Abstract—Can the popular shortest remaining processing time (SRPT) algorithm achieve a constant competitive ratio on multiple servers when server speeds are adjustable (speed scaling) with respect to the flow time plus energy consumption metric? This question has remained open for a while, where a negative result in the absence of speed scaling is well known. The main result of this paper is to show that multi-server SRPT can be constant competitive, with a competitive ratio that only depends on the power-usage function of the servers, but not on the number of jobs/servers or the job sizes (unlike when speed scaling is not allowed). When all job sizes are unity, we show that round-robin routing is optimal and can achieve the same competitive ratio as the best known algorithm for the single server problem. Finally, we show that a class of greedy dispatch policies, including policies that route to the least loaded or the shortest queue, do not admit a constant competitive ratio. When job arrivals are stochastic, with Poisson arrivals and i.i.d. job sizes, we show that random routing and a simple gated-static speed scaling algorithm achieves a constant competitive ratio.

I. INTRODUCTION

How to route and schedule jobs are two of the fundamental problems in multi-processor/multi-server settings, e.g. microprocessors with multiple cores. Microprocessors also have the flexibility of variable speed of operation, called speed scaling, where to operate at speed $s$, the power utilization is $P(s)$; typically, $P(s) = s^\alpha$, with $2 \leq \alpha \leq 3$. Speed scaling is also available in modern queuing systems where servers can operate at variable service rates with an appropriate cost function $P(\cdot)$.

Increasing the speed of the server reduces the response times (completion minus arrival time) but incurs a larger energy cost. Thus, there is a natural tradeoff between the the flow time (defined as the sum of the response times across all jobs) and the total energy cost, and a natural objective is to minimize a linear combination of the flow time and total energy, called flow time plus energy.

In this paper, we consider the online problem of routing, scheduling, and speed scaling in a multi-server setting to minimize the flow time plus energy, where jobs arrive (are released) over time and decisions have to be made causally. On the arrival of a new job, a centralized controller needs to make a causal decision about which jobs to process on which server and at what speed, where preemption and migration is allowed. By migration, we mean that a job can be preempted on one server and restarted on another server later. The model, however, does not allow job splitting, i.e., a job can only be processed on a single server at any time.

For this problem, both the stochastic and worst case analysis is of interest, where in the stochastic model, the input (job sizes and arrival instants) is assumed to follow a distribution, and performance guarantees in expectation are derived. In the worst case analysis, the input can be generated by an adversary, and the performance metric is the competitive ratio, that is defined as the maximum of the ratio of the cost of the online algorithm and the optimal offline algorithm (that knows the entire input sequence ahead of time).

A. Prior Work

1) Single Server: For a single server, it is known that Shortest Remaining Processing Time (SRPT) is an optimal scheduling policy, and the only decision with speed scaling is the optimal dynamic speed choice. There is a large body of work on speed scaling in the single server setting [1]–[8] both in the stochastic as well as worst case settings, where mostly $P(s) = s^\alpha$ is used, under various assumptions, e.g. bounded speed $s \in [0, S]$ [9], with and without deadlines [10]–[12], etc.

In the stochastic model, [6] showed that a simple fixed speed policy (called gated speed) that depends only on the load/utilization and is independent of the current number of unfinished jobs/sizes has a constant multiplicative gap from the ‘unknown’ optimal policy. Further work in this direction can be found in [13], [14], where [14] derived the mean response time under the SRPT algorithm. For the worst case, there are many results [1]–[5], [7]–[12], [15]. A key result in this space was proved in [15], where an SRPT-based speed scaling algorithm is proved to be $(3+\epsilon)$-competitive algorithm for an arbitrary power function $P(\cdot)$. In [8], using essentially the same ideas as in [15], but with a more careful analysis, a slightly modified SRPT-based speed-scaling algorithm is shown to be $(2+\epsilon)$-competitive algorithm, also for an arbitrary power function.

In the worst case setting, when considering speed scaling, two classes of problems are studied: (i) unweighted and (ii) weighted, where in (i) the delay that each job experiences is given equal weight in the flow time computation, while in (ii) it is scaled by a weight that can be arbitrary. The weighted setting is fundamentally harder that the unweighted one, where it is known that constant-competitive online algorithms are not possible [16], even for a single server, while constant competitive algorithms are known for the unweighted case, even for arbitrary energy functions, e.g., the $(2+\epsilon)$-competitive algorithm proposed in [8]. To circumvent the negative result for the weighted case, typically, the online algorithm is allowed a speed augmentation of $1+\epsilon$ compared...
to the optimal offline algorithm, in which case algorithms with $O(1)$ competitive ratios are possible, where $O(1)$ depends on $c$.

2) Multiple Servers: With multiple servers without speed scaling (when the server speeds are fixed), to minimize just the flow time, a well-known negative result from [17] showed that the SRPT algorithm (which always processes the $m$ smallest jobs with $m$ servers) that requires both preemption and job migration has a competitive ratio that grows as the logarithm of the ratio of the largest and the smallest job size. Moreover, [17] also showed that no online algorithm can do better than SRPT when server speeds are fixed.

With multiple servers, one critical aspect is whether job migration is allowed or not. With job migration, a preempted job can be processed by any of the servers and not necessarily by the server where it was partially processed first. Remarkably in [18], a non-migratory algorithm that only requires preemption is proposed that achieves the same competitive ratio as SRPT. A more positive result for SRPT is that if it is allowed a speed augmentation of $(2 - 1/m)$ (respectively, $(1 + \epsilon)$) over the offline optimal algorithm, then it has a constant competitive ratio of 1 (respectively, a constant constant depending on $\epsilon$); see [19], [20].

For the worst-case design, speed scaling with multiple servers to minimize flow time and energy has been studied in [21]–[25]. The homogenous server case was studied in [22], [24], i.e., $P(s)$ is identical for all servers, while the heterogenous case was addressed in [23], [25], where $P(s)$ is allowed to be different for different servers.

For the unweighted flow time and energy problem under the homogenous server case, a variant of the round robin algorithm without migration has been shown to have a competitive ratio of $O(1)$ [21] with $(1 + \epsilon)$ augmentation with bounded server speeds. This result was extended in [22] for the weighted flow time plus energy using a randomized server selection algorithm that also does not use migration.

For the heterogenous server setting with $(1 + \epsilon)$ augmentation, [23], [25], derived algorithms that assigns job to server that cause least increase in the projected future weighted flow and a variant of processor sharing, respectively, that are $O(1)$ competitive in unweighted and weighted flow time plus energy. Moreover, if for server $k$, $P_k(s) = s^{\alpha_k}$, then the algorithm in [23], [26] has a competitive ratio dependent on $\alpha$ without any need for speed augmentation; however, the exact competitive ratio is not provided there.

In the stochastic setting, for multiple servers, the flow time plus energy problem with multiple servers is studied under a fluid model [27], [28] or modelled as a Markov decision process [29], and near optimal policies are derived.

Our focus in this work is on the unweighted flow time plus energy under the homogenous server setting, where in the context of the prior work we want to answer the following open questions: (i) For the worst case design, is it possible to achieve a constant competitive ratio with simpler algorithms without any speed augmentation (compared to algorithms of [23], [25], that are hard to implement)? In particular, can SRPT do so, since it is a widely used and simple to implement algorithm? This question is also directly related to the limitation of SRPT without speed scaling as shown in [17], and whether SRPT with and without speed scaling are fundamentally different. (ii) For the stochastic setting, can simple algorithms achieve near optimal performance without the need of fluid limit approximations?

B. Our Contributions

Let the number of (homogenous) servers be $m$.

