Spectrum Sharing: How Much to Give

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***Abstract*—Spectrum holding per cellular telephony service provider in India is significantly lower than the world average. The spectrum is also severely fragmented across bands and the fragments also have different license conditions. Regulators in India have recently recognized that such spectrum fragmentation is a source of inefficiency for the service providers and have allowed sharing of spectrum among the providers. The genesis of this paper is this regulatory order and it has a three-fold objective.**

We first study an example spectrum allocation. By assuming GSM-like voice telephony service, we analyse the spectrum holding in one service area in some detail. Using simple calculations, we see that complete pooling of resources by the providers may not be stable—one provider may have a lower blocking probability if it does not form a coalition.This leads us to our second objective of developing analytically tractable partial sharing models where the providers do not pool all their resources. For a probabilistic spectrum sharing model, we analyse a simple system and obtain the partial sharing that will make the coalition stable and Pareto efficient. This model is then extended to a larger system and numerical results from the analytical model are used to obtain additional insights. We then consider a deterministic sharing model for which we also present a similar analysis for this system. We also show that the deterministic sharing system can be analysed via a suitably defined circuit multiplexed network that allows us to use Kelly's Erlang fixed point approximation which in turn provides economic insights. The final objective is to develop a game theoretic model for partial sharing. We provide a Nash bargaining framework for partial sharing. We also discuss some revenue sharing mechanisms when the providers' benefits from partial sharing are asymmetric.

I. Introduction

The growth of the cellular telephony network in India has been widely hailed. Among the several hurdles on this growth path has been the spectrum allocation process; spectrum for mobile cellular telephony has been released in a trickle over several phases during the last 20 years with different terms of usage in each phase of the release. Further, the number of service providers in each area has also increased steadily with no consolidation expected to happen in the short and medium term. This has resulted in spectrum holdings of service providers that have a significantly lower average than in most other markets around the world—10MHz per provider and an average Herfindahl-Hirschman Index (HHI) of 0.13 [1].[1] A direct, and deleterious, consequence of this low HHI is the loss of multiplexing efficiencies—lower loads for a given quality of service. As an example, this would mean that in a traditional telephony model, still the dominant traffic on wireless networks and also the dominant source of revenue in India, the Erlangs/MHz at 2% blocking would be significantly lower than if more spectrum was available per provider. If we now also take into consideration the high population densities in India, the unique propagation characteristics of EM waves in areas dense with brick, cement, and concrete construction, the quality of the power supply and the like, we can commiserate with service providers' woes on at least one aspect that governs the quality the cellular telephony service.

That fragmented spectrum is an important source of inefficiency has been recognised by the regulatory authorities and they have recently allowed for spectrum sharing among the service providers. Spectrum sharing policies have been formulated and guidelines issued by the Telecom Regulatory Authority of India in 2014 [2], [3]. The report explicitly states that the "... objective of spectrum sharing is to provide an opportunity to the TSPs to pool their spectrum holdings and thereby improve spectral efficiency. Sharing can also provide additional network capacities in places where there is network congestion due to a spectrum crunch." It goes on to add that it is "essential to ensure that both the licensees pool (combine) their spectrum resources and also use it simultaneously." While sharing of passive infrastructures like towers and antennas had been permitted and is widely used, allowing of sharing spectrum is being discussed only recently. A limitation imposed in [2] is that only two operators can form a 'coalition' to share their spectrum. However, an upside provision is that the operators need not seek the government's permission to form a coalition.

As is to be expected, there is significant debate about the effectiveness of these guidelines and also whether service providers would indeed share the spectrum under these guidelines. Our interest in this paper though is elsewhere. Taking the view that a service provider is defined by the size of its customer-base and its spectral resource, we ask: How will the providers share their resources to improve their QoS, and how will they share the resulting increased revenue, especially if they are asymmetric?

An immediate observation is that since the service providers have their own customer base and also their own spectral resource, full unrestricted pooling of resources, even among two providers, can lead to a loss of QoS for customers of one of the providers. It is reasonable to assume that this would be against the long term interests of the service provider. Thus a partial sharing option should be considered. Then the natural questions for the cooperating providers are how to share and how much to share. In this paper we explore partial spectrum sharing by two service providers by answering these questions through the following model. We assume that the spectrum is divided into several channels and that the number of channels is proportional to the spectrum available with the provider.

[1]HHI measures market concentration. It is calculated as $\sum_{i=1}^{N} x_i^2$ where x_i is the share of provider i and N is the number of providers.

We assume that the it provides circuit multiplexed service via these channels so that we can model the provider as an M/M/m/m loss queueing system. This is a natural model for telephony systems in general and for voice telephony in GSM networks in particular. In this model, spectrum sharing reduces to sharing of the channels which in turn reduces to the rules to accommodate an overflow call from one service provider. We consider two sharing models—probabilistic and deterministic. In probabilistic sharing, an overflow call is accepted by the other provider with a fixed probability if a channel is available. In deterministic sharing each provider gives a fixed number of channels into a common pool and arriving calls are rejected if no channel is available either in the reserved pool of the provider or in the common pool. We develop analytical models for these and determine what sharing mechanisms will be feasible.

