

SUPER-CONVERGENCE FOR MULTIDIMENSIONAL FUNCTIONS WITH A NEW TAYLOR-BERNSTEIN FORM AS INCLUSION FUNCTION

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ABSTRACT. Recently, Lin and Rokne [12] introduced the so-called Taylor-Bernstein form as an inclusion function form for multidimensional functions. This form was theoretically shown to have the super-convergence property. In this paper, we present an improvement of Lin and Rokne's Taylor-Bernstein form to make it more effective in practice. We test and compare the super-convergence behavior of the proposed form with that of Lin and Rokne's Taylor-Bernstein form and also with that of the Taylor model of Berz *et al.* [3]. For the testing, we consider six benchmark examples with dimensions varying from 1 to 6. In all examples, unlike with the Taylor model and Lin and Rokne's Taylor-Bernstein form, we obtain super-convergence of orders up to 9 with the proposed form. Moreover, with the proposed form we quite easily obtain such high orders of super-convergence for up to 5-dim problems.

1. INTRODUCTION

An important problem in interval analysis is the construction of inclusion functions having the property of so-called *super-convergence* (i.e., having a convergence order that is greater than quadratic) for multidimensional functions. Such inclusion functions have applications in the solutions of equations, optimization, quadrature, and others. The first paper in the literature concerning super-convergence is that of Herzberger [9], who shows that super-convergence can be obtained for a certain class of intervals. However, his requirement on the function is unrealistically strong. Cornelius and Lohner [5] propose the interpolation and remainder forms for multidimensional functions that enable any convergence order to be obtained in theory. However, in practice, convergence order of at most 4 or 5 is recommended even for unidimensional functions, see [5] and [20, pg. 9]. The same holds for the improved version of these forms for unidimensional functions, as proposed by Neumaier in [19, sec. 2.4]. Alefeld and Lohner [1] propose centered forms with super-convergence for *unidimensional* functions. However, because of the strong condition on the functional representation, these higher order centered forms have limited practical value [1, pg. 8]. Lin and Rokne [12] propose super-convergent forms that combine Taylor and Bernstein (or B-spline) forms for multidimensional functions. However, for small domains these forms become computationally very demanding, even for unidimensional functions, see [12, pg. 108] and sec. 4 below. Berz *et al.* [3], [14] propose the so-called Taylor models for multidimensional functions. Although the accuracy of the so-called remainder interval part of the Taylor model increases in a super-convergent fashion, the Taylor model itself is known to exhibit only quadratic convergence see Kearfott and Arazyan [11].

In this work, we propose an inclusion function form having the super-convergence property for multidimensional functions. The proposed inclusion function form uses Bernstein polynomials for bounding the range of the polynomial obtained from the Taylor form of the function f . The Bernstein algorithm is combined with the Taylor form to obtain the resulting so-called Taylor-Bernstein form as an inclusion function form of f . The proposed Taylor-Bernstein form has some important differences (in the practical way it is constructed) from the Taylor-Bernstein form of Lin and Rokne [12].

The rest of this paper is organized as follows. In section 2, we give the essentials of the Bernstein form, Taylor form, and the Taylor-Bernstein form of Lin and Rokne [12]. In section 3, we propose a new Taylor-Bernstein form that is more effective in practice. In section 4, we test and compare the super-convergence behavior of the proposed form with that of the Taylor-Bernstein form of Lin and Rokne. We also investigate the performance of the Taylor model of Berz *et al.* as an inclusion function form. For the testing, we consider six benchmark

examples with dimensions varying from 1 to 6, and examine super-convergence for orders up to 9. In section 5, we draw the conclusions of the work.

2. BERNSTEIN, TAYLOR, AND TAYLOR-BERNSTEIN FORMS

2.1. The Bernstein Form. The Bernstein algorithm has established itself as an important tool for finding bounds on the range of multivariate polynomials, see, for instance, [8], [23] and the references cited therein. The salient features of the Bernstein polynomial approach are:

1. The computation of the bounds conveys the information about the sharpness of these bounds.
2. The approach avoids functional evaluations which might be costly if the degree of the polynomial is high.
3. When bisecting a box and applying the Bernstein form to one of the two subboxes to get an enclosure for the range over this subbox, we obtain without any extra cost an enclosure for the range over the other subbox.
4. For sufficiently small boxes the Bernstein form gives the exact range.

2.1.1. Notation and definitions. In this sub-section, we follow the presentation in [8]. Let l be the number of variables and $\mathbf{x} = (x_1, \dots, x_l) \in \mathbb{R}^l$. A multi-index I is an ordered l -tuple of non-negative integers $I = (i_1, \dots, i_l)$. For two given multi-indices I, N we write $I \leq N$ if $0 \leq i_k \leq n_k$, $k = 1, \dots, l$. With $I = (i_1, \dots, i_{r-1}, i_r, \dots, i_{r+1}, \dots, i_l)$ we associate index $I_{r,k}$ given by $I_{r,k} = (i_1, \dots, i_{r-1}, i_r + k, \dots, i_{r+1}, \dots, i_l)$ where $0 \leq i_r + k \leq n_r$. Also, we write $\binom{N}{I}$ for $\binom{n_1}{i_1} \dots \binom{n_l}{i_l}$.

We can expand a given multivariate polynomial into Bernstein polynomials to obtain bounds for its range over an l -dimensional box \mathbf{X} . Without loss of generality, consider the unit box $\mathbf{U} = [0, 1]^l$ since any nonempty box \mathbf{X} of \mathbb{R}^l can be mapped affinely onto this box.

Let $p(\mathbf{x})$ be a multivariate polynomial in l variables with real coefficients. Denote by $N = (n_1, \dots, n_l)$ the tuple of maximum degrees so that n_k is the maximum degree of x_k in $p(\mathbf{x})$ for $k = 1, \dots, l$. Denote by $S = \{I : I \leq N\}$ the set containing all the tuples from \mathbb{R}^l which are ‘smaller than or equal’ to the tuple N of maximum degrees. Then, we can write an arbitrary l -variate polynomial p in the form

$$(1) \quad p(\mathbf{x}) = \sum_{I \in S} a_I \mathbf{x}^I, \quad \mathbf{x} \in \mathbb{R}^l$$

where for $\mathbf{x} = (x_1, \dots, x_l) \in \mathbb{R}^l$ we set $\mathbf{x}^I = x_1^{i_1} x_2^{i_2} \dots x_l^{i_l}$, where $a_I \in \mathbb{R}$ represents the corresponding coefficient to each $\mathbf{x}^I \in \mathbb{R}^l$. We refer to N as the degree of p . The I^{th} Bernstein polynomial of degree N is defined as

$$B_I^N(\mathbf{x}) = B_{i_1}^{n_1}(x_1) \dots B_{i_l}^{n_l}(x_l) \quad \mathbf{x} \in \mathbb{R}^l$$

where, for $i_j = 0, \dots, n_j, j = 1, \dots, l$

$$B_{i_j}^{n_j}(x_j) = \binom{n_j}{i_j} x_j^{i_j} (1 - x_j)^{n_j - i_j}$$

The Bernstein coefficients $b_I(\mathbf{U})$ of p over the unit box \mathbf{U} are given by

$$b_I(\mathbf{U}) = \sum_{J \leq I} \frac{\binom{J}{I}}{\binom{N}{I}} a_J, \quad I \in S$$

Thus, the Bernstein form of a multivariate polynomial p is defined by

$$p(\mathbf{x}) = \sum_{I \in S} b_I(\mathbf{U}) B_I^N(\mathbf{x})$$

The Bernstein coefficients are collected in an array $B(\mathbf{U}) = (b_I(\mathbf{U}))_{I \in S}$, called as a *patch*. Based on the above, we can have an algorithm for finding a patch of Bernstein coefficients.

