Some Applications of an Implicit Duality Theorem to connections of structures of special types including Dirac and Reciprocal structures

H.Narayanan

Department of Electrical Engineering,

Indian Institute of Technology-Bombay

Powai, Mumbai-400076,

India

Keywords: Reciprocity,Duality,Tellegen's theorem, Dirac structure

1 Abstract

In this paper we present some applications of an Implicit Duality Theorem which was originally a folklore result on Ideal Transformers in Electrical Network Theory. We show, among other things, that results on reciprocal networks (due to Tellegen [7]), and Dirac Structures (due to Van der Schaft [8]) follow as a consequence. These results have the form 'interconnection of structures of a particular kind yields a structure of the same kind'. Also discussed is the question 'given smaller structures of a given kind and a desired structure of the same kind, can the former be interconnected to yield the latter?'. We also indicate the range of possible generalizations of this result.

2 Introduction

Results about duality are fundamental to many areas of mathematics including linear algebra, linear inequalities (Farkas' lemma), linear Diophantine equations, linear differential equations etc. In practice, however, it is usual to find that linear systems have 'manifest variables' and 'latent variables', but we wish to exploit the duality involving only the former. Reducing the constraints of the system, which involve both these kinds of variables to one involving only the former, would destroy the natural structure of the system. It would therefore be useful to restate the duality result about 'manifest variables' in such a way that latent variables need not be eliminated. This is achieved by the Implicit Duality Theorem (which was originally a Network Theory folklore result about ideal transformers). Several applications and proofs of this result and its variations may be found in [1], [2], [3], its variations concerning inequality systems and integral systems may be found in [4], its variations concerning modules and the behaviours they annihilate and its variations concerning unimodular and generalized unimodular modules may be found in [5],[6].

In this paper we unify a class of results which can be proved using the vector space version of this theorem. A typical statement of this class is 'if we connect together devices of a particular type (eg., Dirac structure, reciprocal structure, ideal transformer) and make some of the variables manifest, the resulting device would also be of the same type.'

A second question we consider is 'given devices of a particular type, under what conditions can we connect them together to make up another given device of the same type?'

In each case the results can be proved directly, but more tediously, without using this theorem.

We now give examples of results which can be proved using the techniques of this paper. Consider the directed graph of figure 1.

Example 1: Suppose $(e_1, e_2; e_3, e_4; e_5, e_6)$ are the pairs of ports of 2-port ideal transformers i.e. satisfy constraints of the form

$$\begin{aligned} i_{e_1} &= n\, i_{e_2} \\ v_{e_2} &= -n\, v_{e_1} \end{aligned} \tag{1}$$

Observe that the spaces of possible current vectors and possible voltage vectors are complementary orthogonal. Let e_{P_1}, e_{P_2} denote port edges without any device characteristic constraints. Because of the topological relation between all the edges, $i_{P_1}, i_{P_2}, v_{P_1}, v_{P_2}$ can be shown to get constrained to the form

$$\begin{aligned} i_{P_1} &= k\, i_{P_2} \\ v_{P_2} &= -k\, v_{P_1} \end{aligned} \tag{2}$$

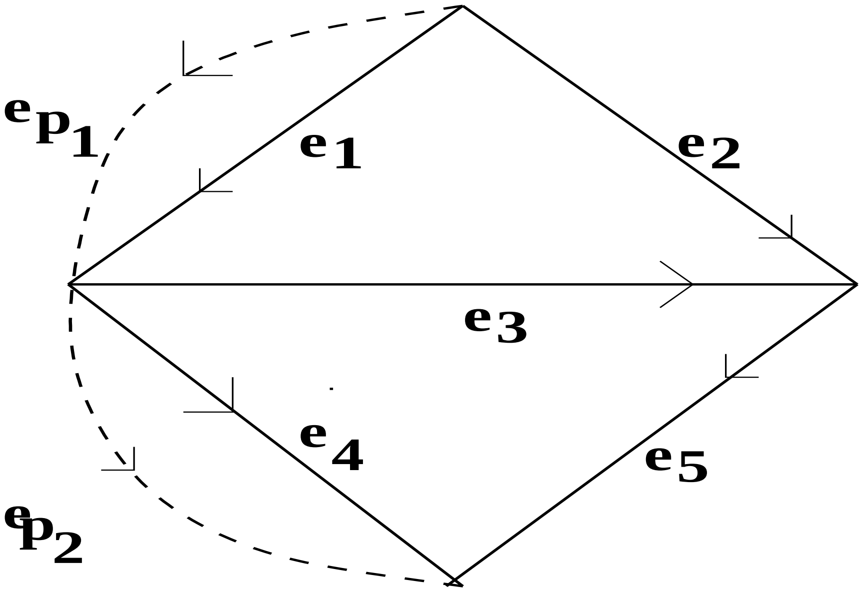

Figure 1: Directed Graph of a Multiport

In general, if P is the set of port branches, the possible i_P vectors form a vector space which can be shown to be complementary orthogonal to the vector space of possible v_P vectors. Thus the current space - voltage space complementary orthogonality of the device is inherited by the port characteristic.

Example 2: Suppose in figure 1, $(e_1, e_2, e_3, e_4, e_5, e_6)$ are resistors (positive or negative) with characteristic

$$v_{e_j} = r_j i_{e_j} \tag{3}$$

Then the possible (v_P^T, i_P^T) vectors which can coexist at the ports can be shown to satisfy an equation of the form

$$\left[\begin{array}{cc} H_1 & H_2 \end{array}\right]\left[\begin{array}{c} v_P \\ i_P \end{array}\right] = 0 \tag{4}$$

where the row rank of the coefficient matrix is the number of ports. Further, if H_1 is the identity matrix then H_2 can be shown to be symmetric. Thus the port characteristic inherits the symmetry property of the constituent devices.

It may be noted that this result (with both positive and negative resistors) is difficult to prove without using the methods of this paper. It is however relatively easy to show directly for positive resistors, if the ports do not form cutsets, that H_1 can be taken to be the identity matrix and H_2 can then be symmetric.