A. *Previous Work*

Spectrum sharing principles and models have been considered in the literature in the context of dynamic spectrum access (DSA) systems [4], [5], [6]. The key difference between DSA and the model that we consider here is that in DSA the primary user has priority over the secondary user in the usage of the spectrum and the sharing mechanism always protects that priority. Also, the timescale for the operation of the protocols considered for DSA is much smaller where the competing users have to determine who is to use the radio resource at a give a time. Ours is a longer timescale model in the sense the users (operators) decide what fraction of the resource is to be shared with others. The closest model in the literature to that considered here is that in [7], [8] which is motivated by sharing of hospital beds among different hospitals. The key difference between the model of [7], [8] and ours is that in the former, sharing is an "all-or-nothing" while we are interested in partial sharing. This will become evident as we proceed in the paper. Thus there does not seem to be much work on either partial resource sharing or on partial spectrum sharing to situate our work.

The rest of the paper is organised as follows. In the next section we analyse the spectrum allocation in the Mumbai local service area. In Section III we describe and analyse a probabilistic sharing model for a coalition of two service providers each of whom have one channel. In Section IV this model is extended to a larger system for which an analytical model is developed and analysed. In Section V a deterministic sharing model is described and analysed. An approximate analysis via Kelly's Erlang fixed point analysis is also shown to be possible. In Section VI we develop a Nash bargaining framework for the deterministic sharing model. Finally, we present a revenue sharing framework in Section VII and conclude with a discussion in Section VIII.

II. Some Ground Truths

Spectrum for cellular telephony is allocated in eight different bands—700, 800, 900, 1800, 1900, 2100, 2300, and 2600 MHz. In India, the 700 and the 2600 MHz have not yet been allocated. The country average for total (uplink+downlink for paired) spectrum from these bands in different countries is shown in Table I [9]. Observe that India has almost the lowest actual and planned allocation.

TABLE I
Assigned and planned spectrum and in different world markets

	USA	Eur	Austr	Braz	China	India
Assigned	608	540–615	478	554	227	247
Planned	55	0–60	230	0	360	15

In India, each local service area (LSA) has at least seven and up to thirteen operators [3] and the spectrum is unevenly divided between the operators. This is illustrated in Table II for the Mumbai LSA.[2] The table lists the amount of spectrum allocated to each of the eight operators in the Mumbai LSA in the four bands that are currently operational. The total spectrum available in Mumbai is 105.7 MHz (paired) and the spectrum held per operator on average is 15.1 MHz, with a standard deviation of 7.7 MHz. The table also lists the number of subscribers with each of the carriers and the number of subscribers per MHz for each of the carriers. Once again note the disparity among the carriers with a range of 57,000–609,000 subscribers per MHz among the carriers. This disparity suggests that different operators require very different cell densities to achieve the same target call drop probability. Equivalently, if we assume comparable cell densities across providers, the data suggests a wide disparity in their call drop probabilities.

We now provide a quantitative feel for potential gains in efficiency that can be effected through spectrum sharing via the following stylised model. The cellular service is viewed as a circuit multiplexed system with $K = 40$ circuits per MHz. This is a simplified view of a GSM network. If we assume a spectrum reuse factor of one and three sectors per cell, then each cell sector can get a third of the spectrum. This is an optimistic estimate but is illustrative. Using this conversion, Table II lists the number of circuits per sector that is available to each operator. Now let us consider two service providers, say Idea and Aircel, with 85 and 59 circuits per sector respectively. A load of 88 Erlangs/sector for Idea would result in a 10% blocking probability while a load of about 70 Erlangs/sector for Aircel would result in a blocking of 20%. If the two just pooled their spectrum and operated as a single entity, we would have a system with a load of 158 Erlangs/sector and 144 circuits. The overall system blocking probability would be slightly more than 10%. In this case, the customers of Aircel would see significant improvement while those of Idea will actually experience a marginally higher blocking. Thus full sharing need not benefit both. On the other hand, if we assume a load of 59 Erlangs/sector for Aircel, then its blocking operating alone would be 9.7%, whereas the blocking in the 'pooled' system would be 5%. Thus, full sharing would benefit both parties in this case.

The preceding calculation essentially illustrates that "full sharing" need not in general be beneficial to both the parties.

[2] User information is obtained from [10] and spectrum information is obtained on 19 July 2015 from http://telecomtalk.info/india-spectrum-data-sheet/134245/.

TABLE II

SPECTRUM ALLOCATIONS IN THE FOUR BANDS TO THE OPERATORS IN MUMBAI LSA.

Operator /Band	900	800	1800	2100	Total	Ckts/Sct	Subs (K)	K Subs/MHz
Airtel	5.0	0.0	15.2	5.0	25.2	336	4,998	198
Vodafone	11.0	0.0	8.2	5.0	24.2	323	8,348	345
Idea	0.0	0.0	6.4	0.0	6.4	85	3,898	609
Reliance	0.0	5.0	5.0	5.0	15.0	200	5,749	383
Aircel	0.0	0.0	4.4	0.0	4.4	59	2,309	525
MTNL	6.2	2.5	6.2	5.0	19.9	265	1,125	57
Tata Docomo	0.0	6.2	4.4	0.0	10.6	142	3,627	342
Total	22.2	13.7	49.8	20.0	105.7	1,410	30,054	284

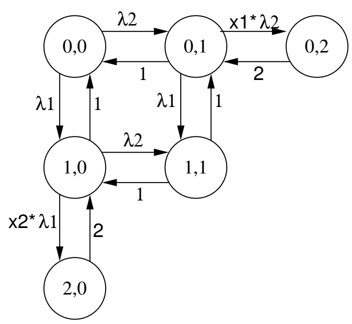

Fig. 1. Markov chain representation of the probabilistic sharing model with each service provider having one channel

This leads us to consider a more general sharing model where the providers share a fraction of their resources with full sharing being a special case.

