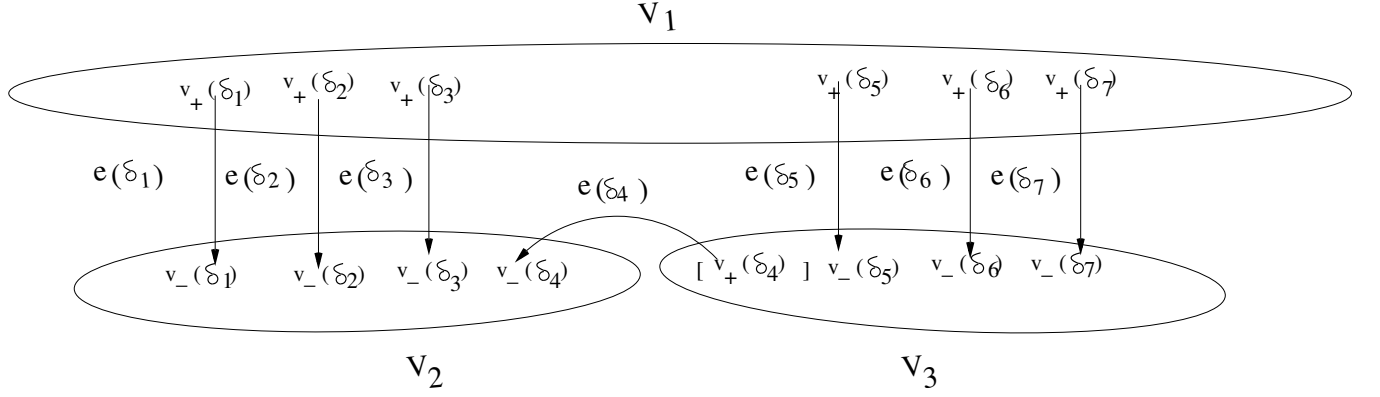


Figure 7: The graph $\mathcal{G}_{comptag(\mathcal{C})}$ for the complex \mathcal{C}

Lemma 34. *Let \mathcal{G} be a directed graph. Then*

1. *The sum of the rows of the incidence matrix $A(\mathcal{G})$ of \mathcal{G} is zero;*
2. *If \mathcal{G} is connected, then the matrix obtained by deleting any row of $A(\mathcal{G})$ has linearly independent rows.*

The incidence matrix $A(\mathcal{G}_{comptag(\mathcal{C})})$ has rows corresponding to nodes of $\mathcal{G}_{comptag(\mathcal{C})}$, i.e., to vertex sets of connected components of $tag(\mathcal{C})$, and columns corresponding to edges of $\mathcal{G}_{comptag(\mathcal{C})}$, i.e., triangles of \mathcal{C} . Let us denote by V_i both a vertex of $\mathcal{G}_{comptag(\mathcal{C})}$ as well as the corresponding vertex set of the connected component of $tag(\mathcal{C})$.

Thus the entry $A(\mathcal{G}_{comptag(\mathcal{C})})_{V_i, \delta_j}$ is zero if neither $v_+(\delta_j)$ nor $v_-(\delta_j)$ belongs to V_i in $tag(\mathcal{C})$, is zero if both of $v_+(\delta_j), v_-(\delta_j)$ belong to V_i in $tag(\mathcal{C})$, is +1 if $v_+(\delta_j)$ belongs to V_i in $tag(\mathcal{C})$, is -1 if $v_-(\delta_j)$ belongs to V_i in $tag(\mathcal{C})$.

The incidence matrix $A(\mathcal{G}_{comptag(\mathcal{C})})$ of the graph $\mathcal{G}_{comptag(\mathcal{C})}$ in Figure 7, constructed from the triangle adjacency graph $tag(\mathcal{C})$, associated with the complex \mathcal{C} in Figure 6 is given below.

	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7
V_1	+1	+1	+1	0	+1	+1	+1
V_2	-1	-1	-1	-1	0	0	0
V_3	0	0	0	+1	-1	-1	-1

We now have the following simple lemma

Lemma 35. *The rows of $A(\mathcal{G}_{comptag(\mathcal{C})})$ are 2-cycles of \mathcal{C} .*

Proof. We will show that the rows of $A(\mathcal{G}_{comptag(\mathcal{C})})$ are orthogonal to the rows of $A^{(2)}(\mathcal{C})$, the 2-coboundary matrix of \mathcal{C} . Consider the ‘ e ’ row corresponding to edge e of \mathcal{C} . Let $\delta_1, \dots, \delta_k$ be the triangles incident at e . Let us, without loss of generality, assume that all these triangles have been oriented to agree with the orientation of e . Thus the row e in $A^{(2)}(\mathcal{C})$ has only 0 and +1 entries. Let us assume further that $\delta_1, \dots, \delta_k, \delta_1$ is the order in which we encounter the triangles as we rotate around e in \mathbb{R}^3 consistent with the orientation of e . (See Figure 1 for the case $k = 3$).