Example 3: Suppose in figure 1 $(e_1, e_2; e_3, e_4; e_5, e_6)$ are the pairs of ports of gyrators with characteristics of the form

$$\begin{aligned} v_{e_1} &= r\ i_{e_2} \\ v_{e_2} &= -r\ i_{e_1} \end{aligned} \tag{5}$$

Then the possible (v_P^T, i_P^T) vectors which can coexist at the ports can be shown to satisfy an equation of the form

$$\left[\begin{array}{cc} H_1 & H_2 \end{array}\right]\left[\begin{array}{c} v_P \\ i_P \end{array}\right] = 0 \tag{6}$$

where the row rank of the coefficient matrix is the number of ports.

Also the space of all possible (i_P^T, v_P^T) which satisfy the port current- voltage constraints can be seen to be complementary orthogonal to the space of all possible (v_P^T, i_P^T) which satisfy the port voltage- current constraints.

Thus we can see that the solution space of $\left[\begin{array}{cc} H_2 & H_1 \end{array}\right]\left[\begin{array}{c} i_P \\ v_P \end{array}\right] = 0$ is complementary orthogonal to the solution space of Equation 6.

3 Preliminaries

In problems arising in Systems/Electrical Network Theory, one is often interested in vectors defined on the edge set or node set of a graph. For instance an electrical network is associated with a directed graph. The vectors which satisfy Kirchhoff's Voltage Law (KVL) are defined on the edge set of this graph. Such a voltage vector

$v(.)$ may therefore be treated as a function defined on the edge set of $E(\mathcal{G})$ of the graph $\mathcal{G}$.

$$v : E(\mathcal{G}) \rightarrow \Re \tag{7}$$

Thus the voltage associated with edge $e_j \in E(\mathcal{G})$ would be $v(e_j)$. This situation concerning vectors is the most natural one for this paper. We therefore define a vector f_S on S over field $\mathcal{F}$ as a function $f_S : S \rightarrow \mathcal{F}$. In a sense we work with situations where there is a natural basis for the underlying *overall vector* space. This would be the set of vectors corresponding to the rows of a unit matrix with columns associated with some predefined set such as the edge set of a graph. We confine ourselves to vectors on finite sets.

The *restriction* of f_S to $T, T \subseteq S$ would be denoted by f_S/T. Given f_S, g_S, we define the *addition* $(f_S + g_S)$ by $(f_S + g_S)(e) = f_S(e) + g_S(e), \forall e \in S$ and the *scalar multiplication* $\lambda f_S, \lambda \in \mathcal{F}$ by $(\lambda f_S)(e) = \lambda(f_S(e))$. *Linear combination, linear dependence, independence* are defined in the usual manner.

A *vector space* on S over $\mathcal{F}$ is a collection of vectors on S over $\mathcal{F}$ closed under addition and scalar multiplication.

The number of vectors in a maximally linearly independent subset (basis) of a collection $\mathcal{K}_S$ of vectors on S is well known to be unique and is called the *rank* of $\mathcal{K}_S$, denoted by $r(\mathcal{K}_S)$.

If the set of rows of a matrix R form a basis for a vector space $\mathcal{V}_S$, we call R, a *representative matrix* of $\mathcal{V}_S$.

A common situation in Networks/Systems is the following (see Figure 2).

We have a device D_P whose manifest variables $x_P(.)$ are indexed by the set P and another device D_{SP} whose manifest variables $x_{SP}(.) \equiv (x_S(.), x_P(.))$ are indexed through the set $S \uplus P$ (S disjoint union P). Now we put D_P and D_{SP} together matching $x_P(.)$ in D_{SP} with $x_P(.)$ in D_P and we examine the possible vector values that x_S can assume.

We are thus led to the following definition

Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be collections of vectors on $S \uplus P$, P respectively. Then the *generalized minor* $\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P$ of $\mathcal{K}_{SP}$ with respect to $\mathcal{K}_P$ is defined as follows :-

$$\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P \equiv \{f_S : \exists f_{SP} \in \mathcal{K}_{SP} \ s.t. \ f_{SP}/P \in \mathcal{K}_P, f_{SP}/S = f_S\}. \tag{8}$$

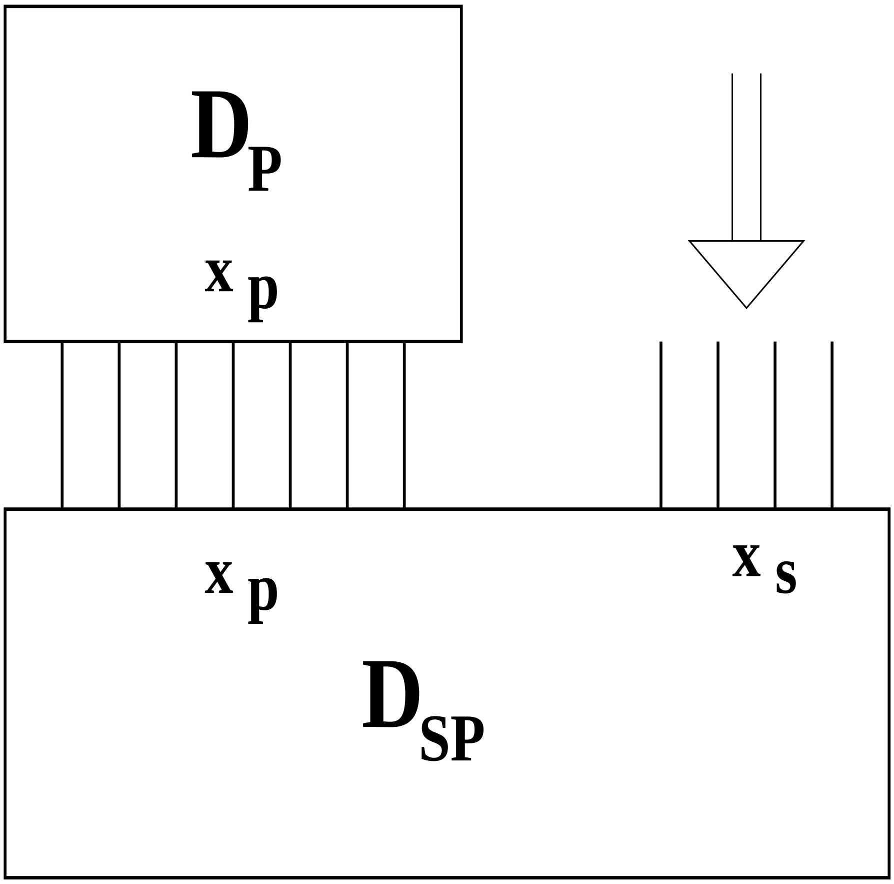

Figure 2: Matching D_{SP} and D_P at x_P

If $\mathcal{K}_{SP}$ and $\mathcal{K}_P$ are vector spaces on $S \uplus P$ and P respectively, it is clear that $\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P$ is a vector space on S.