III. A SIMPLE SHARING MODEL

Consider two service providers labeled S_1 and S_2 each with one channel. Customers of provider S_i arrive according to a Poisson process of rate λ_i and the call holding time is unit mean exponential for both the providers. Probabilistic sharing works as follows. If a call arrives at S_1 (resp. S_2) and its channel is being used by its caller, and if the channel of S_2 (resp. S_1) is free, then the call is accepted with probability x_2 (resp. x_1). For modeling convenience, we will assume that if a call of S_1 (resp. S_2) is using a channel of S_2 (resp. S_1) and if the channel of S_1 (resp. S_2) becomes free, then the call is instantaneously shifted to S_1 (resp. S_2.). This is called call-repacking and has been commonly used in models for cellular systems since they were first introduced in [11]. This model allows us to construct the Markov chain description of the system shown in Fig. 1 where (n_1, n_2) represents the state of the system with n_i being the number of active calls of S_i.

The Markov chain of Fig. 1 can be easily solved to obtain the stationary probabilities. To obtain the blocking probabilities, observe that an arriving call of S_1 is accepted if the system is in states $(0,1)$ or $(0,0)$, blocked with probability $(1-x_2)$ in state $(1,0)$, and blocked with probability 1 if the system is in states $(0,2)$, $(2,0)$ and $(1,1)$. Analogously for calls of S_2. The blocking probabilities, denoted by $B_i(\lambda, x)$, where $x = (x_1, x_2)$ and $\lambda = (\lambda_1, \lambda_2)$, can be shown to be

$$B_1(\lambda, x) = \frac{(1-x_2)\lambda_1 + x_2\lambda_1^2/2 + x_1\lambda_2^2/2 + \lambda_1\lambda_2}{1 + \lambda_1 + \lambda_2 + \lambda_1\lambda_2 + x_1\frac{\lambda_2^2}{2} + x_2\frac{\lambda_1^2}{2}}$$

$$B_2(\lambda, x) = \frac{(1-x_1)\lambda_2 + x_2\lambda_1^2/2 + x_1\lambda_2^2/2 + \lambda_1\lambda_2}{1 + \lambda_1 + \lambda_2 + \lambda_1\lambda_2 + x_1\frac{\lambda_2^2}{2} + x_2\frac{\lambda_1^2}{2}}$$

Substituting $x_1 = x_2 = 0$ in the above gives us

$$B_i(\lambda, 0) = \frac{\lambda_i}{1+\lambda_i},$$

the blocking probabilities when the providers do not share their channels.

We can say that x determines the amount of sharing by the providers and the natural question to ask is how much should each provider share. Clearly, a sharing strategy $x := (x_1, x_2)$ is acceptable to S_i if its blocking probability is less than that without sharing. Further, if the providers are cooperating, then a Pareto efficient sharing strategy that is also stable can be sought. The definition below makes the preceding discussion more precise.

Definition 1: A sharing strategy x is *QoS-stable* if $B_i(\lambda, x) < B_i(\lambda, 0)$ for all i. A sharing strategy x is *Pareto-stable* if it is QoS-stable and if there does not exist another x' for which $B_1(\lambda, x') < B_1(\lambda, x)$ and $B_2(\lambda, x') < B_2(\lambda, x)$.

We are now ready to state our main result.

Theorem 1: For any $\lambda_1, \lambda_2 > 0$, the set of Pareto-stable sharing strategies is non-empty. Further, any Pareto-stable sharing strategy will have $x_i = 1$ for some i.

Proof: First, we characterize the set of QoS-stable sharing strategies. Consider $x \neq 0$ and $\rho \in [0, 1)$. Some simple algebra yields that $B_1(\lambda, x) < B_1(\lambda, \rho x)$ if and only if

$$x_2 > x_1 \frac{\lambda_2^2}{2\lambda_1 + \lambda_1^2},$$

Similarly, $B_2(\lambda, x) < B_2(\lambda, \rho x)$ if and only if

$$x_2 < x_1 \frac{2\lambda_2 + \lambda_2^2}{\lambda_1^2}.$$

Setting $\rho = 0$, we see x is QoS-stable if and only if

$$x_1 \frac{\lambda_2^2}{2\lambda_1 + \lambda_1^2} < x_2 < x_1 \frac{2\lambda_2 + \lambda_2^2}{\lambda_1^2}.$$

Clearly, the set of QoS-stable strategies is non-empty if and only if

$$\frac{\lambda_2^2}{2\lambda_1 + \lambda_1^2} < \frac{2\lambda_2 + \lambda_2^2}{\lambda_1^2}.$$

The preceding condition is equivalent to

$$2\lambda_1\lambda_2 + \lambda_1\lambda_2(\lambda_1+\lambda_2) \ > \ 0,$$

which is always true for any $\lambda_1, \lambda_2 > 0$.

Now that we have characterized the set of QoS-stable strategies, we now analyse the subset of Pareto-stable strategies. We consider the following three cases.

Case 1: $\frac{2\lambda_2+\lambda_2^2}{\lambda_1^2} \leq 1$.