Let us call the ordered pairs $(\delta_1, \delta_2), \dots, (\delta_i, \delta_{i+1}), \dots, (\delta_k, \delta_1)$ ordered pairs of adjacent triangles at e . For each ordered pair (δ_i, δ_{i+1}) we will have an edge in $tag(\mathcal{C})$ between $v_+(\delta_i)$ and $v_-(\delta_{i+1})$. We may ignore all the triangles δ incident at e such that in $tag(\mathcal{C})$ both $v_+(\delta)$ and $v_-(\delta)$ belong to the same component, since these contribute zero entries to the rows of $A(\mathcal{G}_{comptag(\mathcal{C})})$. After this, we can see that of $\delta_1, \dots, \delta_k$ we must have the same number of $v_+(\delta)$ and $v_-(\delta)$ belonging to a given component. Thus in the row of $A(\mathcal{G}_{comptag(\mathcal{C})})$ corresponding to this component we have equal number of +1s and -1s in columns corresponding to triangles

incident at e in \mathcal{C} . But the row e in $A^{(2)}(\mathcal{C})$ has only +1 entries corresponding to triangles incident at it. Therefore, the two vectors are orthogonal. Thus every row of $A(\mathcal{G}_{\text{comptag}(\mathcal{C})})$ is orthogonal to every row of $A^{(2)}(\mathcal{C})$, i.e., rows of $A(\mathcal{G}_{\text{comptag}(\mathcal{C})})$ are 2-cycles of \mathcal{C} . \square

We show later that the rows of $A(\mathcal{G}_{\text{comptag}(\mathcal{C})})$ actually generate the 2-cycle space of \mathcal{C} .

7. Cell dual $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{C}_{\mathcal{T}}$ and the region graph $\mathcal{G}_{\text{region}(\mathcal{C})}$ of \mathcal{C}

The cell dual of the n -cells of an n -complex is a simple concept for studying the interrelationship of the n -cells. In practice, this idea is often used to solve electromagnetic field problems, for instance, in the case of magnetic circuits. Our interest is in the cell dual of the 3-complex $\mathcal{C}_{\mathcal{T}}$ embedded in \mathbb{R}^3 . Section 4 has a description of $\mathcal{C}_{\mathcal{T}}$. We remind the reader that the set S of triangles of \mathcal{C} is contained in the set $S^{(2)}(\mathcal{C}_{\mathcal{T}})$ of triangles of $\mathcal{C}_{\mathcal{T}}$. Further the triangles of \mathcal{C} lie in the interior of the tetrahedron \mathcal{T} so that none of these triangles intersect the subset of triangles of $S^{(2)}(\mathcal{C}_{\mathcal{T}})$ which lie in the boundary of \mathcal{T} . The notions $v_+(\delta), v_-(\delta)$ are defined in the beginning of Section 6. We show in this section that the graph $\mathcal{G}_{\text{region}\mathcal{C}}$, defined below, which captures the interrelationship between connected regions of $\mathcal{T} \setminus \mathcal{C}$, can be obtained from the cell dual of $\mathcal{C}_{\mathcal{T}}$ by contracting edges which correspond to triangles of $\mathcal{C}_{\mathcal{T}}$ which are not in \mathcal{C} .

Definition 36. *Let the triangles in S have the same orientation in \mathcal{C} and $\mathcal{C}_{\mathcal{T}}$. The cell dual $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{C}_{\mathcal{T}}$ is constructed as follows. Put a node v_{ext} in the region external to \mathcal{T} . In the interior of each tetrahedron τ of $\mathcal{C}_{\mathcal{T}}$ put a node v_{τ} . For each triangle δ of $\mathcal{C}_{\mathcal{T}}$ which has $v_+(\delta) \in \tau_i$ and $v_-(\delta) \in \tau_j$ introduce a directed edge e_{δ} from v_{τ_i} to v_{τ_j} . If δ lies in the boundary of \mathcal{T} and in the boundary of the tetrahedron τ , add an edge between v_{ext} and v_{τ} , with direction from $v_+(\delta)$ to $v_-(\delta)$.*

The following lemma is immediate from the definition of the cell dual.

Lemma 37. *$\mathcal{G}_{\mathcal{T}}$ is connected.*

We now have the following lemma.

Lemma 38. *Let $A(\mathcal{G}_{\mathcal{T}})$ be the incidence matrix of $\mathcal{G}_{\mathcal{T}}$. Let A_{red} be the submatrix of $A(\mathcal{G}_{\mathcal{T}})$ obtained by deleting the row corresponding to v_{ext} . Then*

1. A_{red} is the transpose of the 3-coboundary matrix of $\mathcal{C}_{\mathcal{T}}$;
2. in $A(\mathcal{G}_{\mathcal{T}})$, the row corresponding to v_{ext} is the negative of the sum of all the other rows;
3. the row space of $A(\mathcal{G}_{\mathcal{T}}) = \text{row space of } A_{\text{red}} = \text{the 2-cycle space of } \mathcal{C}_{\mathcal{T}}$.

Proof. Parts (1) and (2) are immediate from the definitions of $\mathcal{G}_{\mathcal{T}}, A(\mathcal{G}_{\mathcal{T}}), A_{\text{red}}$ and 3-coboundary matrices.

Part (3) follows from Lemma 37, Lemma 34 and Theorem 15. \square

Definition 39. *The region graph $\mathcal{G}_{\text{region}(\mathcal{C})}$ of \mathcal{C} has nodes corresponding to connected regions $\mathcal{R}_1, \dots, \mathcal{R}_k$ of $\mathcal{T} \setminus \mathcal{C}$ and edges corresponding to triangles of \mathcal{C} . It is built as follows. Put a node v_i in the region \mathcal{R}_i for $i = 1, \dots, k$. For each triangle δ of \mathcal{C} introduce a directed edge e_{δ} from v_i to v_j if $v_+(\delta) \in \mathcal{R}_i$ and $v_-(\delta) \in \mathcal{R}_j$.*

Definition 40. *Let \mathcal{G} be a graph on vertex set V and edge set E . Let $T \subseteq E$. The graph $\mathcal{G} \circ T$, called restriction of \mathcal{G} to T , is obtained from \mathcal{G} by deleting edges in $E \setminus T$ and also deleting any isolated nodes formed in the process.*

The graph $\mathcal{G} \times T$, called contraction of \mathcal{G} to T , is obtained by first fusing the end vertices of edges in $E \setminus T$ and then deleting them. We say, in this case, that the edges of $E \setminus T$ are contracted.