In particular if $\mathcal{V}_{SP}$ and $\mathcal{V}_P$ are respectively the solution spaces of the equations

$$\begin{bmatrix} A_S & A_P \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix} = 0 \tag{9}$$

$$\begin{bmatrix} \hat{A}_P \end{bmatrix} \begin{bmatrix} x_P \end{bmatrix} = 0 \tag{10}$$

then $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ is the projection of the solution space of the combined set of equations onto the axes corresponding to S.

We denote by O_P, the zero vector on P and by A_P, the collection of all vectors on P. $\mathcal{K}_{SP} \longleftrightarrow \{O_P\}$ is called the *contraction* of $\mathcal{K}_{SP}$ on S and is denoted usually by $\mathcal{K}_{SP} \times S$. $\mathcal{K}_{SP} \longleftrightarrow A_P$ is called the *restriction* of $\mathcal{K}_{SP}$ on S and is denoted usually by $\mathcal{K}_{SP} \cdot S$. Thus $\mathcal{K}_{SP} \cdot S$ is obtained by restricting each vector in $\mathcal{K}_{SP}$ to S, while $\mathcal{K}_{SP} \times S$ is obtained by picking all the vectors in $\mathcal{K}_{SP}$ which take zero value on P and restricting these to S.

These two operations are particularly important for electrical networks. Suppose $\mathcal{G}$ is a directed graph with edge set $S \uplus P$ and $\mathcal{V}_{SP}$ is the space of all *voltage* vectors (i.e., vectors which satisfy KVL for $\mathcal{G}$), then we can show that

- $\mathcal{V}_{SP} \cdot S$ is the collection of all voltage vectors for $\mathcal{G} \cdot S$, the graph obtained by deleting all the edges in P but leaving the end points in place.
- $\mathcal{V}_{SP} \times S$ is the collection of all voltage vectors for $\mathcal{G} \times S$, the graph obtained by fusing the end points of P and deleting them.

Contractions, restrictions and the collections obtained through successive applications of their corresponding operations are called *minors of the original collection of vectors*. If S_1 and S_2 are not disjoint and f_{S1}, f_{S2} are vectors on S_1 and S_2 respectively, we define the addition $f_{S1} + f_{S2}$ on $S_1 \cup S_2$ by

$$\begin{aligned} (f_{S1} + f_{S2})(e) &= f_{S1}(e) + f_{S2}(e), e \in S_1 \cap S_2 && (11) \\ &= f_{S1}(e), e \in S_1 - S_2, && (12) \\ &= f_{S2}(e), e \in S_2 - S_1. && (13) \end{aligned}$$

We define $\mathcal{K}_{S1} + \mathcal{K}_{S2}$ to be the collection $\{f_{S1} + f_{S2} : f_{S1} \in \mathcal{K}_{S1}, f_{S2} \in \mathcal{K}_{S2}\}$. When S_1 and S_2 are disjoint, we may write $f_{S1} \oplus f_{S2}$ or (f_{S1}, f_{S2}) in place of $f_{S1} + f_{S2}$. Given vectors f_S and g_S, we define the *dot product* $< f_S, g_S >$ by

$$< f_S, g_S > \equiv \sum_{e \in S} f_S(e) \cdot g_S(e). \qquad (14)$$

We say f_S, g_S are *orthogonal* (or $f_S \perp g_S$) iff $< f_S, g_S > = 0$. We denote by $\mathcal{K}_S^{\perp}$ the collection of all vectors orthogonal to the vectors in $\mathcal{K}_S$. Clearly $\mathcal{K}_S^{\perp}$ is always a vector space even if $\mathcal{K}_S$ is not. When S is finite and $\mathcal{K}_S$ is a vector space, it is well known that $(\mathcal{K}_S^{\perp})^{\perp} = \mathcal{K}_S$. In this case we say that $\mathcal{K}_S$, $\mathcal{K}_S^{\perp}$ are complementary orthogonal.

The fundamental duality result for this paper is the following well known theorem

Theorem 3.1 *The equation $Ax = b$, where entries of A, x, b are from a field $\mathcal{F}$, has a solution iff*

$$A^T\sigma = 0 \Rightarrow b^T\sigma = 0,$$

whenever σ *is a vector over* $\mathcal{F}$*.*

Proof: The equation has a solution iff b belongs to the column space of A i.e., iff b belongs to ((column space of A)$^{\perp}$)$^{\perp}$, i.e., iff b is orthogonal to every vector which is orthogonal to columns of A.

$\square$

Using this result we can prove the following Implicit Duality Theorem [1], [4].

Theorem 3.2 *Let* $\mathcal{V}_{SP}, \mathcal{V}_P$ *be vector spaces on* $S \uplus P, P$ *respectively. Then*

$$(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^{\perp} = \mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp}.$$

We reproduce the proof of the Implicit Duality Theorem given in [4] in the appendix.

A natural question concerning generalized minor is the following:

Given $\mathcal{V}_{SP}, \mathcal{V}_S$ when can we find $\mathcal{V}_P$ s.t. $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_S$?

(In terms of the devices in Figure 2, we are given the devices D_{SP}, the 'desired device' D_S, which is to be obtained by matching x_P of D_{SP} and x_P of some device D_P and asked to find a possible device D_P).

This is answered in the next theorem. [1], [2], [4].