In this case, we argue that the set of Pareto-stable strategies is given by

$$\mathcal{P} := \left\{(x_1,x_2) \mid x_1 = 1, x_2 \in \left(\frac{\lambda_2^2}{2\lambda_1+\lambda_1^2}, \frac{2\lambda_2+\lambda_2^2}{\lambda_1^2}\right)\right\}.$$

To see this, consider $x \in \mathcal{P}$, and any QoS-stable x' such that $x' \neq x$. We need to show that for some $i \in \{1,2\}$, $B_i(\lambda,x) \leq B_i(\lambda,x')$. If $x' \in \mathcal{P}$, this follows easily, since B_1 is strictly decreasing in x_2 whereas B_2 is strictly increasing in x_2. If $x' \notin \mathcal{P}$, there exists $x'' \in \mathcal{P}$ such that $x'' = \rho x'$ for some $\rho \in (0,1)$. Note that there exists $i \in \{1,2\}$ such that $B_i(\lambda,x) \leq B_i(\lambda,x'')$. Moreover, since x'' is QoS-stable, it follows that

$$B_i(\lambda,x'') < B_i(\lambda,\rho x'') = B_i(\lambda,x').$$

Thus, we have

$$B_i(\lambda,x) \leq B_i(\lambda,x').$$

This completes the argument that $\mathcal{P}$ is the set of Pareto-stable strategies.

Case 2: $\frac{\lambda_2^2}{2\lambda_1+\lambda_1^2} \geq 1$.

In this case, an argument similar to that for Case 1 shows that the set of Pareto-stable strategies is given by

$$\left\{(x_1,x_2) \mid x_2 = 1, x_1 \in \left(\frac{\lambda_1^2}{2\lambda_2+\lambda_2^2}, \frac{2\lambda_1+\lambda_1^2}{\lambda_2^2}\right)\right\}.$$

Case 3: $\frac{\lambda_2^2}{2\lambda_1+\lambda_1^2} < 1 < \frac{2\lambda_2+\lambda_2^2}{\lambda_1^2}$.

In this case, an argument similar to that for Case 1 shows that the set of Pareto-stable strategies is given by

$$\left\{(x_1,x_2) \mid x_1 = 1, x_2 \in \left(\frac{\lambda_2^2}{2\lambda_1+\lambda_1^2}, 1\right]\right\} \bigcup \left\{(x_1,x_2) \mid x_2 = 1, x_1 \in \left(\frac{\lambda_1^2}{2\lambda_2+\lambda_2^2}, 1\right]\right\}.$$

Clearly, any Pareto-stable sharing strategy satisfies the property that $x_i = 1$ for some i. This completes the proof. ∎

Although the system considered here is simple in that there is only one channel with each provider, the closed form expressions provide insight on efficient and socially optimum sharing mechanisms. The rather counter intuitive result of Theorem 1 in that it requires that one of the providers has to always accept the others' overflow calls, motivates an analysis of a larger system. In the following section we describe a larger probabilistic sharing system and present its analysis.

IV. Probabilistic Sharing

As in the previous section, calls of S_i arrive according to a Poisson process of rate λ_i and have unit mean exponential holding times. Provider S_i has N_i channels and they are pooled and used as follows. Letting n_i denote the number of active calls of S_i that are present in the system, (n_1,n_2) represents the state of the system. We will use call packing like in the previous section, and have the following admission policy. If a call from S_1 arrives when the system is in state (n_1,n_2) and

- if $n_1 < N_1$, it is admitted with probability 1,
- if $n_1 \geq N_1$ and $n_1+n_2 < N_1+N_2$, it is admitted with probability x_2 and blocked with probability $(1-x_2)$, and
- if $n_1+n_2 = N_1+N_2$, it is blocked with probability 1.

A similar protocol is defined for calls of S_2. It can be seen that $\mathbf{n} = (n_1,n_2)$ evolves as a reversible Markov chain and the stationary distribution, $\pi(\mathbf{n})$ has the following product form structure.

$$\pi(\mathbf{n}) = \frac{1}{G} f_1(n_1) f_2(n_2)$$

where

$$\begin{aligned}
f_1(n) &= \begin{cases} \lambda_1^n/n! & \text{if } n < N_1 \\ \lambda_1^n x_2^{N_1-n}/n! & \text{if } N_1 \leq n \leq N_1+N_2 \end{cases} \\
f_2(n) &= \begin{cases} \lambda_2^n/n! & \text{if } n < N_2 \\ \lambda_2^n x_1^{N_2-n}/n! & \text{if } N_2 \leq n \leq N_1+N_2 \end{cases} \\
G &= \sum_{\mathbf{n}: n_1+n_2 \leq N_1+N_2} f_1(n_1) f_2(n_2)
\end{aligned}$$

Denoting the blocking probability for calls of S_i under a probabilistic sharing strategy x by $B_i^{(p)}(\lambda,x)$, we have,

$$\begin{aligned}
B_1^{(p)}(\lambda,x) = &\sum_{\mathbf{n}: n_1+n_2 = N_1+N_2} \pi(\mathbf{n}) \\
&+ \sum_{\mathbf{n}: n_i \geq N_i, n_1+n_2 < N_1+N_2} \pi(\mathbf{n})(1-x_2), \\
B_2^{(p)}(\lambda,x) = &\sum_{\mathbf{n}: n_1+n_2 = N_1+N_2} \pi(\mathbf{n}) \\
&+ \sum_{\mathbf{n}: n_2 \geq N_2, n_1+n_2 < N_1+N_2} \pi(\mathbf{n})(1-x_1).
\end{aligned}$$

The well known Kaufman-Roberts recursion (see Chapter 6 in [12]) can be adapted to efficiently compute G and the blocking probabilities. We omit the description of the adaptation but provide some numerical results and discuss these. We use $N_1 = 10$, $N_2 = 6$, $\lambda_1 = 7$ and $\lambda_2 = 6$. This is representative of a larger, uncongested S_1 and a smaller, congested S_2. Figure 2 shows how the $B_i^{(p)}(\lambda,x)$ vary with x. We first note that for a fixed x_2, $B_1^{(p)}$ is significantly affected by increasing x_1. Further, the effect of a small change in x_2 is significant, especially for x_2 closer to 1. While the latter effect is present for $B_2^{(p)}$ and is in fact more pronounced, the impact of change in x_2 appears to be minimal. Our extensive numerical analysis indicates that this asymmetry in the effect of changes to x is more pronounced when the relative loads on the two systems are not comparable.