If $T_2 \subseteq T$, we denote $(\mathcal{G} \times T) \times T_2, (\mathcal{G} \times T) \circ T_2, (\mathcal{G} \circ T) \times T_2, (\mathcal{G} \circ T) \circ T_2$, respectively by $\mathcal{G} \times T \times T_2, \mathcal{G} \times T \circ T_2, \mathcal{G} \circ T \times T_2, \mathcal{G} \circ T \circ T_2$.

A graph of the form $\mathcal{G} \circ T \times T_2$ or of the form $\mathcal{G} \times T \circ T_2$ is called a minor of \mathcal{G} .

Definition 41. Let \mathcal{G} be a graph on vertex set V and edge set E . $\mathcal{V}(\mathcal{G})$ denotes the 1-coboundary space on E , i.e., the row space of the incidence matrix of \mathcal{G} .

In the literature, the notation ‘ \circ ’, ‘ \times ’ are used for operations on graphs as above, for operations on vector spaces as in Definition 24 and for operations on matroids as later in Definition 49. This is partly because the operations are related. The context would make clear as to which operation is intended. This is so in the following results which are well known ([19],[11]). (For better readability set difference is denoted $A - B$ rather than $A \setminus B$.)

Theorem 42. Let \mathcal{G} be a graph on vertex set V and edge set E . Let $T_2 \subseteq T$. We have

1. $\mathcal{G} \times T \circ T = \mathcal{G} \times T$,
2. $\mathcal{G} \circ T \times T = \mathcal{G} \circ T$,
3. $\mathcal{G} \times T \times T_2 = \mathcal{G} \times T_2$,
4. $\mathcal{G} \times T \circ T_2 = \mathcal{G} \circ (E - (T - T_2)) \times T_2$,
5. $\mathcal{G} \circ T \circ T_2 = \mathcal{G} \circ T_2$.

Theorem 43. Let \mathcal{G} be a graph on vertex set V and edge set E and let $T \subseteq E$. Let $\mathcal{V}(\mathcal{G}')$ denote the row space of $A(\mathcal{G}')$.

1. $\mathcal{V}(\mathcal{G} \times T) = (\mathcal{V}(\mathcal{G})) \times T$,
2. $\mathcal{V}(\mathcal{G} \circ T) = (\mathcal{V}(\mathcal{G})) \circ T$.

We now have the following simple lemma.

Lemma 44. Let S denote the set of edges of $\mathcal{G}_{\mathcal{T}}$ corresponding to triangles in the complex \mathcal{C} . We have

1. $\mathcal{G}_{region(\mathcal{C})} = \mathcal{G}_{\mathcal{T}} \times S$;
2. $\mathcal{G}_{region(\mathcal{C})}$ is connected.
3. $\mathcal{V}(\mathcal{G}_{region(\mathcal{C})}) = \mathcal{V} \times S$, where \mathcal{V} is the 2-cycle space of $\mathcal{C}_{\mathcal{T}}$ and hence $\mathcal{V}(\mathcal{G}_{region(\mathcal{C})})$ is the 2-cycle space of \mathcal{C} .

Proof. 1. Two tetrahedra τ_0, τ_{end} of $\mathcal{C}_{\mathcal{T}}$ belong to a connected region of $\mathcal{T} \setminus \mathcal{C}$, iff we can find a sequence $\tau_0, \tau_1, \dots, \tau_n, \tau_{end}$, such that each tetrahedron, other than τ_{end} , has a common triangle with the next tetrahedron, with the additional condition that this common triangle does not belong to S . In $\mathcal{G}_{\mathcal{T}}$, these correspond to nodes $v_{\tau_0}, v_{\tau_1}, \dots, v_{\tau_n}, v_{\tau_{end}}$, with successive nodes being joined by an edge e_{δ} with $\delta \notin S$. If we contract all the edges of $\mathcal{G}_{\mathcal{T}}$ corresponding to triangles of $\mathcal{C}_{\mathcal{T}}$ that lie in a connected region of $\mathcal{T} \setminus \mathcal{C}$, this would result in all the nodes of $\mathcal{G}_{\mathcal{T}}$ corresponding to tetrahedra in the connected region being fused to a single node. When this is done repeatedly for all the connected regions of $\mathcal{T} \setminus \mathcal{C}$, we would be left with a graph which has one node $v_{\mathcal{R}_i}$ corresponding to each connected region \mathcal{R}_i of $\mathcal{T} \setminus \mathcal{C}$ and edges between these nodes which correspond to those triangles in S which lie between the connected regions. An edge e_{δ} would be directed from $v_{\mathcal{R}_i}$ to $v_{\mathcal{R}_j}$, if $v_+(\delta)$ lies in \mathcal{R}_i and if $v_-(\delta)$ lies in \mathcal{R}_j . Since the operations we have performed are contractions of edges in $E(\mathcal{G}_{\mathcal{T}}) - S$, in the graph $\mathcal{G}_{\mathcal{T}}$, and this resulting graph is $\mathcal{G}_{region(\mathcal{C})}$, the lemma follows.

2. From Lemma 37 we know that $\mathcal{G}_{\mathcal{T}}$ is connected. Contraction of a connected graph results in a connected graph. The result follows.