Algorithm Patch : $B(\mathbf{U}) = \text{Patch}(\mathbf{X}, a_I)$

Inputs: A box \mathbf{X} , a polynomial p as in (1) of degree N in l -variables with coefficients a_I .

Outputs: A patch $B(\mathbf{U})$ of Bernstein coefficients of p on \mathbf{U} .

BEGIN Algorithm

1. Transform the polynomial p (with coefficients a_I) on \mathbf{X} to a polynomial on \mathbf{U} . Denote the coefficients of the latter as a'_I .
2. Find the Bernstein coefficients of p on \mathbf{U} as

$$b_I(\mathbf{U}) = \sum_{J \leq I} \frac{\binom{I}{J}}{\binom{N}{J}} a'_J, \quad I \in S$$

3. Return the patch $B(\mathbf{U}) = (b_I(\mathbf{U}))_{I \in S}$.

END Algorithm

The following result describes the range enclosure property of the Bernstein coefficients.

Lemma 2.1. [4] : *Let p be a polynomial of degree N . Then, the following property holds for a patch $B(\mathbf{U})$ of Bernstein coefficients :*

$$\bar{p}(\mathbf{X}) \subseteq [\min B(\mathbf{U}), \max B(\mathbf{U})]$$

We can find an enclosure of the range of the multivariate polynomial p on \mathbf{X} by transforming the polynomial into Bernstein form. Then, by Lemma 2.1, the coefficients of the expansion in the Bernstein form provide lower and upper bounds for the range.

2.1.2. Bernstein subdivision process. The obtained range enclosure can be further improved either by degree elevation of the Bernstein polynomial or by subdivision. The subdivision strategy is generally more efficient than the degree elevation strategy [6] and is therefore preferred.

Let \mathbf{D} be any subbox of \mathbf{U} generated by bisection, and suppose the patch $B(\mathbf{D})$ has been already computed. Further suppose \mathbf{D} is bisected along the r -th component direction ($1 \leq r \leq l$) to produce two further subboxes \mathbf{D}_A and \mathbf{D}_B given by

$$\begin{aligned} \mathbf{D}_A &= [\underline{d}_1, \bar{d}_1] \times \dots \times [\underline{d}_r, m(d_r)] \times \dots \times [\underline{d}_l, \bar{d}_l] \\ \mathbf{D}_B &= [\underline{d}_1, \bar{d}_1] \times \dots \times [m(d_r), \bar{d}_r] \times \dots \times [\underline{d}_l, \bar{d}_l] \end{aligned}$$

Then, the patches $B(\mathbf{D}_A)$ and $B(\mathbf{D}_B)$ can be obtained from $B(\mathbf{D})$ by executing the following algorithm.

Algorithm Subdivision : $[B(\mathbf{D}_A), B(\mathbf{D}_B), \mathbf{D}_A, \mathbf{D}_B] = \text{SD}(\mathbf{D}, B(\mathbf{D}), r)$

Inputs: The box $\mathbf{D} \subseteq \mathbf{U}$, its patch $B(\mathbf{D})$, and a component direction r ($1 \leq r \leq l$) in which \mathbf{D} is to be bisected.

Outputs: Subboxes \mathbf{D}_A and \mathbf{D}_B , with respective patches $B(\mathbf{D}_A)$ and $B(\mathbf{D}_B)$

BEGIN Algorithm

1. Bisect \mathbf{D} along the r -th component direction to produce the two subboxes \mathbf{D}_A and \mathbf{D}_B .
2. Compute patch $B(\mathbf{D}_A)$ as follows.
 - (a) Set : $B^{(0)}(\mathbf{D}) \leftarrow B(\mathbf{D})$
 - (b) FOR $k = 1, \dots, n_r$ DO

$$b_I^{(k)}(\mathbf{D}) = \begin{cases} b_I^{(k-1)}(\mathbf{D}) & : i_r < k \\ \frac{1}{2} \{ b_{I_{r,-1}}^{(k-1)}(\mathbf{D}) + b_I^{(k-1)}(\mathbf{D}) \} & : i_r \geq k \end{cases}$$

To obtain the new coefficients, we apply formula given above for $i_j = 0, \dots, n_j, j = 1, \dots, r-1, r+1, \dots, l$.

- (c) Set : $B(\mathbf{D}_A) \leftarrow B^{(n_r)}(\mathbf{D})$
3. Find patch $B(\mathbf{D}_B)$ from intermediate values in above step, as follows
 - (a) FOR $k = 0$ to n_r DO

$$b_{i_1, \dots, n_r-k, \dots, i_l}(\mathbf{D}_B) = b_{i_1, \dots, n_r, \dots, i_l}^{(k)}(\mathbf{D})$$

- (b) Set : $B(\mathbf{D}_B) \leftarrow (b_I(\mathbf{D}_B))_{I \in S}$
4. RETURN $\mathbf{D}_A, \mathbf{D}_B, B(\mathbf{D}_A)$ and $B(\mathbf{D}_B)$

END Algorithm

The following result gives a condition called as *vertex condition*, which can be used to verify if the range enclosure given by the Bernstein coefficients is exact.

Lemma 2.2. [4] : Let p be a polynomial of degree N . Let $B(\mathbf{U})$ be a patch on \mathbf{U} . Then,

$$\{\bar{p}(\mathbf{U}) = [\min B(\mathbf{U}), \max B(\mathbf{U})]\} \Leftrightarrow \{\min B(\mathbf{U}) \text{ resp. } \max B(\mathbf{U}) \text{ occurs at some } I \in S_0\}$$

where, S_0 is a special subset of the index set S defined by

$$S_0 = \{0, n_1\} \times \dots \times \{0, n_l\}$$

The above vertex condition also holds for any subbox $\mathbf{D} \subseteq \mathbf{U}$, see [15]. Combining the tool of Bernstein subdivision and the vertex condition, we can repeatedly improve the bounds till they are exact, i.e., till the vertex condition is satisfied on every subdivision. This leads to the following algorithm for computing exactly the range of p on \mathbf{X} .

Algorithm BernsteinPolynomialBounder : $\bar{p}(\mathbf{X}) = \text{Bounder}(\mathbf{X}, a_I)$

Inputs: A box \mathbf{X} , a polynomial p as in (1) of degree N in l -variables and having coefficients a_I .

Outputs: The exact range $\bar{p}(\mathbf{X})$.