- The SRPT algorithm, with speed chosen as $P^{-1}(\frac{n}{m})$ if $n \geq m$ and $P^{-1}(1)$ if $n < m$, where $n$ is the number of unfinished jobs, is shown to be $c$-competitive, where $$c = P(2 - 1/m) \left(2 + \frac{2}{P^{-1}(1)} \max(1, P(\bar{s}))\right).$$

This means the above algorithm is constant-competitive, with a competitive ratio that is independent of the number of servers as well as the workload sequence. This result is proved under mild regularity assumptions on the power function $P(\cdot)$, which can be further relaxed using standard arguments [15]. For the special case $P(s) = s^\alpha$, where $\alpha \in (1, 2)$, we derive another bound of $3 + \frac{2}{2-c\alpha}$ on the competitive ratio; this bound is tighter than the previous one for $\alpha \ll 2$. Similar to the algorithm proposed in [23], the competitive ratio of the our SRPT-based policy also depends on $\alpha$. However, the algorithm proposed here is much simpler, and comes with a lower implementation complexity.

- An important conclusion to draw from this result is that SRPT with speed scaling is fundamentally different as compared to the case when speed scaling is not allowed; in the latter setting, the competitive ratio depends on the number of jobs and their sizes [17]. Thus, allowing for speed scaling, the ever popular SRPT is shown to be robust in the multiple server setting.

- With speed scaling, we also derive some lower bounds for the immediate dispatch case when the job has to be assigned to a server instantaneously on its arrival and cannot be migrated across servers, though preemption within a server is allowed. Under this setting, we show that greedy routing policies, that assign a new job to the currently least loaded server or to the historically least loaded server have a competitive ratio of at least $\Omega(m^{1-1/\alpha})$. Moreover, even when immediate dispatch is not necessary (i.e., jobs can wait in a common queue), but job migration across servers is not allowed, we show that the competitive ratio of SRPT is at least $\Omega(m^{1-1/\alpha})$.

- For the special case where all jobs have unit size, we show that round robin (RR) routing is optimal, and the best known competitive ratio results on speed scaling to minimize the flow-time plus energy in the single server setting apply in the multiple server setting as well.

- We also consider the stochastic setting, where jobs arrive according to a Poisson process with i.i.d. sizes. This case turns out to be significantly easier than the worst case; we

\footnote{While $m$ does appear in the expression for the competitive ratio, note that $2 - 1/m$ is trivially upper bounded by 2.}
show that with $P(s) = s^\alpha$ ($\alpha > 1$), random routing and a simple gated-static speed scaling algorithm achieves a constant competitive ratio, e.g., $2$ for $P(s) = s^2$.

II. System Model

Let the input consist of $J$ jobs, where job $j$ arrives (is released) at time $a_j$ and has work/size $w_j$. There are $m$ homogenous servers, each with the same power function $P(s)$, where $P(s)$ denotes the power consumed while running at speed $s$. Any job can be processed by any of the $m$ servers.

The speed $s$ is the rate at which work is executed by any of the servers, and $w$ amount of work is completed in time $w/s$ by any server if run at speed $s$ throughout time $w/s$. A job $j$ is defined to be complete at time $c_j$ if $w_j$ amount of work has been completed for it, possibly by different servers. We assume that preemption is allowed, i.e., a job can be suspended and later restarted from the point at which it was suspended. Moreover, we also assume that job migration is allowed, i.e., if a job is preempted it can be processed later at a different server than the one from which it was preempted. Thus, a job can be processed by different servers at different intervals, but at any given time it can be processed by only server, i.e., no job splitting is allowed. The flow time $f_j$ for job $j$ is $f_j = c_j - a_j$ (completion time minus the arrival time) and the overall flow time is $F = \sum j f_j$. From here on we refer to $F$ as just the flow time. Note that $F = \int n(t)dt$, where $n(t)$ is the number of unfinished jobs at time $t$. Thus, flow time can also be interpreted as the cumulative holding cost, where instantaneous holding cost at time $t$ equals $n(t)$.

Let server $k$ run at speed $s_k(t)$ at time $t$. The energy cost is defined as $\sum_{k=1}^m P(s_k(t))$ summed over the flow time. Choosing larger speeds reduces the flow time, however, increases the energy cost, and the natural objective function that has been considered extensively in the literature is the sum of flow time and energy cost, which we define as

$$C = \int n(t)dt + \sum_{k=1}^m P(s_k(t))dt. \tag{1}$$

Any online algorithm only has causal information, i.e., it becomes aware of job $j$ only at time $a_j$. Any online algorithm with multiple servers has to make two causal decisions: routing; that specifies the assignment of jobs to servers, and scheduling; that specifies a job to be processed by each server. For a server with multiple servers, has to make two causal decisions: routing; that specifies the assignment of jobs to servers, and scheduling; that specifies a job to be processed by each server. Moreover, we also assume that job migration is allowed, i.e., if a job is preempted it can be processed later at a different server than the one from which it was preempted. Thus, a job can be processed by different servers at different intervals, but at any given time it can be processed by only server, i.e., no job splitting is allowed. The flow time for job $j$ is $f_j = c_j - a_j$ (completion time minus the arrival time) and the overall flow time is $F = \sum j f_j$. From here on we refer to $F$ as just the flow time. Note that $F = \int n(t)dt$, where $n(t)$ is the number of unfinished jobs at time $t$. Thus, flow time can also be interpreted as the cumulative holding cost, where instantaneous holding cost at time $t$ equals $n(t)$.

Let server $k$ run at speed $s_k(t)$ at time $t$. The energy cost is defined as $\sum_{k=1}^m P(s_k(t))$ summed over the flow time. Choosing larger speeds reduces the flow time, however, increases the energy cost, and the natural objective function that has been considered extensively in the literature is the sum of flow time and energy cost, which we define as

$$C = \int n(t)dt + \sum_{k=1}^m P(s_k(t))dt. \tag{1}$$

Any online algorithm only has causal information, i.e., it becomes aware of job $j$ only at time $a_j$. Any online algorithm with multiple servers has to make two causal decisions: routing; that specifies the assignment of jobs to servers, and scheduling; that specifies a job to be processed by each server. Let the cost (1) of an online algorithm $A$ be $C_A$. Moreover, let the cost of (1) for an offline optimal algorithm that knows the job arrival sequence $A$ (both $a_j$ and $w_j$) in advance be $C_{OFF}$. Then the competitive ratio of the online algorithm $A$ for $A$ is defined as

$$c_A(\sigma) = \frac{C_A(\sigma)}{C_{OFF}(\sigma)}. \tag{2}$$

It is also natural to take the objective to be a linear combination of flow time and energy, i.e., $\int n(t)dt + \beta \int \sum_{k=1}^m P(s_k(t))dt$, where $\beta > 0$ weighs the energy cost relative to the delay cost. However, note that since the factor $\beta$ may be absorbed into the power function, we will work with the objective (1) without loss of generality.

and the objective function considered in this paper is to find an online algorithm that minimizes the worst case competitive ratio

$$c^* = \min_A \max_{\sigma} c_A(\sigma). \tag{3}$$

We will also consider stochastic input $\sigma$ where both $a_j$ and $w_j$ are chosen stochastically, in which case our definition for competitive ratio for $A$ will be

$$c_A = \mathbb{E}[C_A]/\mathbb{E}[C_{OFF}], \tag{4}$$

where the expectation is with respect to the stochastic input; see Section V for the details. Correspondingly, the goal is to come up with an online algorithm that minimizes $c_A$.

In Sections III to IV, we study the worst-case setting, and present the results for the stochastic setting in Section V.

III. Worst Case Competitive Ratio: Upper Bounds

In this section, we present our results on constant competitive policies for scheduling and speed scaling in a multi-server environment. We propose an online policy that performs SRPT scheduling, where the instantaneous speed of each server is a function of the number of outstanding jobs in the system. We prove that this policy is constant competitive for a broad class of power functions. Specifically, the competitive ratio depends only on the power function, but not on the number of jobs, their sizes, or the number of servers.

A. SRPT Algorithm

In this section, we consider the SRPT algorithm for routing, and analyze its competitive ratio when the server speeds are chosen as follows. Let $n(t)$ and $n_0(t)$ denote the number of unfinished jobs with the SRPT algorithm and OPT (the offline optimal algorithm) respectively, at time $t$. Moreover, let $A(t)$ and $OPT(t)$ be the set of active jobs with the SRPT algorithm and OPT, respectively. Recall that the SRPT algorithm maintains a single queue and serves the $\min\{m, n(t)\}$ shortest jobs at any time $t$.

