EE 325: Probability and Random Processes

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Endsem : **40 points** (180 min) April 29, 2013

1. [**2 points**] Explain briefly why the probability space definition includes a σ-field $\mathcal{F}$. Why can we not consider $\mathcal{F}$ to be equal to the set of all subsets of the sample space Ω?

2. [**2 points**] You are given a fair coin (heads and tails are equally likely). Using only this coin, you have to generate a discrete random variable which takes the value 0 with probability $\frac{1}{3}$ and the value 1 with probability $\frac{2}{3}$. Propose a scheme to do this and explain why it works. *Hint: Use conditional events.*

3. [**2 points**] Consider an error correcting code which maps a message bit 0 to the codeword 1101111 and a message bit 1 to the codeword 1010111. If these codewords are passed through a binary symmetric channel with crossover probability $p < \frac{1}{2}$, what is the optimal decoder which minimizes the decision error probability? Assume that the message bits 0 and 1 are equally likely to occur.

4. [**2 points**] Let X be a continuous random variable with probability density function $f_X(x)$. What is the probability density function of $Y = aX + b$ where $a \neq 0$?

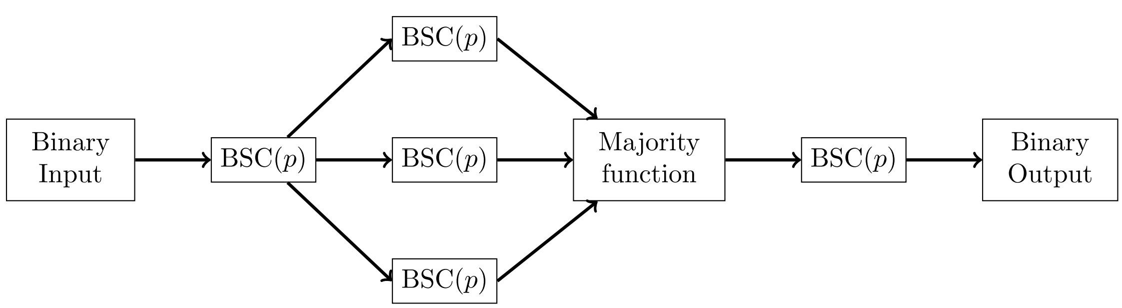
5. [**2 points**] Give an example of uncorrelated Gaussian random variables which are not independent.

6. [**2 points**] Let F and G be the distribution functions of random variables X and Y respectively. Generate a random variable whose distribution function is $\alpha F + (1-\alpha)G$ for a fixed α such that $0 \leq \alpha \leq 1$?

7. [**2 points**] You are given a coin which can be either fair or show heads with probability $\frac{1}{4}$. Devise a scheme which will correctly identify the type of the coin with probability close to one. Explain why your scheme is able to achieve a probability of correct decision close to one.

8. [**2 points**] Give an example of a wide-sense stationary random process which is not strict-sense stationary. Explain your example.

9. [**4 points**] Let BSC(p) denote a binary symmetric channel with crossover probability p. Calculate the crossover probability of the binary symmetric channel which is equivalent to the system below. The majority function block has three binary inputs and one binary output which is equal to the input which is in majority.



Hint: What is the crossover probability of the binary symmetric channel which is equivalent to two binary symmetric channels BSC(p_1) and BSC(p_2) in series?

10. [**4 points**] Suppose we observe a sequence of random variables $Y_1, Y_2, \ldots, Y_n$ given by

$$Y_k = \theta s_k + N_k, \quad k = 1, 2, \ldots, n$$

where the N_k's are independent zero-mean Gaussian random variables with variance σ^2. The sequence $s_1, \ldots, s_n$ is a known signal sequence and θ is an unknown parameter.

(a) Find the maximum likelihood estimate $\hat{\theta}_{ML}(\mathbf{Y})$ of the parameter θ.

(b) Find the mean and variance of $\hat{\theta}_{ML}(\mathbf{Y})$.

11. **[4 points]** Let U be uniformly distributed on $[0, 1)$. For $m = 1, 2, \ldots$ and $j = 1, 2, \ldots, m$, define

$$Y_{m,j} = \begin{cases} 1 & \text{if } U \in [(j-1)/m, j/m) \\ 0 & \text{otherwise.} \end{cases}$$

For any m and a given value of U, exactly one of the $Y_{m,j}$'s is 1 and all the others are 0. Here the sample space is $\Omega = [0, 1)$ and for $\omega \in \Omega$, $U(\omega) = \omega$. The random variables $Y_{m,j}$ map ω to 0 or 1 depending on the value of $U(\omega)$.

Construct the sequence of random variables X_n as

$$\begin{array}{llll} X_1 = Y_{1,1}, & & & \\ X_2 = Y_{2,1}, & X_3 = Y_{2,2}, & & \\ X_4 = Y_{3,1}, & X_5 = Y_{3,2}, & X_6 = Y_{3,3}, & \\ X_7 = Y_{4,1}, & X_8 = Y_{4,2}, & X_9 = Y_{4,3}, & X_{10} = Y_{4,4} \\ \cdots & & & \end{array}$$

(a) Does the sequence X_n converge to zero almost surely? Explain why or why not.

(b) Does the sequence X_n converge to zero in probability? Explain why or why not.

12. **[4 points]** State and prove the weak law of large numbers.

13. **[4 points]** Consider the random process $X(t)$ resulting from sinusoid with random phase.

$$X(t) = A \cos(2\pi f_c t + \Theta)$$

where A and f_c are constants and Θ is uniformly distributed on $[-\pi, \pi]$.

(a) Show that $X(t)$ is wide-sense stationary.

(b) Find the power spectral density of $X(t)$.

14. **[4 points]** Consider the random process $X(t)$ resulting from an amplitude modulated pulse train given by

$$X(t) = \sum_{i=-\infty}^{\infty} A_i p(t - iT)$$

where the A_i's are independent and identically distributed discrete random variables which are equally likely to be ± 1 and $p(t)$ is a unit pulse of duration T

$$p(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

(a) Is the random process $X(t)$ strict-sense stationary? Why or why not?

(b) Is the random process $X(t)$ wide-sense stationary? Why or why not?