BEGIN Algorithm

1. (Compute patch $B(\mathbf{U})$) Execute Algorithm Patch

$$B(\mathbf{U}) = \text{Patch}(\mathbf{X}, a_I)$$

2. (Initialize lists) Set $\mathcal{L} \leftarrow \{(\mathbf{U}, B(\mathbf{U}))\}$, $\mathcal{L}^{sol} \leftarrow \{\}$.
3. (Select item for processing) If \mathcal{L} is empty, go to step 7. Otherwise, pick the first item from \mathcal{L} , denote it as $(\mathbf{D}, B(\mathbf{D}))$, and delete the item entry from \mathcal{L} .
4. (Check vertex condition on patch) If $(\mathbf{D}, B(\mathbf{D}))$ satisfies the vertex condition in Lemma 2.2, that is, if $\min B(\mathbf{D})$ resp. $\max B(\mathbf{D})$ occurs at some $I \in S_0$, enter the item in list \mathcal{L}^{sol} and return to previous step.
5. (Subdivide and find new patches) Execute Algorithm Subdivision

$$[B(\mathbf{D}_A), B(\mathbf{D}_B), \mathbf{D}_A, \mathbf{D}_B] = \text{SD}(\mathbf{D}, B(\mathbf{D}), r)$$

where, r is chosen to vary cyclically¹ from 1 to l .

6. (Add new entries to list) Enter the new items $(\mathbf{D}_A, B(\mathbf{D}_A))$ and $(\mathbf{D}_B, B(\mathbf{D}_B))$ at end of list \mathcal{L} , and return to step 3.
7. (Compute exact polynomial range) Compute the exact range $\bar{p}(\mathbf{X})$ as the minimum to maximum over all the second entries of the items present in list \mathcal{L}^{sol} .
8. RETURN $\bar{p}(\mathbf{X})$.

END Algorithm

2.2. The Taylor form. In this subsection, we first introduce some further notation as in [20]. Let

$$(2) \quad \lambda = \{\lambda_1, \dots, \lambda_l\}, \quad |\lambda| = \lambda_1 + \dots + \lambda_l, \quad \lambda! = \lambda_1! \dots \lambda_l!, \quad D^\lambda f(x) = \frac{\partial^{\lambda_1 + \dots + \lambda_l} f(x)}{\partial x_1^{\lambda_1} \dots \partial x_l^{\lambda_l}}$$

Let $I(\mathbf{X})$ be the set of all boxes contained in \mathbf{X} . Let the width of an interval \mathbf{X} be defined as $w(\mathbf{X}) = \max \mathbf{X} - \min \mathbf{X}$ if $\mathbf{X} \in I(\mathbb{R})$, and as $w(\mathbf{X}) = \max \{w(\mathbf{X}_1), \dots, w(\mathbf{X}_l)\}$, if $\mathbf{X} \in I(\mathbb{R}^l)$. Let the mean of an interval \mathbf{X} be defined as $m(\mathbf{X}) = (\min \mathbf{X} + \max \mathbf{X})/2$ if $\mathbf{X} \in I(\mathbb{R})$, and as $m(\mathbf{X}) = \{m(\mathbf{X}_1), \dots, m(\mathbf{X}_l)\}$, if $\mathbf{X} \in I(\mathbb{R}^l)$. We call a function $F : I(\mathbf{X}) \rightarrow I(\mathbb{R})$ an inclusion function for f , if $\bar{f}(\mathbf{Y}) \subseteq F(\mathbf{Y})$ for all $\mathbf{Y} \in I(\mathbf{X})$. An inclusion function F for f is said to converge of order α , if $w(F(\mathbf{Y})) - w(\bar{f}(\mathbf{Y})) \leq Lw(\mathbf{Y})^\alpha$ for all $\mathbf{Y} \in I(\mathbf{X})$, where L and α are positive constants.

¹That is, r varies starting from 1 through l , and then again from 1 through l , and so on. Besides cyclical, other strategies for subdivision exist, and their efficiency investigated in [7].

Let $f : \mathbf{X} \rightarrow \mathbb{R}$ be a function that is $m+1$ times differentiable on \mathbf{X} . Then, the Taylor expansion of f of order m is given as

$$(3) \quad f(\mathbf{x}) = \underbrace{f(\mathbf{c}) + \sum_{|\lambda|=1}^m \frac{D^\lambda f(\mathbf{c})}{\lambda!} (\mathbf{x} - \mathbf{c})^\lambda}_{p(\mathbf{x})} + \underbrace{\sum_{|\lambda|=m+1} \frac{f^{(\lambda)}(\boldsymbol{\xi})}{\lambda!} (\mathbf{x} - \mathbf{c})^{m+1}}_{r(\mathbf{x})}, \quad \mathbf{x} \in \mathbf{X}$$

where, $\mathbf{c} = m(\mathbf{X})$ and $\boldsymbol{\xi} \in \mathbf{X}$. We shall call $p(\mathbf{x})$ as the polynomial part and $r(\mathbf{x})$ as the remainder part of the Taylor expansion.

Assume an inclusion function of the $(m+1)$ -th derivative of f exists and is bounded, and furthermore that it has the isotonicity property [20]. Then, the corresponding Taylor form of order m , denoted by F_{Taylor} , can be expressed as [12] :

$$(4) \quad F_{Taylor}(\mathbf{X}) = \bar{p}(\mathbf{X}) + R(\mathbf{X})$$

where $\bar{p}(\mathbf{X})$ is the *exact* range of the polynomial part $p(\mathbf{x})$ on \mathbf{X} , and $R(\mathbf{X})$ is any inclusion for the range of the remainder part $r(\mathbf{x})$ on \mathbf{X} . Lin and Rokne [12] show that the Taylor form has convergence order $m+1$.

Theorem 2.3. [12] *Assume that the Taylor form of order m is as defined above. Then,*

$$(5) \quad \begin{aligned} \bar{f}(\mathbf{X}) &\subseteq F_{Taylor}(\mathbf{X}) \\ w(F_{Taylor}(\mathbf{X})) - w(\bar{f}(\mathbf{X})) &= O(w(\mathbf{X})^{m+1}) \end{aligned}$$

2.3. The Taylor-Bernstein form. The Taylor form provides an enclosure for the range of f over \mathbf{X} with convergence order $m+1$. However, it requires the computation of the exact range of a multivariate polynomial $\bar{p}(\mathbf{X})$. Lin and Rokne [12] proposed an algorithm that uses Bernstein form to find a (generally non-sharp) enclosure of $\bar{p}(\mathbf{X})$, so that the resulting combined form, which we shall call as the *Taylor-Bernstein* form, still possesses the property of $m+1$ convergence order given by (5).

We give below the Lin and Rokne algorithm for finding an enclosure of the range of f on \mathbf{X} . Note that this algorithm uses the Taylor form of order m and Bernstein polynomials of sufficiently high degree N' given by (7) below, and that a generally non-sharp enclosure of the exact range of the polynomial part p of Taylor expansion is computed and used.

Algorithm LR [12] : $F_{LR}(\mathbf{X}) = \text{LinRokne}(\mathbf{X}, f, m)$

Inputs: The box \mathbf{X} , an expression for the function f , and the order m of Taylor form to be used.

Output: An enclosure $F_{LR}(\mathbf{X})$ of the range of f on \mathbf{X} .

1. For the given function f , compute (Taylor) coefficients of p in (3) and also the remainder interval $R(\mathbf{X})$.

This may be done automatically on a computer equipped with interval arithmetic using Moore's recursive technique for Taylor coefficients computation, see [16], [17].

2. Relate the obtained Taylor coefficients to those of the power form in (1), and denote the coefficients in this form as a_I .