Theorem 3.3 *Let* $\mathcal{V}_{SP}, \mathcal{V}_S$ *be vector spaces on* $S \uplus P, S$ *respectively.*

1. *If*

$$\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V}_S \supseteq \mathcal{V}_{SP} \times S, \tag{15}$$

then,

$$\mathcal{V}_{SP} \leftrightarrow (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S) = \mathcal{V}_S. \tag{16}$$

2. *There exists a vector space* $\mathcal{V}_P$ *on* P *such that*

$$\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_S \ \ only\ if \tag{17}$$

$$\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V}_S \supseteq \mathcal{V}_{SP} \times S. \tag{18}$$

Proof, given in [4], is reproduced in the appendix.

4 Connection Theorems

In order to illustrate the essential concepts of this paper we will take up the case of reciprocity and sketch a proof of *'connection of resistors results in a reciprocal multiport'*. The proof is essentially the same for *'connection of reciprocal multiports results in a reciprocal multiport'*.

Let us now consider the electrical network in figure 1. The edges $e_1, ..., e_5$ make up the set S. Let us suppose that these have a combined resistive characteristic. The edges e_{P_1}, e_{P_2} make up the set P of port edges and do not have any device characteristic constraints. What are the possible $(i_P^T, v_P^T) \equiv (i_{P_1}, i_{P_2}, v_{P_1}, v_{P_2})$ vectors that can coexist at the ports? For this we need to determine the solution space of the following equations and project it so that we get the space of possible (i_P^T, v_P^T) vectors.

$$\begin{bmatrix} A_S & A_P \end{bmatrix} \begin{bmatrix} i_S \\ i_P \end{bmatrix} = 0 \tag{19}$$

$$\begin{bmatrix} B_S & B_P \end{bmatrix} \begin{bmatrix} v_S \\ v_P \end{bmatrix} = 0 \tag{20}$$

$$\begin{bmatrix} R & -I \end{bmatrix} \begin{bmatrix} i_S \\ v_S \end{bmatrix} = 0. \tag{21}$$

The question we would like to examine is the following:

If the matrix R is symmetric (special case diagonal), what properties do we observe at the ports P_1, P_2?

Let $\mathcal{V}_{iv}(SP)$ be the collection of all vectors $(i_S^T, i_P^T, v_S^T, v_P^T)$ which satisfy the Equations 19,20. Let $\mathcal{V}_{iv}(S)$ be the collection of all vectors (i_S^T, v_S^T) which satisfy the Equations 21.

To obtain the collection of all vectors $\mathcal{V}_{iv}(P)$ which can coexist at the ports we pick vectors $(i_S^T, i_P^T, v_S^T, v_P^T) \in \mathcal{V}_{iv}(SP)$ s.t. $(i_S^T, v_S^T) \in \mathcal{V}_{iv}(S)$ and restrict them down to (i_P^T, v_P^T). Let us denote this operation through which we obtained $\mathcal{V}_{iv}(P)$ from $\mathcal{V}_{iv}(SP)$, $\mathcal{V}_{iv}(S)$ by (abusing notation slightly) $\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S)$. Thus

$$\mathcal{V}_{iv}(P) = \mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S). \tag{22}$$

We observe that when R is symmetric, equation 21 can be rewritten as

$$\begin{bmatrix} I & -R^T \end{bmatrix} \begin{bmatrix} -v_S \\ i_S \end{bmatrix} = 0. \tag{23}$$

The solution space of this equation is complementary orthogonal to that in equation 21.

Therefore if we define a new operation

$$(\mathcal{V}_{iv})^R \equiv \mathcal{V}_{-v,i} \equiv \{(-v, i) \ s.t. \ (i, v) \in \mathcal{V}\} \tag{24}$$

we must have,

$$(\mathcal{V}_{iv}(S))^R = (\mathcal{V}_{iv}(S))^{\perp}. \tag{25}$$

It is easily seen that this property holds also for $\mathcal{V}_{iv}(SP)$. Next it can be seen that

$$(\mathcal{V}_{iv}(SP))^R \leftrightarrow (\mathcal{V}_{iv}(S))^R = (\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S))^R. \tag{26}$$

On the other hand, the *Implicit Duality Theorem* (Theorem 3.2) assures us that

$$(\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S))^{\perp} = (\mathcal{V}_{iv}(SP))^{\perp} \leftrightarrow (\mathcal{V}_{iv}(S))^{\perp}. \tag{27}$$

It thus follows that

$$\begin{aligned}
(\mathcal{V}_{iv}(P))^R &= (\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S))^R \\
&= (\mathcal{V}_{iv}(SP))^R \leftrightarrow (\mathcal{V}_{iv}(S))^R \\
&= (\mathcal{V}_{iv}(SP))^{\perp} \leftrightarrow (\mathcal{V}_{iv}(S))^{\perp} \\
&= (\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S))^{\perp} \\
&= (\mathcal{V}_{iv}(P))^{\perp}.
\end{aligned} \tag{28}$$

Using the argument above we thus have, if $\mathcal{V}_{iv}(P)$ is the solution space of $\begin{bmatrix} I & -R_P \end{bmatrix} \begin{bmatrix} v_P \\ i_P \end{bmatrix} = 0$, then R_P is symmetric (i.e., the port resistance matrix is symmetric). This may be called a *generalized reciprocity theorem.*

Thus we see that when the solution spaces $\mathcal{V}_{iv}(SP)$, $\mathcal{V}_{iv}(S)$ have a certain property, it is also inherited by $\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(P)$, which happens to be the collection of vectors which can coexist at the ports.

The purpose of this paper is to understand the above proof technique in a general form. First consider $(\mathcal{V}_{iv})^R \equiv \{(-v, i) \ s.t. \ (i, v) \in \mathcal{V}_{iv}\}$. What we do here is to take a typical vector (i_S^T, v_S^T), interchange v_S, i_S and change the sign of one of them. Observe that this is really equivalent to performing the linear transformation

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} i_S \\ v_S \end{bmatrix}. \tag{29}$$

What are the natural generalizations of this transformation? First we require $(\mathcal{V}_{iv}^R)^R = \mathcal{V}_{iv}$ since $(\mathcal{V}_{iv}^{\perp})^{\perp} = \mathcal{V}_{iv}$. This would happen only if the transformation L has the property

$$L(L(\begin{bmatrix} i_S \\ v_S \end{bmatrix})) = \pm \begin{bmatrix} i_S \\ v_S \end{bmatrix} \tag{30}$$

Hence $L^2 = \pm I$. Next it is desirable that $(\mathcal{V}_{iv}(SP) \leftrightarrow \mathcal{V}_{iv}(S))^R = \mathcal{V}_{iv}^R(SP) \leftrightarrow \mathcal{V}_{iv}^R(S)$ for *arbitrary subsets* S, P of $S \uplus P$. But this means that for $e_i, e_j \in S$, $e_i \neq e_j$, L should not make i_{e_i}, i_{e_j} interact. So we make L as a direct sum of 2×2 matrices L_e acting on $\begin{bmatrix} i_e \\ v_e \end{bmatrix}$, each L_e having the property $L_e^2 = \pm I$. Further, if any one $L_{e_i}^2 = -I$ then we need to have all $L_{e_j}^2 = -I$.