3. By Lemma 38, the row space of $A(\mathcal{G}_{\mathcal{T}})$ is the 2-cycle space of $\mathcal{C}_{\mathcal{T}}$. By Lemma 27, if \mathcal{V} is the 2-cycle space of $\mathcal{C}_{\mathcal{T}}$, then $\mathcal{V} \times S$ is the 2-cycle space of \mathcal{C} . By Theorem 43, if $\hat{\mathcal{V}}$ is the row space of the incidence matrix of $A(\mathcal{G}_{\mathcal{T}})$, then $\hat{\mathcal{V}} \times S$ is the row space of the incidence matrix of $A(\mathcal{G}_{\mathcal{T}} \times S)$. By part (1) of the present lemma, $\mathcal{G}_{region(\mathcal{C})} = \mathcal{G}_{\mathcal{T}} \times S$. The result follows. □

8. $\mathcal{G}_{region(\mathcal{C})} = \mathcal{G}_{comptag(\mathcal{C})}$ for connected \mathcal{C}

We show in this section that $\mathcal{G}_{region(\mathcal{C})}$, $\mathcal{G}_{comptag(\mathcal{C})}$, are identical if \mathcal{C} is connected (see Definition 7 for definition of connectedness of complexes).

We have seen that the row space of the incidence matrix of $\mathcal{G}_{region(\mathcal{C})}$ is the cycle space of \mathcal{C} . Hence, by the discussion in Subsection 5.4, if $\mathcal{G}_{region(\mathcal{C})}$ can be constructed easily, we have an efficient way of writing equations for the solution of the electrical 2–network based on \mathcal{C} . The definition of $\mathcal{G}_{region(\mathcal{C})}$ requires the notion of path connectedness in $\mathcal{T} \setminus \mathcal{C}$. In practice, determining whether two points in \mathbb{R}^3 can be connected by a path that avoids specified obstacles is cumbersome. On the other hand, as we have seen, building $tag(\mathcal{C})$ and $\mathcal{G}_{comptag(\mathcal{C})}$ from \mathcal{C} embedded in \mathbb{R}^3 is linear time on the size of \mathcal{C} . When \mathcal{C} is disconnected we merely have to repeat the procedure for each connected component of \mathcal{C} . Thus, the fact (which we prove in this section) that $\mathcal{G}_{region(\mathcal{C})} = \mathcal{G}_{comptag(\mathcal{C})}$ for connected \mathcal{C} , is very useful for our purposes.

We will prove the main theorem of this section through a series of lemmas.

Theorem 45. *If \mathcal{C} is connected, then $\mathcal{G}_{region(\mathcal{C})} = \mathcal{G}_{comptag(\mathcal{C})}$.*

Let the edges of $\mathcal{G}_{region(\mathcal{C})}$, $\mathcal{G}_{comptag(\mathcal{C})}$ be named according to the triangles of \mathcal{C} , to which they correspond.

Lemma 46. *The row space of $A(\mathcal{G}_{comptag(\mathcal{C})})$ is contained in the row space of $A(\mathcal{G}_{region(\mathcal{C})})$.*

Proof. By Lemma 35, rows of $A(\mathcal{G}_{comptag(\mathcal{C})})$ are 2–cycles of \mathcal{C} , and by Lemma 44, row space of $A(\mathcal{G}_{region(\mathcal{C})})$ is the space of 2–cycles of \mathcal{C} . The result follows. \square

Lemma 47. 1. *For each row i of $A(\mathcal{G}_{comptag(\mathcal{C})})$ there exists a row i' of $A(\mathcal{G}_{region(\mathcal{C})})$ such that whenever $A(\mathcal{G}_{comptag(\mathcal{C})})(i, j)$ is nonzero, we have $A(\mathcal{G}_{comptag(\mathcal{C})})(i, j) = A(\mathcal{G}_{region(\mathcal{C})})(i', j)$ and further, if δ is a selfloop incident at i in $\mathcal{G}_{comptag(\mathcal{C})}$ it is a selfloop incident at i' in $\mathcal{G}_{region(\mathcal{C})}$.*

2. *When \mathcal{C} is connected, this map $i \rightarrow i'$ is one to one onto and the graphs $\mathcal{G}_{comptag(\mathcal{C})}$, $\mathcal{G}_{region(\mathcal{C})}$ are identical under the renaming of vertex i of $\mathcal{G}_{comptag(\mathcal{C})}$ as vertex i' of $\mathcal{G}_{region(\mathcal{C})}$.*

Proof. 1. By definition, a row of $A(\mathcal{G}_{comptag(\mathcal{C})})$ corresponds to the vertex set of a connected component of $tag(\mathcal{C})$. By Lemma 32, we know that whenever a vertex, say $v_+(\delta)$, is path connected in $tag(\mathcal{C})$ to a vertex, say $v_-(\delta')$, it is also path connected in $\mathcal{T} \setminus \mathcal{C}$ and therefore these vertices belong to the same connected region of $\mathcal{T} \setminus \mathcal{C}$. Each vertex set of a connected component of $tag(\mathcal{C})$ corresponds to a single connected region in which it lies in $\mathcal{T} \setminus \mathcal{C}$. So to every vertex i of $\mathcal{G}_{comptag(\mathcal{C})}$, there corresponds a unique vertex i' of $\mathcal{G}_{region(\mathcal{C})}$, such that whenever an edge δ leaves (enters) a vertex i in $\mathcal{G}_{comptag(\mathcal{C})}$, it also leaves (enters) the corresponding vertex i' of $\mathcal{G}_{region(\mathcal{C})}$. This is also true of selfloops of $\mathcal{G}_{comptag(\mathcal{C})}$. Thus, if selfloop δ is incident at i in $\mathcal{G}_{comptag(\mathcal{C})}$, it is incident at i' in $\mathcal{G}_{region(\mathcal{C})}$.

2. Since, by Lemma 44, $\mathcal{G}_{region(\mathcal{C})}$ is connected, it has no isolated nodes. Therefore, since the edge sets of both graphs are identical, the map $i \rightarrow i'$, from vertex set of $\mathcal{G}_{comptag(\mathcal{C})}$, to vertex set of $\mathcal{G}_{region(\mathcal{C})}$ is onto.