3. Compute the l -tuple of indices D given by

$$(6) \quad D = (d_1, \dots, d_l), \text{ where } d_1, \dots, d_l \geq [1/w(\mathbf{X})]^{m+1}$$

and then the l -tuple of indices N' given by

$$(7) \quad N' = (n'_1, \dots, n'_l), \quad \text{where } n'_k = \max \{n_k, d_k\}, \quad k = 1, \dots, l$$

and construct $S' = \{I : I \leq N'\}$.

4. Find a patch $B(\mathbf{U})$ of Bernstein coefficients of p on \mathbf{U} by executing Algorithm Patch : $B(\mathbf{U}) = \text{Patch}(\mathbf{X}, a_I)$ with S' used in place of S in this Algorithm. Then, compute an enclosure for the range of $\bar{p}(\mathbf{X})$ as

$$(8) \quad B^* = [\min B(\mathbf{U}), \max B(\mathbf{U})]$$

5. Compute an enclosure for the range of f over \mathbf{X} as

$$(9) \quad F_{LR}(\mathbf{X}) = B^* + R(\mathbf{X})$$

6. RETURN $F_{LR}(\mathbf{X})$.

END Algorithm

Lin and Rokne [12] showed that the Taylor-Bernstein form computed in the above algorithm retains the property of $m + 1$ convergence order shown by the Taylor form:

Theorem 2.4. [12] *Let $F_{LR}(\mathbf{X})$ be as computed in Algorithm LR. Then,*

$$\begin{aligned}\bar{f}(\mathbf{X}) &\subseteq F_{LR}(\mathbf{X}) \\ w(F_{LR}(\mathbf{X})) - w(\bar{f}(\mathbf{X})) &= O(w(\mathbf{X})^{m+1})\end{aligned}$$

3. PROPOSED TAYLOR-BERNSTEIN FORM

As seen from (6), D becomes large quite quickly as $w(\mathbf{X})$ becomes smaller, leading to high degrees $N' \gg N$ of the Bernstein polynomials in (7). As a consequence, the Bernstein step of Algorithm LR becomes computationally very intensive as the domain intervals shrink in widths.

We therefore propose an algorithm that uses a different Bernstein step based on Bernstein polynomials of degree N (note that N is the minimum degree of Bernstein polynomials we can possibly use) and is equipped with the tools of subdivision and vertex condition checks.

We further propose to use in step 1 of our algorithm, the Taylor model technique of Berz *et al.* [3], [13] for computing the Taylor coefficients in parallel with the remainder interval. Berz *et al.* have shown that the Taylor model technique is more computationally efficient and gives tighter results than a direct implementation of Moore's recursive techniques.

The algorithm proposed below computes an enclosure for the range of f on \mathbf{X} using the Taylor form of order m and Bernstein polynomials of degree N . We emphasize that the *exact* range of polynomial part of Taylor expansion is computed in this algorithm using Bernstein subdivision, and a vertex condition check is done on every subdivision.

Algorithm TB : $F_{TB}(\mathbf{X}) = TB(\mathbf{X}, f, m)$

Inputs: The box \mathbf{X} , an expression for the function f , and the order m of Taylor form to be used.

Output: An enclosure $F_{TB}(\mathbf{X})$ of the range of f on \mathbf{X} .

1. For the given function f , compute Taylor coefficients of p in (3) in parallel with the remainder interval $R(\mathbf{X})$ using the Taylor model technique of Berz *et al.* [3].
2. Relate the obtained Taylor coefficients to those of the power form in (1), and denote the coefficients in this form as a_I .
3. Find the exact range $\bar{p}(\mathbf{X})$ on \mathbf{X} using Algorithm BernsteinPolynomialBoulder :

$$(10) \quad \bar{p}(\mathbf{X}) = \text{Boulder}(\mathbf{X}, a_I)$$

4. Using $R(\mathbf{X})$ obtained in step 1 and $\bar{p}(\mathbf{X})$ obtained in step 3, compute an enclosure for the range of f over \mathbf{X} as

$$(11) \quad F_{TB}(\mathbf{X}) = \bar{p}(\mathbf{X}) + R(\mathbf{X})$$

5. RETURN $F_{TB}(\mathbf{X})$.

END Algorithm

It is trivial to show that the Taylor-Bernstein form computed in the proposed algorithm also has the property of $m + 1$ convergence order :

Theorem 3.1. *Let $F_{TB}(\mathbf{X})$ be as computed in Algorithm TB. Then,*

$$\begin{aligned}\bar{f}(\mathbf{X}) &\subseteq F_{TB}(\mathbf{X}) \\ w(F_{TB}(\mathbf{X})) - w(\bar{f}(\mathbf{X})) &= O(w(\mathbf{X})^{m+1})\end{aligned}$$

Proof. From (4) and (11), F_{TB} is a Taylor form F_{Taylor} . Now apply Theorem 2.3. ■

4. NUMERICAL RESULTS

We numerically investigate the super-convergence property of the above inclusion function forms on some benchmark examples. The selected examples are of low to medium dimensions. For all our computations, we use a PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al.* [2], [10]. We also investigate the performance of the Taylor model as an inclusion function form in these examples.

In each example, we compute the intervals

$F_{TM}(\mathbf{X})$ - using Taylor model of Berz *et al.* [13], computed with the COSY-INFINITY package.

$F_{LR}(\mathbf{X})$ - using Taylor-Bernstein form of Lin and Rokne, computed with Algorithm LR.

$F_{TB}(\mathbf{X})$ - using the proposed Taylor-Bernstein form, computed with Algorithm TB.

$F_{inner}(\mathbf{X})$ - using *inner* estimates of the range computed with the well-known Moore-Skelboe optimization algorithm of interval analysis (see, for instance, [21]).

Let $\mathbf{X} = [a, b]$, $\mathbf{Y} = [c, d]$ be any two intervals. Then, following [5], as a measure of the overestimation we use the Hausdorff metric

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = |[a, b], [c, d]| = \max\{|a - c|, |b - d|\}$$

Consider a sequence of nested intervals $\{\mathbf{X}^{(i)}\}$. We wish to find and compare for each form, the reduction in overestimation with decrease in the domain interval widths. Consider first the form F_{TM} . Let

$$(12) \quad \mathcal{H}_{TM}(\mathbf{X}^{(i-1)}) := \mathcal{H}(\bar{f}(\mathbf{X}^{(i-1)}), F_{TM}(\mathbf{X}^{(i-1)}))$$

As a measure of the reduction in overestimation obtained with form F_{TM} over successive nested intervals $\mathbf{X}^{(i-1)}$ and $\mathbf{X}^{(i)}$, we use the ratio

$$\mathcal{R}_{TM}(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}) := \frac{\mathcal{H}_{TM}(\mathbf{X}^{(i-1)})}{\mathcal{H}_{TM}(\mathbf{X}^{(i)})} = \frac{\mathcal{H}(\bar{f}(\mathbf{X}^{(i-1)}), F_{TM}(\mathbf{X}^{(i-1)}))}{\mathcal{H}(\bar{f}(\mathbf{X}^{(i)}), F_{TM}(\mathbf{X}^{(i)}))}$$

Define

$$\mathcal{R}^*(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}) := \left(\frac{w(\mathbf{X}^{(i-1)})}{w(\mathbf{X}^{(i)})} \right)^{m+1}$$

If F_{TM} is an inclusion function form having convergence order $m + 1$, then

$$(13) \quad \mathcal{R}_{TM}(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)}) \rightarrow \mathcal{R}^*(\mathbf{X}^{(i-1)}, \mathbf{X}^{(i)})$$

(where the tending is from above) for “small” enough $w(\mathbf{X}^{(i-1)})$, $w(\mathbf{X}^{(i)})$.