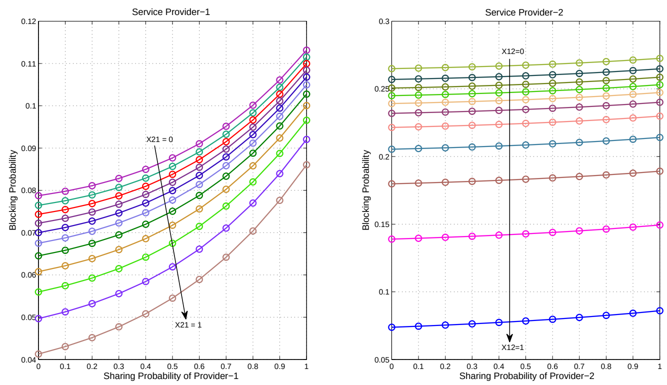

Fig. 2. Plots of $B_1^{(p)}(\lambda, x)$ as a function of x_1 keeping x_2 fixed (left panel) and of $B_2^{(p)}(\lambda, x)$ as a function of x_2 keeping x_1 fixed (right panel). We have used $N_1 = 10$, $N_2 = 6$, $\lambda_1 = 7$ and $\lambda_2 = 6$.

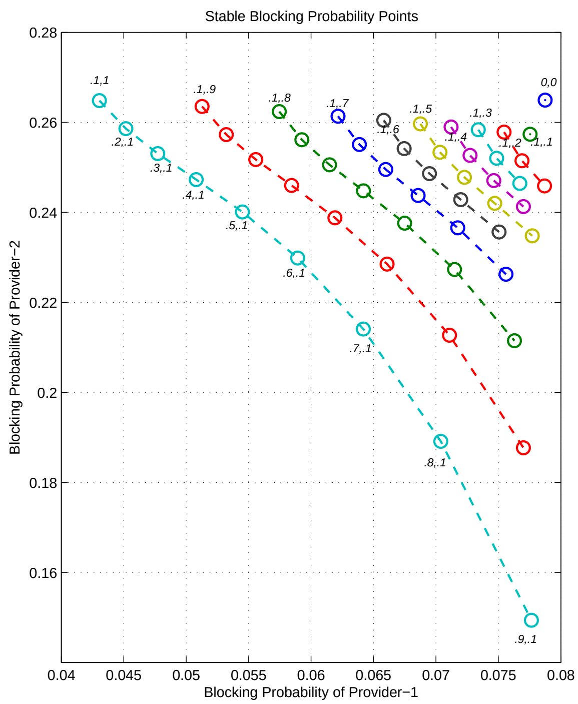

Fig. 3. Figure shows the blocking probabilities, $B_1(\lambda, x)$ and $B_2(\lambda, x)$ for different x with the x indicated by the numbers against the circles.

We now investigate the stable x. Since the closed form expression is hard to analyse, we take recourse to a numerical study. We evaluate $B_i^{(p)}(\lambda, x)$ for different x and in Figure 3, we indicate the set of QoS-stable x and the corresponding blocking probabilities. Observe that the Pareto-stable points seem to satisfy $x_2 = 1$. This, and a more extensive numerical study, seems to indicate that Theorem 1 is possibly more generally true and not just for $N_1 = N_2 = 1$. A better analytical understanding of this model is being pursued.

V. Deterministic Sharing

We now describe a deterministic scheme and its analysis. As in the preceding section, we consider two service providers with N_i, λ_i, n_i and $\mathbf{n}$ having the same interpretation as in the previous section. Rather than probabilistically sharing a channel, service provider S_i provides $k_i \leq N_i$ of its channels to a common pool and the combined system operates as follows. When the system is in state $\mathbf{n}$ and a new call from S_1 arrives then it is admitted if $n_1 + n_2 < N_1 + N_2$ and $n_1 < N_1 + k_2$, otherwise it is dropped. Similarly, a call from S_2 is admitted if $n_1 + n_2 < N_1 + N_2$ and $n_2 < N_2 + k_1$. In this scheme provider S_i shares upto k_i of its channels with the other provider. The set of feasible $\mathbf{n}$, denoted by $\mathcal{S}$, is defined as follows.

$$\mathcal{S} = \{\mathbf{n} : n_1 + n_2 \leq N_1 + N_2, n_1 \leq N_1 + k_2, n_2 \leq N_2 + k_1\}$$

Once again, we can show that $\mathbf{n}$ evolves as a reversible Markov chain over $\mathcal{S}$ and the stationary distribution, denoted by $\pi(\mathbf{n})$ has a product form structure. Specifically, for $\mathbf{n} \in \mathcal{S}$,

$$\pi(\mathbf{n}) = \frac{1}{G(\mathcal{S})} \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{n_2}}{n_2!},$$
$$G(\mathcal{S}) = \sum_{\mathbf{n} \in \mathcal{S}} \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{n_2}}{n_2!}.$$

The blocking states for calls of S_1 is obtained from the protocol description as $\mathcal{S}_{B_1} = \{\mathbf{n} : n_1 + n_2 = N_1 + N_2 \text{ or } n_1 \geq N_1 + k_2\}$. $\mathcal{S}_{B_2}$ is analogously defined. Here too, the Kaufman-Roberts recursion can be adapted to calculate the stationary and blocking probabilities. We will omit the discussion on that adaptation. While the exact result is appealing, an approximate result provides significant insight that is asymptotically exact. We describe this analysis next.