If \mathcal{C} is connected it is easy to see that $\mathcal{G}_{comptag(\mathcal{C})}$ is connected. For connected graphs, by Lemma 34, the row rank of the incidence matrix is one less than the number of nodes of the graph. By Lemma 35, Lemma 44, the row space of $\mathcal{G}_{comptag(\mathcal{C})}$ is contained in the row space of $\mathcal{G}_{region(\mathcal{C})}$. Thus the number of nodes of $\mathcal{G}_{comptag(\mathcal{C})}$ is less or equal to the number of nodes of $\mathcal{G}_{region(\mathcal{C})}$. But the map $i \rightarrow i'$ is onto and therefore we must have the number of nodes the same for both graphs and therefore the map $i \rightarrow i'$ is one to one onto. Further, an edge δ is directed from i to j in $\mathcal{G}_{comptag(\mathcal{C})}$ iff it is directed from i' to j' in $\mathcal{G}_{region(\mathcal{C})}$. Thus, under the renaming of node i of $\mathcal{G}_{comptag(\mathcal{C})}$ as node i' of $\mathcal{G}_{region(\mathcal{C})}$, the two graphs are identical. \square

Remark 4. *If the 2–complex \mathcal{C} is not connected, we first build $\mathcal{G}_{comptag(\mathcal{C}_i)}$ for each of the connected components \mathcal{C}_i of \mathcal{C} . It will turn out that each row of $A(\mathcal{G}_{comptag(\mathcal{C}_i)})$ except one would be identical to a corresponding row of $A(\mathcal{G}_{region(\mathcal{C})})$. If we add these exceptional rows we would get the row corresponding to the external node of $\mathcal{G}_{region(\mathcal{C})}$ and this would complete the construction of $A(\mathcal{G}_{region(\mathcal{C})})$ and therefore of $\mathcal{G}_{region(\mathcal{C})}$.*

9. Matroid dual of the (2, 1) skeleton of \mathcal{C}

In this section, we explore the matroidal relation between the 2–complex \mathcal{C} and the graph $\mathcal{G}_{region(\mathcal{C})}$. Our statements are in general correct for 2–complex \mathcal{C} and $\mathcal{G}_{region(\mathcal{C})}$. If \mathcal{C} is connected $\mathcal{G}_{comptag(\mathcal{C})} = \mathcal{G}_{region(\mathcal{C})}$. Otherwise, we can build $\mathcal{G}_{region(\mathcal{C})}$ from the disconnected $\mathcal{G}_{comptag(\mathcal{C})}$ by fusing the nodes corresponding to the external region in each connected component of the latter.

It is well known that if a graph is nonplanar, its matroid dual will not correspond to a graph. It is a natural question to ask whether the dual of a nonplanar graph has a simple representation. We will sketch arguments to show that any nonplanar graph is the minor of another nonplanar graph whose matroid dual is the (2, 1) skeleton of a 2–complex embedded in \mathbb{R}^3 .

We first discuss some preliminary ideas from matroid theory.

The following two results are fundamental in matroid theory [19], [22], [12].

Theorem 48. *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and let $T \subseteq S$.*

1. *Let $\mathcal{I} \circ T$ be the family of members of \mathcal{I} contained in T . Then $\mathcal{M} \circ T := (T, \mathcal{I} \circ T)$ is a matroid.*
2. *Let $\mathcal{I} \times T$ be the family of all members X of \mathcal{I} contained in T with the property that $X \cup \mathcal{B}_{S-T}$ is a member of \mathcal{I} whenever \mathcal{B}_{S-T} is a base of $\mathcal{M} \circ (S - T)$. Then $\mathcal{M} \times T := (T, \mathcal{I} \times T)$ is a matroid.*
3. *Let \mathcal{I}^* be the family of subsets of S which are subsets of cobases of \mathcal{M} . Then (S, \mathcal{I}^*) is a matroid.*

Definition 49. *Let \mathcal{M} be a matroid on S and let $T \subseteq S$. The matroid $\mathcal{M} \circ T := (T, \mathcal{I} \circ T)$ is called the restriction of \mathcal{M} to T . The matroid $\mathcal{M} \times T := (T, \mathcal{I} \times T)$ is called the contraction of \mathcal{M} to T . The matroid $\mathcal{M}^* := (S, \mathcal{I}^*)$ is called the dual of \mathcal{M} .*

Theorem 50. *Let \mathcal{V} be a vector space on a finite set S and let $T \subseteq S$. Then*

1. $\mathcal{M}(\mathcal{V}^\perp) = (\mathcal{M}(\mathcal{V}))^*$.
2. $\mathcal{M}(\mathcal{V} \circ T) = (\mathcal{M}(\mathcal{V})) \circ T$.
3. $\mathcal{M}(\mathcal{V} \times T) = (\mathcal{M}(\mathcal{V})) \times T$.

We have, by Lemma 44, that the 2–coboundary space of the 2–complex \mathcal{C} and the 1–coboundary space of the graph $\mathcal{G}_{region(\mathcal{C})}$ are complementary orthogonal. The matroid $\mathcal{M}(\mathcal{V}(\mathcal{G}))$ associated with the 1–coboundary space $\mathcal{V}(\mathcal{G})$ of \mathcal{G} is said to be also associated with \mathcal{G} and denoted $\mathcal{M}(\mathcal{G})$. Let us denote by $\mathcal{V}(\mathcal{C})$, the 2–coboundary space of \mathcal{C} and call $\mathcal{M}(\mathcal{V}(\mathcal{C}))$, the matroid $\mathcal{M}(\mathcal{C})$ associated with the (2, 1) skeleton of \mathcal{C} . By Theorem 50, we have that $\mathcal{M}(\mathcal{V}^\perp) = (\mathcal{M}(\mathcal{V}))^*$. So $(\mathcal{M}(\mathcal{C}))^* = \mathcal{M}(\mathcal{G}_{comptag(\mathcal{C})})$. Thus the matroid dual of the (2, 1) skeleton of \mathcal{C} is a graph which may in general be nonplanar.