In practice, the exact range \bar{f} is generally difficult to compute, so the overestimation can be generally found relative only to some *inner* estimate F_{inner} of the range. However, we can easily show that if the $(m + 1)$ -th convergence order property holds relative to F_{inner} , then it implies that the same holds relative to the exact range \bar{f} . That is, it is sufficient if we can show the $(m + 1)$ -th convergence order property with F_{inner} used in place of \bar{f} in above definitions. To avoid introducing more notation, in the sequel we use the quantities given in (12) through (13), with F_{inner} replacing \bar{f} throughout.

Similarly, we can define \mathcal{H}_{LR} , \mathcal{H}_{TB} , \mathcal{R}_{LR} , \mathcal{R}_{TB} for the forms F_{LR} and F_{TB} . For brevity of notation, we drop the arguments $\mathbf{X}^{(i-1)}$, $\mathbf{X}^{(i)}$ of all \mathcal{H} and \mathcal{R} .

Example 4.1. *Gritton's second problem in Chemical Engineering* [11]: The 1 – dim function is

$$\begin{aligned} f(x) = & -371.93625 - 791.2465656 * x + 4044.944143 * x^2 + 978.1375167 * x^3 \\ & -16547.8928 * x^4 + 22140.72827 * x^5 - 9326.549359 * x^6 - 3518.536872 * x^7 \\ & +4782.532296 * x^8 - 1281.47944 * x^9 - 283.4435875 * x^{10} + 202.6270915 * x^{11} \\ & -16.17913459 * x^{12} - 8.88303902 * x^{13} + 1.575580173 * x^{14} + 0.124590848 * x^{15} \\ & -0.03589148622 * x^{16} - 0.0001951095576 * x^{17} + 0.0002274682229 * x^{18} \end{aligned}$$

The domain is $\mathbf{X}^{(i)} = -1 + 2^{-i}[-1, 1]$.

Example 4.2. *Jennrich and Sampson function* [18, problem 6]. The 2 – dim function is

$$f(x) = \sum_{i=1}^{10} f_i(x)^2, \quad f_i(x) = 2 + 2i - (\exp(ix_1) + \exp(ix_2))$$

The domain is $\mathbf{X}^{(i)} = [-1 + 2^{-i}[-1, 1]]^2$.

Example 4.3. *Levy function* [22, Problem L8, pp. 204] The 3– dim function is

$$(14) \quad \begin{aligned} f(x) &= \sum_{i=1}^2 (y_i - 1)^2 (1 + 10 \sin^2(\pi y_{i+1})) + \sin^2(\pi y_1) + (y_3 - 1)^2, \\ y_i &= 1 + \frac{(x_i - 1)}{4} \quad i = 1 \dots 3 \end{aligned}$$

The domain is $\mathbf{X}^{(i)} = [-12 + 2^{-i}[-1, 1]]^3$.

Example 4.4. *Trigonometric function* [18, problem 26]. The 4 – dim function is

$$f(x) = \sum_{i=1}^4 f_i(x)^2, \quad f_i(x) = 4 - \sum_{j=1}^4 \cos x_j + i(1 - \cos x_i) - \sin x_i$$

The domain is $\mathbf{X}^{(i)} = [1.75 + 2^{-i}[-1, 1]]^4$.

Example 4.5. *Griewank function* [22, Problem Griew5, pp. 205] The 5 – dim function is

$$f(x) = \sum_{i=1}^5 \frac{x_i^2}{400} - \prod_{i=1}^5 \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

The domain is $\mathbf{X}^{(i)} = [-600 + 2^{-i}[-1, 1]]^5$.

Example 4.6. *Trigonometric function* [18, problem 26]. The 6 – dim function is

$$f(x) = \sum_{i=1}^6 f_i(x)^2, \quad f_i(x) = 6 - \sum_{j=1}^6 \cos x_j + i(1 - \cos x_i) - \sin x_i$$

The domain is $\mathbf{X}^{(i)} = [1.75 + 2^{-i}[-1, 1]]^6$.

The results for the above examples with the various forms are given² in Tables 1 to 6. The average timings are reported in Table 7. All results shown in the Tables are rounded purely for display purposes.

From the results given in the Tables, we observe that

1. With the Taylor model³ as an inclusion function form, we obtain only *quadratic* convergence in all problems, irrespective of the chosen Taylor order m .
2. With the Lin and Rokne's Taylor-Bernstein form F_{LR} (computed by Algorithm LR) as an inclusion function form, in all cases (except for Taylor orders $m = 2$) we are unable to proceed after just one subdivision, i.e., with $i > 1$, due to the excessive memory requirements arising from high degrees of the associated Bernstein polynomials. For Taylor orders $m = 2$, we are unable to proceed after just two subdivisions, i.e., with $i > 2$, for the same reason. Therefore, as an inclusion form for obtaining super-convergence, the practical utility of F_{LR} is found to be severely restricted.
3. With the proposed Taylor-Bernstein form F_{TB} (computed by Algorithm TB) as an inclusion function form, in problems of up to 4 – dim we quite easily obtain⁴ super-convergence of orders up to 9. In problems of 5 and 6– dim, we do obtain super-convergence of orders up to 9; however, the computational demands are somewhat large for the 5 – dim problem, and become excessive for the 6– dim one.

²In the Tables, a starred entry denotes that the execution is aborted due to excessive memory requirements.

³In version 8.1 of COSY-INFINITY package made available to us, the range of the polynomial part is evaluated by simple interval arithmetic, see also [11].

⁴until we have overestimations of very small magnitudes (of order of $E - 10$ or less).

5. CONCLUSIONS

We proposed a new inclusion form for multidimensional functions. With the proposed form, we could quite easily obtain super-convergence (of orders up to 9) for low to medium dimensional problems. To our knowledge, it is perhaps for the first time that super-convergence of such high orders has actually been demonstrated on multidimensional problems. The new super-convergent form can be constructed on a computer with the fully automated algorithm presented, without any need for hand computations.

For a problem of higher dimensions ($l = 6$), the proposed form was found to be computationally inefficient. This strongly suggests the need for further improvements in the proposed algorithm for dealing with higher dimensional ($l \geq 6$) problems.

Acknowledgments

The authors thank Dr. R. Lohner for strongly suggesting to our group the usage of Bernstein polynomials in range computations. The authors also thank Drs. Berz, Makino, and Hoefkens for providing the COSY-INFINITY software and extending a lot of help regarding the usage of the software. The authors thank Dr. Hoefkens also for his suggestions on improving parts of the paper. The second author would also like to thank Dr. N. D. Jotwani of GCET for motivating, encouraging and providing the required facilities for this work.

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TABLE 1. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 4.1 Gritton ($1 - \dim$).