The speed for job $k \in A(t)$ with the SRPT algorithm is chosen as

$$s_k(t) = \begin{cases} P^{-1} \left( \frac{n(t)}{m} \right) & \text{if } n(t) \geq m, \\ P^{-1}(1), & \text{otherwise.} \end{cases} \tag{5}$$

The above speed scaling rule can be interpreted as follows. Under (5), $\sum_{k \in A(t)} P(s_k(t)) = n(t)$, i.e., the instantaneous power consumption is matched to the instantaneous job holding cost.

Our main result (Theorem 1) is proved under the following assumption on the power function.

Assumption 1. $P : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, strictly increasing, and strictly convex, such that $P(0) = 0$, $\lim_{s \to \infty} P(s) = \infty$, and $s := \inf\{s > 0 \mid P(s) > s\} < \infty$.

Remark 1. It is possible to relax Assumption 1 to allow for almost arbitrary power functions (including non-convex functions and those associated a finite maximum speed) by adapting the arguments in [15]; see, for example, [23]. Since
these arguments are well understood, we do not repeat them here. The main takeaway in the context of the present paper is that Assumption 1 is not restrictive, and that a c-competitive algorithm under Assumption 1 can be extended to obtain a (c + ε)-competitive algorithm under an arbitrary power function for ε > 0.

We are now ready to state our main result, which shows that our SRPT algorithm is constant competitive.

**Theorem 1.** Under Assumption 1, the SRPT algorithm with speed scaling (5) is c-competitive, where

\[ c = P(2 - 1/m) \left( 2 + \frac{2}{P-1} \max(1, P(\bar{s})) \right). \]

Taking P(s) = s^α for α > 1, the competitive ratio equals 4(2 - 1/m)^α. To prove Theorem 1, we use a potential function argument, where the potential function is defined as follows. Let n_o(t, q) and n(t, q) denote the number of unfinished jobs under OPT and the algorithm, respectively, with remaining size at least q. In particular, n_o(t, 0) = n_o(t) and n(t, 0) = n(t). Let

\[ d(t, q) = \max \left\{ 0, \frac{n(t, q) - n_o(t, q)}{m} \right\}. \]

Define

\[ \Phi_1(t) = c_1 \int_0^\infty f(d(t, q)) dq, \]

where \( f(0) = 0 \), and \( \forall i \geq 1 \), \( \Delta(\frac{i}{m}) = f(\frac{i}{m}) - f(\frac{i-1}{m}) = P'(P^{-1}(\frac{i}{m})) \) (this means \( P'(x) \) where \( x = P^{-1}(\frac{i}{m}) \)), and

\[ \Phi_2(t) = c_2 \int_0^\infty (n(t, q) - n_o(t, q)) dq. \]

Consider the potential function

\[ \Phi(t) = \Phi_1(t) + \Phi_2(t). \]

The \( \Phi_1(t) \) part of the potential function is a multi-server generalization of the potential function in [15], while the \( \Phi_2(t) \) part is novel. Let the speed of job \( k \in O(t) \) under OPT at time \( t \) be \( \tilde{s}_k(t) \). Suppose we can show that for any input sequence \( \sigma \),

\[
\int n(t) + \sum_{k \in A(t)} P(s_k(t)) + \frac{d\Phi(t)}{dt} \leq c \left( n_o(t) + \sum_{k \in O(t)} P(\tilde{s}_k(t)) \right)
\]

almost everywhere and that \( \Phi(t) \) satisfies the following boundary conditions (proved in Proposition 5; see Appendix A):

1) Before any job arrives and after all jobs are finished, \( \Phi(t) = 0 \), and
2) \( \Phi(t) \) does not have a positive jump discontinuity at any point of non-differentiability.

Then, integrating (7) with respect to \( t \), we get that

\[
\int n(t) + \sum_{k \in A(t)} P(s_k(t)) \leq \int c \left( n_o(t) + \sum_{k \in O(t)} P(\tilde{s}_k(t)) \right),
\]

which is equivalent to showing that \( C_{SRPT}(\sigma) \leq c C_{OPT}(\sigma) \) for any input \( \sigma \) as required.

The intuition for the form of the competitive ratio in Theorem 1 is as follows.

**Lemma 1.** [19] Without speed scaling, where the speed of each server is fixed to be unity for all times, and the objective is to only minimize the flow-time (total delay), OPT that follows SRPT is \( (2 - 1/m) \)-approximate with respect to OPT.

**Remark 2.** For proving Theorem 1 via showing that (7) is true for some \( c \), we assume that OPT also uses SRPT with arbitrary speeds at time \( t \) that can depend on future job arrivals, since enforcing OPT to use SRPT helps in proving (7). From Lemma 1, it follows that with speed scaling, OPT-SRPT (OPT that is constrained to perform SRPT scheduling) is \( (2 - 1/m) \)-competitive with respect to OPT, since OPT following SRPT can scale the speed up by a factor \( (2 - 1/m) \) at all times, and get exactly the same flow-time as the OPT, by paying an extra multiplicative energy cost of \( P(2 - 1/m) \). Therefore, we show that SRPT with speed scaling as in (5) is c-competitive with respect to OPT-SRPT by showing (7), to get the final result that it is \( c P(2 - 1/m) \)-competitive with respect to OPT itself.

For smaller values of \( \alpha \ll 2 \) the result of Theorem 1 can be further improved for the special case of power-law power functions, as described in the next theorem.

**Theorem 2.** With \( P(s) = s^\alpha \) and for any \( \alpha \in (1, 2) \), the SRPT-based algorithm with speed scaling (5) is c-competitive, where

\[ c = 3 + \frac{2}{2 - \alpha}. \]

The proof of Theorem 2 is similar in spirit to that of Theorem 1, but without assuming that OPT follows SRPT. It also uses the same potential function \( \Phi \) (see (6)), and directly tries to bound the increase in \( \Phi \) because of processing of the jobs by the algorithm and OPT. The limitation on \( \alpha \) appears because without enforcing that OPT follows SRPT, we cannot apply a technical lemma (Lemma 8) jointly on the change made to \( \Phi \) by the algorithm and the OPT, but individually. The improvement in competitive ratio compared to Theorem 1 results because of not enforcing OPT to follow SRPT, thereby saving on the penalty of \( (2 - 1/m) \alpha \). The proof of Theorem 2 is provided in Appendix C, while the remainder of this section is devoted to the proof of Theorem 1.

**Proof of Theorem 1.** In light of Lemma 1 and Remark 2, we assume throughout this proof that OPT performs SRPT scheduling, and additionally include a factor of \( P(2 - 1/m) \) in the competitive ratio. For simplicity, we refer to the OPT-SRPT algorithm as simply OPT throughout this proof.

In the following, we show that (7) is true for a suitable choice of \( c \). To show (7), we bound \( d\Phi/dt \) via individually bounding \( d\Phi_1/dt \) and \( d\Phi_2/dt \) in Lemmas 2 and 3 below. Note that it suffices to show that (7) holds at any instant \( t \) which is not an arrival or departure instant under the algorithm or OPT. For the remainder of this proof, consider any such time instant \( t \). For ease of exposition, we drop the index \( t \) from \( n(t, q), n(t, q_0), n(t), n_o(t), s_k(t) \) and \( \tilde{s}_k(t) \), since only a fixed (though generic) time instant \( t \) is under consideration.
Lemma 2. For \( n \geq m \),
\[
d\Phi_1/dt \leq c_1 n_0 - c_1 n + c_1 \left( \frac{m - 1}{2} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k),
\]
while for \( n < m \),
\[
d\Phi_1/dt \leq c_1 n_o - c_1 \frac{n(n + 1)}{2m} + c_1 \sum_{k \in O} P(\tilde{s}_k)
\]

Lemma 3. \( d\Phi_2/dt \leq -c_2 \min(m, n) P^{-1}(1) + c_2 \sum_{k \in O} \max\{P(\tilde{s}), P(\tilde{s}_k)\} \]

Using Lemmas 2 and 3 (proved in Appendix B), we now prove (7) by considering the following two cases:

[Case 1: \( n \geq m \).] \( n + \sum_{k \in A} P(s_k) + d\Phi(t)/dt \)
\[
\leq c_1 n + c_1 \sum_{k \in O} P(\tilde{s}_k) - c_2 m - 2 \sum_{k \in O} \max\{P(\tilde{s}), P(\tilde{s}_k)\} + c_2 \sum_{k \in O} \max\{P(\tilde{s}), P(\tilde{s}_k)\}
\]
\[
\leq (c_1 + c_2) \min\{P(s), P(s_k)\} (n_o + \sum_{k \in K} P(\tilde{s}_k))
\]

Here, (a) follows from Lemmas 2 and 3, and since \( P(s_k) = n/m \) when \( n \geq m \) (see (5)), while (b) follows by setting \( c_1 = 2 \) and \( c_2 \geq 1/P^{-1}(1) \).