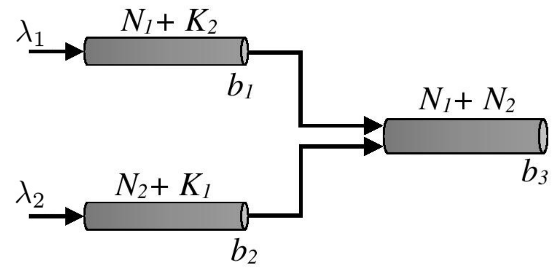

Fig. 4. A circuit multiplexed network whose analysis is the same as the analysis of the deterministic sharing scheme.

Consider the circuit multiplexed network shown in Figure 4. There are two routes in the network—route 1 uses links 1 and 3 while route 2 uses links 2 and 3. The number of circuits on the three links is as shown. A call on route 1 is lost if either of links 1 or 3 are full; similarly, a call on route 2 is lost if either of links 2 or 3 are full. It can be checked that the state space and the stationary distribution for this network is identical to the deterministic sharing system when the offered load is λ_i Erlangs on route i. Further, the blocking probability of route i calls is the same as that of S_i in the deterministic sharing system.

The network of Figure 4 can be analysed using the reduced load approximation and Erlang fixed point method of [14]. A short summary follows.

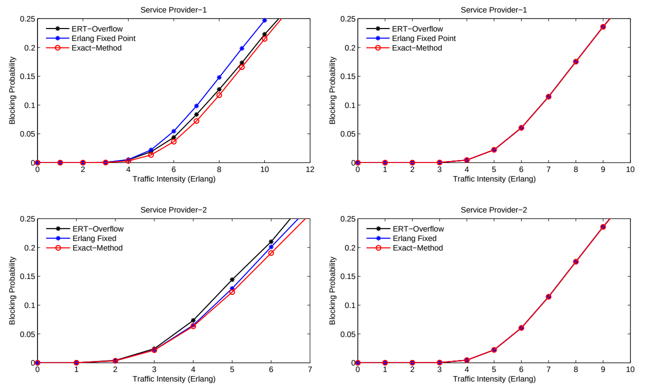

Fig. 5. The two graphs on the left are for $k_1 = k_2 = 1$ and the two on the right are for $k_1 = 10$ and $k_1 = 6$.

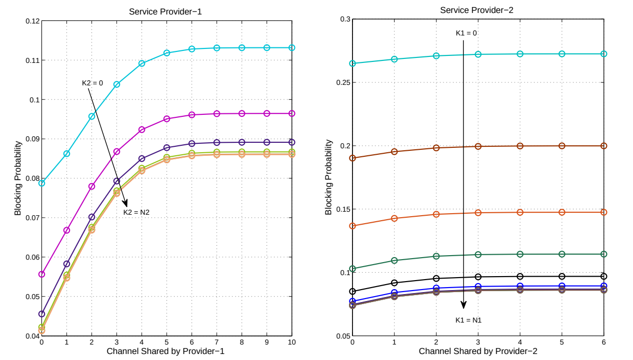

Fig. 6. For a system with $N_1 = 10$, $N_2 = 6$, $\lambda_1 = 7$, and $\lambda_2 = 6$, left panel shows $B_1^{(d)}(\lambda, k)$ as a function of k_1 for a fixed k_2 while the right panel shows $B_2^{(d)}(\lambda, k)$ as a function of k_2 for a fixed k_1.

Let $E(\nu, C)$ be the Erlang-B formula for a load of ν Erlangs on C circuits. Define

$$
\begin{aligned}
b_1 &= E(\lambda_1(1 - b_3), N_1 + K_2) \\
b_2 &= E(\lambda_2(1 - b_3), N_2 + K_1) \\
b_3 &= E(\lambda_1(1 - b_1) + \lambda_2(1 - b_2), N_1 + N_2). \qquad (1)
\end{aligned}
$$

From [14], the set of equations (1) form a vector fixed point equation and has a unique solution. Further, the b_i can be computed via repeated substitution. b_i above are interpreted as the 'link' blocking probabilities. The route blocking probabilities can be obtained as

$$
\begin{aligned}
B_1^{(d)}(\lambda, k) &= 1 - (1 - b_1)(1 - b_3) \\
B_2^{(d)}(\lambda, k) &= 1 - (1 - b_2)(1 - b_3) \qquad (2)
\end{aligned}
$$

Here $k = (k_1, k_2)$ denotes the sharing strategy.

The goodness of the approximation is examined through numerical results. To make comparison easy, we use $\lambda_1 = \lambda_2$ and, as before $N_1 = 10$ and $N_2 = 6$. For two sets of k, Figure 5 plots $B_i^{(d)}(\lambda, k)$ as obtained from the exact analysis, from the Erlang fixed point analysis and from a second kind of approximation that uses moment matching techniques called equivalent random theory (ERT). As can be seen, the approximations do well and can be used to obtain additional insight into the economics of partial sharing. Specifically, the economic interpretation from [16] is directly applicable.