We now introduce notation which makes the above discussion consistent with the notation of the earlier sections. We would like to treat a vector $\begin{bmatrix} i_S^T & v_S^T \end{bmatrix}$ as a single vector on a set rather than as a pair of vectors on S. So we introduce an exact (disjoint) copy $S^{'}$ of S (i.e., there is a bijection between $S, S^{'}$) and replace $\begin{bmatrix} i_S & v_S \end{bmatrix}$ by $\begin{bmatrix} i_S & v_{S'} \end{bmatrix}$. The latter vector is defined on $S \uplus S^{'}$.

If $f_S(g_{S'})$ is a vector on $S(S')$, $f_{S'}(g_S)$ is defined as follows:-

$$f_{S'}(e') \equiv f_S(e), \forall e \in S \quad (g_S(e) \equiv g_{S'}(e'), \forall e' \in S'). \tag{31}$$

Let S, S' be copies, so also P, P'. Further, let S, P, S', P' be pairwise disjoint. For each $e \in S \uplus P$, let L_e denote a linear transformation acting on vectors $f_{ee'}$, defined on {e, e'}.

The transformation $L_{S_1 P_1}$, $S_1 \subseteq S, P_1 \subseteq P$, is defined on vectors $f_{S_1 S_1' P_1 P_1'}$ as follows:

$$L_{S_1 P_1}(f_{S_1 S_1' P_1 P_1'}) \equiv \oplus_{e \in S_1 \uplus P_1} L_e(f_{ee'}), \tag{32}$$

where $f_{ee'} \equiv f_{SS'PP'}/\{e, e'\}$. Clearly,

$$L_{S_1 P_1} = L_{S_2 P_2}/S_1 \uplus S_1' \uplus P_1 \uplus P_1', \tag{33}$$

whenever $S_1 \subseteq S_2 \subseteq S$ and $P_1 \subseteq P_2 \subseteq P$. We will therefore omit the subscript for the transformation. We will call such a linear transformation *copy invariant*.

To illustrate, let $S = \{e_1, e_2, e_3\}$. Let L_{e_1} act on $(i_{e_1}, v_{e'_1})$ to yield (x_1, y_1), L_{e_2} act on $(i_{e_2}, v_{e'_2})$ to yield (x_2, y_2), L_{e_3} act on $(i_{e_3}, v_{e'_3})$ to yield (x_3, y_3). So when $L \equiv L_{e_1} \oplus L_{e_2} \oplus L_{e_3}$ acts on $(i_{e_1}, i_{e_2}, i_{e_3}, v_{e'_1}, v_{e'_2}, v_{e'_3})$ the result is $(x_1, x_2, x_3, y_1, y_2, y_3)$. In the case of *reciprocity* each L_e has a matrix representation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus, when $L \equiv L_{e_1} \oplus L_{e_2} \oplus L_{e_3}$ acts on $(i_{e_1}, i_{e_2}, i_{e_3}, v_{e'_1}, v_{e'_2}, v_{e'_3})$ the result is $(v_{e'_1}, v_{e'_2}, v_{e'_3}, -i_{e_1}, -i_{e_2}, -i_{e_3})$.

Let $\mathcal{K}_{SS'PP'}$ be a collection of vectors on $S \uplus S' \uplus P \uplus P'$. Then we define $\mathcal{K}^L_{SS'PP'}$ as follows:

$$\mathcal{K}^L_{SS'PP'} \equiv \{f_{SS'PP'} : f_{SS'PP'} = L(g_{SS'PP'}), g_{SS'PP'} \in \mathcal{K}_{SS'PP'}\}. \tag{34}$$

In this paper we confine ourselves to the situation where $L \circ L$ is either the identity map or its negative. Equivalently, $L_e \cdot L_e$ may be identity or its negative but the sign remains the same for all $e \in S \uplus P$. Thus for all the cases considered in this paper we have $(\mathcal{K}^L_{SS'PP'})^L = \mathcal{K}_{SS'PP'}$. We now have the following obvious lemma whose proof we omit. The map L is defined as above.

Lemma 4.1 *Let $\mathcal{K}_{SS'PP'}, \mathcal{K}_{PP'}$ be collections of vectors on $S \uplus S' \uplus P \uplus P', P \uplus P'$ respectively with S, S' and P, P' being copies and S, S', P, P' being pairwise disjoint. Then ,*

$$(\mathcal{K}_{SS'PP'} \leftrightarrow \mathcal{K}_{PP'})^L = \mathcal{K}^L_{SS'PP'} \leftrightarrow \mathcal{K}^L_{PP'}. \tag{35}$$

We now prove our main result. We need the following preliminary definition. Let S, S' and P, P' be pairs of copies, S, S', P, P' being pairwise disjoint. Let L be a copy invariant transformation on the collection of all vectors on $S \uplus S' \uplus P \uplus P'$. Let $S_1 \subseteq S, P_1 \subseteq P$. We say that the vector space $\mathcal{V}_{S_1 P_1 S'_1 P'_1}$ is *L-type* iff $(\mathcal{V}_{S_1 P_1 S'_1 P'_1})^L = \mathcal{V}^{\perp}_{S_1 P_1 S'_1 P'_1}$.