The natural question is the converse. Can one build a 2–complex \mathcal{C} whose (2, 1) skeleton is the matroid dual of a nonplanar graph \mathcal{G}_{orig} ? We will sketch an algorithm for building a larger graph \mathcal{G}_{large} which has such a matroid dual and from which by deleting a single vertex and edges incident on it, we can get back \mathcal{G}_{orig} . It is well known that a graph \mathcal{G}_{orig} is nonplanar iff it contains the Kuratowski graphs $K_5, K_{3,3}$ as minors. Therefore if \mathcal{G}_{orig} is nonplanar, so is \mathcal{G}_{large} .

The method we use is to construct regions surrounding the individual nodes in such a way that regions corresponding to two nodes have a common boundary iff there is an edge in \mathcal{G}_{orig} between them. But this method leaves us with an extra ‘external’ region which has common boundary with every one of the regions corresponding to the nodes of \mathcal{G}_{orig} . The 2–complex made up of the boundary triangles of all the regions will turn out to be the matroid dual of a larger graph \mathcal{G}_{large} .

Figure 9 indicates a simplified version of the problem. The original graph \mathcal{G}_{orig} should be thought of as embedded in \mathbb{R}^3 even though for diagrammatic simplicity we have used a planar representation. We surround the nodes by ‘tubes’ $\mathcal{R}_1, \dots, \mathcal{R}_6$, whose surfaces are triangulated but not shown. The tubes have certain common triangles with other tubes. These are indicated as e'_1, \dots, e'_6 . In addition, there is an external region \mathcal{R}_{ext} which has common triangles with all the tubes. Let the complex \mathcal{C} be composed of all surface triangles, their boundary edges and vertices. The resulting region graph, \mathcal{G}_{large} of \mathcal{C} , is shown alongside. The dotted lines indicate sets of parallel edges corresponding to triangles common to the external region and the other ‘tubes’. The 2–coboundary space $\mathcal{V}(\mathcal{C})$ and the 2–cycle space $\mathcal{V}(\mathcal{C})^\perp$ respectively of \mathcal{C} would be the 1–cycle space and 1–coboundary space of \mathcal{G}_{large} if the orientations of triangles \mathcal{C} and the edges of \mathcal{G}_{large} are consistent.

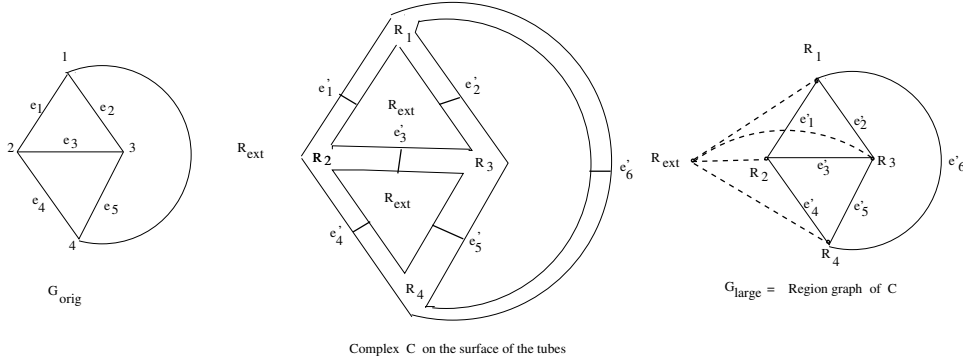


Figure 8: From \mathcal{G}_{orig} to \mathcal{G}_{large}

Remark 5. *The matroid dual has to be distinguished from the Lefschetz/Poincaré duality of complexes (see [7], [10]). In general the latter do not lead to matroid duality. For instance, when a graph is drawn on a torus without edges crossing each other, the coboundary space of its Lefschetz dual will have dimension less (by genus of torus) than the dimension of the cycle space of the original graph. In general when a 2-complex \mathcal{C} is embedded in a three dimensional space, say \mathcal{S} , the region graph will be the Lefschetz dual but unless the space \mathcal{S} is contractible, the region graph may not be the matroid dual of \mathcal{C} and will not allow us to write equations for an electrical 2-network based on \mathcal{C} .*

10. Dual of an electrical 2-network

From the discussions in the previous sections, it is reasonable to expect that if an electrical 2-network is built on a 2-complex \mathcal{C} , we could build a ‘dual’ electrical network of the graph based kind, the solution of which would yield a solution of the 2-network. We spell out the details in this section.

First we dualize constraints K_1, K_2 of Subsection 5.1.

Constraint K_1 (Generalized KVL) can be simplified to read

“the net outward flux ϕ_S through 2-cells of \mathcal{C} which are boundaries of connected regions of tetrahedra of $\mathcal{C}_{\mathcal{T}}$ is zero.”

Constraint *Dual* K_1 (KCL) should read

“the net current ϕ_S away from a node of $\mathcal{G}_{region(\mathcal{C})}$ is zero.”

Constraint K_2 (Generalized KCL) can be simplified to read

“the current adjusted mmf vector m_S is a 2-cycle of \mathcal{C} .”

By Lemma 44 we know that 2-cycles of \mathcal{C} and 1-coboundaries of $\mathcal{G}_{region(\mathcal{C})}$ are identical. So

Constraint *Dual* K_2 (KVL) should read

“the vector m_S is a 1-coboundary of $\mathcal{G}_{region(\mathcal{C})}$.”