We now explore the effect of changing k on the blocking probabilities of the two providers. Figure 6 shows the effect of k on the blocking probabilities. We see that the results are much like that in Figure 2 except that the role of x is now played by k. A good mapping between x and k is not straightforward and is subject of future work.

Turning to our key interest—analysing stable k, Figure 7 shows the QoS-stable blocking probabilities for different k when $N_1 = 10$, $N_2 = 6$, $\lambda_1 = 7$ and $\lambda_2 = 6$. It is interesting to compare this with Figure 3. Once again, it appears that a Pareto-stable sharing would require S_2 to put all its channels into the common pool.

VI. Sharing Agreements

In the previous sections, we have studied two spectrum sharing models: a deterministic sharing model and a probabilistic sharing model. For these models, we characterised the call blocking probability of each provider under all possible sharing configurations. In this section, we turn to the game theoretic question of which sharing configuration (if any) the two providers would agree upon, given that each provider's interest is to minimise its own call blocking probability.

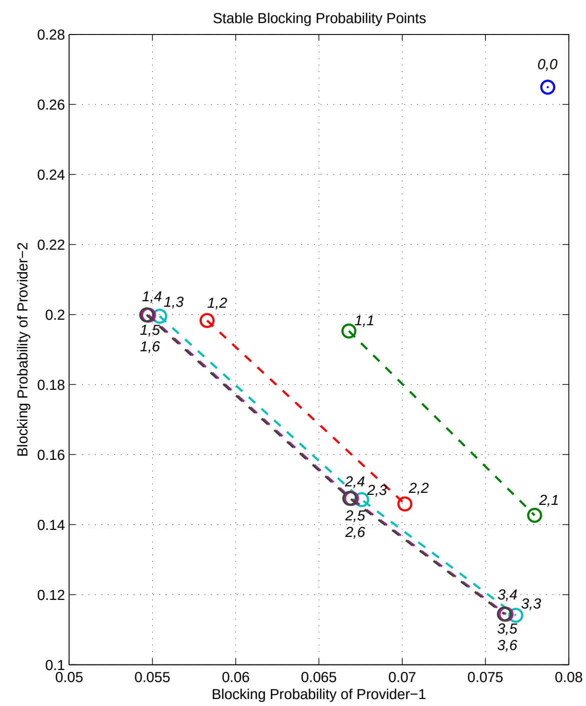

Fig. 7. Figure shows the blocking probabilities for different k with the k indicated by the numbers against the circles.

A natural game theoretic framework for capturing a sharing agreement between the service providers is the Nash bargaining solution [15]. In the following, we describe how the sharing models proposed in the previous sections enable us to apply the Nash bargaining framework to the spectrum sharing problem. To keep the discussion brief, we focus only on the deterministic sharing model here.

We begin with a brief description of the Nash bargaining solution, following by its application to the deterministic sharing model.

A. Nash bargaining solution

A bargaining problem seeks to capture situations where the utility of each player depends on the actions of all players, and the players can 'bargain' and agree on a mutually beneficial action profile. A two-player bargaining problem is described as follows. For $i \in \{1, 2\}$, let u_i denote the utility of Player i. The bargaining problem is defined by the tuple (U, d). $U \subset \mathbb{R}^2$ is the set of feasible utility pairs (u_1, u_2), i.e., the set of utility pairs that can be achieved over all action profiles. It is assumed that the set U is convex and compact. $d = (d_1, d_2) \in U$ is the disagreement outcome; d_i is the utility of Player i if the players fail to reach an agreement. A bargaining solution is a function $f(U, d)$ that specifies an outcome $(f_1(U, d), f_2(U, d)) \in U$ corresponding to the 'bargaining' between the two players.

Nash [15] proved that the following four axioms uniquely specify a bargaining solution $f^N(U, d)$, called the Nash bargaining solution (NBS).

1) **Pareto efficiency:** The solution is Pareto efficient, i.e., there does not exist a $(v_1, v_2) \in U$ such that $v_i \geq f_i(U, d)$ for all i and $v_i > f_i(U, d)$ for some i.
2) **Symmetry:** If $(v_1, v_2) \in U$ implies that $(v_2, v_1) \in U$, and $d_1 = d_2$, then $f_1(U, d) = f_2(U, d)$.
3) **Invariance to equivalent utility representations:** Given a bargaining problem (U, d) and constants $\alpha > 0$, $\beta \in \mathbb{R}$, solution corresponding to the transformed problem $(\tilde{U}, \tilde{d})$, where
$$\tilde{U} = \{(\alpha u_1 + \beta, \alpha u_2 + \beta) : (u_1, u_2) \in U\},$$
$$\tilde{d} = (\alpha d_1 + \beta, \alpha d_2 + \beta),$$
satisfies $f_i(\tilde{U}, \tilde{d}) = \alpha f_i(U, d) + \beta$ for each i.
4) **Invariance to irrelevant outcomes:** Given two bargaining problems (U, d) and $(\tilde{U}, d)$, where $U \subseteq \tilde{U}$, if $f(\tilde{U}, d) \in U$, then $f(U, d) = f(\tilde{U}, d)$.