Theorem 4.1 *Let $\mathcal{V}_{SS'PP'}, \mathcal{V}_{PP'}$ be L-type vector spaces. Then $\mathcal{V}_{SS'PP'} \leftrightarrow \mathcal{V}_{PP'}$ is also an L-type vector space.*

Proof: *We need to show that*

$$(\mathcal{V}_{SS'PP'} \leftrightarrow \mathcal{V}_{PP'})^{\perp} = (\mathcal{V}_{SS'PP'} \leftrightarrow \mathcal{V}_{PP'})^L. \tag{36}$$

By the Implicit Duality Theorem (Theorem 3.2), the LHS equals $\mathcal{V}^{\perp}_{SPS'P'} \leftrightarrow \mathcal{V}^{\perp}_{PP'}$ and by Lemma 4.1, the RHS equals $\mathcal{V}^L_{SS'PP'} \leftrightarrow \mathcal{V}^L_{PP'}$. Since $\mathcal{V}_{SPS'P'}, \mathcal{V}_{PP'}$ are L-type spaces, this latter is the same as $\mathcal{V}^{\perp}_{SPS'P'} \leftrightarrow \mathcal{V}^{\perp}_{PP'}$ and the result follows.

□

We next extend Theorem 3.3.

Theorem 4.2 *Let $\mathcal{V}_{SPS'P'}, \mathcal{V}_{SS'}$ be L-type vector spaces. Then there exists an L-type vector space $\mathcal{V}_{PP'}$ such that*

$$\mathcal{V}_{SPS'P'} \leftrightarrow \mathcal{V}_{PP'} = \mathcal{V}_{SS'} \tag{37}$$

iff $\mathcal{V}_{SPS'P'} \cdot (S \uplus S') \supseteq \mathcal{V}_{SS'} \supseteq \mathcal{V}_{SPS'P'} \times (S \uplus S')$.

Proof: By Theorem 3.3, there exists a $\mathcal{V}_{PP'}$ such that $\mathcal{V}_{SPS'P'} \leftrightarrow \mathcal{V}_{PP'} = \mathcal{V}_{SS'}$ iff $\mathcal{V}_{SPS'P'} \cdot (S \uplus S') \supseteq \mathcal{V}_{SS'} \supseteq \mathcal{V}_{SPS'P'} \times (S \uplus S')$, and if such $\mathcal{V}_{PP'}$ exists we may, without loss of generality, take $\mathcal{V}_{PP'} = \mathcal{V}_{SPS'P'} \leftrightarrow \mathcal{V}_{SS'}$. But in this latter case since $\mathcal{V}_{SPS'P'}$ and $\mathcal{V}_{SS'}$ are L-type we would also have $\mathcal{V}_{PP'}$ as L-type. The theorem follows.

□

Remark 1: The generalized minor operation $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ is asymmetrical since $P \subset (S \uplus P)$ if S is nonvoid. One can define the symmetrical operation (matched sum)

$$\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ} \equiv \{(f_S, f_Q) : (f_S, f_P) \in \mathcal{V}_{SP}, (f_P, f_Q) \in \mathcal{V}_{PQ}\} \tag{38}$$

and the operation (skewed sum)

$$\mathcal{V}_{SP} \rightleftharpoons \mathcal{V}_{PQ} \equiv \{(f_S, f_Q) : ((f_S, f_P) \in \mathcal{V}_{SP}, (-f_P, f_Q) \in \mathcal{V}_{PQ}\}. \tag{39}$$

We can then show that ([4]), $(\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_{PQ})^{\perp} = \mathcal{V}_{SP}^{\perp} \rightleftharpoons \mathcal{V}_{PQ}^{\perp}$.
When we deal with L-type vector spaces, the above fact (matched sum transforming to skewed sum when we orthogonalize) prevents us from giving a single connection result for different choices of L. But in the trivial case where Q is void matched sum and skewed sum are identical. We therefore choose the lesser evil and work with $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ rather than with its symmetrical version.
Remark 2: Theorem 4.1 is open to two interpretations. We may regard one of $\mathcal{V}_{SPS'P'}, \mathcal{V}_{PP'}$ as corresponding to devices and the other to the connection constraints. If $\mathcal{V}_{PP'}$ is made up of several device characteristics, it would be a direct sum of the form $\oplus_i \mathcal{V}_{P_i P_i'}$. In this case $\mathcal{V}_{SPS'P'}$ would be the solution space of the connection constraints and the result of the operation would be a vector space $\mathcal{V}_{SS'}$

on $S \uplus S'$. On the other hand $\mathcal{V}_{SPS'P'}$ could be the direct sum $\oplus_i \mathcal{V}_{S_iP_iS'_iP'_i}$ each $\mathcal{V}_{S_iP_iS'_iP'_i}$ being a device. In this case $\mathcal{V}_{PP'}$ would be the solution space of connection constraints. Once again the result would be $\mathcal{V}_{SS'}$ on $S \uplus S'$.

5 Specializations of the connection theorems

We now consider some special L-type spaces which have been considered in the literature.

5.1 Reciprocal Structures

In this case, $L_e = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so that vector $(x_S, y_{S'}) \in \mathcal{V}_{SS'}$ iff $(-y_S, x_{S'}) \in \mathcal{V}^L_{SS'}$ (taking $x_{S'}(e') \equiv x_S(e)$ and $y_S(e) \equiv y_{S'}(e')$). Thus, $\mathcal{V}_{SS'}$ is a *reciprocal structure* iff $\mathcal{V}^{\perp}_{SS'} = \mathcal{V}^L_{SS'}$ with $\mathcal{V}^L_{SS'}$ defined as above.

Example 1: A multiport with conductance representation $[I \quad -G] \begin{bmatrix} i_{S_1} \\ v_{S'_1} \end{bmatrix} = 0$ would be reciprocal iff the solution space of the above operation with typical vector $(i_{S_1}, v_{S'_1})$ is complementary orthogonal to the space of all $(-v_{S_1}, i_{S'_1})$ vectors i.e., iff the row space of $[I \quad -G]$ is complementary orthogonal to the row space $[G \ I]$ i.e., iff row space of $[G^T \ I]$ is identical to the row space of $[G \ I]$, i.e., iff $G = G^T$.