For Taylor order $m = 2$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$1E + 6$	$1E + 5$	$2E + 4$	$4E + 3$	$9E + 2$	$2E + 2$	$6E + 1$	$1E + 1$
\mathcal{H}_{LR}	$1E + 6$	$8E + 4$	$5E + 3$	*	*	*	*	*
\mathcal{H}_{TB}	$1E + 6$	$8E + 4$	$5E + 3$	$5E + 2$	$5E + 1$	$6E + 0$	$7E - 1$	$1E - 1$
\mathcal{R}^*	—	8	8	8	8	8	8	8
\mathcal{R}_{TM}	—	14.2	5.8	4.5	4.2	4.1	4.0	4.0
\mathcal{R}_{LR}	—	17.9	15.5	—	—	—	—	—
\mathcal{R}_{TB}	—	19.0	14.6	11.2	9.6	8.8	8.4	8.2

For Taylor order $m = 4$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$8E + 5$	$8E + 4$	$1E + 4$	$4E + 3$	$9E + 2$	$2E + 2$	$2E + 1$	$1E + 1$
\mathcal{H}_{LR}	$7E + 5$	$2E + 4$	*	*	*	*	*	*
\mathcal{H}_{TB}	$7E + 5$	$2E + 4$	$6E + 2$	$2E + 1$	$5E - 1$	$1E - 2$	$5E - 4$	$1E - 5$
\mathcal{R}^*	—	32	32	32	32	32	32	32
\mathcal{R}_{TM}	—	10.8	5.0	4.1	4.0	4.0	4.0	4.0
\mathcal{R}_{LR}	—	38.0	—	—	—	—	—	—
\mathcal{R}_{TB}	—	38.0	35.9	33.9	32.9	32.4	32.2	32.1

For Taylor order $m = 6$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$7E + 5$	$7E + 4$	$1E + 4$	$4E + 3$	$9E + 2$	$2E + 2$	$6E + 1$	$1E + 1$
\mathcal{H}_{LR}	$5E + 5$	$3E + 2$	*	*	*	*	*	*
\mathcal{H}_{TB}	$1E + 5$	$3E + 2$	$2E + 0$	$1E - 2$	$8E - 5$	$6E - 7$	$5E - 9$	$2E - 10$
\mathcal{R}^*	—	128	128	128	128	128	128	128
\mathcal{R}_{TM}	—	9.9	5.0	4.1	4.0	4.0	4.0	4.0
\mathcal{R}_{LR}	—	1375.0	—	—	—	—	—	—
\mathcal{R}_{TB}	—	291.0	179.6	158.8	145.3	137.2	128.5	22.0

For Taylor order $m = 8$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$7E + 5$	$7E + 4$	$1E + 4$	$4E + 3$	$9E + 2$	$2E + 2$	$6E + 1$	$1E + 1$
\mathcal{H}_{LR}	$2E + 5$	$3E + 0$	*	*	*	*	*	*
\mathcal{H}_{TB}	$2E + 4$	$3E + 0$	$4E - 3$	$6E - 6$	$1E - 8$	$2E - 10$	$2E - 10$	$2E - 10$
\mathcal{R}^*	—	512	512	512	512	512	512	512
\mathcal{R}_{TM}	—	9.3	4.9	4.1	4.0	4.0	4.0	4.0
\mathcal{R}_{LR}	—	$6E + 4$	—	—	—	—	—	—
\mathcal{R}_{TB}	—	5170.6	805.8	671.7	587.7	518.4	12.1	0.9

TABLE 2. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 4.2 Jennrich and Sampson (2-dim).

For Taylor order $m = 2$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$3E + 5$	$1E + 2$	$4E + 0$	$5E - 1$	$1E - 1$	$3E - 2$	$7E - 3$	$2E - 3$
\mathcal{H}_{LR}	$3E + 5$	$1E + 2$	$2E + 0$	*	*	*	*	*
\mathcal{H}_{TB}	$3E + 5$	$1E + 2$	$2E + 0$	$8E - 2$	$5E - 3$	$3E - 4$	$2E - 5$	$2E - 6$
\mathcal{R}^*	—	8	8	8	8	8	8	8
\mathcal{R}_{TM}	—	$3E + 3$	25.6	6.7	4.5	4.1	4.0	4.0
\mathcal{R}_{LR}	—	$3E + 3$	50.2	—	—	—	—	—
\mathcal{R}_{TB}	—	$3E + 3$	48.3	22.6	17.1	14.6	12.7	11.1
For Taylor order $m = 4$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$4E + 6$	$7E + 1$	$2E + 0$	$5E - 1$	$1E - 1$	$3E - 2$	$7E - 3$	$2E - 3$
\mathcal{H}_{LR}	$4E + 6$	$6E + 1$	*	*	*	*	*	*
\mathcal{H}_{TB}	$4E + 6$	$6E + 1$	$2E - 1$	$2E - 3$	$3E - 5$	$5E - 7$	$9E - 9$	$2E - 10$
\mathcal{R}^*	—	32	32	32	32	32	32	32
\mathcal{R}_{TM}	—	$6E + 4$	29.9	4.7	4.1	4.0	4.0	4.0
\mathcal{R}_{LR}	—	$7E + 4$	—	—	—	—	—	—
\mathcal{R}_{TB}	—	$7E + 4$	251.7	104.4	74.4	61.9	53.7	47.6
For Taylor order $m = 6$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$2E + 7$	$4E + 1$	$2E + 0$	$5E - 1$	$1E - 1$	$3E - 2$	$7E - 3$	$2E - 3$
\mathcal{H}_{LR}	$2E + 7$	$2E + 1$	*	*	*	*	*	*
\mathcal{H}_{TB}	$2E + 7$	$2E + 1$	$2E - 2$	$4E - 5$	$1E - 7$	$5E - 10$	$1E - 12$	$3E - 12$
\mathcal{R}^*	—	128	128	128	128	128	128	128
\mathcal{R}_{TM}	—	$5E + 5$	16.7	4.4	4.1	4.0	4.0	4.0
\mathcal{R}_{LR}	—	$8E + 5$	—	—	—	—	—	—
\mathcal{R}_{TB}	—	$8E + 5$	$1E + 3$	456.8	311.2	256.2	495.6	0.3
For Taylor order $m = 8$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$4E + 7$	$2E + 1$	$2E + 0$	$5E - 1$	$1E - 1$	$3E - 2$	$7E - 3$	$2E - 3$
\mathcal{H}_{LR}	$4E + 7$	$7E + 0$	*	*	*	*	*	*
\mathcal{H}_{TB}	$4E + 7$	$7E + 0$	$1E - 3$	$6E - 7$	$5E - 10$	$2E - 12$	$2E - 12$	$3E - 12$
\mathcal{R}^*	—	512	512	512	512	512	512	512
\mathcal{R}_{TM}	—	$2E + 6$	8.6	4.3	4.1	4.0	4.0	4.0
\mathcal{R}_{LR}	—	$5E + 6$	—	—	—	—	—	—
\mathcal{R}_{TB}	—	$5E + 6$	$5E + 3$	$2E + 3$	$1E + 3$	205.6	1.0	0.7

TABLE 3. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 4.3 Levy ($3 - dim$).