[Case 2: \( n < m \).] \( n + \sum_{k \in A} P(s_k) + d\Phi(t)/dt \)
\[
\leq n + c_1 n_o - c_1 \frac{n(n + 1)}{2m} + c_1 \sum_{k \in O} P(\tilde{s}_k) - c_2 m P^{-1}(1)
\]

This proves (7) for
\[
e = c_1 + c_2 \max\{1, P(\tilde{s})\} = \left( 2 + \frac{2}{P^{-1}(1)} \right) \max\{1, P(\tilde{s})\}.
\]

In the next section, we consider a special case when all jobs have unit size, but their arrival instants are still worst case, for which we can improve the competitive ratio guarantees.

B. Equal Sized Jobs

Assume that all jobs have equal size, which is taken to be 1 without loss of generality. There are \( m \) servers and jobs are assigned on arrival to one of the \( m \) servers for service. OPT refers to the offline optimal policy. We propose the following policy \( \mathcal{U} \). Each job on its arrival is assigned to servers in a round-robin fashion, and each server \( k \) uses speed \( s_k(t) = P^{-1}(n_k(t)) \), where \( n_k(t) \) is the number of unfinished jobs that have been assigned to server \( k \).

**Theorem 3.** With unit job sizes, under Assumption 1, \( \mathcal{U} \) is 2-competitive.

**Proof.** In Proposition 1, we show that when all jobs are of unit size, OPT follows round robin scheduling. Thus, \( \mathcal{U} \) and OPT see the same set of arrivals on each server. The result follows from [8], which shows that choosing speed \( s_k(t) = P^{-1}(n_k(t)) \) for a single server system is a 2-competitive.

**Proposition 1.** With unit job sizes, under Assumption 1, OPT performs round robin dispatch across servers.

**Proof.** Let us assume that OPT can hold arriving jobs in a central queue before dispatch to one of the \( m \) servers. It suffices to show that even in this expanded space of policies, OPT can be assumed to perform round robin dispatch without loss of optimality (WLO).

1) From the convexity of the power function, it follows that OPT serves each job at a constant speed. Labeling jobs in the order of their arrival, let \( s_j \) denote the speed at which job \( j \) is served.

2) WLO, we may assume that OPT dispatches jobs for service in a FCFS manner.

**Claim 1:** WLO, OPT completes jobs in the order of their arrival.

It follows from Claim 1 that OPT can be assumed to perform round robin WLO.

**Proof of Claim 1:** Let \( a_i \) denote the time when job \( l \) begins service and let \( d_i \) denote the time when the same job completes service. Suppose the claim does not hold, i.e., there exist \( i, j \) where \( i < j \) such that \( d_j < d_i \). We now demonstrate an alternative power allocation that is strictly better for OPT.

Note that \( a_i \leq a_j \). Let \( r \leq 1 \) denote the remaining work \( j \) at time \( a_j \). Clearly, \( d_j < d_i \) implies that \( s_j > s_i \). Fix \( \delta \in (0, 1/s_j] \) such that
\[
s_j \delta + s_i \left( \frac{1}{s_j} - \delta \right) = r. \tag{8}
\]

Consider the following power allocation:

1) Starting at time \( a_j \), job \( i \) is served at speed \( s_j \) for \( \delta \) time units, and at speed \( s_i \) for \( \frac{1}{s_j} - \delta \) time units.

2) Starting at time \( a_j \), job \( j \) is served at speed \( s_j \) for \( \frac{1}{s_j} - \delta \) time units, and at speed \( s_i \) for \( \frac{1}{s_j} - \delta \) time units.

From (8), it is not hard to see that under this new power allocation, the departure instants of jobs \( i \) and \( j \) are unchanged, i.e., job \( i \) completes at time \( d_i \), whereas job \( j \) completes at time \( d_i \). Moreover, under the above power allocation, the cost of OPT remains unchanged. Indeed, the increase in the delay cost of job \( j \) is exactly compensated by the decrease
in the delay cost of job \(i\). Moreover, the energy cost remains unchanged, and the cost associated with all remaining jobs remains unchanged as well (we simply interchange all subsequent dispatches between the servers serving jobs \(i\) and \(j\)).

Now, from the convexity of the power function, it follows that we can strictly decrease the energy cost of \(\text{OPT}\) by running jobs \(i\) and \(j\) at constant speeds from time \(a_j\), such that the completion times remain unchanged.

This gives us a contradiction, and completes the proof of the claim.

\[\square\]

IV. Worst Case Competitive Ratio: Lower Bounds

In the previous section, we showed that while SRPT scheduling is not constant-competitive in a multi-server environment without speed scaling, it can be made constant-competitive when speed scaling is allowed. However, one issue with implementing SRPT on multiple servers is the need for job migration. In this section, we show that a broad class of greedy non-migratory policies is not constant-competitive.

We begin by stating the following preliminary result.

**Lemma 4.** On a single server, consider a single burst of \(n\) jobs, with sizes \(x_n \leq x_{n-1} \leq \cdots \leq x_1\). The cost incurred by \(\text{OPT}\) in processing this burst equals \(c \sum_{k=1}^{n} x_k k^{1-1/\alpha}\), where the constant \(c\) depends on \(\alpha\).

The proof of Lemma 4 follows by direct computation of the optimal speeds for each job that minimize the flow time plus energy cost (1).

A. Greedy algorithms

**Lemma 5.** Consider the class of policies that routes an incoming job to a server with the least amount of unfinished workload. All policies in this class have a competitive ratio that is \(\Omega(m^{1-1/\alpha})\).

**Proof.** Consider the following instance: A burst of \(m-1\) jobs, each having size \(w\) arrives at time 0, and another burst of \(w\) jobs, each having size 1 arrives at time \(0^+\).

Any workload-based greedy policy would assign the first \(m-1\) jobs of size \(w\) to \(m-1\) different servers, and the \(w\) jobs of size 1 to the remaining server. By Lemma 4, the cost incurred by any such algorithm is at least

\[c(m-1)w + c \sum_{k=1}^{w} k^{1-1/\alpha} \geq c(m-1)w + c'w^{2-1/\alpha}.\]

Consider now an algorithm \(A\) that assigns the first \(m-1\) jobs of size \(w\) to \(m-1\) different servers and then distributes the \(w\) jobs of size 1 uniformly among all \(m\) servers. The algorithm \(A\) then performs scheduling and speed scaling on each server as per single server OPT. The cost incurred by \(A\) (which upper bounds the cost under OPT) equals (using Lemma 4)

\[c m \left( \frac{w}{m} \right)^{2-1/\alpha} + cmw \leq c m (\frac{w}{m})^{2-1/\alpha} + cmw.\]

Now, setting \(w = m^d\) for large enough \(d\), we see that the competitive ratio of any workload-based greedy policy is \(\Omega(m^{1-1/\alpha})\).

\[\square\]

It follows from the proof of Lemma 5 that the competitive ratio of any policy that routes an incoming job to a server that has been assigned the least aggregate workload so far (including completed as well as queued workload) is also \(\Omega(m^{1-1/\alpha})\).