Example 2: Consider the solution space of $\begin{bmatrix} A \ 0 \\ 0 \ B \end{bmatrix} \begin{bmatrix} i_S \\ v_{S'} \end{bmatrix} = 0$, where the row spaces of A, B are complementary orthogonal. (This is the case for the current space and voltage space associated with a directed graph.) This is the definition of an *ideal transformer*. In this case again we have a reciprocal structure since the row space of $\begin{bmatrix} 0 \ A \\ -B \ 0 \end{bmatrix}$ is clearly complementary orthogonal to the row space of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Now consider reciprocal structures $\mathcal{V}_{S_1S'_1}, \ldots, \mathcal{V}_{S_nS'_n}$, all the S_i, S'_j being pairwise disjoint. Let us now connect these by using the reciprocal structure $\mathcal{V}_{S_1P_1\ldots S_nP_nS'_1P'_1\ldots S'_nP'_n}$ (which we denote by $\mathcal{V}_{SPS'P'}$ in short), where all the S_i, S'_j, P_k, P'_l ,are pairwise disjoint. In other words we perform the operation

$$\mathcal{V}_{SPS'P'} \leftrightarrow (\mathcal{V}_{S_1S'_1} \oplus \ldots \oplus \mathcal{V}_{S_nS'_n}) \tag{40}$$

which by theorem 4.1 yields another reciprocal structure. Observe that this structure may be denoted $\mathcal{V}_{PP'}$ where $P = \cup P_i, P' = \cup P'_i$. In particular, if we have reciprocal multiports $\mathcal{V}_{S_1S'_1}, \ldots, \mathcal{V}_{S_nS'_n}$ and put them together according to the constraints of a graph $\begin{bmatrix} A\,0 \\ 0\,B \end{bmatrix} \begin{bmatrix} i_{S\uplus P} \\ v_{S'\uplus P'} \end{bmatrix} = 0$, then on the exposed ports P, we have a reciprocal structure. This is a generalization of a result due to Tellegen originally stated for 2-ports.

5.2 Dirac Structures

In this case $L_e = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so that vector $(x_S, y_{S'}) \in \mathcal{V}_{SS'}$ iff $(y_S, x_{S'}) \in \mathcal{V}^L_{SS'}$ (taking $x'_S(e') \equiv x_S(e)$ and $y_S(e) \equiv y_{S'}(e')$) Thus $\mathcal{V}_{SS'}$ is a *Dirac structure* iff $\mathcal{V}^{\perp}_{SS'} = \mathcal{V}^L_{SS'}$ with $\mathcal{V}^L_{SS'}$ defined as above.

Example 1: A multiport with a resistance representation

$$[-R\ I] \begin{bmatrix} i_{S_1} \\ v_{S'_1} \end{bmatrix} = 0 \tag{41}$$

would be Dirac iff the solution space of the above equation with typical vector (i_{S_1}, v_{S_1}) is complementary orthogonal to the space of all$(v_{S_1}, i_{S'_1})$ vectors, i.e., iff the row space of $[-R\ I]$ is complementary orthogonal to the row space $[I\ \ -R]$ i.e., iff the row space of $[I\ R^T]$ identical to the row space of $[I\ \ -R]$ i.e., iff $-R = R^T$. A gyrator ($\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$) is thus an example of a Dirac structure.

Example 2 : Consider the ideal transformer, i.e., the solution space of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} i_S \\ v'_S \end{bmatrix} = 0$, where the row spaces of A, B are complementary orthogonal. This is a Dirac structure since row space of $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is complementary orthogonal to the row space of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

5.3 Ideal Transformer

In the previous sections, we have seen that the ideal transformer is both a reciprocal and a Dirac structure.

Suppose next that $\mathcal{V}_{SS'}$ is both reciprocal and Dirac. We then have

$$\mathcal{V}_{SS'} = \{(g_S, f_{S'}) : (f_S, g_{S'}) \in \mathcal{V}_{SS'}\} = \{(-g_S, f_{S'}) : (f_S, g_{S'}) \in \mathcal{V}_{SS'}\}. \qquad (42)$$

Since $(\mathcal{V}_{SS'}^{\perp})^{\perp} = \mathcal{V}_{SS'}$, it is clear that $(f_S, g_{S'}) \in \mathcal{V}_{SS'}$ iff $((f_S, -g_{S'}) \in \mathcal{V}_{SS'}$. From this we conclude that $(f_S, g_{S'}) \in \mathcal{V}_{SS'}$ iff $(f_S, O_{S'}) \in \mathcal{V}_{SS'}$ and $(O_S, g_{S'}) \in \mathcal{V}_{SS'}$. Thus $\mathcal{V}_{SS'}$ can be expressed as $\mathcal{V}_S \oplus \mathcal{V}_{S'}$, for appropriate spaces .

We may thus write $\mathcal{V}_{SS'}$ as the solution space of the equation,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} i_S \\ v'_S \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (43)$$

Since the structure is Dirac it would follow that row spaces of $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ are complementary orthogonal i.e., that row spaces of A and B are complementary orthogonal. We thus see that a space is an ideal transformer iff it is both reciprocal and Dirac. In [1] connection theorems (Theorem 4.1, Theorem 4.2) are proved for ideal transformers.

5.4 Other Examples

Let us take $L_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that, $(x_s, y_s) \in \mathcal{V}_{SS'}$ iff $(x_s, -y_s) \in \mathcal{V}_{SS'}^{\perp}$. Let $\mathcal{V}_{SS'}^{L}$ be an L-structure, i.e., $\mathcal{V}_{SS'}^{L} = \mathcal{V}_{SS'}^{\perp}$. Let $\mathcal{V}_{SS'}$ be the solution space of the equation,

$$[I \ P] \begin{bmatrix} v_S \\ i_{S'} \end{bmatrix} = 0. \qquad (44)$$

The space of all $(v_S, -i_{S'})$, where $(v_s, +i_{S'}) \in \mathcal{V}_{SS'}$ is complementary orthogonal to $\mathcal{V}_{SS'}$. Hence, row space of $[I \ -P]$ is complementary orthogonal to row space of [$I \ P$]. Thus, $(-P^T \ I)$ is row equivalent to $[I \ -P]$. When P is invertible, it follows that $P^{-1} = P^T$.