We defined an electrical 2-network \mathcal{N} as a pair $(\mathcal{C}_S, \mathcal{D}_S)$, where

\mathcal{C}_S is a 2-complex embedded in \mathbb{R}^3 with S as its 2-cells,

\mathcal{D}_S , the device characteristic, is a collection of ordered pairs (ϕ_S, m_S) of vectors on S over \mathbb{R} .

Therefore, we may define the dual network \mathcal{N}^d to \mathcal{N} as the pair $(\mathcal{G}_{region(\mathcal{C})}, \mathcal{D}_S)$.

A solution of \mathcal{N}^d is an ordered pair (ϕ_S, m_S) satisfying the following constraints

1. ϕ_S satisfies Constraint *Dual* K_1 .
2. m_S satisfies Constraint *Dual* K_2 .
3. $(\phi_S, m_S) \in \mathcal{D}_S$.

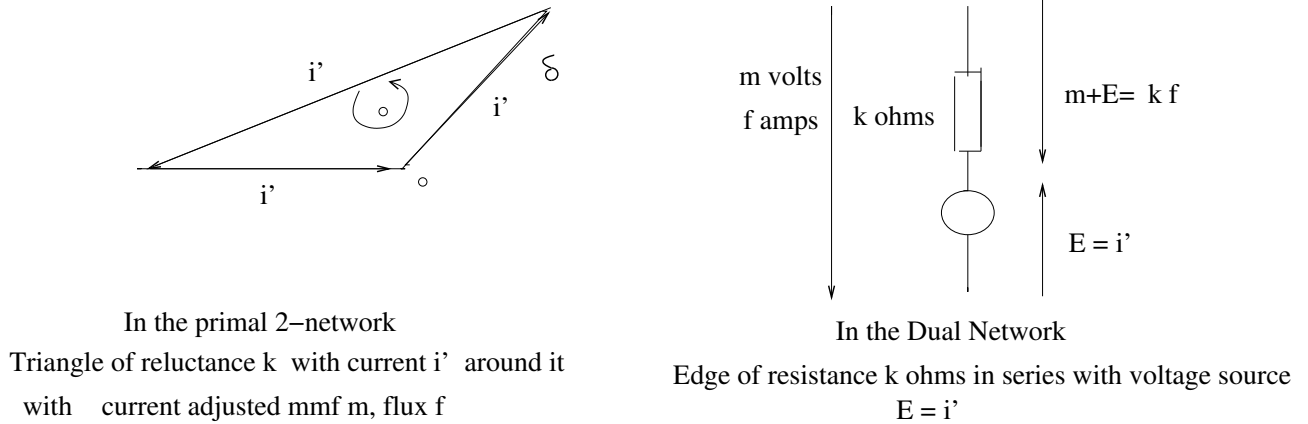


Figure 9: Triangle in a 2-network and the corresponding edge in the dual network

Observe that ϕ_S satisfies Constraint K_1 iff it satisfies Constraint *Dual* K_1 and m_S satisfies Constraint K_2 iff it satisfies Constraint *Dual* K_2 . Thus (ϕ_S, m_S) is a solution of \mathcal{N} iff it is a solution of \mathcal{N}^d .

Subsection 5.4 contains a description of cycle based solution of \mathcal{N} . This corresponds to coboundary based solution of \mathcal{N}^d . In particular, when the device characteristic permits ϕ_S to be expressed as an affine function of m_S , one could use nodal analysis based on node potentials for \mathcal{N}^d .

It is of interest to know how results of the ‘standard’ electrical networks carry over to electrical 2-networks.

We have already seen that Tellegen’s Theorem carries through (Theorem 30). Ideas of topological network theory such as multiport decomposition and topological transformation [11] go through essentially unchanged because we are able to build the dual nonplanar graph to the complex and therefore the dual electrical network.

Kirchhoff’s tree theorem [14] states that the equivalent conductance seen across a pair of terminals a, b of a network composed of resistors equals

$$\frac{\text{sum of all tree conductance products}}{\text{sum of all 2-tree conductance products separating } a, b}.$$

Here a 2-tree separating a, b is a loop free set obtained from a tree by removing an edge so that a, b occur in different connected components of the tree and a tree (2-tree) conductance product is the product of the conductances in a tree (2-tree) of the resistive network. This theorem clearly has an analogue for the 2-network \mathcal{N} . Let exactly one of the triangles, say δ , carry a current i' around it consistent with the orientation. After solving \mathcal{N}^d , we can obtain $f_\delta := \phi_\delta = gi'$. To compute g by Kirchhoff’s tree formula, note that the triangle δ , and the current i' translate in the dual network to an edge e_δ composed of a resistor (of resistance value = reluctance of triangle) in series with a voltage source of value $E := i'$ (see Figure 10).

In the dual network, after solution, we have the relation, current $f := f_\delta = gE$. The quantity g is the equivalent conductance as seen by the source E when no other source is present. We can write an expression for the inverse reluctance g using Kirchhoff’s tree formula for the dual network. This can be translated to an expression involving the cobases of $\mathcal{M}(\mathcal{C})$ and subcobases which become a cobase when δ is added. Let us call the latter ‘ δ -friendly’ subcobases. Then the equivalent inverse reluctance seen by the current loop i' , would be

$$\frac{\text{sum of all cobase (reluctance)}^{-1} \text{products}}{\text{sum of all } \delta\text{-friendly subcobase (reluctance)}^{-1} \text{products}}.$$

11. Generalizations

We discuss in this section how the ideas of this paper go through if we have an $(n - 1)$ -complex \mathcal{C} embedded in \mathbb{R}^n .