**Lemma 6.** Consider the class of policies that route an incoming job to a server with the least number of queued jobs (join the shortest queue (JSQ)). All policies in this class have a competitive ratio that is \(\Omega(m^{1-1/\alpha})\).

**Proof.** Consider the following instance: \(m^2\) jobs arrive in quick succession, causing any JSQ-based policy to perform round robin routing. Every \(m^2\) arriving job has size \(w\), while all remaining jobs have size 1.

Thus, under any JSQ-based policy, one server would get \(m\) jobs of size \(w\) routed to it, whereas all other servers would get \(m\) jobs of size 1. Thus, the cost under any such policy is at least \(cw \sum_{k=1}^{m} k^{1-1/\alpha} + c(m-1) \sum_{k=1}^{m} k^{1-1/\alpha} \geq c w m^{2-1/\alpha} + c'(m-1)m m^{2-1/\alpha}\).

Consider an algorithm \(A\) that routes the jobs uniformly across the servers, such that each server gets \(m-1\) jobs of size 1, and one job of size \(w\). Post routing, \(A\) performs scheduling and speed scaling on each server as per single server OPT. The cost incurred by \(A\) (which upper bounds the cost of OPT) is thus \(m \left[ cw + c' \sum_{k=1}^{m-1} k^{1-1/\alpha} \right] \leq c'' \left[ mw + m m^{3-1/\alpha} \right] \).

Now, setting \(w = m^d\) for large enough \(d\), we see that the competitive ratio of any JSQ-based policy is \(\Omega(m^{1-1/\alpha})\).

\[\square\]

It is also clear from the above proof that any policy that performs round robin routing would have a competitive ratio that is \(\Omega(m^{1-1/\alpha})\).

B. SRPT-based algorithms

In this section, we consider the following class of non-migratory SRPT-based policies: Let \(y_j(t)\) denote the least remaining processing time among all jobs queued at server \(j\). If server \(j\) is idle at time \(t\), then set \(y_j(t) = 0\). Consider now a job of size \(x\) arriving into the system at time \(t\). If the set \(\{j : y_j(t) > x\}\) is non-empty, then the job is assigned to a server from this set. Else, the job is assigned to any server, or held in a central queue. Each server may preempt between jobs queued at that server. But jobs once assigned to a certain server must complete service at that server, i.e., migration is not allowed.

**Lemma 7.** Consider the class of non-migratory SRPT-based policies described above. All policies in this class have a competitive ratio that is \(\Omega(m^{1-1/\alpha})\).

**Proof.** Consider the following instance: \(m-1\) jobs of size 1 arrive at time 0, and \(m\) jobs of sizes \(w, w-\epsilon, \cdots, w-(m-1)\epsilon\) arrive in quick succession right after.

Any non-migratory SRPT-based policy would route the \(m-1\) jobs of unit size to \(m-1\) different servers, and the next \(m\)
jobs to the remaining server. Thus, the cost incurred is at least
\((w - m)e c \sum_{k=1}^{m} k^{1-\alpha} \geq c'(w - m)e m^{2-\alpha}\).

Consider next a policy \(A\) that routes the first \(m - 1\) unit sized jobs to \(m - 1\) different servers, and distributes the next \(m\) jobs across all servers. Post routing, \(A\) performs scheduling and speed scaling on each server as per single server OPT. The cost incurred by \(A\) (which upper bounds the cost of OPT) is thus at most \(mc(w + 2^{1-\alpha})\). Now, setting \(w = m^d\) for large enough \(d\), we see that the competitive ratio of any non-migratory SRPT-based policy is \(\Omega(m^{1-\alpha})\).

\[\square\]

V. Stochastic Input

In this section, we consider a stochastic model for the job arrivals. Jobs arrive according to a Poisson process of rate \(\lambda\), and have i.i.d. sizes. Let \(X\) denote a generic job size. We assume that \(E[X] < \infty\). The load, which is the rate at which work is submitted to the system, is given by \(\Lambda = \lambda E[X]\).

The performance metric under consideration is the stationary variant of the flow time plus energy metric considered for the worst-case analysis, i.e.,
\[C = E[T] + E[E],\]
where \(T\) denotes the steady state response time, and \(E\) denotes the energy required to serve a job in steady state.\(^3\) In the present section, we restrict attention to power functions of the form \(P(s) = s^\alpha\), where \(\alpha > 1\).

In the following, we generalise a result proved in [6] for the single server setting to the multi-server setting. Specifically, we show that a policy that routes each job randomly, and runs each server at a constant speed \(s^*(\Lambda)\) when active, is constant competitive. Note that the speed chosen depends on the load \(\Lambda\), which needs to be known or learnt. Policies of this type are referred to in [6] as \textit{gated} policies.

Specifically, the proposed algorithm \(S\) is the following: Arriving jobs are routed to any server uniformly at random. Each server performs processor sharing (PS) scheduling using a fixed speed \(s^*(\Lambda)\), which is the optimal static speed to minimize the metric (9) on that (single) server.

We begin our analysis by deriving a lower bound on the performance of any routing and speed scaling policy.

A. Lower Bound

Let \(s_i\) denote the time-averaged speed of server \(i\). We have
\[\lambda C \geq \lambda E[E] = \sum_{i=1}^{m} E[P(s_i)]\]

\[\geq \sum_{i=1}^{m} P(E[s_i]) \geq \sum_{i=1}^{m} P(\Lambda/m)\]

\[= \frac{\Lambda^\alpha}{m^{\alpha-1}}.\]

The first inequality above is an application of Jensen’s inequality, while the second exploits the convexity of the power function, given that \(\sum_{i=1}^{m} E[s_i] = \Lambda\) (for stability).

\(^3\)Of course, for this metric to be meaningful, we restrict attention to policies that are regenerative, and thus have a meaningful steady state behavior. We also note that it is straightforward to extend the results of this section to a metric that is a linear combination of \(E[T]\) and \(E[E]\).

Next, we derive an alternate lower bound on \(C\). Consider the case when only a single job of size \(X\) arrives. This job is run at a constant speed \(s^*\) that minimizes its response time plus energy consumption, i.e., \(s^* = \arg \inf_{s > 0} \frac{X}{s} + \frac{\Lambda}{s} P(s) = \left(\frac{1}{\alpha-1}\right)^{1/\alpha}\). This yields the following lower bound on the performance of any algorithm
\[\lambda C \geq \Lambda^\alpha (\alpha - 1) \left(\frac{1}{\alpha-1}\right).\] Combining (10) and (11) gives us
\[\lambda C \geq \max \left(\Lambda^\alpha (\alpha - 1) \left(\frac{1}{\alpha-1}\right), \frac{\Lambda^\alpha}{m^{\alpha-1}}\right).\] Thus, \(s^*(\Lambda) = \arg \min_{s > 0} c(s)\), and the performance of the algorithm \(S\) is given by \(c(s^*(\Lambda))\).

\textbf{Theorem 4.} In the stochastic input setting, the competitive ratio of the algorithm \(S\) is a constant that depends on \(\alpha\) but not on \(\lambda\), the job size distribution, or \(m\).

\textbf{Proof.} The proof follows by comparing the performance \(S\) with the lower bound (12) that holds for any algorithm. Indeed, for any algorithm \(A\),
\[\frac{C_S}{C_A} \leq \frac{\lambda c(s^*(\Lambda))}{\max \left(\Lambda^\alpha (\alpha - 1) \left(\frac{1}{\alpha-1}\right), \frac{\Lambda^\alpha}{m^{\alpha-1}}\right)}\]

\[\leq \frac{\lambda (1 + \Lambda/m)}{\max \left(\Lambda^\alpha (\alpha - 1) \left(\frac{1}{\alpha-1}\right), \frac{\Lambda^\alpha}{m^{\alpha-1}}\right)\min(1, \alpha - 1) \left(\frac{1}{\alpha-1}\right)\max(\Lambda, \frac{\Lambda^\alpha}{m^{\alpha-1}})}\]

\[\leq \frac{\Lambda + \Lambda (1 + \Lambda/m)^{\alpha-1}}{\min(1, \alpha - 1) \left(\frac{1}{\alpha-1}\right)\max(\Lambda, \frac{\Lambda^\alpha}{m^{\alpha-1}}) + 2^{\alpha-1}}\]

\(\square\)

The above bound can be tightened for the case \(\alpha = 2\), since \(s^*(\Lambda)\) can be computed explicitly in this case.