Formally, Nash [15] proved the following result: There is a unique bargaining solution $f^N(U, d)$ that satisfies the above axioms. Moreover, $f^N(U, d)$ is the unique solution of the following optimization.

$$\begin{aligned} \max .\ & (u_1 - d_1)(u_2 - d_2) \\ \text{s.t.}\ & (u_1, u_2) \in U \\ & u_1 \geq d_1,\ u_2 \geq d_2 \end{aligned}$$

B. Applying the Nash bargaining framework to spectrum sharing

We now discuss how spectrum sharing problem can be cast into the Nash bargaining framework. Consider the deterministic sharing model introduced in Section V. Let $B_i^{(d)}(k)$ denote the blocking probability of Provider i, corresponding to the sharing configuration $k = (k_1, k_2)$. Define

$$\tilde{U} = \{(-B_1^{(d)}(k), -B_2^{(d)}(k)) \ : \ 0 \leq k_i \leq N_i,\ i = 1, 2\}.$$

λ_2	Nash bargaining solution		
	(k_1, k_2)	p	(k_1', k_2')
2	(10,1)	0.761	(10,2)
4	(10,4)	1	–
6	(10,10)	1	–
8	(2,10)	0.934	(3,10)
10	(1,10)	0.244	(2,10)

TABLE III
$N_1 = N_2 = 10, \lambda_1 = 6$

The bargaining problem (U, d) is defined as follows:

$$U = \text{conv}(U'),$$
$$d = (-B_1^{(d)}(0, 0), -B_2^{(d)}(0, 0)).$$

Here $\text{conv}(A)$ denotes the convex hull of A. Note that we are taking the utility of a provider to be (-1) times its blocking probability. Indeed, the U' is the set of all utility pairs achievable under different (static) sharing configurations. By taking U to be convex hull of U', we are allowing for agreements involving time sharing between different spectrum sharing configurations k. For example, we allow for agreements of the following form: configuration (k_1, k_2) is followed for a fraction of time p, and configuration (k_1', k_2') is followed for fraction of time $1 - p$, where $p \in [0, 1]$. It is easy to see that the NBS corresponds to a convex combination of at most two (static) configurations.

C. Numerical experiments

In this section, we present some numerical experiments, computing the spectrum sharing agreement postulated by the NBS in various scenarios. For our experiments, we set $N_1 = N_2 = 10$, $\lambda_1 = 6$, $\mu = 1$. For different values of λ_2, the sharing agreement corresponding to the Nash bargaining solution (NBS) is tabulated in Table III. Note that in general, the sharing agreement is of the following form: (k_1, k_2) for a fraction of time p, and (k_1', k_2') for a fraction of time $1 - p$, with $p \in [0, 1]$.

We note that time-shared as well as 'static' configurations may emerge as the NBS. When λ_2 is much smaller than λ_1, the NBS corresponds to Provider 1 pooling all its channels and Provider 2 pooling only a few. In the symmetric scenario, i.e., $\lambda_2 = \lambda_1$, the NBS corresponds to both providers pooling all their channels. Finally, when λ_2 is much larger than λ_1, the NBS corresponds to Provider 2 pooling all its channels and Provider 1 pooling only a few. Interestingly, note that each NBS configuration involves at least one of the providers pooling all its channels.

VII. Revenue Sharing

In the previous sections, we study spectrum sharing between two providers, with each provider being interested solely in minimizing its call blocking probability. An alternative setting is where the providers can also make side payments to one another. This is equivalent to the scenario wherein the two providers form a coalition and the total revenue of the coalition is split between the providers in a suitable manner. In this section, we briefly discuss this scenario.

Since there is partial sharing in the coalition, the key question is how revenue should be shared among the members, and hence several revenue sharing models can be envisaged. Let λ_i be the offered load to S_i, B_i the blocking probability experienced by its callers and r_i be the revenue per carried Erlang. Let $\hat{B}_i$ be the blocking experienced in S_i without spectrum sharing. The Shapley value is usually used to share the revenue or costs in a coalition and we will use that here. Before proceeding, note that the revenue for S_i without the coalition, denoted by $\hat{R}_i$ is given by

$$\hat{R}_i = r_i \lambda_i (1 - \hat{B}_i)$$

A simple model would be to pool the total revenue and use the Shapley value to share this revenue. The total revenue for the coalition, R_t, is

$$R_t = r_1 \lambda_1 (1 - B_1) + r_2 \lambda_2 (1 - B_2),$$

and provider S_i would receive

$$\begin{aligned} R_1 &= \frac{1}{2}\left\{ \hat{R}_1 + R_t - \hat{R}_2 \right\} \\ R_2 &= \frac{1}{2}\left\{ \hat{R}_2 + R_t - \hat{R}_1 \right\} \end{aligned}$$

An alternative would be to guarantee each provider $\hat{R}_i$ and share only the surplus arising from the coalition,

$$S_R = r_1 (\hat{B}_1 - B_1) \lambda_1 + r_2 (\hat{B}_2 - B_2) \lambda_2$$

in the Shapley ratio. Note that this is the revenue earned by the system in the states made possible by the coalition. It can be shown that both these schemes result in the same revenue for the providers.

VIII. Discussion

In this paper we explored the notion of partial sharing of resources using two analytically tractable models. Of course these are simple models and several variations are immediately possible. For example, in probabilistic sharing, x_i could be state dependent. Similarly, each provider could release a channel into the common pool based on the current state of the system. We could also develop a spot market where the price of a free channel is dynamically decided and dependent on the state of the system. These models will need to be explored.

A key question that needs to be expressed is the network formation. In probabilistic sharing this would mean determining $x_{i,j}$, the probability with which an overflow call from S_j is accepted by S_i. The stable $x_{i,j}$ would be of interest.

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