For Taylor order $m = 2$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$2E + 2$	$3E + 1$	$5E + 0$	$1E + 0$	$2E - 1$	$5E - 2$	$1E - 2$	$3E - 3$
\mathcal{H}_{LR}	$2E + 2$	$2E + 1$	$2E + 0$	*	*	*	*	*
\mathcal{H}_{TB}	$2E + 2$	$2E + 1$	$2E + 0$	$3E - 1$	$3E - 2$	$4E - 3$	$5E - 4$	$7E - 5$
\mathcal{R}^*	—	8	8	8	8	8	8	8
\mathcal{R}_{TM}	—	6.7	5.8	5.0	4.6	4.3	4.1	4.0
\mathcal{R}_{LR}	—	9.3	8.7	—	—	—	—	—
\mathcal{R}_{TB}	—	9.3	8.8	8.4	8.2	8.1	8.1	8.0

For Taylor order $m = 4$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$2E + 2$	$3E + 1$	$5E + 0$	$1E + 0$	$2E - 1$	$5E - 2$	$1E - 2$	$3E - 3$
\mathcal{H}_{LR}	$2E + 1$	$1E - 1$	*	*	*	*	*	*
\mathcal{H}_{TB}	$7E + 0$	$1E - 1$	$2E - 3$	$4E - 5$	$1E - 6$	$3E - 8$	$8E - 10$	$2E - 11$
\mathcal{R}^*	—	32	32	32	32	32	32	32
\mathcal{R}_{TM}	—	6.5	5.6	5.0	4.6	4.3	4.1	4.1
\mathcal{R}_{LR}	—	203.0	—	—	—	—	—	—
\mathcal{R}_{TB}	—	72.0	53.2	46.1	40.6	36.8	34.9	51.5

For Taylor order $m = 6$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$2E + 2$	$3E + 1$	$5E + 0$	$1E + 0$	$2E - 1$	$5E - 2$	$1E - 2$	$3E - 3$
\mathcal{H}_{LR}	$1E + 0$	$8E - 3$	*	*	*	*	*	*
\mathcal{H}_{TB}	$1E + 0$	$8E - 3$	$76E - 5$	$5E - 7$	$4E - 9$	$2E - 11$	$9E - 12$	$9E - 12$
\mathcal{R}^*	—	128	128	128	128	128	128	128
\mathcal{R}_{TM}	—	6.4	5.6	5.0	4.6	4.3	4.1	4.0
\mathcal{R}_{LR}	—	143.4	—	—	—	—	—	—
\mathcal{R}_{TB}	—	143.4	135.7	132.0	130.3	185.4	2.26	0.96

For Taylor order $m = 8$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$2E + 2$	$3E + 1$	$5E + 0$	$1E + 0$	$2E - 1$	$5E - 2$	$1E - 2$	$3E - 3$
\mathcal{H}_{LR}	$1E - 2$	$2E - 5$	*	*	*	*	*	*
\mathcal{H}_{TB}	$1E - 2$	$2E - 5$	$3E - 8$	$4E - 11$	$9E - 12$	$10E - 12$	$9E - 12$	$9E - 12$
\mathcal{R}^*	—	512	512	512	512	512	512	512
\mathcal{R}_{TM}	—	6.4	5.6	5.0	4.6	4.3	4.1	4.1
\mathcal{R}_{LR}	—	738.8	—	—	—	—	—	—
\mathcal{R}_{TB}	—	738.9	704.2	643.3	4.6	0.9	1.0	0.9

TABLE 4. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 4.4 Trigonometric ($4 - \dim$).

For Taylor order $m = 2$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$4E + 2$	$9E + 1$	$2E + 1$	$5E + 0$	$1E + 0$	$3E - 1$	$7E - 2$	$2E - 2$
\mathcal{H}_{LR}	$3E + 2$	$3E + 1$	$3E + 0$	*	*	*	*	*
\mathcal{H}_{TB}	$3E + 2$	$3E + 1$	$3E + 0$	$3E - 1$	$3E - 2$	$4E - 3$	$5E - 4$	$7E - 5$
\mathcal{R}^*	—	8	8	8	8	8	8	8
\mathcal{R}_{TM}	—	4.9	4.5	4.2	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	10.5	9.5	—	—	—	—	—
\mathcal{R}_{TB}	—	10.5	9.5	8.8	8.4	8.2	8.1	8.1

For Taylor order $m = 4$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$4E + 2$	$9E + 1$	$2E + 1$	$5E + 0$	$1E + 0$	$3E - 1$	$7E - 2$	$2E - 2$
\mathcal{H}_{LR}	$1E + 1$	$2E - 1$	*	*	*	*	*	*
\mathcal{H}_{TB}	$1E + 1$	$2E - 1$	$5E - 3$	$1E - 4$	$3E - 6$	$8E - 8$	$2E - 9$	$7E - 11$
\mathcal{R}^*	—	32	32	32	32	32	32	32
\mathcal{R}_{TM}	—	4.9	4.4	4.2	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	56.3	—	—	—	—	—	—
\mathcal{R}_{TB}	—	56.3	50.4	44.5	39.8	36.5	34.3	30.6

For Taylor order $m = 6$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$4E + 2$	$9E + 1$	$2E + 1$	$5E + 0$	$1E + 0$	$3E - 1$	$7E - 2$	$2E - 2$
\mathcal{H}_{LR}	$2E + 0$	$9E - 3$	*	*	*	*	*	*
\mathcal{H}_{TB}	$2E + 0$	$9E - 3$	$6E - 5$	$4E - 7$	$3E - 9$	$3E - 11$	$7E - 12$	$7E - 12$
\mathcal{R}^*	—	128	128	128	128	128	128	128
\mathcal{R}_{TM}	—	4.9	4.4	4.2	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	189.0	—	—	—	—	—	—
\mathcal{R}_{TB}	—	189.0	167.6	151.3	140.5	99.2	3.6	1.0

For Taylor order $m = 8$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$4E + 2$	$9E + 1$	$2E + 1$	$5E + 0$	$1E + 0$	$3E - 1$	$7E - 2$	$2E - 2$
\mathcal{H}_{LR}	$5E - 1$	$6E - 5$	*	*	*	*	*	*
\mathcal{H}_{TB}	$5E - 1$	$6E - 5$	$8E - 8$	$1E - 10$	$8E - 12$	$7E - 12$	$7E - 12$	$7E - 12$
\mathcal{R}^*	—	512	512	512	512	512	512	512
\mathcal{R}_{TM}	—	4.9	4.4	4.2	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	828.6	—	—	—	—	—	—
\mathcal{R}_{TB}	—	828.6	734.6	623.2	17.2	1.1	1.0	0.9

TABLE 5. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 4.5 Griewank ($5 - \dim$).