\textbf{Corollary 1.} For \(\alpha = 2\), in the stochastic input setting, the competitive ratio of the algorithm \(S\) is at most 2.

\textbf{Proof.} For \(\alpha = 2\), from (13), we get \(s^*(\Lambda) = 1 + \frac{\Lambda}{m}\), and thus, the performance \(C_S\) under the algorithm satisfies
\[\lambda C(S) = \Lambda^2/m + 2\Lambda.\]

Now, from (12), \(\lambda C_A \geq \max\{\Lambda^2/m, 2\Lambda\}\) under any algorithm \(A\), which implies the statement of the corollary. \(\square\)
VI. CONCLUDING REMARKS

In this paper, we show that SRPT can be made constant competitive in the multi-server speed scaling environment with respect to the flow time plus energy metric. This presents an interesting contrast to the case when server speeds are constant, where it is known that SRPT has an unbounded competitive ratio with respect to the flow time metric. We also show that the multi-server speed scaling problem is easy in the absence of job size variability; simple round robin dispatch in conjunction with a single-server speed scaling rule is near-optimal. Finally, we show that a broad class of policies based on greedy non-migratory dispatch rules do not admit a constant competitive ratio.

In contrast, in the stochastic setting, we show that random routing, along with a gated static speed setting is constant competitive. However, the required speed is a function of the load, which needs to be learnt.

While SRPT is a well studied scheduling policy in the multiple server setting, one issue with implementing SRPT in practice is the need for migration. Considering that there is a cost associated with migration of a job across servers in practice, a natural generalization would be to include this cost of migration in the performance metric. How to optimally tradeoff flow time, energy consumption, and migration costs is an interesting open problem for the future. However, it is easy to bound the performance of the SRPT-based speed scaling algorithm proposed in this paper accounting for migration costs. Indeed, in a job sequence consisting of $J$ jobs, SRPT performs at most $J$ migrations. Thus, assuming a fixed cost of each migration, our SRPT-based algorithm remains constant competitive with respect to the flow time plus energy plus migration cost metric if one assumes a lower bound on the size of each job; in this case, the migration cost is at most a constant factor of the flow time.

Finally, we note that while there is a considerable literature on speed scaling in parallel multi-server environments, we are not aware of any work on speed scaling in tandem queueing systems, and more generally, on a queueing network. Coming up with constant competitive speed scaling algorithms in these settings is an interesting avenue for future work.

APPENDIX A
PROOF OF PROPOSITION 5

**Proposition 5.** $\Phi(\cdot)$ as defined in (6) satisfies boundary conditions (1) and (2).

*Proof.* Note that Condition (1) is satisfied; before any job is released and after all jobs are finished, $\Phi(t) = 0$, since $d(t, q) = 0$ and $n(t, q) = n_o(t, q) = 0$ for all $q$. Whenever a new job arrives/is released, $d(t, q)$ and $n(t, q) - n_o(t, q)$ does not change for any $q$, so $\Phi$ remains unchanged. Similarly, whenever a job is completed by the algorithm or OPT, $d(t, q)$ or $n(t, q) - n_o(t, q)$ is changed for only a single point of $q = 0$, which does not introduce a discontinuity in $\Phi(t)$. Thus, Condition (2) is also satisfied.

APPENDIX B
PROOF OF LEMMATA 2 AND 3

To prove Lemmas 2 and 3 we need the following technical lemma from [15].

**Lemma 8.** [Lemma 3.1 in [15]] For $s_k, \tilde{s}_k, x \geq 0$,

$$\Delta(x)(-s_k + \tilde{s}_k) \leq (-s_k + P^{-1}(x)\Delta(x) + P(\tilde{s}_k) - x.$$  

*Proof of Lemma 2.* Throughout we assume that OPT is following SRPT. Let $g(i)$ and $q_o(i)$ denote, respectively, the size of the $i^{th}$ shortest job in service under the algorithm and OPT.

**Case 1:** $n \geq m$. Suppose that OPT is serving $r$ jobs, where $r \leq m$. Define $\tilde{n}(q) = \max(n(q), n - r)$, and $\tilde{n}_o(q) = \max(n_o(q), n_o - r)$. The function $g(q) := \tilde{n}(q) / \tilde{n}_o(q)$ satisfies the following properties.

1) $g(0) = n - n_o$, $g(q) \to n - n_0$ as $q \to \infty$.

2) $g$ is piecewise constant and left-continuous, with a downward jump of 1 at $q = q(i)$, $1 \leq i \leq r$, and an upward jump of 1 at $q = q_o(i)$, $1 \leq i \leq r$.

Consider the change in $\Phi_1$ due to OPT $(n_o(q) \to n_o(q) - 1)$ for $q = q_o(1), \cdots, q_o(m))$:

$$d\Phi_1 = c_1 \sum_{i=1}^m \left[ f \left( \frac{n_o(q_o(i))}{m} - n_o(q_o(i)) \right) + 1 \right] s_i dt - f \left( \frac{n(q_o(i)) - n_o(q_o(i))}{m} \right) \tilde{s}_i dt$$

$$\leq c_1 \sum_{i=1}^m \Delta \left( \frac{n(q_o(i)) - n_o(q_o(i)) + 1}{m} \right) \tilde{s}_i dt$$

$$\leq c_1 \sum_{i=1}^m \Delta \left( \frac{g(q_o(i)) + 1}{m} \right) \tilde{s}_i dt$$

In writing (a) we take $\Delta(i/m) = 0$ for $i \leq 0$. (b) holds since $\tilde{n}_o(q_o(i)) = n_o(q_o(i))$ for $1 \leq i \leq r$, and $\tilde{n}_o(q) \geq n(q)$ $\forall q$.

Next, consider the change in $\Phi_1$ due to the algorithm $(n(q) \to n(q) - 1$ for $q = q(1), \cdots, q(m))$:

$$d\Phi_1 = c_1 \sum_{i=1}^m \left[ f \left( \frac{n(q(i)) - n_o(q(i))}{m} \right) - f \left( \frac{n(q(i)) - n_o(q(i))}{m} \right) \right] s_i dt$$

$$\leq - c_1 \sum_{i=1}^m \Delta \left( \frac{n(q(i)) - n_o(q(i))}{m} \right) s_i dt$$

$$\leq - c_1 \sum_{i=1}^m \Delta \left( \frac{n(q(i)) - n_o(q(i))}{m} \right) s_i dt$$

$$\leq - c_1 \sum_{i=1}^m \Delta \left( \frac{g(q(i))}{m} \right) s_i dt$$

This assumes all jobs being served by the algorithm and OPT have distinct remaining sizes. If, for example, $k$ jobs under OPT have the same remaining size $\hat{q}$, then $g$ would have an upward jump of $k$ at $\hat{q}$.
Here, (a) holds because \( n(q(i)) = \tilde{n}(q(i)) \) for \( 1 \leq i \leq r \), and \( \tilde{n}_o(q) \geq n_o(q) \) for all \( q \). (b) follows since \( n(q(i)) \geq n - i + 1 \).

We now combine (14) and (15) to capture the overall change in \( \Phi_1 \). In doing so, we make the following crucial observation.

**Claim 1:** For each \( i \in \{1, 2, \cdots, r \} \), one can find a unique \( j \in \{1, 2, \cdots, r \} \) such that \( g(q(i)) \geq g(q(j)) + 1 \).