We could generalize the above as well as Dirac structures by taking

$$L_e \equiv \begin{pmatrix} cos(\theta_e) & sin(\theta_e) \\ sin(\theta_e) & -cos(\theta_e) \end{pmatrix} \qquad (45)$$

choosing any θ_e for each e.

6 Conclusion

In this paper we have defined an $L-$structure to be a vector space $\mathcal{V}_{SS'}$ whose image $\mathcal{V}^L_{SS'}$ under an appropriate linear transformation L is equal to $\mathcal{V}^{\perp}_{SS'}$. Using the Implicit Duality Theorem, we have shown that the generalized minor of an L-structure with respect to another L-structure yields an L structure. As a consequence we are able to show, in particular, that connecting together Dirac (reciprocal) structures through Dirac (reciprocal) structures yields a Dirac (reciprocal) structure. We are also able to show in a unified manner necessary and sufficient conditions under which a number of L-structures may be connected together using L-structures to yield a specified L-structure.

References

[1] H. Narayanan: On the decomposition of vector spaces. *Linear algebra and its Applications* **76** (1986) 61-98.

[2] H. Narayanan: A unified construction of adjoint systems and networks. *International Journal of Circuit Theory and its Applications* **14** (1986) 263-276.

[3] H. Narayanan: Topological transformations of Electrical Networks. *International Journal of Circuit Theory and its Applications* **15** (1987) 211-233.

[4] H. Narayanan: Submodular Functions and Electrical Networks. *Annals of Discrete Mathematics* **54** (North Holland, London, New York, Amsterdam, 1997).

[5] H.Narayanan: On the Duality between Controllability and Observability in Behavioural Systems Theory. *Proc. International Conference on Communications Control and Signal Proc. CCSP 2000*, Bangalore, July 25-July 28,2000, 183-186.

[6] H.Narayanan: Matroids representable over Modules,Electrical Network Topology and Behavioural Systems Theory. *Research Report, EE Dept.,IIT Bombay*,October 2000.

[7] B.D.H. Tellegen: A General Network Theorem, with Applications. *Proc. Inst. Radio Engrs. Australia* **14** (1953) 265-270.

[8] A.van der Schaft: Interconnection and Geometry. in *J.W.Polderman and H.L.Trentelman (ed.): From Intelligent Control to Behavioural Systems*, (Printed at the Univ. of Groningen Press,1999).

7 Appendix

Proof of the Implicit Duality Theorem (Theorem 3.2)

Let

$$(\overset{S}{\mathbf{A}_S}\ \overset{P}{\mathbf{A}_P}),\quad (\overset{P}{\hat{\mathbf{A}}_P})$$

be the representative matrices of $\mathcal{V}_{SP}$, $\mathcal{V}_P$ respectively. A vector $\mathbf{x}_S$ belongs to $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P$ iff there exist vectors λ_1, λ_2 s.t.

$$\begin{bmatrix} {\mathbf{A}_S}^T & \mathbf{0} \\ {\mathbf{A}_P}^T & \hat{\mathbf{A}}_P^T \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_S \\ \mathbf{0} \end{bmatrix}.$$

By Theorem 3.1, this happens

iff

$$\begin{pmatrix} \mathbf{A}_S & \mathbf{A}_P \\ \mathbf{0} & \hat{\mathbf{A}}_P \end{pmatrix} \begin{bmatrix} \mathbf{y}_S \\ \mathbf{y}_P \end{bmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \Rightarrow (\mathbf{x}_S^T \mathbf{y}_S = 0)$$

i.e., iff

$$(\mathbf{y}_S \in (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})) \Rightarrow (\mathbf{x}_S^T \mathbf{y}_S = 0).$$

Thus, $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})^{\perp}$. It is clear that $(\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})$ is a vector space. By using the fact that for any vector space $\mathcal{V}$ on a finite set, $\mathcal{V}^{\perp\perp} = \mathcal{V}$, we get

$$\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp} = (\mathcal{V}_{SP}^{\perp} \leftrightarrow \mathcal{V}_P^{\perp})^{\perp\perp} = (\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P)^{\perp}.$$

□

Proof of Theorem 3.3

Suppose $\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V}_S \supseteq \mathcal{V}_{SP} \times S$.

Let $\mathcal{V}_P$ equal $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_S$, i.e., equal the collection of all vectors f_P s.t. for some vector $f_S \in \mathcal{V}_S$, $f_S \oplus f_P \in \mathcal{V}_{SP}$.

Clearly $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P \supseteq \mathcal{V}_S$. If $f_S{}' \in \mathcal{V}_{SP} \times S$ then $f_S{}' \in \mathcal{V}_S$ and $f_S{}' \oplus \mathbf{0}_P \in \mathcal{V}_{SP}$. Hence, by definition of $\mathcal{V}_P, \mathbf{0_P} \in \mathcal{V}_P$.

Let f_S be s.t. $f_S \oplus f_P \in \mathcal{V}_{SP}$ for some $f_P \in \mathcal{V}_P$. We know that there exists $f_S{}' \in \mathcal{V}_S$

s.t. ${f_S}'\oplus f_P \in \mathcal{V}_{SP}$. Since $\mathcal{V}_{SP}$ is a vector space, we must have, $(f_S - {f_S}')\oplus \mathbf{0}_P \in \mathcal{V}_{SP}$. Hence, $f_S - {f_S}' \in \mathcal{V}_{SP} \times S \subseteq \mathcal{V}_S$. Since $\mathcal{V}_S$ is closed under addition and ${f_S}' \in \mathcal{V}_S$, it follows that $(f_S - {f_S}') + {f_S}' = f_S$ also belong to $\mathcal{V}_S$. Thus, $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P \subseteq \mathcal{V}_S$. Therefore $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_S$.

Next suppose $\mathcal{V}_{SP} \leftrightarrow \mathcal{V}_P = \mathcal{V}_S$. It is clear from the definition of the generalized minor operation that $\mathcal{V}_{SP} \cdot S \supseteq \mathcal{V}_S$. Since $\mathbf{0} \in \mathcal{V}_P$, every vector f_S s.t. $f_S \oplus \mathbf{0}_P \in \mathcal{V}_{SP}$ must belong to $\mathcal{V}_S$. Thus, $\mathcal{V}_{SP} \times S \subseteq \mathcal{V}_S$.

□