For Taylor order $m = 2$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$9E - 1$	$1E - 1$	$2E - 2$	$5E - 3$	$1E - 3$	$3E - 4$	$7E - 5$	$2E - 5$
\mathcal{H}_{LR}	$9E - 1$	$6E - 2$	$5E - 3$	*	*	*	*	*
\mathcal{H}_{TB}	$9E - 1$	$6E - 2$	$5E - 3$	$6E - 4$	$6E - 5$	$7E - 6$	$9E - 7$	$1E - 7$
\mathcal{R}^*	—	8	8	8	8	8	8	8
\mathcal{R}_{TM}	—	8.2	5.0	4.4	4.2	4.1	4.0	4.0
\mathcal{R}_{LR}	—	14.3	11.2	—	—	—	—	—
\mathcal{R}_{TB}	—	14.3	11.2	9.7	8.9	8.5	8.2	8.1

For Taylor order $m = 4$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$9E - 1$	$1E - 1$	$2E - 2$	$5E - 3$	$1E - 3$	$3E - 4$	$7E - 5$	$2E - 5$
\mathcal{H}_{LR}	$4E - 1$	$9E - 3$	*	*	*	*	*	*
\mathcal{H}_{TB}	$4E - 1$	$9E - 3$	$2E - 4$	$7E - 6$	$2E - 7$	$6E - 9$	$2E - 10$	$3E - 12$
\mathcal{R}^*	—	32	32	32	32	32	32	32
\mathcal{R}_{TM}	—	8.3	5.0	4.4	4.2	4.1	4.0	4.0
\mathcal{R}_{LR}	—	43.0	—	—	—	—	—	—
\mathcal{R}_{TB}	—	43.0	37.8	35.1	33.6	32.9	32.6	67.3

For Taylor order $m = 6$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$9E - 1$	$1E - 1$	$2E - 2$	$5E - 3$	$1E - 3$	$3E - 4$	$7E - 5$	$1E - 5$
\mathcal{H}_{LR}	$6E - 2$	$3E - 4$	*	*	*	*	*	*
\mathcal{H}_{TB}	$6E - 2$	$3E - 4$	$2E - 6$	$1E - 8$	$9E - 11$	$6E - 12$	$1E - 11$	$9E - 12$
\mathcal{R}^*	—	128	128	128	128	128	128	128
\mathcal{R}_{TM}	—	8.1	5.1	4.4	4.2	4.1	4.0	4.0
\mathcal{R}_{LR}	—	196.1	—	—	—	—	—	—
\mathcal{R}_{TB}	—	196.1	166.2	148.6	143.4	14.3	0.53	1.3

For Taylor order $m = 8$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$9E - 1$	$1E - 1$	$2E - 2$	$5E - 3$	$1E - 3$	$3E - 4$	$7E - 5$	$1E - 5$
\mathcal{H}_{LR}	$1E - 2$	$2E - 5$	*	*	*	*	*	*
\mathcal{H}_{TB}	$1E - 2$	$2E - 5$	$4E - 8$	$7E - 11$	$1E - 11$	$7E - 12$	$1E - 11$	$9E - 12$
\mathcal{R}^*	—	512	512	512	512	512	512	512
\mathcal{R}_{TM}	—	8.1	5.1	4.4	4.2	4.1	4.1	4.0
\mathcal{R}_{LR}	—	599.2	—	—	—	—	—	—
\mathcal{R}_{TB}	—	599.2	583.4	557.5	7.2	1.4	0.6	1.3

TABLE 6. Overestimations and their reduction ratios for various Taylor orders obtained with Taylor model, Algorithm LR, and Algorithm TB in Example 4.6 Trigonometric ($6 - \dim$).

For Taylor order $m = 2$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$1E + 3$	$3E + 2$	$6E + 1$	$2E + 1$	$3E + 0$	$1E + 0$	$2E - 1$	$5E - 2$
\mathcal{H}_{LR}	$9E + 2$	$9E + 1$	$9E + 0$	*	*	*	*	*
\mathcal{H}_{TB}	$9E + 2$	$9E + 1$	$9E + 0$	$1E + 0$	$1E - 1$	$2E - 2$	$2E - 3$	$2E - 4$
\mathcal{R}^*	—	8	8	8	8	8	8	8
\mathcal{R}_{TM}	—	5.0	4.5	4.3	4.1	4.0	4.0	4.0
\mathcal{R}_{LR}	—	10.1	10.1	—	—	—	—	—
\mathcal{R}_{TB}	—	10.1	9.3	8.7	8.4	8.2	8.1	8.1

For Taylor order $m = 4$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$1E + 3$	$3E + 2$	$6E + 1$	$1E + 1$	$3E + 0$	$1E + 0$	$2E - 1$	$5E - 2$
\mathcal{H}_{LR}	$4E + 1$	$7E - 1$	*	*	*	*	*	*
\mathcal{H}_{TB}	$4E + 1$	$7E - 1$	$1E - 2$	$2E - 4$	$5E - 6$	$2E - 7$	$3E - 9$	$1E - 10$
\mathcal{R}^*	—	32	32	32	32	32	32	32
\mathcal{R}_{TM}	—	5.0	4.5	4.2	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	61.3	—	—	—	—	—	—
\mathcal{R}_{TB}	—	61.3	56.9	51.6	46.2	41.2	37.1	28.5

For Taylor order $m = 6$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$1E + 3$	$3E + 2$	$6E + 1$	$1E + 1$	$3E + 0$	$1E + 0$	$2E - 1$	$5E - 2$
\mathcal{H}_{LR}	$7E + 0$	$4E - 2$	*	*	*	*	*	*
\mathcal{H}_{TB}	$7E + 0$	$4E - 2$	$2E - 4$	$2E - 6$	$1E - 8$	$1E - 10$	$3E - 11$	$3E - 11$
\mathcal{R}^*	—	128	128	128	128	128	128	128
\mathcal{R}_{TM}	—	5.0	4.5	4.3	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	176.4	—	—	—	—	—	—
\mathcal{R}_{TB}	—	176.4	157.8	144.9	136.8	103.7	4.4	1.0

For Taylor order $m = 8$:								
i	0	1	2	3	4	5	6	7
$w(\mathbf{X}^{(i)})$	$2 * 2^{-0}$	$2 * 2^{-1}$	$2 * 2^{-2}$	$2 * 2^{-3}$	$2 * 2^{-4}$	$2 * 2^{-5}$	$2 * 2^{-6}$	$2 * 2^{-7}$
\mathcal{H}_{TM}	$1E + 3$	$3E + 2$	$6E + 1$	$1E + 0$	$3E + 0$	$1E + 0$	$2E - 1$	$5E - 2$
\mathcal{H}_{LR}	$1E - 1$	$2E - 4$	*	*	*	*	*	*
\mathcal{H}_{TB}	$1E - 1$	$2E - 4$	$2E - 7$	$3E - 10$	$2E - 11$	$3E - 11$	$3E - 11$	$3E - 11$
\mathcal{R}^*	—	512	512	512	512	512	512	512
\mathcal{R}_{TM}	—	5.0	4.5	4.3	4.1	4.1	4.0	4.0
\mathcal{R}_{LR}	—	931.7	—	—	—	—	—	—
\mathcal{R}_{TB}	—	931.7	852.5	700.7	10.7	0.9	0.8	1.2

TABLE 7. Average execution times with various algorithms. The time is in seconds, unless otherwise stated. The average is taken over all i , and all Taylor orders m . Note that with Algorithm LR, mostly only one subdivision ($i = 1$) is found possible in the problems.

Example	Name	dim	Average Execution Time		
			Taylor model	Algorithm LR	Algorithm TB
4.1	Gritton	1	0.01	0.07	0.04
4.2	Jennrich & Sampson	2	0.11	0.01	0.15
4.3	Levy	3	0.12	0.09	0.20
4.4	Trigonometric	4	0.12	0.35	3.60
4.5	Griewank	5	0.20	3	183
4.6	Trigonometric	6	0.23	6	3 hours