To see that this claim is true, note that at each job with remaining size \( q(k) \) (\( 1 \leq k \leq r \)) being served by the algorithm contributes a down-tick of magnitude 1 in \( g(q(k)) \). Similarly, each job with remaining size \( q_o(k) \) (\( 1 \leq k \leq r \)) being served by \( \text{OPT} \) contributes an up-tick of magnitude 1 in \( g \) at \( q_o(k) \).

It is therefore clear that each down-tick from \( 1 \) to \( l - 1 \) can be mapped to an unique up-tick from \( l - 1 \) to \( l \). Moreover, at the downtick, say at \( q(i) \), we have \( g(q(i)) \geq l \) (because \( g \) is left-continuous), and at the corresponding up-tick, say at \( q_o(j) \), we have \( g(q_o(j)) \leq l - 1 \), implying \( g(q_o(j)) + 1 \leq l \), (again, because \( g \) is left-continuous). This proves the claim.

Based on the above observation, combining (14) and (15), we can now bound the overall change in \( \Phi_1 \) as follows.

\[
\frac{d\Phi_1}{dt} \leq c_1 \sum_{i=1}^{r} \Delta \left( \frac{g(q(i))}{m} \right) (s_i - \tilde{s}_i) - c_1 \sum_{i=r+1}^{m} \Delta \left( \frac{n - i + 1 - n_o}{m} \right) s_i.
\] (16)

Invoking Lemma 8, terms in the first summation of (16) can be bounded as

\[
\Delta \left( \frac{g(q(i))}{m} \right) (s_i - \tilde{s}_i) \leq P(\tilde{s}_i) - \frac{g(q(i))}{m},
\]

since \( s_k = P^{-1} \left( \frac{n}{m} \right) \geq P^{-1} \left( \frac{g(q(i))}{m} \right) \). Lemma 8 can also be used to bound the terms of the second summation of (16) as

\[
-\Delta \left( \frac{n - i + 1 - n_o}{m} \right) s_i \leq -\frac{n - i + 1 - n_o}{m}.
\]

(taking \( \tilde{s}_i \) in the statement of Lemma 8 to be zero). Combining the above bounds, we arrive at

\[
\frac{d\Phi_1}{dt} \leq c_1 \sum_{i=1}^{r} P(\tilde{s}_i) - \frac{g(q(i))}{m} - c_1 \sum_{i=r+1}^{m} \left( \frac{n - i + 1 - n_o}{m} \right).
\]

Finally, noting that \( g(q(i)) \geq n - i + 1 - n_o \) for \( 1 \leq i \leq r \), we conclude that

\[
\frac{d\Phi_1}{dt} \leq c_1 \sum_{i=1}^{r} P(\tilde{s}_i) - c_1 \sum_{i=r+1}^{m} \left( \frac{n - i + 1 - n_o}{m} \right),
\]

\[
= c_1 n - c_1 \sum_{i=1}^{r} P(\tilde{s}_i) + \sum_{i=r+1}^{m} \left( \frac{n - i + 1 - n_o}{m} \right).
\]

**Case 2:** \( n < m \). Let \( r \) denote the number of jobs in service under \( \text{OPT} \). Define \( h(q) := n(q) - n_o(q) \). As before, the rate of change of \( \Phi_1 \) can be expressed as follows:

\[
n(\tilde{n}(q(i)) = n - i + 1 \text{ if the algorithm has exactly one job with remaining size } q(i). \text{ If multiple jobs have the same remaining size } q(i) \text{ under the algorithm, then we have } n(q(i)) \geq n - i + 1.
\]

The magnitude of the discontinuous in \( g \) at \( q \) thus equals \( |j \in R : q_o(j) = q| - |j \in R : g(j) = q| \), where \( R = \{1, 2, \cdots, r \} \).

\[
\frac{d\Phi_1}{dt} = c_1 \sum_{i=1}^{r} \Delta \left( \frac{h(q_o(i)) + 1}{m} \right) \tilde{s}_i - c_1 \sum_{i=1}^{n} \Delta \left( \frac{h(q(i))}{m} \right) s_i
\]

\[
\leq c_1 \sum_{i=1}^{r} \Delta \left( \frac{h(q_o(i)) + 1}{m} \right) \tilde{s}_i - c_1 \sum_{i=1}^{n} \Delta \left( \frac{h(q(i))}{m} \right) s_i
\]

**Claim 2:** For each \( i \in \{1, 2, \cdots, n \} \) such that \( h(q_o(i)) \geq 0 \), one can find a unique \( j \in \{1, 2, \cdots, n \} \) such that \( h(q(j)) \geq h(q_o(i)) + 1 \).

The proof of the above claim follows along the same lines as the proof of Claim 1 for \( n \geq m \). Note that \( h \) is piecewise constant and left-continuous, with \( h(0) = n - n_o \), \( h(q) = 0 \) for large enough \( q \), has upward jumps at \( q_o(i) \) \( (i \leq n_o) \) at downward jumps at \( q(i) \) \( (i \leq n) \). Thus, any uptick in \( h(\cdot) \) from \( l - 1 \) to \( l \) for \( l \geq 1 \) can be mapped to a unique downtick from \( l \) to \( l - 1 \). The rest of the argument is identical to that of Claim 1.

Based on the above observation, suppose that a subset \( J \) of algorithm terms are matched with \( \text{OPT} \) terms.

\[
\frac{d\Phi_1}{dt} \leq c_1 \sum_{i \in J} \Delta \left( \frac{h(q(i))}{m} \right) (-s_i + \tilde{s}_i) + c_1 \sum_{i \notin J} \Delta \left( \frac{h(q(i))}{m} \right) (-s_i)
\]

Applying Lemma 8 as before,

\[
\frac{d\Phi_1}{dt} \leq c_1 \left( \sum_{i \in J} P(\tilde{s}_i) - \frac{h(q(i))}{m} \right) - c_1 \sum_{i \notin J} h(q(i)) \]

\[
\leq c_1 \sum_{i \in J} P(\tilde{s}_i) - c_1 \sum_{i \in J} \frac{n - i + 1 - n_o}{m}
\]

\[
\leq c_1 \sum_{i \in J} P(\tilde{s}_i) + c_1 n_o - c_1 \frac{n(n+1)}{2m}
\]

**Proof of Lemma 3.** The rate of change in \( \Phi_2 \) is

\[
\frac{d\Phi_2}{dt} = -c_2 \sum_{k \in A} \tilde{s}_k + c_2 \sum_{k \in \tilde{O}} \tilde{s}_k
\]

\[
\leq -c_2 \min(n, m) P^{-1}(1) + c_2 \sum_{k \in \tilde{O}} \max(P(\tilde{s}), P(\tilde{s}_k))
\]

The bounding of the first term above uses \( s_k \geq P^{-1}(1) \). The bounding of the second term is based on: (i) \( \tilde{s}_k \leq \tilde{s} \) when \( \tilde{s}_k \leq \tilde{s} \), and (ii) \( \tilde{s}_k \leq P(\tilde{s}_k) \) when \( \tilde{s}_k > \tilde{s} \).

**APPENDIX C**

**PROOF OF THEOREM 2**

Unlike in the proof of Theorem 1, we now make no assumptions on the scheduling of \( \text{OPT} \). We use the same potential function \( \Phi \) as before (see (6)), and show that (7) holds for a suitable \( c \). Note that it suffices to show that (7) holds at any instant \( t \) which is not an arrival or departure instant.
under the algorithm or OPT. For the remainder of this proof, consider any such time instant \( t \). For ease of exposition, we drop the index \( t \) from \( n(t, q), n(t, q_0), n(t), n_o(t), s_k(t) \) and \( \tilde{s}_k(t) \), since only a fixed (but generic) time instant \( t \) is under consideration.

Our proof is based on the following lemmas.