As before, we will assume that there is a large n -simplex \mathcal{T} , whose interior contains \mathcal{C} . The n -simplex \mathcal{T} is decomposed into a set of n -simplices whose interiors do not intersect and which together with their faces constitute the n -complex $\mathcal{C}_{\mathcal{T}}$. We assume \mathcal{C} to be a subcomplex of $\mathcal{C}_{\mathcal{T}}$. Next, because \mathcal{T} is contractible we have, by Theorem 14, that for $0 < j \leq n$, every $(j - 1)$ -cycle of the complex is the boundary of a j -chain. When $n > 3$, we have to consider two cases, $j = n$ and $j < n$.

The ideas of this paper go through essentially unchanged for the $j = n$ case.

We will then have the $(n - 1)$ -coboundary space of \mathcal{C} as regular. The electrical $(n - 1)$ -circuit would be defined essentially identically, with the constraints K_1, K_2 modified by replacing 2-cycle and 2-coboundary spaces by $(n - 1)$ -cycle and $(n - 1)$ -coboundary spaces respectively. The generalized Tellegen's Theorem has an identical proof replacing 2 by $(n - 1)$ at appropriate places. The $(n - 1)$ -dimensional analogue of the notion of triangles coming together at an edge requires some careful handling. But finally we will have a matroid dual which is the region graph. This region graph can again be constructed by an analogue of the sliding algorithm.

11.1. Sketch of the $(n - 1)$ -complex solution

The analogue of an edge is an $(n - 2)$ -dimensional simplex. The collection of triangles coming together at an edge corresponds to a collection Σ of $(n - 1)$ -dimensional simplices. We describe a general way of reducing this problem to line segments coming together at a node which will also work in the case considered in this paper.

Consider an $(n - 2)$ -dimensional simplex e contained in \mathbb{R}^n . Suppose the barycentre of e is the origin. Let \mathcal{V} be the two dimensional vector space that is the orthogonal complement of the linear span of e . Let e be a face of each member of a collection Σ of $(n - 1)$ -dimensional simplices $\delta_1, \dots, \delta_k$. Consider \mathcal{V} intersected with the union of all the simplices in Σ . This set is the union of line segments l_{δ_i} having the origin (representing e) as one endpoint, as can be seen from dimensional considerations. Let all the simplices δ_i be oriented so that e appears with a positive sign in $\partial(\delta_i)$. We could take our convention as clockwise rotation around the origin for fixing the order in which the l_{δ_i} are encountered. The vertices $v_-(l_{\delta_i}), v_+(l_{\delta_i})$ are to either side of l_{δ_i} so that as we move clockwise about the origin, we encounter $v_-(l_{\delta_i})$ before crossing l_{δ_i} to encounter $v_+(l_{\delta_i})$.

The construction of $\text{tag}(\mathcal{C}), \mathcal{G}_{\text{comptag}(\mathcal{C}), \mathcal{G}_{\text{region}(\mathcal{C})}$ is as in the 2-complex case and it will turn out that $\mathcal{G}_{\text{comptag}(\mathcal{C})} = \mathcal{G}_{\text{region}(\mathcal{C})}$ when \mathcal{C} is connected and that the $(n - 1)$ -cycle space of \mathcal{C} is the row space of the incidence matrix of $\mathcal{G}_{\text{comptag}(\mathcal{C})}$.

So in this case of an $(n - 1)$ -complex \mathcal{C} embedded in \mathbb{R}^n , the electrical $(n - 1)$ -network problem can be solved as a graph based electrical network problem.

11.2. $j < n$ case

In this case \mathcal{C} is a $(j - 1)$ -complex. Constraint K_1 would read "the flux vector ϕ_S is orthogonal to $(j - 1)$ -cycles of \mathcal{C} which are boundaries of j -chains of $\mathcal{C}_{\mathcal{T}}$." Because of contractibility of \mathcal{T} in \mathbb{R}^n , $(j - 1)$ -cycles of \mathcal{C} would reduce to boundaries of j -chains of $\mathcal{C}_{\mathcal{T}}$. However, these j -chains cannot be regarded as regions of $\mathcal{C}_{\mathcal{T}}$. There would still be a non trivial generalized Tellegen's Theorem stating that the space of ϕ_S and the space of m_S are complementary orthogonal. Complete unimodularity would not hold for the $(j - 1)$ -coboundary or cycle space. Nor can we use ideas like region graph. The computation of a maximal linearly independent set of rows of the $(j - 1)$ -coboundary matrix will involve linear algebraic computations (see Subsection 5.3).

12. Conclusion

In this paper we discussed a generalization of Kirchhoff's laws on graphs, to the case of 2-complexes. We were led to this generalization while attempting to solve a physical problem involving fluxes and mmfs in Euclidean space. We showed that there was a non trivial generalization of Tellegen's Theorem on the complementary orthogonality of voltage and current spaces, to that of flux and mmf spaces in the new context. This helped us to define an electrical 2-network on a 2-complex \mathcal{C} and discuss procedures for

its solution. We gave linear time algorithms for building auxiliary graphs $tag(\mathcal{C})$ and $\mathcal{G}_{comptag(\mathcal{C})}$. When \mathcal{C} was connected, we showed that $\mathcal{G}_{comptag(\mathcal{C})}$ was the matroid dual of \mathcal{C} , i.e., that the column matroid on the incidence matrix of $\mathcal{G}_{comptag(\mathcal{C})}$ was dual to the column matroid of the 2-coboundary matrix of \mathcal{C} . Using this duality, we showed how to build in linear time, a graph based dual network to the given 2-network, whose solutions were identical to solutions of the latter after appropriate renaming of variables. We inferred that preprocessing for the new class of networks for parallelization, using methods such as multiport decomposition and topological transformation, was as easy as it was for graph based networks. Finally, we discussed generalizations of networks on 2-complexes embedded in \mathbb{R}^3 , first to $(n-1)$ -complexes and then to j -complexes, $0 < j < n - 1$, embedded in \mathbb{R}^n .

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