EE 325: Probability and Random Processes

Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay
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Assignment 3 — Due Date: March 15, 2013

1. Suppose an encoder maps a 0 bit to a binary codeword $\mathbf{v}_0$ of length n and maps a 1 bit to a binary codeword $\mathbf{v}_1$ of length n. The codewords are passed through a binary symmetric channel with crossover probability p. Suppose $\mathbf{r}$ is the received word corresponding to a single transmitted codeword. If $\mathbf{v}_0$ and $\mathbf{v}_1$ share the same prefix[1] of length $k < n$, show that the optimal decoder can ignore the first k bits in the received word $\mathbf{r}$.

2. Let X and Y be random variables with means μ_x and μ_y respectively. Let σ_x^2 and σ_y^2 be their respective variances. Define the correlation coefficient of X and Y is defined to be
$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}.$$
Show that $\rho_{X,Y}$ always lies between -1 and 1. *Hint:* $\text{cov}(X,Y) = E\left[(X-\mu_x)(Y-\mu_y)\right]$.

3. Let X and Y be independent uniform random variables between 0 and 1. For $Z = X + Y$, find
   (a) $E[Z|X]$
   (b) $E[ZX|X]$
   (c) $E[X|Z]$
   (d) $E[ZX|Z]$

   Hint: $W = X, Z = X + Y$ is a one-to-one transformation of X and Y.

4. If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent, show that $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

5. Suppose we observe the value of a random variable $Y = X + Z$ where Z is a Gaussian random variable with mean 0 and variance σ^2 and X is a discrete random variable which is equally likely to be $\pm A$. The value of A is known. We want to use the maximum likelihood rule to decide the value of X. The rule decides $X = A$ if the conditional density at $Y = y$ given $X = A$ is greater than the conditional density at $Y = y$ given $X = -A$. Otherwise it decides $X = -A$. Show that the decisions made by the rule do not depend on the values of A or σ^2.

6. Let F and G be the distribution functions of random variables X and Y respectively. How can we generate a random variable whose distribution function is $\alpha F + (1-\alpha)G$ for a fixed α such that $0 \leq \alpha \leq 1$?

[1] For instance, codewords 01011 and 01001 share a prefix of length 3