**Lemma 9.** For \( n \geq m \),
\[
d\Phi_1/dt \leq c_1 n_o - c_1 (2 - \alpha) n + c_1 (2 - \alpha) \left( \frac{m - 1}{2} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k),
\]
while for \( n < m \),
\[
d\Phi_1/dt \leq c_1 n_o + c_1 (2 - \alpha) n/2 + c_1 \sum_{k \in O} P(\tilde{s}_k).
\]

**Lemma 10.** \( d\Phi_2/dt \leq -c_2 \min(m, n) + c_2 \sum_{k \in O} P(\tilde{s}_k) \)

Using Lemmas 9 and 10, we now prove (7) by considering the following two cases:

**Case 1:** \( n \geq m \).
\[
n + \sum_{k \in A} P(s_k) + d\Phi(t)/dt
\leq n + n_o - c_1 (2 - \alpha) n + c_1 (2 - \alpha) \left( \frac{m - 1}{2} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k) + c_2 m + c_2 \sum_{k \in O} P(\tilde{s}_k)
\leq (c_1 + c_2) (n_o + \sum_{k \in O} P(\tilde{s}_k)) + n [2 - c_1 (2 - \alpha)]
\quad + \left[ c_1 (2 - \alpha) \left( \frac{m - 1}{2} \right) - c_2 m \right]
\leq (c_1 + c_2) (n_o + \sum_{k \in O} P(\tilde{s}_k)).
\]

Here, (a) follows by setting \( c_1 = \frac{2}{1 - \alpha} \), and \( c_2 \geq 1 \).

**Case 2:** \( n < m \).
\[
n + \sum_{k \in A} P(s_k) + d\Phi(t)/dt
\leq 2n + c_1 n_o + c_1 (2 - \alpha) n/2 + c_1 \sum_{k \in O} P(\tilde{s}_k) - c_2 n
\quad + c_2 \sum_{k \in O} P(\tilde{s}_k)
\leq (c_1 + c_2) (n_o + \sum_{k \in O} P(\tilde{s}_k)) + n (2 + c_1 (2 - \alpha))/2 - c_2
\quad \leq (c_1 + c_2) (n_o + \sum_{k \in O} P(\tilde{s}_k)).
\]

Here, (a) follows setting \( c_1 = \frac{2}{1 - \alpha} \), and \( c_2 = 3 \).
This proves (7) for \( c = c_1 + c_2 = 3 + \frac{2}{2 - \alpha} \). It now remains to prove Lemmas 9 and 10.

**Proof of Lemma 9.** Let \( q(i) \) denote the size of the \( i \)-th shortest job in service under the algorithm. Note that since the algorithm performs STPT scheduling, \( q(i) \) is also the size of the \( i \)-th shortest job in the system under the algorithm. Let \( q_o(i) \) denote the size of the \( i \)-th largest job in service under OPT.

**Case 1:** \( n \geq m \). When \( n \geq m \), since the algorithm processes the \( m \) shortest jobs, the change in \( \Phi_1 \) because of the algorithm \( (n(q) \rightarrow n(q) - 1) \) for \( q = (q(1), \ldots, q(m)) \) is
\[
d\Phi_1 = c_1 \sum_{k=1}^m \left[ f \left( \frac{n(q(k)) - n_o(q(k))}{m} \right) s_k dt \right.
\quad - c_1 f \left( \frac{n(q(k)) - n_o(q(k))}{m} \right) s_k dt\]
\[\leq c_1 \sum_{k=1}^m \Delta \left( \frac{n(q(k)) - n_o(q(k))}{m} \right) s_k dt \tag{17}\]
\[\leq -c_1 \sum_{k=1}^m \Delta \left( \frac{n - k + 1 - n_o}{m} \right) s_k dt \tag{18}\]
In writing (a) we take \( \Delta(i/m) = 0 \) for \( i \leq 0 \). (b) follows since \( n(q(k)) \geq n - k + 1 \), and \( n_o(q(k)) \leq n_o \) for all \( k \). Next, we bound the terms of (18) using Lemma 8. For those terms where the argument of \( \Delta(\cdot) \) is non-negative, Lemma 8 implies that
\[
\Delta \left( \frac{n - k + 1 - n_o}{m} \right) s_k \leq - \left( \frac{n - k + 1 - n_o}{m} \right) s_k,
\]
take \( \tilde{s}_k = 0 \) and note that
\[
s_k = P^{-1} \left( \frac{n}{m} \right) > P^{-1} \left( \frac{n - k + 1 - n_o}{m} \right).
\]
Of course, the same bound is trivial for the terms where the argument of \( \Delta(\cdot) \) is negative. This yields the bound
\[
d\Phi_1 \leq -c_1 \sum_{k=1}^m \left( \frac{n - k + 1 - n_o}{m} \right) dt. \tag{19}\]

We now consider the change in \( \Phi_1 \) due to OPT.
\[
d\Phi_1 = c_1 \sum_{k \in O} \Delta \left( \frac{n(q_o(k)) - n_o(q_o(k))}{m} \right) \tilde{s}_k dt
\leq c_1 \sum_{k \in O} \left( \frac{n + 1 - k}{m} \right) \tilde{s}_k dt, \tag{20}\]
where (a) follows since \( n(q_o(k)) \leq n \), and \( n_o(q_o(k)) \geq k \).\footnote{The reader should verify that the bound on \( n_o(q(i)) \) applies even if multiple active jobs under OPT have identical sizes.}

Applying Lemma 8 with \( s_k = 0 \), we get that the change in \( \Phi_1 \) because of OPT is
\[
d\Phi_1/dt \leq c_1 (\alpha - 1) \sum_{k=1}^{|O|} \left( \frac{n + 1 - k}{m} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k),
\]
since \( P^{-1}(i) P'(P^{-1}(i)) = \alpha \epsilon - i \). Finally, since \( |O| \leq m \), we have that the change in \( \Phi_1 \) because of OPT satisfies
\[
d\Phi_1/dt \leq c_1 (\alpha - 1) \sum_{k=1}^m \left( \frac{n - k + 1}{m} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k). \tag{21}\]
In the above bound, we use the fact that for component of $d \Phi$ that the summations in (17) and (18) only run from $\Phi_n < m$.

Case 2: $n < m$. Our approach in capturing the change in $\Phi_1$ due to the algorithm is the same as that in Case 1, except that the summations in (17) and (18) only run from $k = 1$ to $k = n$. An application of Lemma 8 as before implies that the component of $d \Phi_1/ dt$ because of the algorithm satisfies

$$d \Phi_1 / dt \leq c_1 n_o - c_1 (2 - \alpha) n + c_1 (2 - \alpha) \left( \frac{m - 1}{2} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k).$$

Combining (19) and (21), the overall change in $\Phi_1$ satisfies

$$d \Phi_1 / dt \leq -c_1 \sum_{k=1}^{n} \left( \frac{n - k + 1 - n_o}{m} \right) dt. \quad (22)$$

The analysis of the impact of OPT on $\Phi_1$ also proceeds as in Case 1, except that the summation in (20) only runs from $k = 1$ to $k = n$; note that the remaining terms in the sum are zero. An application of Lemma 8 as before implies that the component of $d \Phi_1 / dt$ because of OPT satisfies

$$d \Phi_1 / dt \leq c_1 (\alpha - 1) \sum_{k=1}^{n} \left( \frac{n - k + 1}{m} \right) + c_1 \sum_{k \in O} P(\tilde{s}_k) \quad (23)$$

Combining (22) and (23), the overall change in $\Phi_1$ is bounded as

$$d \Phi_1 / dt \leq -c_1 n_o + c_1 \frac{(2 - \alpha) n}{2} + c_1 \sum_{k \in O} P(\tilde{s}_k).$$

Proof of Lemma 10. The proof is similar to that of Lemma 3, except that we exploit the specific form of the power function.

$$d \Phi_2 / dt = -c_2 \sum_{k \in A} s_k + c_2 \sum_{k \in O} \tilde{s}_k \leq -c_2 \min(n, m) + c_2 \sum_{k \in O} P(\tilde{s}_k)$$

In the above bound, we use the fact that for $P(s) = s^\alpha$, the minimum speed utilized by any algorithm is $P^{-1}(1) = 1$. Thus, $s_k, \tilde{s}_k \geq 1$, and $\tilde{s}_k \leq P(\tilde{s}_k)$.

REFERENCES


